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STABILIZATION OF WAVE EQUATIONS WITH VARIABLE COEFFICIENTS AND INTERNAL MEMORY

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Abstract. In this article, we consider the stabilization of a wave equation with variable coefficients and internal memory in an open bounded domain, by the Riemannian geometry approach. For the wave equation with a locally distributed memory with a kernel, we obtain exponential decay of the energy under some geometric conditions. In addition, for the wave equation with nonlinear internal time-varying delay without upper bound, we obtain uniform decay of the energy.

1. Introduction and statement of main results

Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. Define

$$\mathscr{A}u = -\operatorname{div} A(x)\nabla u \quad \text{for } u \in H^1(\Omega), \tag{1.1}$$

where $A(x) = (a_{ij}(x))_{n\times n}$ is a symmetric, positively definite matric for each $x \in \mathbb{R}^n$ and $a_{ij}(x)$ is a smooth function on $\mathbb{R}^n$ for each $1 \le i, j \le n$.

We consider the stabilization of the problem

$$\begin{gathered} u_{tt} + \mathscr{A}u + a(x)[\mu_1 u_t(x,t) + \mu_2 \int_0^\infty k(\rho)u_t(x,t-\rho)d\rho] = 0 \\ (x,t) \in \Omega \times (0,+\infty), \\ u(x,t)\big|_\Gamma = 0 \quad t \in (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega, \\ u_t(x,t) = f_0(x,t) \quad (x,t) \in \Omega \times (-\infty, 0), \end{gathered} \tag{1.2}$$

where $a(x) \in C^1(\overline{\Omega})$ is a nonnegative function and the kernel $k(\cdot)$ satisfies

$$\int_0^\infty |k(\rho)|d\rho = 1. \tag{1.3}$$

Moreover, $\mu_1, \mu_2 > 0$, and the initial data (u_0, u_1, f_0) belongs to a suitable space.

Stability results for system (1.2) in the case of $\mu_2 = 0$; that is, without memory, were obtained by some authors. See [6, 12, 26].

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Time delays often arise in many physical, chemical, biological and economical phenomena. In recent years, different equations with time delay effects have become an active area of research. In particular, as is well-known that an arbitrarily small delay may be the source of instability and some dissipative mechanism need to be introduced to against the instabilities, the control and stabilization of the wave equations with time delay have been extensively studied by several authors (see for example, [1, 2, 10, 13, 18, 19, 20, 21, 22, 23, 24, 25] and many others.) To be specific, with a internal or boundary constant delay term, the stability and instability results of the constant coefficient wave equation are given by [1, 18, 25]. The results in [18] have been extended to the variable coefficient wave equation in [22, 24]. Besides, with a time-varying delay term in the boundary or interior, the uniform decay results of the energy of the constant coefficient wave equation are obtained by [2, 10, 13, 19, 20, 21, 23].

The following system was studied in [17].

$$\begin{gathered} u_{tt} + \mathscr{A} u = 0 \quad (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0 \quad (x,t) \in \Gamma_2 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu_{\mathscr{A}}} + b u_t(x,t) + \int_0^t k(t-\rho) u_t(x,\rho) d\rho = 0 \quad (x,t) \in \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega, \\ u_t(x,t) = f_0(x,t) \quad (x,t) \in \Gamma_1 \times (-\infty, 0) \end{gathered} \tag{1.4}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$ and $\Gamma_2 \neq \emptyset$. b is a positive constant and $\frac{\partial u}{\partial \nu_{\mathscr{A}}} = \langle A(x)u, \nu \rangle$ is the co-normal derivative, $\langle \cdot, \cdot \rangle$ denotes the standard metric of the Euclidean space $\mathbb{R}^n$ and $\nu(x)$ is the outside unit normal vector for each $x \in \Gamma$.

The exponential decay of the energy of system (1.4) is obtained under the following assumptions: the kernel $k(\cdot)$ satisfies

$$k(t) \geq 0, \quad k'(t) \leq -\gamma_0 k(t), \quad k''(t) \geq -\gamma_1 k'(t), \tag{1.5}$$

where γ_0, γ_1 are positive constants, and there exists a vector field H on $\overline{\Omega}$ and a constant $\rho_0 > 0$ such that

$$D_g H(X,X) \geq \rho_0 |X|_g^2 \quad \text{for} \quad X \in \mathbb{R}^n_x \quad x \in \overline{\Omega}, \tag{1.6}$$

$$\sup_{x\in\overline{\Omega}} \operatorname{div} H < \inf_{x\in\overline{\Omega}} \operatorname{div} H + 2\rho_0, \tag{1.7}$$

$$H \cdot \nu \leq 0 \quad x \in \Gamma_2 \quad \text{and} \quad H \cdot \nu \geq \delta \quad x \in \Gamma_1, \tag{1.8}$$

where δ is a positive constant.

Note that the initial memory of u_t is zero and $\int_0^t k(t-\rho) u_t(x,\rho) d\rho = \int_0^{-t} k(\rho)$ $u_t(x, t+\rho) d\rho$ in (1.4). Our objective in this paper is to study the exponential decay of the energy of system (1.2) with a nonzero initial memory of u_t, a more general kernel $k(\cdot)$ and vector field H than (1.4).

To obtain our stabilization result, we assume that

$$\mu_2 < \mu_1. \tag{1.9}$$

Let

$$G(h) = \int_h^{+\infty} |k(\rho)| d\rho \quad for \quad h \geq 0. \tag{1.10}$$

Define the energy of system (1.2) by

$$\begin{aligned} E(t) = \frac{1}{2}\int_\Omega \Big(u_t^2 + \sum_{i,j=1}^n a_{ij}u_{x_i}u_{x_j}\Big)dx \\ + \xi \int_0^\infty \int_\Omega a(x)G(\rho)u_t^2(x,t-\rho)\,dx\,d\rho, \end{aligned} \tag{1.11}$$

where ξ is a positive constant satisfying

$$\mu_2 < 2\xi < 2\mu_1 - \mu_2. \tag{1.12}$$

As in [29, 15], we define

$$g = A^{-1}(x) \quad \text{for } x \in \mathbb{R}^n \tag{1.13}$$

as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold. For each $x \in \mathbb{R}^n$, the metric g introduces an inner product and the norm on the tangent space on $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$\langle X, Y\rangle_g = \langle A^{-1}(x)X, Y\rangle, \quad |X|_g^2 = \langle X, X\rangle_g \quad X,\, Y \in \mathbb{R}^n_x,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard dot metric. Let $f \in C^1(\mathbb{R}^n)$, we define the gradient $\nabla_g f$ of f in the Riemannian metric g by

$$X(f) = \langle \nabla_g f, X\rangle_g, \tag{1.14}$$

where X is any vector field on $(\mathbb{R}^n, g)$.

We denote the Levi-Civita connection of the metric g by D_g. For the system (1.2), our main assumption is as follows:

(A1) There exist $\varepsilon, \rho_0 > 0, \Omega_i \subset \Omega$ with smooth boundary $\partial\Omega_i$ and C^2 vector fields H^i on $\overline{\Omega_i}, i = 1, 2, \dots, m$ such that $\Omega_i \cap \Omega_j = \emptyset, 0 \le i < j \le m$ and

$$D_g H^i(X, X) \ge \rho_0 |X|_g^2 \quad \text{for} \quad X \in \mathbb{R}^n_x \quad x \in \overline{\Omega_i}, \tag{1.15}$$

$$a(x) \ge a_0, \quad \text{for } x \in \overline{V_1} \cap \overline{\Omega}, \tag{1.16}$$

where m is a positive integer and

$$V_1 = \overline{\Omega} \cap \aleph_\varepsilon(\cup_{i=1}^m \Gamma_0^i \cup (\Omega \backslash \cup_{i=1}^m \Omega_i)), \tag{1.17}$$

where

$$\begin{gathered} \aleph_\varepsilon(S) = \cup_{x\in S}\{y \in \mathbb{R}^n \,|\, |y - x| < \varepsilon\}, \quad S \subset \mathbb{R}^n, \\ \Gamma_0^i = \{x \in \partial\Omega_i \,|\, H^i(x)\cdot\nu^i(x) > \varepsilon_0\}, \end{gathered} \tag{1.18}$$

with $\nu^i(x)$ the unit normal of $\partial\Omega_i$ in the Euclidean space $\mathbb{R}^n$, pointing towards the exterior of Ω_i, and ε_0 is a nonnegative constant satisfying

$$\Gamma \supset \cup_{i=1}^m \{x \in \partial\Omega_i \,|\, 0 < H^i(x)\cdot\nu^i(x) \le \varepsilon_0\}. \tag{1.19}$$

For $0 < \varepsilon' < \varepsilon$, we set

$$V_2 = \aleph_{\varepsilon'}(\cup_{i=1}^m \Gamma_0^i \cup (\Omega \backslash \cup_{i=1}^m \Omega_i)). \tag{1.20}$$

Assumption (1.15) is a verifiable condition used in [29] to establish the controllability of the wave equation with variable coefficients. There some examples of the global existence of such vector fields are given by using the Riemann curvature theory. So far, it has been widely used in the study of control and stabilization of many variable-coefficient systems, see for example [4, 7, 9, 11, 16, 27, 28].

If $\varepsilon_0 = 0$ in (1.18), Assumption (A1) is used in [5, 6] to study the locally distributed control and stabilization of the wave equation with variable coefficients. If $\varepsilon_0 \neq 0$, Assumption (A1) is a weaker than the geometric conditions in [5, 6].

If $J = 1$ and $\Omega_1 = \Omega$, then from (1.17), we have

$$V_1\big|_{\varepsilon_0\neq 0} \subset \Gamma_0 \subset V_1\big|_{\varepsilon_0=0}, \tag{1.21}$$

where

$$\Gamma_0 = \{x \in \partial\Omega | H^1(x)\cdot \nu(x) > 0\}. \tag{1.22}$$

Γ_0 is widely used to study the control and stabilization of the wave equations with boundary feedbacks. See [8, 17, 23, 29].

In what follows, we denote by C or C_i any positive constant which may be different from line to line. The following is the stability results of system (1.2).

Theorem 1.1. *Assumption* (A1) *holds, and that ε_0 is sufficiently small and there are positive constants $\lambda > 1$ and $T_0 > 0$ such that*

$$G(\rho) \geq \lambda G(\rho + T_0) \quad \forall \rho \geq 0. \tag{1.23}$$

Then there exist constants $C_1, C_2 > 0$, such that

$$E(t) \leq C_1 e^{-C_2 t} E(0), \quad \forall t > 0. \tag{1.24}$$

Example 1.2. Let $G(h) = \int_h^{+\infty} e^{-\rho} d\rho = e^{-h}, h \geq 0$, where $|k(\rho)| = e^{-\rho}$ satisfies (1.3). Since $G'(h) = -e^{-h} < 0$, condition (1.23) holds naturally.

In this paper, we also consider the stabilization of the problem

$$\begin{gathered} u_{tt} + \mathscr{A}u + a_1 g_1(u_t(x,t)) + a_2 g_2(u_t(x, t-\tau(t))) = 0 \\ (x,t) \in \Omega \times (0, +\infty), \\ u(x,t)\big|_{\Gamma} = 0 \quad t \in (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega, \\ u_t(x,t) = h_0(x,t) \quad (x,t) \in \Omega \times (-\tau(0), 0), \end{gathered} \tag{1.25}$$

where $a_1 > 0$, $a_2 \in \mathbb{R}$ are constants and $\tau(t)$ satisfies

$$\tau(t) \geq 0 \quad \text{and} \quad \tau'(t) \leq d < 1 \quad \forall t \geq 0, \tag{1.26}$$

where d is a constant. And there exist positive constants $c_1, p \geq 1$ such that $g_1, g_2 \in C(\mathbb{R})$ satisfy

$$g_1(0) = 0, \quad s g_1(s) \geq \max\{|s|^2, (g_2(s))^2\} \quad \text{for } s \in \mathbb{R}, \tag{1.27}$$

$$|g_1(s)| \leq c_1 |s| \quad \text{for } |s| > 1. \tag{1.28}$$

In [2, 10, 13, 19, 20, 21, 23], the well-posedness and stabilization of the wave equation with a time-varying delay was studied under the assumption that $\tau(t)$ has a upper bound. While in this paper, we will consider the stabilization of system (1.25) with a more general $\tau(t)$, that is, $\tau(t)$ does not need to have a upper bound (See (1.26)). To obtain our stabilization result, we assume that

$$\frac{|a_2|}{\sqrt{1-d}} < a_1. \tag{1.29}$$

We define the energy of system (1.25) as

$$\begin{aligned} J(t) = &\frac{1}{2}\int_\Omega \Big(u_t^2 + \sum_{i,j=1}^n a_{ij}u_{x_i}u_{x_j}\Big)dx \\ &+ \eta\int_t^{\phi(t)}\int_\Omega u_t(x,\varphi(\rho))g_1(u_t(x,\varphi(\rho)))\,dx\,d\rho, \end{aligned} \tag{1.30}$$

where η is a positive constant satisfying

$$\frac{|a_2|}{2\sqrt{1-d}} < \frac{\eta}{1-d} < a_1 - \frac{|a_2|}{2\sqrt{1-d}}, \tag{1.31}$$

and $\phi(t),\varphi(t)$ satisfy

$$\varphi(t) = t-\tau(t), \quad \forall t\ge 0, \quad \phi(t) = \varphi^{-1}(t), \quad \forall t \ge -\tau(0). \tag{1.32}$$

Since $\varphi'(t) = 1-\tau'(t) \ge 1-d > 0$, $\varphi(t)$ and $\phi(t)$ are strictly increasing functions satisfying

$$\lim_{t\to+\infty}\varphi(t) = +\infty, \quad \lim_{t\to+\infty}\phi(t) = +\infty. \tag{1.33}$$

As in [3, 14], we let $h \in C([0,+\infty))$ be a concave increasing function such that

$$h(0) = 0, \quad s^2 + (g_1(s))^2 \le h(sg_1(s)) \quad \text{for } |s|\le 1. \tag{1.34}$$

We define

$$F(t) = \sup\{\tau(\rho)+1 | 0\le\rho\le t\} \quad \forall t\ge 0. \tag{1.35}$$

The following is the stability results of system (1.25).

Theorem 1.3. *(a) Assume that*

$$|g_1(s)| \le c_2|s| \quad \textit{for } |s|\le 1. \tag{1.36}$$

Then there exist constants $C_1, C_2 > 0$, such that

$$J(t) \le \frac{C_1 J(0)}{t^{C_2}}, \quad \forall t > 0. \tag{1.37}$$

(b) Assume that

$$\lim_{t\to+\infty}\frac{F(t)}{t} = 0. \tag{1.38}$$

Then

$$\lim_{t\to+\infty}\frac{F(\phi(t))}{t} = 0. \tag{1.39}$$

Also there exist constants $C_1, C_2 > 0$, such that

$$J(t) \le C_1 h\Big(\frac{C_2}{\phi(t)}J(0)\Big) + C_1\frac{F(\phi(t))}{t}J(0), \forall t > 0. \tag{1.40}$$

Example 1.4. Let $\tau(t) = \frac{d^2}{t+1}$, then $\tau'(t) = -\frac{d^2}{(t+1)^2}$, which implies relation (1.26). Since

$$\lim_{t\to+\infty}\frac{F(t)}{t} = \lim_{t\to+\infty}\frac{sup\{d^2/(\rho+1)+1|0\le\rho\le t\}}{t} = 0, \tag{1.41}$$

condition (1.38) is satisfied.

This article is organized as follows. Section 2 is devoted to presenting the well-posedness of systems (1.2) and (1.25). The technical details of the proof for Theorems 1.1 and 1.3 are given in Section 3 and Section 4, respectively.

2. Well-posedness

To obtain the well-posedness of system (1.2), we define

$$L_a^2(\Omega\times(-\infty,t))=\Big\{u:\int_0^\infty\int_\Omega a(x)G(\rho)u^2(x,t-\rho)\,dx\,d\rho<+\infty\Big\}. \tag{2.1}$$

$$\begin{aligned}&L_a^2(\Omega,H^1(-\infty,t))\\&=\Big\{u:\int_0^\infty\int_\Omega a(x)G(\rho)(u^2(x,t-\rho)+u_\rho^2(x,t-\rho))\,dx\,d\rho<+\infty\Big\}.\end{aligned} \tag{2.2}$$

Note that system (1.2) is a linear equation and the kernel $k(\cdot)$ defined on $[0,+\infty)$ does not change with time t. Using the methods in [18], by a similar proof, we obtain the following well-posedness result.

Theorem 2.1. *For any initial datum $(u_0,u_1,f_0)\in H_0^1(\Omega)\times L^2(\Omega)\times L_a^2(\Omega\times(-\infty,0))$, there exists a unique solution u of system (1.2) satisfying*

$$u\in C^1([0,+\infty),L^2(\Omega))\cap C([0,+\infty),H_0^1(\Omega)),\quad u_t\in C([0,+\infty),L_a^2(\Omega\times(-\infty,t))).$$

Moreover, if $(u_0,u_1,f_0)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times L_a^2(\Omega,H^1(-\infty,0))$ satisfies the compatibility condition $f_0(\cdot,0)=u_1$, then the unique solution u satisfies

$$u\in C^1([0,+\infty),H_0^1(\Omega))\cap C([0,+\infty),H^2(\Omega)\cap H_0^1(\Omega))$$

and $u_t\in C([0,+\infty),L_a^2(\Omega,H^1(-\infty,t)))$.

To obtain the stabilization of system (1.25), we assume system (1.25) is well-posed such that

$$\int_0^{\phi(0)}\int_\Omega h_0(x,\varphi(t))g_1(h_0(x,\varphi(t)))\,dx\,dt<+\infty \tag{2.3}$$

and $u\in C^1([0,+\infty),L^2(\Omega))\cap C([0,+\infty),H^1_{\Gamma_2}(\Omega))$.

3. Proofs of Theorem 1.1

The following lemma is given in [29, Lemma 2.1] to introduce the relations between the standard dot metric $\langle\cdot,\cdot\rangle$ and the Riemannian metric $g=\langle\cdot,\cdot\rangle_g$.

Lemma 3.1. *Let $x=(x_1,\cdots,x_n)$ be the natural coordinate system in $\mathbb{R}^n$. Let f, h be functions and let H, X be vector fields. Then*

(a)
$$\langle H(x),A(x)X(x)\rangle_g=\langle H(x),X(x)\rangle,\quad x\in\mathbb{R}^n; \tag{3.1}$$

(b)
$$\nabla_g f=\sum_{i=1}^n\Big(\sum_{j=1}^n a_{ij}(x)f_{x_j}\Big)\frac{\partial}{\partial x_i}=A(x)\nabla f,\quad x\in\mathbb{R}^n, \tag{3.2}$$

where ∇f is the gradient of f in the standard metric;

(c)
$$\nabla_g f(h)=\langle\nabla_g f,\nabla_g h\rangle_g=\langle\nabla f,\ A(x)\nabla h\rangle,\quad x\in\mathbb{R}^n, \tag{3.3}$$

where the matrix $A(x)$ is given in the formula (1.1).

Let

$$E_0(t)=\frac{1}{2}\int_\Omega \big(u_t^2+|\nabla_g u|_g^2\big)dx\,. \tag{3.4}$$

Using (1.11) and (1.30), we have

$$E(t)=E_0(t)+\xi\int_0^\infty\int_\Omega a(x)G(\rho)u_t^2(x,t-\rho)\,dx\,d\rho, \tag{3.5}$$

$$J(t)=E_0(t)+\eta\int_t^{\phi(t)}\int_\Omega u_t(x,\varphi(\rho))g_1(u_t(x,\varphi(\rho)))\,dx\,d\rho. \tag{3.6}$$

Let $\hat\Omega$ be a subset of Ω, we define

$$E_0(\hat\Omega,t)=\frac{1}{2}\int_{\hat\Omega}\big(u_t^2+|\nabla_g u|_g^2\big)dx, \tag{3.7}$$

$$E(\hat\Omega,t)=E_0(\hat\Omega,t)+\xi\int_0^\infty\int_{\hat\Omega} a(x)G(\rho)u_t^2(x,t-\rho)\,dx\,d\rho. \tag{3.8}$$

Lemma 3.2. *Suppose that* (1.12) *holds. Let $u(x,t)$ be the solution of* (1.2). *Then there exist constants $C_1,C_2>0$ such that*

$$E(0)-E(T)\ge C_1\int_0^T\int_\Omega a(x)\Big(u_t^2(x,t)+\int_0^\infty|k(\rho)|u_t^2(x,t-\rho)d\rho\Big)\,dx\,dt, \tag{3.9}$$

$$E(0)-E(T)\le C_2\int_0^T\int_\Omega a(x)\Big(u_t^2(x,t)+\int_0^\infty|k(\rho)|u_t^2(x,t-\rho)d\rho\Big)\,dx\,dt, \tag{3.10}$$

where $T>0$. The assertion (3.9) *implies that $E(t)$ is decreasing.*

Proof. Differentiating (1.11), we obtain

$$\begin{aligned}E'(t)&=\int_\Omega(u_tu_{tt}+\nabla_g u\cdot\nabla u_t)\,dx\\&\quad+2\xi\int_0^{+\infty}\int_\Omega a(x)G(\rho)u_{tt}(x,t-\rho)u_t(x,t-\rho)\,dx\,d\rho.\end{aligned} \tag{3.11}$$

Applying Green's formula, the fact that

$$u_t(x,t-\rho)=-u_\rho(x,t-\rho),\quad u_{tt}(x,t-\rho)=u_{\rho\rho}(x,t-\rho),$$

and integrating by parts, we obtain

$$\begin{aligned}E'(t)=\int_\Omega a(x)\Big[&\Big(-\mu_1u_t^2(x,t)-\mu_2u_t(x,t)\int_0^\infty k(\rho)u_t(x,t-\rho)d\rho\Big)\\&+\xi\Big(u_t^2(x,t)-\int_0^\infty|k(\rho)|u_t^2(x,t-\rho)d\rho\Big)\Big]dx.\end{aligned} \tag{3.12}$$

With (1.3) we deduce that

$$\begin{aligned}&\Big|u_t(x,t)\int_0^\infty k(\rho)u_t(x,t-\rho)d\rho\Big|\\&\le\frac{1}{2}u_t^2(x,t)+\frac{1}{2}\Big(\int_0^\infty k(\rho)u_t(x,t-\rho)d\rho\Big)^2\\&\le\frac{1}{2}u_t^2(x,t)+\frac{1}{2}\int_0^\infty|k(\rho)|d\rho\int_0^\infty|k(\rho)|u_t^2(x,t-\rho)d\rho\\&=\frac{1}{2}u_t^2(x,t)+\frac{1}{2}\int_0^\infty|k(\rho)|u_t^2(x,t-\rho)d\rho.\end{aligned}$$

Then, it follows from (1.12) and (3.12) that

$$E'(t) \le -C_1 \int_\Omega a(x)\Big(u_t^2(x,t) + \int_0^\infty |k(\rho)| u_t^2(x,t-\rho)d\rho\Big)dx, \tag{3.13}$$

$$E'(t) \ge -C_2 \int_\Omega a(x)\Big(u_t^2(x,t) + \int_0^\infty |k(\rho)| u_t^2(x,t-\rho)d\rho\Big)dx, \tag{3.14}$$

where $C_1, C_2 > 0$ and C_1 satisfies

$$C_1 = \min\big\{\xi - \frac{\mu_2}{2},\ \mu_1 - \frac{\mu_2}{2} - \xi\big\}.$$

Then inequalities (3.9) and (3.10) follow directly from (3.13) and (3.14) by integrating from 0 to T. $\square$

By a similar proof as in [29, Proposition 2.1], we have the following identities.

Lemma 3.3. *Suppose that $u(x,t)$ solves the equation*

$$u_{tt} + \mathscr{A}u + a(x)\Big[\mu_1 u_t(x,t) + \mu_2 \int_0^\infty k(\rho) u_t(x,t-\rho)d\rho\Big] = 0 \tag{3.15}$$

for $(x,t) \in \Omega \times (0,+\infty)$, and that $\mathcal{H}$ is a vector field defined on $\overline{\Omega}$. Then

$$\begin{aligned}
&\int_0^T \int_\Gamma \frac{\partial u}{\partial \nu_{\mathscr{A}}} \mathcal{H}(u) d\Gamma dt + \frac{1}{2}\int_0^T \int_\Gamma \big(u_t^2 - |\nabla_g u|_g^2\big)\mathcal{H}\cdot \nu d\Gamma dt \\
&= (u_t, \mathcal{H}(u))\big|_0^T + \int_0^T \int_\Omega a(x)\mathcal{H}(u)[\mu_1 u_t(x,t) + \mu_2 \int_0^\infty k(\rho)u_t(x,t-\rho)d\rho]\,dx\,dt \\
&\quad + \int_0^T \int_\Omega D_g\mathcal{H}(\nabla_g u, \nabla_g u)\,dx\,dt + \frac{1}{2}\int_0^T\int_\Omega \Big(u_t^2 - |\nabla_g u|_g^2\Big)\operatorname{div}\mathcal{H}\,dx\,dt.
\end{aligned} \tag{3.16}$$

Moreover, assuming that $P \in C^1(\overline{\Omega})$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \big(u_t^2 - |\nabla_g u|_g^2\big) P\,dx\,dt \\
&= (u_t, uP)\big|_0^T + \frac{1}{2}\int_0^T\int_\Omega \nabla_g P(u^2)\,dx\,dt - \int_0^T\int_\Gamma Pu\frac{\partial u}{\partial \nu_{\mathscr{A}}} d\Gamma dt \\
&\quad + \int_0^T\int_\Omega a(x)Pu[\mu_1 u_t(x,t) + \mu_2\int_0^\infty k(\rho)u_t(x,t-\rho)d\rho]\,dx\,dt.
\end{aligned} \tag{3.17}$$

The following is the energy inequality for our stabilization problem.

Lemma 3.4. *Let the conditions in Theorem 1.1 hold, and let $u(x,t)$ be the solution of system (1.2). Then there exists $\overline{T} > 0$ such that, for $T > \overline{T}$, there exists a positive constant C_T such that*

$$E(0) \le C_T \int_0^T \int_\Omega a(x)\Big[u_t^2(x,t) + \int_0^\infty |k(\rho)|u_t^2(x,t-\rho)d\rho\Big]\,dx\,dt. \tag{3.18}$$

Proof. Let $\phi^i \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \phi^i \le 1$ and

$$\phi^i = \begin{cases} 1, & x \in \Omega_i \backslash V_1 \\ 0, & x \in V_2, \end{cases} \tag{3.19}$$

for $1 \le i \le m$. Set

$$\mathcal{H} = \phi^i H^i, \quad P = \frac{1}{2}\big(\operatorname{div}(\phi^i H^i) - \rho_0\big), \quad \Omega_i = \Omega.$$

Substituting (3.17) into (3.16), we have

$$\begin{aligned}
&\Pi_{\partial\Omega_i}\\
&= \big(u_t, \phi^i H^i(u) + Pu\big)\big|_0^T + \frac{\rho_0}{2}\int_0^T E_0(\Omega_i, t)dt\\
&\quad + \int_0^T\int_{\Omega_i}\Big(D_g(\phi^i H^i)(\nabla_g u, \nabla_g u) - \rho_0|\nabla_g u|_g^2 + \frac{1}{2}\nabla_g P(u^2)\Big)\,dx\,dt\\
&\quad + \int_0^T\int_{\Omega_i} a(x)\phi^i H^i(u)[\mu_1 u_t(x,t) + \mu_2\int_0^\infty k(\rho)u_t(x, t-\rho)d\rho]\,dx\,dt\\
&\quad + \int_0^T\int_{\Omega_i} a(x)Pu[\mu_1 u_t(x,t) + \mu_2\int_0^\infty k(\rho)u_t(x,t-\rho)d\rho]\,dx\,dt,
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
\Pi_{\partial\Omega_i} &= \int_0^T\int_{\partial\Omega_i}\frac{\partial u}{\partial\nu_{\mathscr{A}}}\big(\phi^i H^i(u) + uP\big)\,d\Gamma dt\\
&\quad + \frac{1}{2}\int_0^T\int_{\partial\Omega_i}\big(u_t^2 - |\nabla_g u|_g^2\big)\,\phi^i H^i\cdot\nu d\Gamma dt.
\end{aligned} \tag{3.21}$$

Note that $\partial\Omega_i \in V_2 \cup \Gamma_1^i$, where $\Gamma_1^i = \{x \in \partial\Omega_i \cap \Gamma | H^i(x)\cdot\nu(x) \le 0\}$. We decompose $\Pi_{\partial\Omega_i}$ as

$$\Pi_{\partial\Omega_i} = \Pi_{\partial\Omega_i\cap V_2} + \Pi_{(\partial\Omega_i\cap\Gamma_1^i)\setminus V_2}, \tag{3.22}$$

where

$$\begin{aligned}
\Pi_{\partial\Omega_i\cap V_2} &= \int_0^T\int_{\partial\Omega_i\cap V_2}\frac{\partial u}{\partial\nu_{\mathscr{A}}}\big(\phi^i H^i(u) + uP\big)\,d\Gamma dt\\
&\quad + \frac{1}{2}\int_0^T\int_{\partial\Omega_i\cap V_2}\big(u_t^2 - |\nabla_g u|_g^2\big)\,\phi^i H^i\cdot\nu d\Gamma dt,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\Pi_{(\partial\Omega_i\cap\Gamma_1^i)\setminus V_2} &= \int_0^T\int_{(\partial\Omega_i\cap\Gamma_1^i)\setminus V_2}\frac{\partial u}{\partial\nu_{\mathscr{A}}}\big(\phi^i H^i(u) + uP\big)\,d\Gamma dt\\
&\quad + \frac{1}{2}\int_0^T\int_{(\partial\Omega_i\cap\Gamma_1^i)\setminus V_2}\big(u_t^2 - |\nabla_g u|_g^2\big)\,\phi^i H^i\cdot\nu d\Gamma dt.
\end{aligned} \tag{3.24}$$

From (3.19), we have

$$\Pi_{\partial\Omega_i\cap V_2} = 0. \tag{3.25}$$

Since $u\big|_{(\partial\Omega_i\cap\Gamma_1^i)\setminus V_2} = 0$, we obtain $\nabla_{\Gamma_g} u\big|_{(\partial\Omega_i\cap\Gamma_1^i)\setminus V_2} = 0$; that is,

$$\nabla_g u = \frac{\partial u}{\partial\nu_{\mathscr{A}}}\frac{\nu_{\mathscr{A}}}{|\nu_{\mathscr{A}}|_g^2} \quad \text{for} \quad x \in (\partial\Omega_i\cap\Gamma_1^i)\setminus V_2. \tag{3.26}$$

Similarly, we have

$$H(u) = \langle H, \nabla_g u\rangle_g = \frac{\partial u}{\partial\nu_{\mathscr{A}}}\frac{H\cdot\nu}{|\nu_{\mathscr{A}}|_g^2} \quad \text{for } x \in (\partial\Omega_i\cap\Gamma_1^i)\setminus V_2. \tag{3.27}$$

Using formulas (3.26) and (3.27) in (3.24), with (1.19) and (1.20), we obtain

$$\begin{aligned}
\Pi_{(\partial\Omega_i\cap\Gamma_1^i)\setminus V_2} &= \frac{1}{2}\int_0^T\int_{(\partial\Omega_i\cap\Gamma_1^i)\setminus V_2}\Big(\frac{\partial u}{\partial\nu_{\mathscr{A}}}\Big)^2\frac{H\cdot\nu}{|\nu_{\mathscr{A}}|_g^2}d\Gamma dt\\
&\le C\varepsilon_0\int_0^T\int_\Gamma\Big(\frac{\partial u}{\partial\nu_{\mathscr{A}}}\Big)^2 d\Gamma dt.
\end{aligned} \tag{3.28}$$

Let H_1 be a C^1 vector field on $\overline{\Omega}$ satisfying

$$H_1 = \frac{\nu_{\mathscr{A}}}{|\nu_{\mathscr{A}}|_g^2} \quad x \in \Gamma. \tag{3.29}$$

Replacing $\mathcal{H}$ with H_1 in (3.16) and noting that $u\big|_\Gamma = 0$, by a similar discussion on Γ with (3.26) $\sim$ (3.28) we have

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_\Gamma \Big(\frac{\partial u}{\partial \nu_{\mathscr{A}}}\Big)^2 d\Gamma dt \\
&= (u_t, H_1(u))\big|_0^T + \int_0^T dt \int_\Omega D_g H_1(\nabla_g u, \nabla_g u) dx \\
&\quad + \frac{1}{2}\int_0^T dt \int_\Omega \big(u_t^2 - |\nabla_g u|_g^2\big) \operatorname{div} H_1 dx \\
&\quad + \int_0^T dt \int_\Omega a(x) H_1(u) \Big[\mu_1 u_t(x,t) + \mu_2 \int_0^\infty k(\rho) u_t(x, t-\rho) d\rho\Big] dx \\
&\le C_1(E(0) + E(T)) + C_2 \int_0^T E_0(t) dt \\
&\quad + C_3 \int_0^T \int_\Omega a(x) \big[u_t^2(x,t) + \int_0^\infty |k(\rho)| u_t^2(x, t-\rho) d\rho\big]\, dx\, dt.
\end{aligned} \tag{3.30}$$

Substituting (3.25), (3.28) and (3.30) into (3.20), and using (1.15) and (1.16), we obtain

$$\begin{aligned}
&\int_0^T E_0(\Omega_i \backslash V_1, t) dt \\
&\le C_4(E(0) + E(T)) + C_5 \varepsilon_0 \int_0^T E_0(t) dt + \int_0^T \int_{\Omega_i} (C_\alpha u^2 + \alpha |\nabla_g u|_g^2)\, dx\, dt \\
&\quad + C_6 \int_0^T \int_\Omega a(x) [u_t^2(x,t) + \int_0^\infty |k(\rho)| u_t^2(x, t-\rho) d\rho + |\nabla_g u|_g^2]\, dx\, dt,
\end{aligned} \tag{3.31}$$

where α is sufficiently small. Then, noting that $\Omega \subset (\bigcup_{i=1}^m \Omega_i \cup V_1)$, we have

$$\begin{aligned}
&\int_0^T E_0(\Omega \backslash V_1, t) dt \\
&\le C_4 m(E(0) + E(T)) + C_5 m \varepsilon_0 \int_0^T E_0(t) dt \\
&\quad + \sum_{i=1}^m \int_0^T \int_{\Omega_i} (C_\alpha u^2 + \alpha |\nabla_g u|_g^2)\, dx\, dt + C_6 m \int_0^T \int_\Omega a(x) [u_t^2(x,t) \\
&\quad + \int_0^\infty |k(\rho)| u_t^2(x, t-\rho) d\rho + |\nabla_g u|_g^2]\, dx\, dt \\
&\le C_4 m(E(0) + E(T)) + C_5 m \varepsilon_0 \int_0^T E_0(t) dt + C_7 \int_0^T \int_\Omega u^2\, dx\, dt \\
&\quad + C_8 \int_0^T \int_\Omega a(x) [u_t^2(x,t) + \int_0^\infty |k(\rho)| u_t^2(x, t-\rho) d\rho + |\nabla_g u|_g^2]\, dx\, dt.
\end{aligned} \tag{3.32}$$

Then, using (1.16), we have

$$\begin{aligned}
&\int_0^T E_0(t)dt \\
&\le C_9(E(0)+E(T)) + C_{10}\int_0^T\int_\Omega u^2\,dx\,dt \\
&\quad + C_{11}\int_0^T\int_\Omega a(x)[u_t^2(x,t)+\int_0^\infty |k(\rho)|u_t^2(x,t-\rho)d\rho + |\nabla_g u|_g^2]\,dx\,dt.
\end{aligned} \tag{3.33}$$

Set $P=a(x)$ and substituting identity (3.17) into identity (3.33), we obtain

$$\begin{aligned}
\int_0^T E_0(t)dt &\le C_{12}(E(0)+E(T)) + C_{13}\int_0^T\int_\Omega u^2\,dx\,dt \\
&\quad + C_{14}\int_0^T\int_\Omega a(x)[u_t^2(x,t)+\int_0^\infty |k(\rho)|u_t^2(x,t-\rho)d\rho]\,dx\,dt.
\end{aligned} \tag{3.34}$$

From (1.23), we have

$$\begin{aligned}
&\int_0^{T_0}\int_0^\infty\int_\Omega a(x)|k(\rho)|u_t^2(x,t-\rho)\,dx\,d\rho\,dt \\
&= \int_0^{T_0}\int_{-t}^\infty\int_\Omega a(x)|k(t+\rho)|u_t^2(x,-\rho)\,dx\,d\rho\,dt \\
&\ge \int_0^\infty\int_\Omega a(x)\int_0^{T_0}|k(t+\rho)|u_t^2(x,-\rho)\,dx\,d\rho\,dt \\
&\ge \big(1-\frac{1}{\lambda}\big)\int_0^\infty\int_\Omega a(x)G(\rho)u_t^2(x,-\rho)\,dx\,d\rho\,dt.
\end{aligned} \tag{3.35}$$

Then, for $T\ge T_0$, with (3.5) and (3.9) we obtain

$$\begin{aligned}
&C_{12}(E(T)+E(0)) + E(0) \le (2C_{12}+1)E(0) \\
&= (2C_{12}+1)E_0(0) + (2C_{12}+1)\xi\int_0^\infty\int_\Omega a(x)G(\rho)u_t^2(x,t-\rho)\,dx\,d\rho\,dt \\
&\le (2C_{12}+1)E_0(0) \\
&\quad + (2C_{12}+1)\big(\frac{\lambda}{\lambda-1}\big)\xi\int_0^T\int_0^\infty\int_\Omega a(x)|k(\rho)|u_t^2(x,t-\rho)\,dx\,d\rho\,dt.
\end{aligned} \tag{3.36}$$

Note that for $T\ge 2C_{12}+1$,

$$\begin{aligned}
(2C_{12}+1)E_0(0) &\le \int_0^T E_0(t)dt + \int_0^{2C_{12}+1}(E_0(0)-E_0(t))dt \\
&= -\int_0^{2C_{12}+1}\int_0^t\int_\Omega a(x)u_t(x,t')[\mu_1 u_t(x,t') \\
&\quad + \mu_2\int_0^\infty k(\rho)u_t(x,t'-\rho)d\rho]\,dx\,dt'dt + \int_0^T E_0(t)dt \\
&\le (2C_{12}+1)(\mu_1+\frac{\mu_2}{2})\int_0^{2C_{12}+1}\int_\Omega a(x)[u_t^2(x,t) \\
&\quad + \int_0^\infty |k(\rho)|u_t^2(x,t-\rho)d\rho]\,dx\,dt + \int_0^T E_0(t)dt.
\end{aligned} \tag{3.37}$$

Substituting (3.36)-(3.37) into (3.34), for sufficiently large T, we obtain

$$\begin{aligned} E(0) \le C_{15} \int_0^T \int_\Omega a(x)\Big(u_t^2(x,t) + \int_0^\infty |k(\rho)| u_t^2(x,t-\rho) d\rho\Big)\, dx\, dt \\ + C_{13} \int_0^T \int_\Omega u^2\, dx\, dt. \end{aligned} \tag{3.38}$$

Estimate (3.18) follows from the inequality (3.38) by a compactness-uniqueness argument as in [24]. $\square$

Proof of Theorem 1.1. Let $\overline{T} > 0$ be given by Lemma 3.4. Then it follows from (3.9) and (3.18) that, for $T > \overline{T}$,

$$\begin{aligned} E(0) &\le C_T \int_0^T \int_\Omega a(x)(u_t^2(x,t) + \int_0^\infty |k(\rho)| u_t^2(x,t-\rho) d\rho)\, dx\, dt \\ &\le C_T C_1^{-1}(E(0) - E(T)). \end{aligned} \tag{3.39}$$

Then

$$E(T) \le \frac{C_T C_1^{-1} - 1}{C_T C_1^{-1}} E(0). \tag{3.40}$$

Estimate (1.24) follows from the inequality (3.40). $\square$

4. Proof of Theorem 1.3

Lemma 4.1. *Suppose that* (1.29) *holds, and let $u(x,t)$ be the solution of* (1.25). *Then there exist constants $C_1, C_2 > 0$ such that*

$$\begin{aligned} & J(T_1) - J(T_2) \\ & \ge C_1 \int_{T_1}^{T_2} \int_\Omega (u_t(x,t) g_1(u_t(x,t)) + u_t(x,\varphi(t)) g_1(u_t(x,\varphi(t))))\, dx\, dt, \end{aligned} \tag{4.1}$$

$$\begin{aligned} & J(T_1) - J(T_2) \\ & \le C_2 \int_{T_1}^{T_2} \int_\Omega (u_t(x,t) g_1(u_t(x,t)) + u_t(x,\varphi(t)) g_1(u_t(x,\varphi(t))))\, dx\, dt, \end{aligned} \tag{4.2}$$

where $T_2 > T_1 \ge 0$. Assertion (4.1) *implies that $J(t)$ is decreasing.*

Proof. Differentiating (1.30), with (1.32), we obtain

$$\begin{aligned} J'(t) = \int_\Omega (u_t u_{tt} + \nabla_g u \cdot \nabla u_t)\, dx + \eta \phi'(t) \int_\Omega u_t(x,t) g_1(u_t(x,t)) dx \\ - \eta \int_\Omega u_t(x,\varphi(t)) g_1(u_t(x,\varphi(t))) dx. \end{aligned} \tag{4.3}$$

Note that

$$\phi'(t) = \frac{1}{\varphi'(\phi(t))} = \frac{1}{1 - \tau'(\phi(t))} \le \frac{1}{1-d}, \tag{4.4}$$

by Green's formula, we deduce that

$$
\begin{aligned}
J'(t) = \eta\phi'(t) &\int_\Omega u_t(x,t) g_1(u_t(x,t))dx \\
&- \eta \int_\Omega u_t(x,\varphi(t)) g_1(u_t(x,\varphi(t)))dx \\
&+ \int_\Omega [-a_1 u_t g_1(u_t) - a_2 u_t g_2(u_t(x,\varphi(t)))]dx \\
\le &\int_\Omega \Big[-\eta u_t(x,\varphi(t)) g_1(u_t(x,\varphi(t))) + \frac{\sqrt{1-d}|a_2|}{2} g_2^2(u_t(x,\varphi(t)))\Big] dx \\
&+ \int_\Omega \Big(-a_1 u_t g_1(u_t) + \frac{|a_2|}{2\sqrt{1-d}} u_t^2 + \frac{\eta}{1-d} u_t g_1(u_t)\Big) dx\,.
\end{aligned} \tag{4.5}
$$

From (1.27), (1.29) and (1.31), we obtain

$$
J'(t) \le -C_1 \int_\Omega [u_t(x,t) g_1(u_t(x,t)) + u_t(x,\varphi(t)) g_1(u_t(x,\varphi(t)))]dx, \tag{4.6}
$$

where $C_1 > 0$ satisfies

$$
C_1 = \min\big\{a_1 - \frac{|a_2|}{2\sqrt{1-d}} - \frac{\eta}{1-d}, \quad \eta - \frac{\sqrt{1-d}|a_2|}{2}\big\}.
$$

Note that

$$
\phi'(t) \ge 0 \quad \forall t \ge 0. \tag{4.7}
$$

From the first step of (4.5), with (1.27) we obtain

$$
J'(t) \ge -C_2 \int_\Omega [u_t(x,t) g_1(u_t(x,t)) + u_t(x,\varphi(t)) g_1(u_t(x,\varphi(t)))]dx, \tag{4.8}
$$

where C_2 is a positive constant. Then the inequality (4.1)/(4.2) follows directly from (4.6)/(4.8) integrating from T_1 to T_2. □

Proof of Theorem 1.3. Let $T_2 > T_1 \ge 0$. Multiplying (1.25) by u and integrating from T_1 to T_2, we have

$$
\begin{aligned}
&\int_{T_1}^{T_2} \int_\Omega \Big(u_t^2 - |\nabla_g u|_g^2 \Big)\, dx\, dt \\
&= (u_t, u)\big|_{T_1}^{T_2} + \int_{T_1}^{T_2} \int_\Omega u \left(a_1 g_1(u_t(x,t)) + a_2 g_2(u_t(x,t-\tau(t)))\right) dx\, dt.
\end{aligned} \tag{4.9}
$$

Then

$$
\begin{aligned}
&\int_{T_1}^{T_2} E_0(t)dt \\
&= 2\int_{T_1}^{T_2}\int_\Omega u_t^2\,dx\,dt - \int_{T_1}^{T_2}\int_\Omega \left(u_t^2 - |\nabla_g u|_g^2\right)dx\,dt \\
&= 2\int_{T_1}^{T_2}\int_\Omega u_t^2\,dx\,dt \\
&\quad - (u_t,u)\big|_{T_1}^{T_2} - \int_{T_1}^{T_2}\int_\Omega u\left(a_1 g_1(u_t) + a_2 g_2(u_t(\varphi(t)))\right)\,dx\,dt \\
&\le 2\int_{T_1}^{T_2}\int_\Omega u_t^2 dx + C(J(T_1)+J(T_2)) \\
&\quad + \varepsilon\int_{T_1}^{T_2}\int_\Omega u^2\,dx\,dt + C_\varepsilon\int_{T_1}^{T_2}\int_\Omega [g_1^2(u_t(x,t)) + g_2^2(u_t(x,\varphi(t)))]\,dx\,dt.
\end{aligned}
\tag{4.10}
$$

From (1.27), we have

$$
g_1(s) \ge |s| \quad for \quad s\in\mathbb{R}. \tag{4.11}
$$

Then, from (4.10) it follows that

$$
\begin{aligned}
\int_{T_1}^{T_2} E_0(t)dt \le \widetilde{C}(J(T_1)+J(T_2)) + C\int_{T_1}^{T_2}\int_\Omega \Big[g_1^2(u_t(x,t)) \\
+ u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))\Big]\,dx\,dt,
\end{aligned}
\tag{4.12}
$$

where $\widetilde{C}$ is a positive constant.

Proof of (a). From (1.28) and (1.36) we have

$$
g_1^2(s) \le \max\{c_1,c_2\} s g_1(s) \quad \text{for } s\in\mathbb{R}. \tag{4.13}
$$

Then, from (4.12) it follows that

$$
\begin{aligned}
\int_{T_1}^{T_2} E_0(t)dt \le \widetilde{C}(J(T_1)+J(T_2)) + C\int_{T_1}^{T_2}\int_\Omega \Big[u_t(x,t)g_1(u_t(x,t)) \\
+ u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))\Big]\,dx\,dt.
\end{aligned}
\tag{4.14}
$$

Since $J(t)$ is decreasing, from (3.6) we deduce that

$$
\begin{aligned}
&\widetilde{C}(J(T_1)+J(T_2)) + J(T_1) \\
&\le (2\widetilde{C}+1)J(T_1) \\
&= (2\widetilde{C}+1)E_0(T_1) + (2\widetilde{C}+1)\eta\int_{T_1}^{\phi(T_1)}\int_\Omega u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))d\Gamma dt.
\end{aligned}
\tag{4.15}
$$

From (1.27), we deduce that

$$
\begin{aligned}
&(2\widetilde{C}+1)E_0(T_1)\\
&= \int_{T_1}^{T_1+2\widetilde{C}+1} E_0(t)dt + \int_{T_1}^{T_1+2\widetilde{C}+1} (E_0(T_1)-E_0(t))dt\\
&= \int_{T_1}^{T_1+2\widetilde{C}+1} E_0(t)dt + \int_{T_1}^{T_1+2\widetilde{C}+1} \int_0^t \int_\Omega u_t(x,t')\big[a_1 g_1(u_t(x,t'))\\
&\quad + a_2 g_2(u_t(x,\varphi(t')))\big]\big]\,dx\,dt'dt\\
&\le \int_{T_1}^{T_1+2\widetilde{C}+1} E_0(t)dt + (2\widetilde{C}+1)\big(a_1+\frac{|a_2|}{2}\big)\\
&\quad\times \int_{T_1}^{T_1+2\widetilde{C}+1} \int_\Omega [u_t(x,t)g_1(u_t(x,t)) + g_2^2(u_t(x,\varphi(t)))]\,dx\,dt\\
&\le \int_{T_1}^{T_1+2\widetilde{C}+1} E_0(t)dt + (2\widetilde{C}+1)\big(a_1+\frac{|a_2|}{2}\big)\int_{T_1}^{T_1+2\widetilde{C}+1} \int_\Omega [u_t(x,t)g_1(u_t(x,t))\\
&\quad + u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))]\,dx\,dt.
\end{aligned} \tag{4.16}
$$

Substituting (4.15) and (4.16) into (4.14), for $T_2 \ge \max\{T_1+2\widetilde{C}+1, \phi(T_1)\}$, with (4.1) we have

$$
\begin{aligned}
J(T_1) &\le C\int_{T_1}^{T_2}\int_\Omega [u_t(x,t)g_1(u_t(x,t)) + u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))]\,dx\,dt\\
&\le CC_1^{-1}(J(T_1)-J(T_2)).
\end{aligned} \tag{4.17}
$$

Then

$$
J(T_2) \le \lambda J(T_1), \tag{4.18}
$$

where $0<\lambda<1$ is a constant and $T_2 \ge \max\{T_1+2\widetilde{C}+1, \phi(T_1)\}$.

From (1.32), we have

$$
\phi'(t) = \frac{1}{\varphi'(\phi(t))} = \frac{1}{1-\tau'(\phi(t))} \le \frac{1}{1-d}, \quad \forall t \ge -\tau(0). \tag{4.19}
$$

Then

$$
\phi(t) \le \frac{t+\tau(0)}{1-d} - \tau(0) \quad \forall t \ge -\tau(0). \tag{4.20}
$$

Let $d<d_1<1$ and T_0 be positive constants satisfying

$$
\frac{d}{1-d}(t+\tau(0)) \le \frac{t}{1-d_1} =: M(t) \quad \forall t \ge T_0, \tag{4.21}
$$

$$
M(T_0) - T_0 \ge 2\widetilde{C}+1, \tag{4.22}
$$

where $\widetilde{C}$ is given by (4.12). From (4.20), we have

$$
M(T) \ge \phi(T) \quad \textit{and} \quad M(T) \ge T+2\widetilde{C}+1 \quad \forall T \ge T_0. \tag{4.23}
$$

From (4.18) and (4.23), we have

$$
J\Big(\big(\frac{1}{1-d_1}\big)^{k-1}T_0\Big) \le \lambda^{k-1}J(T_0) \le \lambda^{k-1}J(0). \tag{4.24}
$$

Noting that $J(t)$ is decreasing, the estimate (1.37) holds.

Proof of (b). From (1.38) and (4.20), we have

$$\lim_{t\to+\infty}\frac{F(\phi(t))}{t}=0. \tag{4.25}$$

Then estimate (1.39) holds.

We let T_1 in (4.12) be a positive constant satisfying $T_1\geq\tau(T_1)$, with (1.32) and (3.6) we deduce that

$$\begin{aligned}
&\int_{T_1}^{T_2}J(t)dt\\
&=\int_{T_1}^{T_2}E_0(t)dt+\eta\int_{T_1}^{T_2}\int_t^{\phi(t)}\int_\Omega u_t(x,\varphi(\rho))g_1(u_t(x,\varphi(\rho)))\,dx\,d\rho\,dt\\
&\leq\int_{T_1}^{T_2}E_0(t)dt+\eta\int_{T_1}^{T_2}\int_t^{\phi(t)}\int_\Omega u_t(x,\varphi(\rho))g_1(u_t(x,\varphi(\rho)))\,dx\,d\rho\,dt\\
&\quad+\eta\int_{\phi^{-1}(T_1)}^{T_1}\int_{T_1}^{\phi(t)}\int_\Omega u_t(x,\varphi(\rho))g_1(u_t(x,\varphi(\rho)))\,dx\,d\rho\,dt\\
&\quad+\eta\int_{T_2}^{\phi(T_2)}\int_t^{\phi(T_2)}\int_\Omega u_t(x,\varphi(\rho))g_1(u_t(x,\varphi(\rho)))\,dx\,d\rho\,dt\\
&=\int_{T_1}^{T_2}E_0(t)dt+\eta\int_{T_1}^{\phi(T_2)}\int_{\phi^{-1}(\rho)}^{\rho}\int_\Omega u_t(x,\varphi(\rho))g_1(u_t(x,\varphi(\rho)))\,dx\,dtd\rho\\
&=\int_{T_1}^{T_2}E_0(t)dt+\eta\int_{T_1}^{\phi(T_2)}(\rho-\varphi(\rho))\int_\Omega u_t(x,\varphi(\rho))g_1(u_t(x,\varphi(\rho)))\,dx\,dt\\
&=\int_{T_1}^{T_2}E_0(t)dt+\eta\int_{T_1}^{\phi(T_2)}\tau(t)\int_\Omega u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))\,dx\,dt.
\end{aligned}\tag{4.26}$$

Substituting (4.2) and (4.26) into (4.12), we have

$$\begin{aligned}
\int_{T_1}^{T_2}J(t)dt\leq C\int_{T_1}^{\phi(T_2)}\int_\Omega\Big[g_1^2(u_t(x,t))\\
+F(t)u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))\Big]\,dx\,dt+\widetilde{C}J(T_2),
\end{aligned}\tag{4.27}$$

where $\widetilde{C}$ is a positive constant.

Since $J(t)$ is decreasing, we deduce that

$$\int_{T_1}^{T_2}J(t)dt\geq(T_2-T_1)J(T_2). \tag{4.28}$$

Substituting (4.28) into (4.27), for $T_2\geq T_1+2\widetilde{C}$, we have

$$\begin{aligned}
&(T_2-T_1)J(T_2)\\
&\leq C\int_{T_1}^{\phi(T_2)}\int_\Omega[g_1^2(u_t(x,t))+F(t)u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))]\,dx\,dt\\
&\leq C\int_{T_1}^{\phi(T_2)}\int_{\{x\in\Omega|u_t(x,t)|\leq1\}}[g_1^2(u_t(x,t))+u_t^2(x,t)]\,dx\,dt\\
&\quad+C\int_{T_1}^{\phi(T_2)}\int_\Omega F(t)[u_t(x,t)g_1(u_t(x,t)+u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))]\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{T_1}^{\phi(T_2)}\int_{\{x\in\Omega|u_t(x,t)|\le 1\}} h(u_t g_1(u_t))\,dx\,dt\\
&\quad + CF(\phi(T_2))\int_{T_1}^{\phi(T_2)}\int_\Omega [u_t(x,t)g_1(u_t(x,t)+u_t(x,\varphi(t))g_1(u_t(x,\varphi(t)))]\,dx\,dt\\
&\le C\int_{T_1}^{\phi(T_2)}\int_\Omega h(u_t g_1(u_t))\,dx\,dt + CF(\phi(T_2))(J(T_1)-J(\phi(T_2))\\
&\le C(\phi(T_2)-T_1)\operatorname{meas}(\Omega)\\
&\quad\times h\Big(\frac{1}{(\phi(T_2)-T_1)\operatorname{meas}(\Omega)}\int_{T_1}^{\phi(T_2)}\int_\Omega u_t g_1(u_t)\,dx\,dt\Big) + CF(\phi(T_2))J(T_1)\\
&\le C(\phi(T_2)-T_1)\operatorname{meas}(\Omega)h\Big(\frac{1}{(\phi(T_2)-T_1)\operatorname{meas}(\Omega)}J(T_1)\Big)\\
&\quad + CF(\phi(T_2))J(T_1).
\end{aligned}
$$

Noting that T_1 is a constant, for sufficiently large T_2, with (4.20) we have

$$
J(T_2)\le C_1 h\Big(\frac{C_2}{\phi(T_2)}J(0)\Big) + C_1\frac{F\big(\phi(T_2)\big)}{T_2}J(0).
$$

Since $J(t)$ is decreasing, estimate (1.40) holds. $\square$

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