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OSCILLATION OF SECOND-ORDER EMDEN-FOWLER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this article, we establish new oscillation criteria for the second-order Emden-Fowler neutral delay differential equation

$$\Big(r(t)|z'(t)|^{\alpha-1}z'(t)\Big)' + q(t)|x(\sigma(t))|^{\beta-1}x\big(\sigma(t)\big) = 0,$$

where $z(t) = x(t) + p(t)x(\tau(t)), \alpha > 0$ and $\beta > 0$. Our results improve some well-known results which were published recently in the literature. Some illustrative examples are also provided to show the significance of our results.

1. Introduction

In this article, we consider the second-order Emden-Fowler neutral delay differential equation

$$\Big(r(t)|z'(t)|^{\alpha-1}z'(t)\Big)' + q(t)|x\big(\sigma(t)\big)|^{\beta-1}x\big(\sigma(t)\big) = 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $t \geq t_0 > 0$, $\alpha > 0$, and $\beta > 0$. Here we use the following assumptions:

- (A1) $r, \sigma \in C^1([t_0,\infty),(0,\infty))$, $r(t) > 0$, $r'(t) \geq 0$ $\sigma(t) \leq t$, $\sigma'(t) > 0$, and $\lim_{t\to\infty}\sigma(t) = \infty$;
- (A2) $p, q, \tau \in C([t_0,\infty),R)$, $0 \leq p(t) < 1$, $q(t) \geq 0$, $\tau(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \infty$.

A function $x(t) \in C^1([T_x,\infty),R)$, $T_x \geq t_0$ is called a solution of (1.1) if it satisfies the property $r(t)|z'(t)|^{\alpha-1}z'(t) \in C^1([T_x,\infty),R)$ and (1.1) on $[T_x,\infty)$. In this article, we only consider the nontrivial solutions of (1.1), which ensure $\sup\{|x(t)| : t \geq T\} > 0$ for the condition $T \geq T_x$. A solution of (1.1) is said to be oscillatory if it has an arbitrarily large zero point on $[T_x,\infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Recently, there have been a large number of papers that devoted to the oscillation of the neutral differential equations. We refer the readers to the articles [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

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Candan [5] studied the oscillation for second-order neutral differential equations with distributed deviating arguments

$$\Big(r(t)|z'(t)|^{\alpha-1}z'(t)\Big)' + \int_c^d f\big(t, x\big(\sigma(t,\xi)\big)\big)d\xi = 0, \tag{1.2}$$

where $z(t) = x(t) + \int_a^b p(t,\xi)x(\tau(t,\xi))d\xi$, $|f(t,u)| \geq q(t,\xi)|u^\alpha|$, and $\alpha > 0$.

In [5] the following results are presented, with the notation $Q(t) = \int_c^d [1 - p(\sigma(t,\xi))]^\alpha p(t,\xi)d\xi$, $\bar{Q}(t) = \int_t^\infty Q(s)ds$, $\bar{R}(t) = \frac{\alpha\sigma_1'(t)}{r^{1/\alpha}(\sigma_1(t))}$, and $\sigma_1(t) = \sigma(t,\alpha)$.

Theorem 1.1 ([5, Theorem 2.1]). *Assume that*

$$\int_{t_0}^\infty \frac{1}{r^{1/\alpha}(t)}\,dt = \infty, \tag{1.3}$$

$$\int_{t_0}^\infty Q(t)\,dt = \infty, \tag{1.4}$$

then (1.2) *is oscillatory.*

Theorem 1.2 ([5, Theorem 2.3]). *Assume that* (1.3) *holds and*

$$\int_{t_0}^\infty Q(t)\,dt < \infty. \tag{1.5}$$

If

$$\liminf_{t\to\infty} \frac{1}{\bar{Q}(t)} \int_t^\infty \bar{Q}^{\frac{\alpha+1}{\alpha}}(s)\bar{R}(s)ds > \frac{\alpha}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}, \tag{1.6}$$

then (1.2) *is oscillatory.*

In 2011, Li et al. [11] studied the oscillatory behavior of the second order Emden-Fowler delay differential equation of the neutral type

$$(r(t)(x(t) + p(t)x(t-\tau))')' + q(t)x^\beta\big(\sigma(t)\big) = 0, \tag{1.7}$$

where $\tau \geq 0, \beta \geq 1$, and $r(t)$ satisfies

$$\int_{t_0}^\infty \frac{1}{r(t)}dt < \infty, \tag{1.8}$$

and they presented the following result.

Theorem 1.3 ([11, Theorem 2.1]). *Suppose* (1.8) *holds. If there exists a function* $\rho \in C^1([t_0,\infty), R)$, $\rho(t) \geq t$, $\rho'(t) > 0$, $\sigma(t) \leq \rho(t) - \tau$ *such that for all sufficiently large* t_1 *and any* $M > 0$ *and* $L > 0$, *it holds*

$$\int^\infty \Big[q(t)(1 - p\big(\sigma(t)\big))^\beta R^\beta\big(\sigma(t)\big) - \frac{\beta M^{1-\beta}\sigma'(t)R^{\beta-1}\big(\sigma(t)\big)}{r\big(\sigma(t)\big)\int_{t_1}^t \frac{\sigma'(s)}{r(\sigma(s))}\,ds}\Big]dt = \infty, \tag{1.9}$$

$$\int^\infty \big[q(t)(\frac{1}{1+p(\rho(t))})^\beta \delta^\beta(t) - \frac{\beta\rho'(t)}{L^{\beta-1}\delta(t)r(\rho(t))}\big]dt = \infty, \tag{1.10}$$

where $R(t) = \int_{t_0}^t r^{-1}(s)ds$ *and* $\delta(t) = \int_{\rho(t)}^t r^{-1}(s)ds$, *then* (1.7) *is oscillatory.*

In 2016, Agarwal et al. [1] considered the oscillation criteria for second order half-linear neutral delay differential equation

$$(r(t)[(x(t) + p(t)x(\tau(t)))']^\alpha)' + q(t)x^\alpha\big(\sigma(t)\big) = 0, t \geq t_0, \tag{1.11}$$

where $\alpha \geq 1$ is a quotient of odd positive integers. A new oscillation criterion is given as follow.

Theorem 1.4 ([1, Theorem 2.2]). *Assume that*

$$\pi(t_0) < \infty, \quad \text{where } \pi(t) = \int_t^\infty r^{-1/\alpha}(s)ds. \tag{1.12}$$

If there exist the functions $\rho, \delta \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \Big[\rho(s)q(s)(1-p(\sigma(s)))^\alpha - \frac{(\rho'_+(s))^{\alpha+1}r(\sigma(s))}{(\alpha+1)^{\alpha+1}(\rho(s)\sigma'(s))^\alpha}]ds = \infty, \tag{1.13}$$

$$\limsup_{t\to\infty} \int_{t_0}^t [\psi(s) - \frac{\delta(s)r(s)(\varphi_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}]ds = \infty, \tag{1.14}$$

where

$$\psi(t) = \delta(t)[q(t)(1-p(\sigma(s))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))})^\alpha + \frac{1-\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)}],$$

$$p(t) < \frac{\pi(t)}{\pi(\tau(t))}, \quad \varphi(t) = \frac{\delta'(t)}{\delta(t)} + \frac{1+\alpha}{r^{1/\alpha}(t)\pi(t)},$$

$\rho'_+(t) = \max\{0,\rho'(t)\}$, *and* $\varphi_+(t) = \max\{0,\varphi(t)\}$, *then* (1.11) *is oscillatory.*

We see that the neutral delay Emden-Fowler equation (1.7) and neutral delay half-linear equation (1.11) are not mutually inclusive each other. However, equations (1.7) and (1.11) are included in the (1.1). Therefor, it will be of great interest to find some oscillation criteria for the neutral differential equation (1.1).

Our aim in this article is to establish some new sufficient conditions for the oscillation of (1.1), by using generalized Riccati inequalities. To the best of our knowledge, very little is known regarding the oscillation criterion of (1.1). The relevance of our theorems becomes clear in the carefully selected examples.

The rest of article is organized as follows. In Section 2, we state and prove our main results. In Section 3, we show several examples.

2. Main Results

The following inequalities contain the variable t, in which we assume that they hold for the sufficiently large t, if there is no other statement.

Theorem 2.1. *Assume that*

$$\int_{t_0}^\infty (\frac{1}{r(t)})^{1/\alpha}dt = \infty, \tag{2.1}$$

$$\int_{t_0}^\infty [1-p(\sigma(t))]^\beta q(t)dt = \infty\,. \tag{2.2}$$

Then (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). We assume without loss of generality that $x(t)$ is eventually positive, that is, there exists a $t_0 \geq 0$ such that $x(t) > 0$ for $t \geq t_0$ and thus there exists a $t_1 \geq t_0$ such that $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. If $x(t)$ is an eventually negative solution, it can be proved by the similar manner. From (1.1), we have

$$\Big(r(t)|z'(t)|^{\alpha-1}z'(t)\Big)' \leq -q(t)x^\beta\big(\sigma(t)\big) \leq 0. \tag{2.3}$$

Hence, $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing. Thus, we have two possible cases for $z'(t)$.

Case I. $z'(t)<0$ for $t\ge t_1$. Using the decreasing property of $r(t)|z'(t)|^{\alpha-1}z'(t)$, we obtain

$$r(t)|z'(t)|^{\alpha-1}z'(t)\le r(t_2)|z'(t_2)|^{\alpha-1}z'(t_2),\quad t\ge t_2\ge t_1. \tag{2.4}$$

Dividing both sides of (2.4) by $r(t)$, integrating from t_2 to t and using (2.1), we have

$$z(t)\le z(t_2)-r^{1/\alpha}(t_2)|z'(t_2)|\int_{t_2}^t r^{-1/\alpha}(s)ds\to-\infty,\text{ as } t\to\infty,$$

which contradicts positivity of $z(t)$.

Case II. $z'(t)>0$ for $t>t_1$. Since $z(t)>x(t)$ and $z(t)$ is increasing, we have

$$z(t)=x(t)+p(t)x(\tau(t))\le x(t)+p(t)z(\tau(t))\le x(t)+p(t)z(t).$$

Thus,

$$(1-p(t))z(t)\le x(t), t\ge t_2^*\ge t_1$$

or

$$[1-p\big(\sigma(t)\big)]^\beta z^\beta\big(\sigma(t)\big)\le x^\beta\big(\sigma(t)\big), t\ge t_3\ge t_2^*. \tag{2.5}$$

Substituting (2.5) into (2.3), we have

$$(r(t)(z'(t))^\alpha)'\le -q(t)[1-p\big(\sigma(t)\big)]^\beta z^\beta\big(\sigma(t)\big). \tag{2.6}$$

On the other hand, since $r(t)(z'(t))^\alpha$ is decreasing, we have

$$r(t)(z'(t))^\alpha\le r(\sigma(t))(z'\big(\sigma(t)\big))^\alpha$$

or

$$\Big(\frac{r(t)}{r\big(\sigma(t)\big)}\Big)^{1/\alpha}\le\frac{z'\big(\sigma(t)\big)}{z'(t)}. \tag{2.7}$$

Set the function

$$w(t):=\frac{r(t)(z'(t))^\alpha}{z^\beta\big(\sigma(t)\big)},\quad t\ge t_3. \tag{2.8}$$

It is obvious that $w(t)>0$. Taking the derivative of $w(t)$, using (2.6), (2.7) and (2.8), we have

$$\begin{aligned} w'(t)&=\frac{(r(t)(z'(t))^\alpha)'}{z^\beta\big(\sigma(t)\big)}-\frac{\beta r(t)(z'(t))^\alpha z'\big(\sigma(t)\big)\sigma'(t)}{z^{\beta+1}\big(\sigma(t)\big)}\\ &\le -q(t)[1-p\big(\sigma(t)\big)]^\beta-\frac{\beta\sigma'(t)(r^{1/\alpha}(t)z'(t))^{\alpha+1}}{r^{1/\alpha}\big(\sigma(t)\big)z^{\beta+1}\big(\sigma(t)\big)}. \end{aligned} \tag{2.9}$$

In view of the positivity of $z(t)$ and $z'(t)$, we obtain

$$w'(t)+q(t)[1-p(\sigma(t))]^\beta\le 0. \tag{2.10}$$

Integrating both sides of (2.10) from t_3 to t and using (2.2), we obtain

$$w(t)\le w(t_3)-\int_{t_3}^t q(s)[1-p(\sigma(s))]^\beta\,ds\to-\infty,\quad\text{as } t\to\infty.$$

which contradicts the fact $w(t)>0$. The proof is complete. □

Note that Theorem 2.1 is an improvement of [8, Theorem 1].

Lemma 2.2. *Assume that $x(t)$ is an eventually positive solution of* (1.1)*, and $w(t)$ is defined by* (2.8)*. Then*

$$w'(t) \le -q(t)(1-p(\sigma(t)))^{\beta} - \frac{\xi K\sigma'(t)}{r^{1/\xi}(\theta(t))} w^{\frac{\xi+1}{\xi}}(t), \tag{2.11}$$

where $\xi = \min\{\alpha,\beta\}$ and

$$K = \begin{cases} 1, & \alpha=\beta \\ \text{const} > 0, & \alpha \ne \beta, \end{cases} \quad \theta(t) = \begin{cases} t, & \alpha > \beta \\ \sigma(t), & \alpha \le \beta. \end{cases}$$

Proof. Proceeding as in the proof of Theorem 2.1, we obtain (2.9); that is

$$\begin{aligned} w'(t) &\le -q(t)[1-p(\sigma(t))]^{\beta} - \frac{\beta\sigma'(t)[r^{1/\alpha}(t)z'(t)]^{\alpha+1}}{r^{1/\alpha}(\sigma(t))z^{\beta}(\sigma(t))} \\ &\le -q(t)[1-p(\sigma(t))]^{\beta} - \frac{\beta\sigma'(t)}{r^{1/\alpha}(\sigma(t))}[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}} w^{\frac{\alpha+1}{\alpha}}(t). \end{aligned} \tag{2.12}$$

If $\beta \ge \alpha$, in view of $z(\sigma(t))$ being increasing, then there exist constants $K_1 > 0$ and $t_4 \ge t_3$ such that $[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}} \ge K_1$ for $t \ge t_4$. Thus, (2.12) gives

$$w'(t) \le -q(t)[1-p(\sigma(t))]^{\beta} - \frac{\alpha K_1\sigma'(t)}{r^{1/\alpha}(\sigma(t))} w^{\frac{\alpha+1}{\alpha}}(t). \tag{2.13}$$

It is easy to check that $K_1 = 1$ for $\alpha = \beta$.

Next, if $\alpha > \beta$, since $(r(t)(z'(t))^{\alpha})' \le 0$ and $r'(t) \ge 0$, we obtain $z''(t) \le 0$, which implies that $z'(t)$ is decreasing and $[z'(t)]^{\frac{\beta-\alpha}{\beta}}$ is increasing. Then there exist constant $K_2 > 0$ and $t_5 \ge t_4$ such that $[z'(t)]^{\frac{\beta-\alpha}{\beta}} \ge K_2$ for $t \ge t_5$. Hence, by (2.12) it has

$$\begin{aligned} w'(t) &\le -q(t)[1-p(\sigma(t))]^{\beta} - \frac{\beta\sigma'(t)}{r^{1/\beta}(t)}[z'(t)]^{\frac{\beta-\alpha}{\beta}} w^{\frac{\beta+1}{\beta}}(t) \\ &\le -q(t)[1-p(\sigma(t))]^{\beta} - \frac{\beta K_2\sigma'(t)}{r^{1/\beta}(t)} w^{\frac{\beta+1}{\beta}}(t), t \ge t_5. \end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14), we have that inequality (2.11) holds for all $\alpha > 0$ and $\beta > 0$. □

We now consider the case when (2.2) does not hold. We use the following notation for simplicity:

$$Q(t) = \int_t^{\infty} q(s)[1-p(\sigma(s))]^{\beta} ds, \quad A(t) = \frac{\xi K\sigma'(t)}{r^{1/\xi}(\theta(t))}. \tag{2.15}$$

Define a sequence of functions $\{y_n(t)\}_{n=0}^{\infty}$ by

$$y_0(t) = Q(t), \quad t \ge t_0$$

and

$$y_n(t) = \int_t^{\infty} A(s) y_{n-1}^{\frac{\xi+1}{\xi}}(s) ds + y_0(t), \quad t \ge t_0, \ n = 1,2,3,\dots. \tag{2.16}$$

By induction we see that $y_0 \le y_{n+1}(t), t \ge t_0, n = 1,2,3,\dots$

Lemma 2.3. *Assume that $x(t)$ is an eventually positive solution of* (1.1)*. Then $y_n(t) \le w(t)$, where $w(t)$ and $y_n(t)$ are defined by* (2.8) *and* (2.16)*, respectively. Also, there exits a positive function $y(t)$ on $[T, \infty)$, such that $\lim_{n\to\infty} y_n(t) = y(t)$ for $t \ge T \ge t_0$ and*

$$y(t) = \int_t^\infty A(s) y^{\frac{\xi+1}{\xi}}(s) ds + y_0(s), \quad t \ge T. \tag{2.17}$$

Proof. Proceeding as in the proof of Lemma 2.2, we have inequality (2.11) or

$$w'(t) \le -q(t)[1 - p(\sigma(t))]^\beta - A(t) w^{\frac{\xi+1}{\xi}}(t). \tag{2.18}$$

Integrating both sides of (2.18) from t to t', we obtain

$$w(t') - w(t) + \int_t^{t'} q(s)[1 - p(\sigma(s))]^\beta ds - \int_t^{t'} w^{\frac{\xi+1}{\xi}}(s) A(s) ds \le 0. \tag{2.19}$$

Then it is clear that

$$w(t') - w(t) + \int_t^{t'} w^{\frac{\xi+1}{\xi}}(s) A(s) ds \le 0. \tag{2.20}$$

It follows that

$$\int_t^\infty w^{\frac{\xi+1}{\xi}}(s) A(s) ds < \infty, \quad t \ge T. \tag{2.21}$$

Otherwise, $w(t') \le w(t) - \int_t^{t'} w^{\frac{\xi+1}{\xi}}(s) A(s) ds \to -\infty$ as $t' \to \infty$, which contradicts to the fact that $w(t) > 0$. Since $w(t)$ is positive and decreasing $\lim_{t\to\infty} w(t) = l \ge 0$. By (2.21), we have $l = 0$. Thus, from (2.19), we have

$$w(t) \ge Q(t) + \int_t^\infty w^{\frac{\xi+1}{\xi}}(s) A(s) ds = y_0(t) + \int_t^\infty w^{\frac{\xi+1}{\xi}}(s) A(s) ds,$$

i.e.

$$w(t) \ge Q(t) = y_0(t). \tag{2.22}$$

Moreover, by induction we can also see that $w(t) \ge y_n(t)$ for $t \ge t_0$, $n = 1, 2, 3 \dots$ Thus, since the sequence $\{y_n(t)\}_{n=0}^\infty$ monotone increasing and bounded above, it converges to $y(t)$. Letting $n \to \infty$ in (2.16) and using Lebesgue's monotone convergence theorem, we obtain (2.17). $\square$

The following theorem provides a new oscillation criterion of (1.1) with respect to that the condition (2.2) of Theorem 2.1 does not hold.

Theorem 2.4. *Assume that* (2.1) *holds and* (2.2) *is not valid. If*

$$\liminf_{t\to\infty} \frac{1}{Q(t)} \int_t^\infty Q^{\frac{\xi+1}{\xi}}(s) A(s) ds > \frac{\xi}{(\xi+1)^{\frac{\xi+1}{\xi}}}, \tag{2.23}$$

where $\xi, Q(t)$ and $A(t)$ are defined by (2.11) *and* (2.15)*, then* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Proceeding as in the proof of Lemma 2.2 and Lemma 2.3, we obtain (2.22) and have

$$\frac{w(t)}{Q(t)} \ge 1 + \frac{1}{Q(t)} \int_t^\infty A(s) Q^{\frac{\xi+1}{\xi}}(s) \Big(\frac{w(s)}{Q(s)}\Big)^{\frac{\xi+1}{\xi}} ds, \quad t \ge T. \tag{2.24}$$

Let $\lambda = \inf_{t\ge T} \frac{w(t)}{Q(t)}$, then obviously $\lambda \ge 1$.

On the other hand, from (2.23) we know that there exists a constant $C > 0$ such that

$$\liminf_{t\to\infty} \frac{1}{Q(t)} \int_t^\infty Q^{\frac{\xi+1}{\xi}}(s)A(s)ds > C > \frac{\xi}{(\xi+1)^{\frac{\xi+1}{\xi}}}. \tag{2.25}$$

Then, from (2.24) and (2.25), we see that

$$\lambda \geq 1 + \lambda^{\frac{\xi+1}{\xi}} C. \tag{2.26}$$

Using the inequality

$$Bu - Au^{\frac{\xi+1}{\xi}} \leq \frac{\xi^\xi}{(\xi+1)^{\xi+1}} \frac{B^{\xi+1}}{A^\xi},$$

where $A > 0$, $B \geq 0$ and $\xi > 0$. We get

$$\lambda - C\lambda^{\frac{\xi+1}{\xi}} \leq \frac{\xi^\xi}{(\xi+1)^{\xi+1}} \frac{1}{C^\xi}. \tag{2.27}$$

Combining (2.25) and (2.27), we see that

$$\lambda < 1 + C\lambda^{\frac{\xi+1}{\xi}},$$

which contradicts with (2.26). The proof is complete. $\square$

Theorem 2.4 improves Theorem 1.2 and the corresponding result in [6]. In the following, we establish new oscillation criteria of (1.1) with respect to that the condition (2.1) of Theorem 2.1 is invalid.

Theorem 2.5. *Assume that* (1.12) *holds. If there exists a function* ρ *in the space* $C^1([T_0,\infty),(0,\infty))$ *such that for all sufficiently large* T *and any* $K > 0$, $M > 0$, *it holds*

$$\limsup_{t\to\infty} \int_T^t \Big[\rho(s)q(s)(1-p(\sigma(s)))^\beta - \frac{(\rho'_+(s))^{\xi+1} r(\theta(s))}{(\xi+1)^{\xi+1}(K\rho(s)\sigma'(s))^\xi}\Big] ds = \infty, \tag{2.28}$$

$$\limsup_{t\to\infty} \int_T^t [\pi^\eta(s)q(s)(1-p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))})^\beta - \frac{\mu}{\pi(s)r^{1/\alpha}(s)}] ds = \infty, \tag{2.29}$$

where $p(t) < \frac{\pi(t)}{\pi(\tau(t))}, \xi = \min\{\alpha,\beta\}$, $\eta = \max\{\alpha,\beta\}$,

$$\theta(t) = \begin{cases} t, & \alpha > \beta, \\ \sigma(t), & \alpha \leq \beta, \end{cases}$$

$\rho'_+(t) = \max\{0,\rho'(t)\}$, *and* $\mu = (\frac{\eta}{\eta+1})^{\eta+1}(\frac{\eta}{M})^\eta$ *(when* $\alpha = \beta$, $K = 1, M = \alpha$*), then* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists a $t_1 \geq t_0 > 0$ such that $x(t) > 0, x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Hence, $z(t) > 0, t \geq t_1$. On the other hand, from (1.1) we see that

$$\Big(r(t)|z'(t)|^{\alpha-1}z'(t)\Big)' \leq 0, \quad t \geq t_1, \tag{2.30}$$

which implies that $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing. Hence, $z'(t)$ does not eventually change signs, that is, there exists a $t_2 \geq t_1$ such that either $z'(t) > 0$ or $z'(t) < 0$ for all $t \geq t_2$.

Case I. $z'(t) > 0$ for $t \geq t_2$. It follows from the definition of $z(t)$ that

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - p(t)z(\tau(t)) \geq (1-p(t))z(t). \tag{2.31}$$

It follows from equations (1.1) and (2.31) that

$$(r(t)(z'(t))^{\alpha})' + q(t)(1-p(\sigma(t)))^{\beta} z^{\beta}(\sigma(t)) \le 0, t \ge t_3 \ge t_2. \tag{2.32}$$

Define a function $u(t)$ by

$$u(t) := \rho(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(\sigma(t))}, \quad t \ge t_3. \tag{2.33}$$

Then, $u(t) > 0, t \ge t_3$. Taking differentiation on both sides of (2.33), we have

$$u'(t) \le -\rho(t)q(t)(1-p(\sigma(t)))^{\beta} + \frac{\rho'(t)}{\rho(t)}u(t) - \frac{\rho(t)r(t)(z'(t))^{\alpha}\beta\sigma'(t)z'(\sigma(t))}{z^{\beta+1}(\sigma(t))}. \tag{2.34}$$

For this inequality, if $\alpha \le \beta$, in view of $r^{1/\alpha}(t)z'(t) \le r^{1/\alpha}(\sigma(t))z'(\sigma(t))$, we see that

$$u'(t) \le -\rho(t)q(t)(1-p(\sigma(t)))^{\beta} + \frac{\rho'(t)}{\rho(t)}u(t) - \frac{\beta\sigma'(t)}{(\rho(t)r(\sigma(t)))^{1/\alpha}}[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}} u^{\frac{\alpha+1}{\alpha}}(t).$$

Because $z(\sigma(t))$ is increasing, there exists constants $K_1 > 0$ and $t_4 \ge t_3$ such that $[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}} \ge K_1,\ t \ge t_4$. Thus, the above inequality gives

$$u'(t) \le -\rho(t)q(t)(1-p(\sigma(t)))^{\beta} + \frac{\rho'(t)}{\rho(t)}u(t) - \frac{\alpha K_1\sigma'(t)}{(\rho(t)r(\sigma(t)))^{1/\alpha}}u^{\frac{\alpha+1}{\alpha}}(t). \tag{2.35}$$

Obviously, if $\alpha = \beta$, then $K_1 = 1$.

If $\alpha > \beta$, since $(r(t)(z'(t))^{\alpha})' \le 0$ and $r'(t) \ge 0$, we obtain $z''(t) \le 0$, which implies that $z'(t)$ is decreasing and $[z'(t)]^{\frac{\beta-\alpha}{\beta}}$ is increasing. Then there exist constants $K_2 > 0$, $t_5 \ge t_4$ such that $[z'(t)]^{\frac{\beta-\alpha}{\beta}} \ge K_2$, $t \ge t_5$. Thus, inequality (2.34) becomes

$$\begin{aligned} u'(t) &\le -\rho(t)q(t)(1-p(\sigma(t)))^{\beta} + \frac{\rho'(t)}{\rho(t)}u(t) \\ &\quad - \frac{\beta\sigma'(t)}{(\rho(t)r(t))^{1/\beta}}[z'(t)]^{\frac{\beta-\alpha}{\beta}} u^{\frac{\beta+1}{\beta}}(t) \\ &\le -\rho(t)q(t)(1-p(\sigma(t)))^{\beta} + \frac{\rho'(t)}{\rho(t)}u(t) \\ &\quad - \frac{\beta K_2\sigma'(t)}{(\rho(t)r(t))^{1/\beta}}u^{\frac{\beta+1}{\beta}}(t), \quad t \ge t_5. \end{aligned} \tag{2.36}$$

Combining (2.35) and (2.36), we obtain for any $\alpha > 0$ and $\beta > 0$ that

$$u'(t) \le -\rho(t)q(t)(1-p(\sigma(t)))^{\beta} + \frac{\rho'(t)}{\rho(t)}u(t) - \frac{\xi K\sigma'(t)}{(\rho(t)r(\theta(t)))^{1/\xi}}u^{\frac{\xi+1}{\xi}}(t), \tag{2.37}$$

for $t \ge t_5$, where $\xi = \min\{\alpha, \beta\}$, and

$$K = \begin{cases} 1, & \alpha = \beta \\ K > 0, & \alpha \ne \beta, \end{cases} \quad \theta(t) = \begin{cases} t, & \alpha > \beta \\ \sigma(t), & \alpha \le \beta. \end{cases}$$

Let $y = u(t), D = \frac{\rho'(t)}{\rho(t)}$, and $C = \frac{\xi K\sigma'(t)}{(\rho(t)r(\theta(t)))^{1/\xi}}$. By (2.37) and the inequality

$$Dy - Cy^{\frac{\xi+1}{\xi}} \le \frac{\xi^{\xi}}{(\xi+1)^{\xi+1}}\frac{D_+^{\xi+1}}{C^{\xi}}, \tag{2.38}$$

where $C > 0, y \geq 0$, and $D_+ = \max\{0, D\}$, we obtain

$$u'(t) \leq -\rho(t)q(t)(1-p(\sigma(t)))^\beta + \frac{(\rho'_+(t))^{\xi+1} r(\theta(t))}{(\xi+1)^{\xi+1}(K\rho(t)\sigma'(t))^\xi}. \tag{2.39}$$

Integrating both sides of (2.39) from $T > t_5$ to t, we obtain

$$u(t) \leq u(T) - \int_T^t \Big[\rho(s)q(s)(1-p(\sigma(s)))^\beta - \frac{(\rho'_+(s))^{\xi+1} r(\theta(s))}{(\xi+1)^{\xi+1}(K\rho(s)\sigma'(s))^\xi}\Big] ds. \tag{2.40}$$

Letting $t \to \infty$ in the above inequality, we obtain a contradiction with (2.28).

Case II. $z'(t) < 0$ for $t > t_2$. By (2.30) we have

$$(r(t)(-z'(t))^\alpha)' \geq 0, \quad t \geq t_2. \tag{2.41}$$

Then, $r^{1/\alpha}(t)(-z'(t))$ is an increasing function and thus

$$z'(s) \leq \Big(\frac{r(t)}{r(s)}\Big)^{1/\alpha} z'(t), \quad s \geq t \geq t_2. \tag{2.42}$$

Integrating the above inequality from t to l, we obtain

$$z(l) \leq z(t) + r^{1/\alpha}(t)z'(t) \int_t^l r^{-1/\alpha}(s) ds, l \geq t \geq t_2.$$

Letting $t \to \infty$, we then have

$$z(t) \geq \pi(t) r^{1/\alpha}(t)(-z'(t)), t \geq t_2. \tag{2.43}$$

It follows that

$$z^\alpha(t) \geq \pi^\alpha(t) r(t)(-z'(t))^\alpha, \quad t \geq T_1 \geq t_2. \tag{2.44}$$

If $\alpha \geq \beta$, then $z^{\alpha-\beta}(t)$ is a decreasing function and thus there exists a constant $l_1 > 0$ such that $z^{\alpha-\beta}(t) \leq l_1$ and $t \geq T_1$.

Define a function $V(t)$ by

$$V(t) := \frac{r(t)(-z'(t))^\alpha}{z^\beta(t)}, \quad t \geq T_1. \tag{2.45}$$

Hence, $V(t) > 0$, $t \geq T_1$ and we have

$$l_1 \geq z^{\alpha-\beta}(t) \geq \pi^\alpha(t) V(t), \quad \alpha \geq \beta. \tag{2.46}$$

On the other hand, from (2.43) it follows that

$$z^\beta(t) \geq \pi^\beta(t)\Big(r^{1/\alpha}(t)(-z'(t))\Big)^{\beta-\alpha+\alpha}. \tag{2.47}$$

Note that $\big(r^{1/\alpha}(t)(-z'(t))\big)^{\beta-\alpha}$ is an increasing function for $\beta > \alpha$. Then there exists a constant $l_2 > 0$ such that

$$l_2 \geq \Big(r^{1/\alpha}(t)(-z'(t))\Big)^{\alpha-\beta} \geq \pi^\beta(t) V(t), \quad \beta > \alpha. \tag{2.48}$$

Combining (2.46) and (2.48), we have

$$0 < \pi^\eta(t) V(t) \leq l, \tag{2.49}$$

where $\eta = \max\{\alpha, \beta\}$ and $l = \max\{l_1, l_2\}$.

We further observe that (2.43) gives $(\frac{z(t)}{\pi(t)})' \geq 0$ for $t \geq t_2$. Then $\frac{z(t)}{\pi(t)}$ is an increasing function and thus

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - p(t)z(\tau(t)) \geq \Big(1 - p(t)\frac{\pi(\tau(t))}{\pi(t)}\Big) z(t).$$

Note that $z'(t) < 0$. Hence we find

$$x^{\beta}(\sigma(t)) \geq \Big(1 - p(\sigma(t))\frac{\pi(\tau(t))}{\pi(\sigma(t))}\Big)^{\beta} z^{\beta}(t). \tag{2.50}$$

Combining (1.1) and (2.50), we obtain

$$(r(t)(-z'(t))^{\alpha})' - Q(t)z^{\beta}(t) \geq 0, t \geq T_1 \geq t_2, \tag{2.51}$$

where

$$Q(t) = q(t)\Big(1 - p(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\Big)^{\beta}. \tag{2.52}$$

Differentiating on both sides of (2.45), using (2.51), we obtain

$$V'(t) \geq Q(t) + \frac{\beta r(t)(-z'(t))^{\alpha+1}}{z^{\beta+1}(t)}, \quad t \geq T_1. \tag{2.53}$$

For this inequality, if $\alpha \geq \beta$, because $[z'(t)]^{\frac{\beta-\alpha}{\alpha}}$ is an increasing function, there exist constants $M_1 > 0$, $T_2 \geq T_1$, such that $[z'(t)]^{\frac{\beta-\alpha}{\alpha}} \geq M_1, t \geq T_2$. From(2.53), we obtain

$$V'(t) \geq Q(t)+\frac{\beta}{r^{1/\alpha}(t)}[z(t)]^{\frac{\beta-\alpha}{\alpha}}V^{\frac{\alpha+1}{\alpha}}(t) \geq Q(t)+\frac{\beta M_1}{r^{1/\alpha}(t)}V^{\frac{\alpha+1}{\alpha}}(t), \quad t \geq T_2. \tag{2.54}$$

Note that if $\alpha = \beta$, then $M_1 = 1$.

Now if $\alpha < \beta$, $[r^{1/\alpha}(t)(-z'(t))]^{\frac{\beta-\alpha}{\beta}}$ is an increasing function and there exist constants $M_2 > 0$ and $T > T_2$, such that $[r^{1/\alpha}(t)(-z'(t))]^{\frac{\beta-\alpha}{\beta}} > M_2, t \geq T$. By (2.53), we have

$$\begin{aligned} V'(t) &\geq Q(t) + \frac{\beta}{r^{1/\alpha}(t)}\big[r^{1/\alpha}(t)(-z'(t))\big]^{\frac{\beta-\alpha}{\beta}}V^{\frac{\beta+1}{\beta}}(t) \\ &\geq Q(t) + \frac{\beta M_2}{r^{1/\alpha}(t)}V^{\frac{\beta+1}{\beta}}(t), \quad t \geq T. \end{aligned} \tag{2.55}$$

Combining (2.54) and (2.55), we obtain

$$V'(t) \geq Q(t) + \frac{M}{r^{1/\alpha}(t)}V^{\frac{\eta+1}{\eta}}(t), \quad t \geq T, \tag{2.56}$$

where $\eta = \max\{\alpha, \beta\}$, and $M = \begin{cases} \alpha, & \alpha = \beta \\ K > 0, & \alpha \neq \beta. \end{cases}$

Multiplying both sides of (2.56) by $\pi^{\eta}(t)$ and integrating from T to t, yields

$$\begin{aligned} \int_T^t \pi^{\eta}(s)Q(s)ds \leq &\int_T^t \pi^{\eta-1}(s)r^{-1/\alpha}(s)[\eta V(s) - M\pi(s)V^{\frac{\eta+1}{\eta}}(s)]ds \\ &+ \pi^{\eta}(t)V(t) - \pi^{\eta}(T)V(T). \end{aligned} \tag{2.57}$$

Let $y = V(s)$, $D = \eta$ and $C = M\pi(s)$. Again by the inequality (2.38), we have

$$\int_T^t \pi^{\eta}(s)Q(s)ds \leq \int_T^t \frac{\mu}{\pi(s)r^{1/\alpha}(s)}ds + \pi^{\eta}(t)V(t) - \pi^{\eta}(T)V(T). \tag{2.58}$$

Combining (2.58), (2.52), and (2.49), we have

$$\int_T^t [\pi^{\eta}(s)q(s)(1 - p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))})^{\beta} - \frac{\mu}{\pi(s)r^{1/\alpha}(s)}]ds \leq l, \tag{2.59}$$

where $\mu = (\frac{\eta}{\eta+1})^{\eta+1}(\frac{\eta}{M})^{\eta}$, which contradicts condition (2.29). The proof is complete. $\square$

Setting $\alpha = \beta$ in (1.1), by Theorem 1.3 we immediately have the following result.

Corollary 2.6. *Suppose that $\alpha = \beta$ and (1.12) holds. If there exists a function ρ in the space $C^1([t_0,\infty),(0,\infty))$ such that for all sufficiently large $T, T \geq t_0$, it holds that*

$$\limsup_{t\to\infty}\int_T^t [\rho(s)q(s)(1-p(\sigma(s)))^{\alpha} - \frac{(\rho'_+(s))^{\alpha+1}r(\sigma(s))}{(\alpha+1)^{\alpha+1}(\rho(s)\sigma'(s))^{\alpha}}]ds = \infty, \tag{2.60}$$

$$\limsup_{t\to\infty}\int_T^t [\pi^{\alpha}(s)q(s)(1-p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))})^{\alpha} - \frac{\varepsilon}{\pi(s)r^{1/\alpha}(s)}]ds = \infty, \tag{2.61}$$

where $p(t) < \frac{\pi(t)}{\pi(\tau(t))}$, $\varepsilon = (\frac{\alpha}{\alpha+1})^{\alpha+1}$ and $\rho'_+(t) = \max\{0,\rho'(t)\}$, then (1.1) is oscillatory.

Corollary 2.6 holds for any $\alpha > 0$ while Theorem 1.4 holds for $\alpha \geq 1$, which is a quotient of odd positive integers. On the other hand, condition (2.61) is more general than condition (1.14) of Theorem 1.4. We shall illustrate this in Example 3.3, given in next section.

Note that in (1.1), if $\alpha = 1$ and $\beta > 1$, then (1.1) is super-linear and Theorem 2.5 has the following corollary.

Corollary 2.7. *Suppose (1.8) holds. If there exists a function ρ in the space $C^1([t_0,\infty),(0,\infty))$, and the constants $K > 0$ and $M > 0$, such that for all sufficiently large $T \geq t_0$, it holds*

$$\limsup_{t\to\infty}\int_T^t \Big[\rho(s)q(s)(1-p(\sigma(s)))^{\beta} - \frac{(\rho'_+(s))^2 r(\sigma(s))}{4(K\rho(s)\sigma'(s))}\Big]ds = \infty, \tag{2.62}$$

$$\limsup_{t\to\infty}\int_T^t [\pi^{\beta}(s)q(s)(1-p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))})^{\beta} - \frac{\mu_1}{\pi(s)r(s)}]ds = \infty, \tag{2.63}$$

where $p(t) < \frac{\pi(t)}{\pi(\tau(t))}$, $\rho'_+(t) = \max\{0,\rho'(t)\}$, $\mu_1 = \big(\frac{\beta}{\beta+1}\big)^{\beta+1}(\frac{\beta}{M})^{\beta}$, and $\pi(t) = \int_t^{\infty}\frac{1}{r(s)}ds$, then (1.7) is oscillatory.

Note that in equation (1.1), if $\alpha = 1$ and $0 < \beta < 1$, then (1.1) is sub-linear and Theorem 2.5 has the following corollary.

Corollary 2.8. *Suppose (1.8) holds. If there exists a function ρ in the space $C^1([t_0,\infty),(0,\infty))$, and the constants $K > 0$ and $M > 0$, such that for all sufficiently large $T \geq t_0$, we have*

$$\limsup_{t\to\infty}\int_T^t \Big[\rho(s)q(s)(1-p(\sigma(s)))^{\beta} - \frac{(\rho'_+(s))^{\beta+1}r(s)}{(\beta+1)^{\beta+1}(K\rho(s)\sigma'(s))^{\beta}}\Big]ds = \infty\,, \tag{2.64}$$

$$\limsup_{t\to\infty}\int_T^t \Big[\pi(s)q(s)(1-p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))})^{\beta} - \frac{\mu_2}{\pi(s)r(s)}\Big]ds = \infty, \tag{2.65}$$

where $\mu_2 = 1/(4M)$, then (1.7) is oscillatory.

3. Examples

In this section, we provide some examples to illustrate our results.

Example 3.1. Consider the neutral delay differential equation

$$\Big(|z'(t)|^{\alpha-1}z'(t)\Big)' + e^{\mu\beta t}(t)|x(\mu t)|^{\beta-1}x(\mu t) = 0, \tag{3.1}$$

where $z(t) = x(t) + (1-e^{-t})x(t-1)$, $\alpha > 0$, $\beta > 0$, and $0 < \mu < 1$.

Comparing (3.1) with (1.1), we see that $r(t) = 1$, $q(t) = e^{\mu\beta t}$, $\sigma(t) = \mu t$, and $p(t) = 1 - e^{-t}$, then $q(t)[1-p\big(\sigma(t)\big)]^{\beta} = 1$. Clearly one can see that conditions of Theorem 2.1 are satisfied. Hence, (3.1) is oscillatory.

Example 3.2. Consider the neutral differential equation

$$\Big(e^{-(\frac{\alpha}{2}+\frac{\alpha}{\xi})t}|z'(t)|^{\alpha-1}z'(t)\Big)' + e^{-t}|x(t-2)|^{\beta-1}x(t-2) = 0, \tag{3.2}$$

where $z(t) = x(t) + \frac{1}{2}x(t-\tau)$, $\tau > 0$, $\alpha > 0$, $\beta > 0$, and $\xi = \min\{\alpha, \beta\}$.

Comparing the (3.2) with (1.1), we see that $r(t) = e^{-(\frac{\alpha}{2}+\frac{\alpha}{\xi})t}, q(t) = e^{-t}$, then

$$\begin{gathered}
Q(t) = \int_t^\infty q(s)[1-p(\sigma(s))]^{\beta}ds = (\frac{1}{2})^{\beta}e^{-t},\\
A(t) = \frac{\xi K\sigma'(t)}{r^{1/\alpha}(\theta(t))} = \xi K e^{(\frac{1}{2}+\frac{1}{\xi})\theta(t)}.
\end{gathered}$$

In view of $\theta(t) = \begin{cases} t, & \alpha > \beta \\ \sigma(t), & \alpha \le \beta, \end{cases}$ we have that for $\alpha > \beta$,

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{Q(t)} \int_t^\infty [Q(s)]^{\frac{\xi+1}{\xi}} A(s)ds\\
&= \liminf_{t\to\infty} 2^{\beta}e^{t} \int_t^\infty \big[(\frac{1}{2})^{\beta}e^{-s}\big]^{\frac{\xi+1}{\xi}} \xi K e^{(\frac{1}{2}+\frac{1}{\xi})s}ds\\
&= \liminf_{t\to\infty} \xi K 2^{-\frac{\beta}{\xi}} e^{t} \int_t^\infty e^{-\frac{s}{2}}ds = \infty.
\end{aligned}$$

If $\alpha \le \beta$ we have

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{Q(t)} \int_t^\infty [Q(s)]^{\frac{\xi+1}{\xi}} A(s)ds\\
&= \liminf_{t\to\infty} 2^{\beta}e^{t} \int_t^\infty \big[(\frac{1}{2})^{\beta}e^{-s}\big]^{\frac{\xi+1}{\xi}} \xi K e^{(\frac{1}{2}+\frac{1}{\xi})(s-2)}ds\\
&= \liminf_{t\to\infty} \xi K 2^{-\frac{\beta}{\xi}} e^{-(\frac{\xi+1}{\xi})} e^{t} \int_t^\infty e^{-\frac{s}{2}}ds = \infty.
\end{aligned}$$

Clearly one can see that all conditions of Theorem 2.4 are satisfied, therefore, (3.2) is oscillatory.

Example 3.3. Consider the half-linear delay differential equation of neutral type

$$\Big(t^6\big[\big(x(t) + \frac{1}{6}x(\frac{t}{3})\big)'\big]^3\Big)' + Kt^2x^3(\frac{t}{2}) = 0, t \ge 1. \tag{3.3}$$

We claim that this equation satisfies the conditions of Corollary 2.6. First, in (3.3), $\alpha = \beta = 3, K > 0$. If we choose $\rho(t) = 1$ then $\rho'(t) = 0$, and we have

$$\rho(t)q(t)(1 - p(\sigma(s)))^\alpha = \frac{125K}{216}t^2,$$

then condition (2.60) is satisfied.

By (1.12), we have $\pi(t) = \frac{1}{t}$, and condition (1.12) holds. Notice that $\tau(t) = \frac{t}{3}, \sigma(t) = \frac{t}{2}$, and thus $\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} = 3$; then $\pi^\alpha(t)q(t)(1 - p(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))})^\alpha = \frac{K}{8t}$, where $\varepsilon = (\frac{3}{4})^4$, $\frac{\varepsilon}{\pi(t)r^{1/\alpha}(t)} = \frac{81}{256t}$.

If we set $K > \frac{81}{32}$, condition (2.61) is satisfied. By Corollary 2.6, equation (3.3) is oscillatory for $K > 81/32$.

Now if we use Theorem 1.4 to work through this example, we need to satisfy condition (1.14). However, (1.11) requires the function $\psi(t) > 0$, but where

$$q(t)(1 - p(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))})^\alpha = \frac{K}{8}t^2, \frac{1-\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} = -2t^2,$$

then $\psi(t) = \delta(t)t^2(\frac{K}{8} - 2)$. Hence, $\psi(t) > 0$ holds for $K > 16$. However Corollary 2.6 only requires $K > \frac{81}{32}$. Consequently, Corollary 2.6 improves Theorem 1.4.

Example 3.4. Consider the neutral-type equation

$$\Big(t^8|z'(t)|^{\alpha-1}z'(t)\Big)' + t^5|x(\frac{t}{3})|^{\beta-1}x(\frac{t}{3}) = 0, t \geq 1, \tag{3.4}$$

where $z(t) = x(t) + \frac{1}{4}x(\frac{t}{2}), \alpha = 4, \beta = 2$.

We now use Theorem 2.5 to show that this equation is oscillatory. Notice that $\pi(t) = \frac{1}{t}$ in (3.4), then (1.12) holds. If we choose $\rho(t) = 1$, then (2.28) is satisfied. To verify condition (2.29), we have

$$\begin{aligned}
&\limsup_{t\to\infty}\int_T^t \Big[\pi^\eta(s)q(s)\Big(1 - p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))}\Big)^\beta - \frac{\mu}{\pi(s)r^{1/\alpha}(s)}\Big]ds \\
&= \limsup_{t\to\infty}\int_T^t \Big[\frac{1}{s^4}s^5(\frac{1}{2})^2 - \frac{(\frac{4}{5})^5(\frac{4}{M})^4}{s}\Big]ds = \infty.
\end{aligned}$$

Then (2.29) holds. Hence, by Theorem 2.5, equation (3.4) is oscillatory.

Note that Theorem 1.4 cannot be applied to the oscillation of (3.4).

Example 3.5. Consider the super-linear Emden-Fowler equation

$$\Big(t^2\Big(x(t) + \frac{1}{8}x(\frac{t}{4})\Big)'\Big)' + t|x(\frac{t}{5})|^\beta \operatorname{sgn} x(\frac{t}{5}) = 0, \quad t \geq 1. \tag{3.5}$$

In this example, $\alpha = 1$, $\beta = \frac{3}{2} > 1$, and $\pi(t) = \frac{1}{t}$; as a result, (1.12) holds. By Letting $\rho(t) = 1$, condition (2.62) is satisfied. On the other hand,

$$\begin{aligned}
&\limsup_{t\to\infty}\int_T^t \Big[\pi^\eta(s)q(s)\Big(1 - p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))}\Big)^\beta - \frac{\mu_1}{\pi(s)r(s)}\Big]ds \\
&= \limsup_{t\to\infty}\int_T^t \Big[\frac{1}{\sqrt{s}}(\frac{1}{2})^{3/2} - (\frac{3}{5})^{5/2}(\frac{3}{2M})^{3/2}\frac{1}{s}\Big]ds = \infty.
\end{aligned}$$

This shows that (2.63) holds. Then by Corollary 2.7, equation (3.5) is oscillatory.

Example 3.6. Consider the sub-linear Emden-Fowler equation

$$\Big(t^2\Big(x(t)+\frac{1}{8}x(\frac{t}{4})\Big)'\Big)'+t|x(\frac{t}{5})|^\beta \operatorname{sgn} x(\frac{t}{5})=0, \quad t\geq 1, \tag{3.6}$$

where $0<\beta=1/2<1$.

In this example, it is easy to find that (1.12) and (2.64) are satisfied. We also see that

$$\begin{aligned}
&\limsup_{t\to\infty}\int_T^t \Big[\pi(s)q(s)\Big(1-p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))}\Big)^\beta-\frac{\mu_2}{\pi(s)r(s)}\Big]ds\\
&=\limsup_{t\to\infty}\int_T^t \Big[(\frac{1}{2})^{1/2}-\frac{1}{4M}\frac{1}{s}\Big]ds=\infty,
\end{aligned}$$

which shows that (2.65) is satisfied. By Corollary 2.8, we can say that (3.6) is oscillatory.

However, Theorem 1.3 cannot be applied to this example because it requires $\beta\geq 1$.

Example 3.7. Consider the linear neutral-type equation

$$\Big(t^2\Big(x(t)+px(\frac{t}{m})\Big)'\Big)'+qx(t)=0, \tag{3.7}$$

where $m>1$, $0\leq p<\frac{1}{m}$, $q>0$, and $\alpha=\beta=1$.

Observe that (1.12) and (2.28) of Theorem 2.5 are satisfied. Because $\pi(t)=\frac{1}{t}, r(t)=t^2, \mu=\frac{1}{4}$, by condition (2.29) we have

$$\begin{aligned}
&\limsup_{t\to\infty}\int_T^t \Big[\pi(s)q(s)\Big(1-p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))}\Big)^\beta-\frac{\mu}{\pi(s)r(s)}\Big]ds\\
&=\limsup_{t\to\infty}\int_T^t \big[\frac{q}{s}(1-mp)-\frac{1}{4s}\big]ds.
\end{aligned}$$

Hence, if $q(1-mp)>1/4$, then condition (2.29) holds. According to Theorem 2.5, the neutral-type equation (3.7) is oscillatory. If we set $p=0$ in (3.7), then the second-order Euler equation $(t^2x'(t))'+qx(t)=0$ is oscillatory as $q>1/4$.

We remark that Theorem 2.5 can be applied to the linear equation (3.7), the half-linear equation (3.3), the super-linear equation (3.5), and the sub-linear equation (3.6). This gives four types of equations with uniform oscillation criterion and improves the results in the literature such as [1, 2, 11, 14].

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