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EXISTENCE, BLOW-UP AND EXPONENTIAL DECAY FOR KIRCHHOFF-LOVE EQUATIONS WITH DIRICHLET CONDITIONS

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Abstract. The article concerns the initial boundary value problem for a nonlinear Kirchhoff-Love equation. First, by applying the Faedo-Galerkin, we prove existence and uniqueness of a solution. Next, by constructing Lyapunov functional, we prove a blow-up of the solution with a negative initial energy, and establish a sufficient condition for the exponential decay of weak solutions.

1. Introduction

In this article, we consider the initial boundary value problem with homogeneous Dirichlet boundary conditions

$$\begin{aligned}
&u_{tt} - \frac{\partial}{\partial x}\big[B\big(x,t,u,\|u\|^2,\|u_x\|^2,\|u_t\|^2,\|u_{xt}\|^2\big)\big(u_x + \lambda_1 u_{xt} + u_{xtt}\big)\big] + \lambda u_t \\
&= F\big(x,t,u,u_x,u_t,u_{xt},\|u(t)\|^2,\|u_x(t)\|^2,\|u_t(t)\|^2,\|u_{xt}(t)\|^2\big) \\
&\quad - \frac{\partial}{\partial x}\big[G\big(x,t,u,u_x,u_t,u_{xt},\|u(t)\|^2,\|u_x(t)\|^2,\|u_t(t)\|^2,\|u_{xt}(t)\|^2\big)\big] \\
&\quad + f(x,t), \quad x \in \Omega = (0,1),\ 0 < t < T,
\end{aligned} \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where $\lambda > 0$, $\lambda_1 > 0$ are constants and $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$; f, F and G are given functions that assumptions stated later.

This problem has the so called model of Kirchhoff-Love type because it connects Kirchhoff and Love equation, this type is also introduced in [17]. More precisely (1.1) has its origin in the nonlinear vibration of an elastic string (Kirchhoff [5]), for which the associated equation is

$$\rho h u_{tt} = \Big(P_0 + \frac{Eh}{2L}\int_0^L |\frac{\partial u}{\partial y}(y,t)|^2 dy\Big) u_{xx}, \tag{1.4}$$

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here u is the lateral deflection, L is the length of the string, h is the cross sectional area, E is Young's modulus, ρ is the mass density, and P_0 is the initial tension. On the other hand, (1.1) arises from the Love equation

$$u_{tt} - \frac{E}{\rho} u_{xx} - 2\mu^2\omega^2 u_{xxtt} = 0, \tag{1.5}$$

presented by Radochová [14]. This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy functional

$$\int_0^T \int_0^L \big[\frac{1}{2}F\rho(u_t^2 + \mu^2\omega^2 u_{tx}^2) - \frac{1}{2}F(Eu_x^2 + \rho\mu^2\omega^2 u_x u_{xtt})\big]\, dx\, dt\,, \tag{1.6}$$

where u is the displacement, L is the length of the rod, F is the area of cross-section, ω is the cross-section radius, E is the Young modulus of the material and ρ is the mass density.

It is well known that the existence, global existence, decay properties and blow-up of solutions to the initial boundary value problem for Kirchhoff type models under different types of hypotheses in have been extensively studied by many authors, for example, we refer to [2, 3, 4, 13, 15, 18, 19], and references therein.

In [3], the authors studied the existence of global solutions and exponential decay for a Kirchhoff-Carrier model with viscosity.

In [15], the authors discussed the global well-posedness and uniform exponential stability for the Kirchhoff equation in $\mathbb{R}^n$. Here, the global solvability is proved when the initial data is taken small enough and the exponential decay of the energy is obtained in the strong topology $H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$.

In [13], the author investigated the global existence, decay properties, and blow-up of solutions to the initial boundary value problem for the nonlinear Kirchhoff type.

In [18], the viscoelastic equation of Kirchhoff type was considered and the authors established a new blow-up result for arbitrary positive initial energy, by using simple analysis techniques.

The purpose of this paper is establishing the existence, blow up and exponential decay of weak solutions for(1.1)–(1.3). To our knowledge, there is no decay or blow up result for equations of this type. However, the existence and exponential decay of solutions or blow up results for Love equation were studied in [12]. Here, by combining the linearization method for the nonlinear term, the Faedo-Galerkin method and the weak compactness method, the existence of a unique weak solution of a Dirichlet problem for the nonlinear Love equation

$$\begin{aligned} &u_{tt} - u_{xx} - u_{xxtt} - \lambda_1 u_{xxt} + \lambda u_t \\ &= F(x,t,u,u_x,u_t,u_{xt}) - \frac{\partial}{\partial x}[G(x,t,u,u_x,u_t,u_{xt})] + f(x,t), \end{aligned} \tag{1.7}$$

for $0 < x < 1$ and $t > 0$, has been proved. When $F = F(u) = a|u|^{p-2}u$, $G = G(u_x) = b|u_x|^{p-2}u_x$, $a,\ b \in \mathbb{R}$, $p > 2$, the blow up and exponential decay of solutions were established. For details, in case of $a > 0$, $b > 0$; $f(x,t) \equiv 0$, with negative initial energy, the solution of (1.7) blows up in finite time. In case of $a > 0$, $b < 0$, if $\|\tilde{u}_{0x}\|^2 - a\|\tilde{u}_0\|_{L^p}^p > 0$ and $f \in L^2((0,1)\times\mathbb{R}_+)$, $\|f(t)\| \le Ce^{-\gamma_0 t}$, such that $f(t)$ decays exponentially as $t \to +\infty$, the energy of the solution decays exponentially as $t \to +\infty$. Finally, in case of $a < 0$, $b < 0$ and $\|f(t)\|$ is small enough, (1.7) has a unique global solution with energy decaying exponentially as $t \to +\infty$, without the initial data $(\tilde{u}_0, \tilde{u}_1)$ small enough.

Our model was inspired in the above mentioned works and motivated by the results in [12], we study the existence, blow-up and exponential decay estimates for (1.1)–(1.3). This article is organized as follows. Section 2 is devoted to preliminaries and an existence result for (1.1)–(1.3) in case F, $G \in C^1([0,1] \times [0,T] \times \mathbb{R}^4 \times \mathbb{R}^4_+)$; $B \in C^1([0,1] \times [0,T] \times \mathbb{R} \times \mathbb{R}^4_+)$ with $B(x,t,y,z) \geq b_0 > 0$, $\forall (x,t) \in [0,1] \times [0,T]$, for all $y \in \mathbb{R}$, for all $z \in \mathbb{R}^4_+$. Since f, G, B are arbitrary, we need to combine the linearization method, the Faedo-Galerkin method and the weak compactness method.

In Sections 3, 4, Problem (1.1)–(1.3) is considered in the case $B = B(x,t)$ and $F = F(u, u_x)$, $G = G(u, u_x)$ such that $(F, G) = (\frac{\partial \mathcal{F}}{\partial u}, \frac{\partial \mathcal{F}}{\partial v})$. More details, in Section 3, with $f(x,t) \equiv 0$ and a negative initial energy, we prove that the solution of (1.1)–(1.3) blows up in finite time. In Section 4, we give a sufficient condition, in which the initial energy is positive and small, to guarantee the global existence and exponential decay of weak solutions. In the proof, a suitable Lyapunov functional is constructed. The results obtained here may be considered as the generalizations of those in [7, 12, 17], based on the main tool in [17] and the techniques in [12].

2. Existence of a weak solution

First, we set the preliminary as follows.

Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space, $\| \cdot \|$ be the norm in L^2 and $\| \cdot \|_X$ be the norm in the Banach space X. Let $L^p(0,T;X)$, $1 \leq p \leq \infty$ be the Banach space of the real functions $u : (0,T) \to X$ measurable, with

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^p dt \Big)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Denote $u(t) = u(x,t)$, $u'(t) = u_t(t) = \frac{\partial u}{\partial t}(x,t)$, $u''(t) = u_{tt}(t) = \frac{\partial^2 u}{\partial t^2}(x,t)$, $u_x(t) = \frac{\partial u}{\partial x}(x,t)$, $u_{xx}(t) = \frac{\partial^2 u}{\partial x^2}(x,t)$.

With $F \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4 \times \mathbb{R}^4_+)$, $F = F(x, t, y_1, \dots, y_4, z_1, \dots, z_4)$, we put $D_1 F = \frac{\partial F}{\partial x}$, $D_2 F = \frac{\partial F}{\partial t}$, $D_{i+2} F = \frac{\partial F}{\partial y_i}$, $D_{i+6} F = \frac{\partial F}{\partial z_i}$, with $i = 1, \dots, 4$ and $D^\alpha F = D_1^{\alpha_1} \dots D_{10}^{\alpha_{10}} F$, $\alpha = (\alpha_1, \dots, \alpha_{10}) \in \mathbb{Z}^{10}_+$, $|\alpha| = \alpha_1 + \dots + \alpha_{10} \leq k$, $D^{(0,\dots,0)} F = F$.

Similarly, with $B \in C^k([0,1] \times [0,T] \times \mathbb{R} \times \mathbb{R}^4_+)$, $B = B(x,t,y,z_1,\dots,z_4)$, we put $D_1 B = \frac{\partial B}{\partial x}$, $D_2 B = \frac{\partial B}{\partial t}$, $D_3 B = \frac{\partial B}{\partial y}$, $D_{i+3} B = \frac{\partial B}{\partial z_i}$, with $i = 1, \dots, 4$ and $D^\beta B = D_1^{\beta_1} \dots D_7^{\beta_7} B$, $\beta = (\beta_1, \dots, \beta_7) \in \mathbb{Z}^7_+$, $|\beta| = \beta_1 + \dots + \beta_7 \leq k$, $D^{(0,\dots,0)} B = B$.

We recall the following properties related to the usual spaces $C([0,1])$, H^1, and $H^1_0 = \{v \in H^1 : v(1) = v(0) = 0\}$.

Lemma 2.1. *(i) The imbedding* $= H^1 \hookrightarrow C([0,1])$ *is compact and*

$$\|v\|_{C[0,1]} \leq \sqrt{2} \big(\|v\|^2 + \|v_x\|^2 \big)^{1/2}, \quad \forall v \in H^1. \tag{2.1}$$

(ii) On H_0^1, the norms $\|v_x\|$ and $\|v\|_{H^1} = \big(\|v\|^2 + \|v_x\|^2\big)^{1/2}$ are equivalent. On the other hand

$$\|v\|_{C([0,1])} \le \|v_x\| \quad \text{for all } v \in H_0^1. \tag{2.2}$$

Now, we consider the existence of a local solution for (1.1)–(1.3), with $\lambda, \lambda_1 \in \mathbb{R}, \lambda_1 > 0$. Without loss of generality, by the fact that F contains the variable u_t and λ is arbitrary, we can suppose that $\lambda = 0$. The weak formulation of (1.1)–(1.3) can be given in as follows: Find $u \in \widetilde{W} = \{u \in L^\infty(0, T_*; H_0^1 \cap H^2) : u',\ u'' \in L^\infty(0, T_*; H_0^1 \cap H^2)\}$, such that u satisfies the variational equation

$$\begin{aligned} &\langle u''(t), w\rangle + \langle B[u](t)(u_x(t) + \lambda_1 u_x'(t) + u_x''(t)), w_x\rangle \\ &= \langle f(t), w\rangle + \langle F[u](t), w\rangle + \langle G[u](t), w_x\rangle, \end{aligned} \tag{2.3}$$

for all $w \in H_0^1$, a.e., $t \in (0, T)$, with the initial conditions

$$u(0) = \tilde{u}_0, \quad u_t(0) = \tilde{u}_1, \tag{2.4}$$

where

$$\begin{aligned} B[u](x,t) &= B\big(x, t, u(x,t), \|u(t)\|^2, \|u_x(t)\|^2, \|u'(t)\|^2, \|u_x'(t)\|^2\big), \\ F[u](x,t) &= F\Big(x, t, u(x,t), u_x(x,t), u'(x,t), u_x'(x,t), \|u(t)\|^2, \|u_x(t)\|^2, \\ &\quad \|u'(t)\|^2, \|u_x'(t)\|^2\Big), \\ G[u](x,t) &= G\Big(x, t, u(x,t), u_x(x,t), u'(x,t), u_x'(x,t), \|u(t)\|^2, \|u_x(t)\|^2, \\ &\quad \|u'(t)\|^2, \|u_x'(t)\|^2\Big). \end{aligned} \tag{2.5}$$

We use the following assumptions:

- (H1) $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$;
- (H2) $f, f' \in L^2(Q_T)$, $Q_T = (0,1) \times (0,T)$;
- (H3) $B \in C^1([0,1] \times [0,T] \times \mathbb{R} \times \mathbb{R}_+^4)$ and there exists a constant $b_0 > 0$ such that $B(x,t,y,z) \ge b_0$, for all $(x,t) \in [0,1] \times [0,T]$, for all $y \in \mathbb{R}$, for all $z \in \mathbb{R}_+^4$;
- (H4) $F \in C^1([0,1] \times [0,T] \times \mathbb{R}^4 \times \mathbb{R}_+^4)$;
- (H5) $G \in C^1([0,1] \times [0,T] \times \mathbb{R}^4 \times \mathbb{R}_+^4)$.

Theorem 2.2. *Let* (H1)–(H5) *hold. Then Problem* (1.1)–(1.3) *has a unique local solution u and*

$$\begin{gathered} u \in L^\infty(0, T_*; H_0^1 \cap H^2), \quad u' \in L^\infty(0, T_*; H_0^1 \cap H^2), \\ u'' \in L^\infty(0, T_*; H_0^1 \cap H^2), \end{gathered} \tag{2.6}$$

for $T_* > 0$ small enough.

Remark 2.3. Thanks to the regularity obtained by (2.6), Problem (1.1)–(1.3) has a unique strong solution

$$u \in C^1([0, T_*]; H_0^1 \cap H^2), \quad u'' \in L^\infty(0, T_*; H_0^1 \cap H^2). \tag{2.7}$$

Proof of Theorem 2.2. We have two steps. Using linearization, step 1 constructs a linear recurrent sequence $\{u_m\}$. Step 2 shows that $\{u_m\}$ converges to u and u is exactly a unique local solution of (1.1)–(1.3).

Step 1. Consider $T > 0$ fixed, let $M > 0$, we put

$$K_M(f) = (\|f\|_{L^2(Q_T)}^2 + \|f'\|_{L^2(Q_T)}^2)^{1/2}, \tag{2.8}$$

$$\|B\|_{C^0(\tilde{A}_M)} = \sup_{(x,t,y,z_1,\dots,z_4)\in\tilde{A}_M} |B(x,t,y,z_1,\dots,z_4)|,$$

with

$$\begin{gathered}
\tilde{A}_M = [0,1]\times[0,T]\times[-M,M]\times[0,M^2]^4,\\
\bar{B}_M = \|B\|_{C^1(\tilde{A}_M)} = \|B\|_{C^0(\tilde{A}_M)} + \sum_{i=1}^{7}\|D_iB\|_{C^0(\tilde{A}_M)},
\end{gathered}$$

$$\|F\|_{C^0(A_M)} = \sup_x, t, y_1,\dots,y_4,z_1,\dots,z_4) \in A_M |F(x,t,y_1,\dots,y_4,z_1,\dots,z_4)|,$$

with

$$\begin{gathered}
A_M = [0,1]\times[0,T]\times[-M,M]^4\times[0,M^2]^4,\\
\bar{F}_M = \|F\|_{C^1(A_M)} = \|F\|_{C^0(A_M)} + \sum_{i=1}^{10}\|D_iF\|_{C^0(A_M)},\\
\bar{G}_M = \|G\|_{C^1(A_M)} = \|G\|_{C^0(A_M)} + \sum_{i=1}^{10}\|D_iG\|_{C^0(A_M)}.
\end{gathered}$$

For each $T_* \in (0,T]$ and $M>0$, we put

$$\begin{aligned}
W(M,T_*) = \Big\{ & v\in L^\infty(0,T_*;H_0^1\cap H^2) : v'\in L^\infty(0,T_*;H_0^1\cap H^2),\\
& v''\in L^\infty(0,T_*;H_0^1),\ \text{with } \|v\|_{L^\infty(0,T_*;H_0^1\cap H^2)},\\
& \|v'\|_{L^\infty(0,T_*;H_0^1\cap H^2)},\ \|v''\|_{L^\infty(0,T_*;H_0^1)} \le M\Big\},\\
W_1(M,T_*) = \{ & v\in W(M,T_*) : v''\in L^\infty(0,T_*;H_0^1\cap H^2)\},
\end{aligned}\tag{2.9}$$

where $Q_{T_*} = \Omega\times(0,T_*)$.

We establish the linear recurrent sequence $\{u_m\}$ as follows. We choose the first term $u_0\equiv 0$, suppose that

$$u_{m-1}\in W_1(M,T_*),\tag{2.10}$$

and associate with problem (1.1)–(1.3) the following problem.

Find $u_m\in W_1(M,T_*)$ $(m\ge 1)$ which satisfies the linear variational problem

$$\begin{gathered}
\langle u_m''(t),w\rangle + \langle B_m(t)(u_{mx}(t)+\lambda_1 u_{mx}'(t)+u_{mx}''(t)),w_x\rangle\\
= \langle f(t),w\rangle + \langle F_m(t),w\rangle + \langle G_m(t),w_x\rangle,\quad \forall w\in H_0^1,\\
u_m(0)=\tilde{u}_0,\quad u_m'(0)=\tilde{u}_1,
\end{gathered}\tag{2.11}$$

where

$$\begin{aligned}
& B_m(x,t) = B[u_{m-1}](x,t)\\
& = B\Big(x,t,u_{m-1}(x,t),\|u_{m-1}(t)\|^2,\|\nabla u_{m-1}(t)\|^2,\|u_{m-1}'(t)\|^2,\|\nabla u_{m-1}'(t)\|^2\Big),\\
& F_m(x,t) = F[u_{m-1}](x,t)\\
&\qquad = F\Big(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t),u_{m-1}'(x,t),\nabla u_{m-1}'(x,t),\\
&\qquad \|u_{m-1}(t)\|^2,\|\nabla u_{m-1}(t)\|^2,\|u_{m-1}'(t)\|^2,\|\nabla u_{m-1}'(t)\|^2\Big),
\end{aligned}\tag{2.12}$$

$$
\begin{aligned}
G_m(x,t) &= G[u_{m-1}](x,t) \\
&= G\Big(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t),u'_{m-1}(x,t),\nabla u'_{m-1}(x,t), \\
&\quad \|u_{m-1}(t)\|^2,\|\nabla u_{m-1}(t)\|^2,\|u'_{m-1}(t)\|^2,\|\nabla u'_{m-1}(t)\|^2\Big).
\end{aligned}
$$

Lemma 2.4. *Let* (H1)–(H5) *hold. Then there exist positive constants* M*,* $T_*>0$ *such that, for* $u_0\equiv 0$*, there exists a recurrent sequence* $\{u_m\}\subset W_1(M,T_*)$ *defined by* (2.10)–(2.12).

Proof. The proof consists of several steps.

(i) The Faedo-Galerkin approximation (introduced by Lions [6]). Consider a special orthonormal basis $\{w_j\}$ on H_0^1 : $w_j(x)=\sqrt{2}\sin(j\pi x)$, $j\in\mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta=-\frac{\partial^2}{\partial x^2}$. It is clear to see that there exists $c_{mj}^{(k)}(t)$, $1\le j\le k$, on interval $[0,T]$ such that if we have expression in form

$$
u_m^{(k)}(t)=\sum_{j=1}^k c_{mj}^{(k)}(t)w_j, \tag{2.13}
$$

then $u_m^{(k)}(t)$ satisfies

$$
\begin{gathered}
\langle \ddot{u}_m^{(k)}(t),w_j\rangle+\langle B_m(t)\big(u_{mx}^{(k)}(t)+\lambda_1\dot{u}_{mx}^{(k)}(t)+\ddot{u}_{mx}^{(k)}(t)\big),w_{jx}\rangle \\
=\langle f(t),w_j\rangle+\langle F_m(t),w_j\rangle+\langle G_m(t),w_{jx}\rangle,\quad 1\le j\le k, \\
u_m^{(k)}(0)=\tilde{u}_{0k},\quad \dot{u}_m^{(k)}(0)=\tilde{u}_{1k},
\end{gathered}
\tag{2.14}
$$

in which

$$
\begin{gathered}
\tilde{u}_{0k}=\sum_{j=1}^k\alpha_j^{(k)}w_j\to\tilde{u}_0 \quad \text{strongly in } H_0^1\cap H^2, \\
\tilde{u}_{1k}=\sum_{j=1}^k\beta_j^{(k)}w_j\to\tilde{u}_1 \quad \text{strongly in } H_0^1\cap H^2.
\end{gathered}
\tag{2.15}
$$

Indeed, (2.14) leads to an equivalent form of system (2.14) as follows

$$
\begin{gathered}
\ddot{c}_{mi}^{(k)}(t)+\sum_{j=1}^k b_{ij}^{(m)}(t)(\ddot{c}_{mj}^{(k)}(t)+\lambda_1\dot{c}_{mj}^{(k)}(t)+c_{mj}^{(k)}(t))=f_{mi}(t), \\
c_{mi}^{(k)}(0)=\alpha_i^{(k)},\quad \dot{c}_{mi}^{(k)}(0)=\beta_i^{(k)},\quad 1\le i\le k,
\end{gathered}
\tag{2.16}
$$

where

$$
\begin{gathered}
f_{mj}(t)=\langle f(t),w_j\rangle+\langle F_m(t),w_j\rangle+\langle G_m(t),w_{jx}\rangle, \\
b_{ij}^{(m)}(t)=\langle B_m(t)w_{ix},w_{jx}\rangle,\quad 1\le i,\ j\le k.
\end{gathered}
\tag{2.17}
$$

System (2.16), (2.17) has a unique solution $c_{mj}^{(k)}(t)$, $1\le j\le k$ on interval $[0,T]$, the proof is obtained through (2.10) and normal argument (see [1]).

(ii) A priori estimates. We shall give a priori estimates to show that there exist positive constants M, $T_* > 0$ such that $u_m^{(k)} \in W(M, T_*)$, for all m and k. Put

$$\begin{aligned} S_m^{(k)}(t) = &\|\sqrt{B_m(t)}u_{mx}^{(k)}(t)\|^2 + \|\sqrt{B_m(t)}\Delta u_m^{(k)}(t)\|^2 \\ &+ \|\dot{u}_m^{(k)}(t)\|^2 + \|\dot{u}_{mx}^{(k)}(t)\|^2 \\ &+ 2\|\sqrt{B_m(t)}\dot{u}_{mx}^{(k)}(t)\|^2 + \|\sqrt{B_m(t)}\Delta\dot{u}_m^{(k)}(t)\|^2 \\ &+ \|\ddot{u}_m^{(k)}(t)\|^2 + \|\sqrt{B_m(t)}\ddot{u}_{mx}^{(k)}(t)\|^2 \\ &+ 2\lambda_1\int_0^t \Big[\|\sqrt{B_m(s)}\dot{u}_{mx}^{(k)}(s)\|^2 + \|\sqrt{B_m(s)}\Delta\dot{u}_m^{(k)}(s)\|^2 \\ &+ \|\sqrt{B_m(s)}\ddot{u}_{mx}^{(k)}(s)\|^2\Big]\,ds. \end{aligned} \tag{2.18}$$

It follows from (2.14) and (2.18) that

$$\begin{aligned} &S_m^{(k)}(t) \\ &= S_m^{(k)}(0) + 2\int_0^t \langle f(s), \dot{u}_m^{(k)}(s)\rangle ds - 2\int_0^t \langle f(s), \Delta\dot{u}_m^{(k)}(s)\rangle ds \\ &\quad + 2\int_0^t \langle f'(s), \ddot{u}_m^{(k)}(s)\rangle\, ds + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle\, ds \\ &\quad - 2\int_0^t \langle F_m(s), \Delta\dot{u}_m^{(k)}(s)\rangle\, ds + 2\int_0^t \langle G_m(s), \dot{u}_{mx}^{(k)}(s)\rangle\, ds \\ &\quad + 2\int_0^t \langle F_m'(s), \ddot{u}_m^{(k)}(s)\rangle ds + 2\int_0^t \langle G_m'(s), \ddot{u}_{mx}^{(k)}(s)\rangle\, ds \\ &\quad + 2\int_0^t \langle G_{mx}(s), \triangle\dot{u}_m^{(k)}(s)\rangle ds + \int_0^t ds\int_0^1 B_m'(x,s)\Big[|u_{mx}^{(k)}(x,s)|^2 + |\Delta u_m^{(k)}(x,s)|^2 \\ &\quad + 2|\dot{u}_{mx}^{(k)}(x,s)|^2 + |\Delta\dot{u}_m^{(k)}(x,s)|^2 - |\ddot{u}_{mx}^{(k)}(x,s)|^2\Big]\,dx \\ &\quad - 2\int_0^t \langle B_m'(s)(u_{mx}^{(k)}(s) + \lambda_1\dot{u}_{mx}^{(k)}(s)), \ddot{u}_{mx}^{(k)}(s)\rangle ds \\ &\quad - 2\int_0^t \langle B_{mx}(s)(u_{mx}^{(k)}(s) + \lambda_1\dot{u}_{mx}^{(k)}(s) + \ddot{u}_{mx}^{(k)}(s)), \Delta\dot{u}_m^{(k)}(s)\rangle ds \\ &= S_m^{(k)}(0) + \sum_{j=1}^{12} I_j. \end{aligned} \tag{2.19}$$

First, we need to estimate $\xi_m^{(k)} = \|\ddot{u}_m^{(k)}(0)\|^2 + \|\sqrt{B_m(0)}\ddot{u}_{mx}^{(k)}(0)\|^2$. Letting $t \to 0_+$ in $(2.14)_1$, multiplying the result by $\ddot{c}_{mj}^{(k)}(0)$, it gives

$$\begin{aligned} &\|\ddot{u}_m^{(k)}(0)\|^2 + \|\sqrt{B_m(0)}\ddot{u}_{mx}^{(k)}(0)\|^2 \\ &\quad + \langle B_m(0)(\lambda_1\tilde{u}_{1kx} + \tilde{u}_{0kx}), \ddot{u}_{mx}^{(k)}(0)\rangle \\ &= \langle f(0), \ddot{u}_m^{(k)}(0)\rangle + \langle F_m(0), \ddot{u}_m^{(k)}(0)\rangle + \langle G_m(0), \ddot{u}_{mx}^{(k)}(0)\rangle. \end{aligned}$$

Then

$$\begin{aligned} \xi_m^{(k)} &= \|\ddot{u}_m^{(k)}(0)\|^2 + \|\sqrt{B_m(0)}\ddot{u}_{mx}^{(k)}(0)\|^2 \\ &\le \big[\lambda_1\|\sqrt{B_m(0)}\tilde{u}_{1kx}\| + \|\sqrt{B_m(0)}\tilde{u}_{0kx}\|\big]\|\sqrt{B_m(0)}\ddot{u}_{mx}^{(k)}(0)\| \end{aligned}$$

$$
\begin{aligned}
&\quad + [\|f(0)\| + \|F_m(0)\|\ ]\|\ddot{u}_m^{(k)}(0)\| + \|G_m(0)\|\|\ddot{u}_{mx}^{(k)}(0)\| \\
&\leq [\lambda_1\|\sqrt{B_m(0)}\tilde{u}_{1kx}\| + \|\sqrt{B_m(0)}\tilde{u}_{0kx}\|\ ]\sqrt{\xi_m^{(k)}} \\
&\quad + [\|f(0)\| + \|F_m(0)\|]\sqrt{\xi_m^{(k)}} + \|G_m(0)\|\sqrt{\frac{\xi_m^{(k)}}{b_0}} \\
&\leq [\lambda_1\|\sqrt{B_m(0)}\tilde{u}_{1kx}\| + \|\sqrt{B_m(0)}\tilde{u}_{0kx}\| + \|f(0)\| + \|F_m(0)\| + \frac{1}{\sqrt{b_0}}\|G_m(0)\|]^2.
\end{aligned}
$$

On the other hand, $B_m(x,0) = B(x,0,\tilde{u}_0,\|\tilde{u}_0\|^2,\|\tilde{u}_{0x}\|^2,\|\tilde{u}_1\|^2,\|\tilde{u}_{1x}\|^2)$ is independent of m and the constant $\|F_m(0)\| + \|G_m(0)\|/\sqrt{b_0}$ is also independent of m, because

$$
\begin{aligned}
&\|F_m(0)\| + \frac{\|G_m(0)\|}{\sqrt{b_0}} \\
&= \|F\big(\cdot,0,\tilde{u}_0,\tilde{u}_{0x},\tilde{u}_1,\tilde{u}_{1x},\|\tilde{u}_0\|^2,\|\tilde{u}_{0x}\|^2,\|\tilde{u}_1\|^2,\|\tilde{u}_{1x}\|^2\big)\| \\
&\quad + \frac{1}{\sqrt{b_0}}\|G\big(\cdot,0,\tilde{u}_0,\tilde{u}_{0x},\tilde{u}_1,\tilde{u}_{1x},\|\tilde{u}_0\|^2,\|\tilde{u}_{0x}\|^2,\|\tilde{u}_1\|^2,\|\tilde{u}_{1x}\|^2\big)\|.
\end{aligned}
$$

Therefore,

$$
\xi_m^{(k)} \leq \bar{S}_0, \quad \text{for all } m,\ k, \tag{2.20}
$$

where $\bar{S}_0$ is a constant depending only on f, $\tilde{u}_0$, $\tilde{u}_1$, B, F, G and λ_1.

Equations (2.15), (2.18) and (2.20) imply that

$$
\begin{aligned}
S_m^{(k)}(0) &= \|\sqrt{B_m(0)}\tilde{u}_{0kx}\|^2 + \|\sqrt{B_m(0)}\Delta\tilde{u}_{0k}\|^2 + \|\tilde{u}_{1k}\|^2 + \|\tilde{u}_{1kx}\|^2 \\
&\quad + 2\|\sqrt{B_m(0)}\tilde{u}_{1kx}\|^2 + \|\sqrt{B_m(0)}\Delta\tilde{u}_{1k}\|^2 + \xi_m^{(k)} \\
&\leq S_0, \quad \text{for all } m,\ k \in \mathbb{N},
\end{aligned}
$$

where S_0 is also a constant depending only on f, $\tilde{u}_0$, $\tilde{u}_1$, B, F, G and λ_1.

We estimate the terms I_j of (2.19). By the Cauchy - Schwartz inequality, we obtain

$$
\begin{aligned}
I_1 &= 2\int_0^t \langle f(s),\dot{u}_m^{(k)}(s)\rangle\,ds \leq \|f\|^2_{L^2(Q_T)} + \int_0^t \|\dot{u}_m^{(k)}(s)\|^2 ds; \\
I_2 &= -2\int_0^t \langle f(s),\Delta\dot{u}_m^{(k)}(s)\rangle ds \leq \|f\|^2_{L^2(Q_T)} + \int_0^t \|\Delta\dot{u}_m^{(k)}(s)\|^2 ds; \\
I_3 &= 2\int_0^t \langle f'(s),\ddot{u}_m^{(k)}(s)\rangle ds \leq \|f'\|^2_{L^2(Q_T)} + \int_0^t \|\ddot{u}_m^{(k)}(s)\|^2 ds.
\end{aligned}
$$

Note that

$$
\begin{aligned}
S_m^{(k)}(t) &\geq \|\sqrt{B_m(t)}\dot{u}_{mx}^{(k)}(t)\|^2 + \|\sqrt{B_m(t)}\Delta\dot{u}_m^{(k)}(t)\|^2 + \|\sqrt{B_m(t)}\ddot{u}_{mx}^{(k)}(t)\|^2 \\
&\geq b_0\big(\|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\Delta\dot{u}_m^{(k)}(t)\|^2 + \|\ddot{u}_{mx}^{(k)}(t)\|^2\big),
\end{aligned}
$$

so

$$
I_1 + I_2 + I_3 \leq 2K_M^2(f) + \frac{1}{b_0}\int_0^t S_m^{(k)}(s)ds. \tag{2.21}
$$

Because

$$
|F_m(x,t)| \leq \bar{F}_M, \quad |G_m(x,t)| \leq \bar{G}_M, \tag{2.22}
$$

we have

$$\begin{aligned} I_4 &= 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds \le T_*\bar{F}_M^2 + \int_0^t \|\dot{u}_m^{(k)}(s)\|^2 ds; \\ I_5 &= -2\int_0^t \langle F_m(s), \Delta\dot{u}_m^{(k)}(s)\rangle ds \le T_*\bar{F}_M^2 + \int_0^t \|\Delta\dot{u}_m^{(k)}(s)\|^2 ds; \\ I_6 &= 2\int_0^t \langle G_m(s), \dot{u}_{mx}^{(k)}(s)\rangle ds \le T_*\bar{G}_M^2 + \int_0^t \|\dot{u}_{mx}^{(k)}(s)\|^2 ds. \end{aligned}$$

By

$$\begin{aligned} S_m^{(k)}(t) &\ge 2\|\sqrt{B_m(t)}\dot{u}_{mx}^{(k)}(t)\|^2 + \|\sqrt{B_m(t)}\Delta\dot{u}_m^{(k)}(t)\|^2 \\ &\ge b_0(\|\dot{u}_m^{(k)}(t)\|^2 + \|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\Delta\dot{u}_m^{(k)}(t)\|^2), \end{aligned}$$

we have

$$I_4 + I_5 + I_6 \le 2T_*(\bar{F}_M^2 + \bar{G}_M^2) + \frac{1}{b_0}\int_0^t S_m^{(k)}(s)ds. \tag{2.23}$$

We remark that

$$\begin{aligned} F_m'(t) = {} & D_2F[u_{m-1}] + D_3F[u_{m-1}]u_{m-1}' + D_4F[u_{m-1}]\nabla u_{m-1}' \\ & + D_5F[u_{m-1}]u_{m-1}'' + D_6F[u_{m-1}]\nabla u_{m-1}'' \\ & + 2D_7F[u_{m-1}]\langle u_{m-1}(t), u_{m-1}'(t)\rangle + 2D_8F[u_{m-1}]\langle \nabla u_{m-1}(t), \nabla u_{m-1}'(t)\rangle \\ & + 2D_9F[u_{m-1}]\langle u_{m-1}'(t), u_{m-1}''(t)\rangle + 2D_{10}F[u_{m-1}]\langle \nabla u_{m-1}'(t), \nabla u_{m-1}''(t)\rangle \end{aligned}$$

yields

$$\|F_m'(t)\| \le (1 + 4M + 8M^2)\bar{F}_M \equiv \tilde{F}_M. \tag{2.24}$$

Thus

$$I_7 = 2\int_0^t \langle F_m'(s), \ddot{u}_m^{(k)}(s)\rangle ds \le T_*\tilde{F}_M^2 + \int_0^t \|\ddot{u}_m^{(k)}(s)\|^2 ds. \tag{2.25}$$

In a similar way, we obtain the estimate

$$I_8 = 2\int_0^t \langle G_m'(s), \ddot{u}_{mx}^{(k)}(s)\rangle ds \le T_*\tilde{G}_M^2 + \int_0^t \|\ddot{u}_{mx}^{(k)}(s)\|^2 ds, \tag{2.26}$$

with $\tilde{G}_M = (1 + 4M + 8M^2)\bar{G}_M$. From

$$G_{mx}(t) = D_1G[u_{m-1}] + D_3G[u_{m-1}]\nabla u_{m-1} + D_4G[u_{m-1}]\Delta u_{m-1} \tag{2.27}$$
$$+ D_5G[u_{m-1}]\nabla u_{m-1}' + D_6G[u_{m-1}]\Delta u_{m-1}', \tag{2.28}$$

we obtain

$$\|G_{mx}(t)\| \le (1 + 4M)\bar{G}_M \le \tilde{G}_M. \tag{2.29}$$

Hence

$$I_9 = 2\int_0^t \langle G_{mx}(s), \triangle\dot{u}_m^{(k)}(s)\rangle ds \le T_*\tilde{G}_M^2 + \int_0^t \|\triangle\dot{u}_m^{(k)}(s)\|^2 ds. \tag{2.30}$$

On the other hand

$$\begin{aligned} 2S_m^{(k)}(t) &\ge 2\|\sqrt{B_m(t)}\ddot{u}_{mx}^{(k)}(t)\|^2 + 2\|\sqrt{B_m(t)}\Delta\dot{u}_m^{(k)}(t)\|^2 \\ &\ge b_0(2\|\ddot{u}_{mx}^{(k)}(t)\|^2 + 2\|\Delta\dot{u}_m^{(k)}(t)\|^2) \\ &\ge b_0(\|\ddot{u}_m^{(k)}(t)\|^2 + \|\ddot{u}_{mx}^{(k)}(t)\|^2 + \|\Delta\dot{u}_m^{(k)}(t)\|^2). \end{aligned}$$

We have verified that

$$
\begin{aligned}
I_7 + I_8 + I_9 &\le 2T_*(\tilde{F}_M^2 + \tilde{G}_M^2) \\
&\quad + \int_0^t [\|\ddot{u}_m^{(k)}(s)\|^2 + \|\ddot{u}_{mx}^{(k)}(s)\|^2 + \|\triangle \dot{u}_m^{(k)}(s)\|^2] ds \\
&\le 2T_*(\tilde{F}_M^2 + \tilde{G}_M^2) + \frac{2}{b_0} \int_0^t S_m^{(k)}(s) ds.
\end{aligned}
\tag{2.31}
$$

It is known that

$$
\begin{aligned}
B_m'(t) &= D_2 B[u_{m-1}] + D_3 B[u_{m-1}] u_{m-1}' \\
&\quad + 2D_4 B[u_{m-1}] \langle u_{m-1}(t), u_{m-1}'(t)\rangle \\
&\quad + 2D_5 B[u_{m-1}] \langle \nabla u_{m-1}(t), \nabla u_{m-1}'(t)\rangle \\
&\quad + 2D_6 B[u_{m-1}] \langle u_{m-1}'(t), u_{m-1}''(t)\rangle \\
&\quad + 2D_7 B[u_{m-1}] \langle \nabla u_{m-1}'(t), \nabla u_{m-1}''(t)\rangle,
\end{aligned}
\tag{2.32}
$$

so

$$
|B_m'(x,t)| \le (1 + M + 8M^2)\bar{B}_M \equiv \tilde{B}_M. \tag{2.33}
$$

We also have

$$
\begin{aligned}
S_m^{(k)}(t) &\ge \|\sqrt{B_m(t)} u_{mx}^{(k)}(t)\|^2 + \|\sqrt{B_m(t)} \Delta u_m^{(k)}(t)\|^2 + 2\|\sqrt{B_m(t)} \dot{u}_{mx}^{(k)}(t)\|^2 \\
&\quad + \|\sqrt{B_m(t)} \Delta \dot{u}_m^{(k)}(t)\|^2 + \|\sqrt{B_m(t)} \ddot{u}_{mx}^{(k)}(t)\|^2 \\
&\ge b_0 [\|u_{mx}^{(k)}(t)\|^2 + \|\Delta u_m^{(k)}(t)\|^2 + 2\|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\Delta \dot{u}_m^{(k)}(t)\|^2 + \|\ddot{u}_{mx}^{(k)}(t)\|^2],
\end{aligned}
$$

hence

$$
\begin{aligned}
|I_{10}| &= \Big| \int_0^t ds \int_0^1 B_m'(x,s) [|u_{mx}^{(k)}(x,s)|^2 + |\Delta u_m^{(k)}(x,s)|^2 + 2|\dot{u}_{mx}^{(k)}(x,s)|^2 \\
&\quad + |\Delta \dot{u}_m^{(k)}(x,s)|^2 - |\ddot{u}_{mx}^{(k)}(x,s)|^2] dx \Big| \\
&\le \tilde{B}_M \int_0^t \Big[\|u_{mx}^{(k)}(s)\|^2 + \|\Delta u_m^{(k)}(s)\|^2 + 2\|\dot{u}_{mx}^{(k)}(s)\|^2 + \|\Delta \dot{u}_m^{(k)}(s)\|^2 \\
&\quad + \|\ddot{u}_{mx}^{(k)}(s)\|^2 \Big] ds \\
&\le \frac{\tilde{B}_M}{b_0} \int_0^t S_m^{(k)}(s) ds.
\end{aligned}
\tag{2.34}
$$

Note that

$$
\begin{aligned}
S_m^{(k)}(t) &\ge \|\sqrt{B_m(t)} u_{mx}^{(k)}(t)\|^2 + 2\|\sqrt{B_m(t)} \dot{u}_{mx}^{(k)}(t)\|^2 + \|\sqrt{B_m(t)} \ddot{u}_{mx}^{(k)}(t)\|^2 \\
&\ge b_0 [\|u_{mx}^{(k)}(t)\|^2 + 2\|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\ddot{u}_{mx}^{(k)}(t)\|^2],
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
|I_{11}| &= \big| 2 \int_0^t \langle B_m'(s)(u_{mx}^{(k)}(s) + \lambda_1 \dot{u}_{mx}^{(k)}(s)), \ddot{u}_{mx}^{(k)}(s)\rangle ds \big| \\
&\le 2\tilde{B}_M \int_0^t (\|u_{mx}^{(k)}(s)\| + \lambda_1 \|\dot{u}_{mx}^{(k)}(s)\|) \|\ddot{u}_{mx}^{(k)}(s)\| ds \\
&\le \frac{\tilde{B}_M}{b_0} (2 + \lambda_1) \int_0^t S_m^{(k)}(s) ds.
\end{aligned}
\tag{2.35}
$$

Because of

$$\begin{gathered}
B_{mx}(x,t) = D_1 B[u_{m-1}] + D_3 B[u_{m-1}]\nabla u_{m-1},\\
|B_{mx}(x,t)| \le \bar{B}_M(1+2M) \equiv \hat{B}_M,
\end{gathered}$$

$$\begin{aligned}
S_m^{(k)}(t) &\ge \|\sqrt{B_m(t)}u_{mx}^{(k)}(t)\|^2 + 2\|\sqrt{B_m(t)}\dot{u}_{mx}^{(k)}(t)\|^2\\
&\quad + \|\sqrt{B_m(t)}\ddot{u}_{mx}^{(k)}(t)\|^2 + \|\sqrt{B_m(t)}\Delta\dot{u}_m^{(k)}(t)\|^2\\
&\ge b_0\Big(\|u_{mx}^{(k)}(t)\|^2 + 2\|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\ddot{u}_{mx}^{(k)}(t)\|^2 + \|\Delta\dot{u}_m^{(k)}(t)\|^2\Big),
\end{aligned}$$

we have the estimate

$$\begin{aligned}
I_{12} &= 2\int_0^t \langle B_{mx}(s)(u_{mx}^{(k)}(s) + \lambda_1\dot{u}_{mx}^{(k)}(s) + \ddot{u}_{mx}^{(k)}(s)), \Delta\dot{u}_m^{(k)}(s)\rangle ds\\
&\le 2\hat{B}_M\int_0^t (\|u_{mx}^{(k)}(s)\| + \lambda_1\|\dot{u}_{mx}^{(k)}(s)\| + \|\ddot{u}_{mx}^{(k)}(s)\|)\|\Delta\dot{u}_m^{(k)}(s)\| ds \qquad (2.36)\\
&\le \frac{\hat{B}_M}{b_0}(4+\lambda_1)\int_0^t S_m^{(k)}(s) ds.
\end{aligned}$$

Consequently, estimates (2.19), (2), (2.21), (2.23), (2.31), (2.34), (2.35) and (2.36) show that

$$\begin{aligned}
S_m^{(k)}(t) &\le S_0 + 2K_M^2(f) + 4T_*(\bar{F}_M^2 + \bar{G}_M^2)\\
&\quad + \frac{1}{b_0}[4 + (7+2\lambda_1)\tilde{B}_M]\int_0^t S_m^{(k)}(s) ds.
\end{aligned} \tag{2.37}$$

We choose $M > 0$ sufficiently large such that

$$S_0 + 2K_M^2(f) \le \frac{1}{2}M^2, \tag{2.38}$$

and then choose $T_* \in (0, T]$ small enough such that

$$\big(\frac{1}{2}M^2 + 4T_*(\bar{F}_M^2 + \bar{G}_M^2)\big)\exp[\frac{T_*}{b_0}[4 + (7+2\lambda_1)\tilde{B}_M]] \le M^2, \tag{2.39}$$

and

$$k_{T_*} = 2\sqrt{\bar{D}_M}\sqrt{T_*}\exp[T_*(1 + \frac{\tilde{B}_M}{2b_0})] < 1, \tag{2.40}$$

with

$$\bar{D}_M = \frac{1}{b_0}[4(1+2M)^2(\bar{F}_M + \bar{G}_M)^2 + (2+\lambda_1)^2(1+4M)^2M^2\bar{B}_M^2].$$

From (2.37)–(2.39), we have

$$\begin{aligned}
S_m^{(k)}(t) &\le \exp[\frac{-T_*}{b_0}[4 + (7+2\lambda_1)\tilde{B}_M]]M^2\\
&\quad + \frac{1}{b_0}[4 + (7+2\lambda_1)\tilde{B}_M]\int_0^t S_m^{(k)}(s) ds.
\end{aligned} \tag{2.41}$$

Using Gronwall's Lemma, (2.41) leads to

$$S_m^{(k)}(t) \le \exp[\frac{-T_*}{b_0}[4 + (7+2\lambda_1)\tilde{B}_M]]M^2\exp[\frac{-t}{b_0}[4 + (7+2\lambda_1)\tilde{B}_M]] \le M^2, \tag{2.42}$$

for all $t \in [0, T_*]$, for all m and k, so

$$u_m^{(k)} \in W(M, T_*), \text{ for all } m \text{ and } k. \tag{2.43}$$

(iii) Limiting process. By (2.42), there exists a subsequence of $\{u_m^{(k)}\}$ with a same notation, such that

$$\begin{gathered} u_m^{(k)} \to u_m \quad \text{in } L^\infty(0,T_*;H_0^1\cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' \quad \text{in } L^\infty(0,T_*;H_0^1\cap H^2) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to u_m'' \quad \text{in } L^\infty(0,T_*;H_0^1) \text{ weakly*}, \\ u_m \in W(M,T_*). \end{gathered} \tag{2.44}$$

Passing to limit in (2.14), (2.15), it is clear to see that u_m is satisfying (2.11), (2.12) in $L^2(0,T_*)$. Furthermore, $(2.11)_1$ and $(2.44)_4$ imply that

$$\begin{aligned} B_m(t)\Delta u_m''(t) &= -B_m(t)[\Delta u_m(t) + \lambda_1\Delta u_m'(t)] - B_{mx}(t)\big(u_{mx}(t) + \lambda_1 u_{mx}'(t) \\ &\quad + u_{mx}''(t)\big) + u_m''(t) - f(t) - F_m(t) + G_{mx}(t) \\ &\equiv \Psi_m \in L^\infty(0,T_*;L^2). \end{aligned}$$

We have

$$b_0\|\Delta u_m''(t)\| \le \|B_m(t)\Delta u_m''(t)\| = \|\Psi_m(t)\| \le \|\Psi_m\|_{L^\infty(0,T_*;L^2)}.$$

Hence $u_m'' \in L^\infty(0,T_*;H_0^1\cap H^2)$, so we obtain $u_m \in W_1(M,T_*)$, Lemma 2.4 is proved. It means that step 1 is done. $\square$

Step 2. We state the following lemma.

Lemma 2.5. *Let* (H1)–(H5) *hold. Then*

- (i) *Problem* (1.1)–(1.3) *has a unique weak solution $u \in W_1(M,T_*)$, where $M > 0$ and $T_* > 0$ are chosen constants as in Lemma 2.4.*
- (ii) *The linear recurrent sequence $\{u_m\}$ defined by* (2.10)–(2.12) *converges to the solution u of* (1.1)–(1.3) *strongly in the space*

$$W_1(T_*) = \{v \in L^\infty(0,T_*;H_0^1) : v' \in L^\infty(0,T_*;H_0^1)\}. \tag{2.45}$$

Proof. We use the result obtained in Lemma 2.4 and the compact imbedding theorems to prove Lemma 2.5. It means that the existence and uniqueness of a weak solution of Prob. (1.1)–(1.3) is proved.

(i) Existence. It is well known that $W_1(T_*)$ is a Banach space (see Lions [6]), with respect to the norm

$$\|v\|_{W_1(T_*)} = \|v\|_{L^\infty(0,T_*;H_0^1)} + \|v'\|_{L^\infty(0,T_*;H_0^1)}. \tag{2.46}$$

It is clear that $\{u_m\}$ is a Cauchy sequence in $W_1(T_*)$. Indeed, let $w_m = u_{m+1} - u_m$, we have

$$\begin{gathered} \begin{aligned} &\langle w_m''(t), w\rangle + \langle B_{m+1}(t)(w_{mx}(t) + \lambda_1 w_{mx}'(t) + w_{mx}''(t)), w_x\rangle \\ &= \langle F_{m+1}(t) - F_m(t), w\rangle + \langle G_{m+1}(t) - G_m(t), w_x\rangle \\ &\quad - \langle [B_{m+1}(t) - B_m(t)](u_{mx}(t) + \lambda_1 u_{mx}'(t) + u_{mx}''(t)), w_x\rangle, \quad \forall w \in H_0^1, \end{aligned} \\ w_m(0) = w_m'(0) = 0. \end{gathered} \tag{2.47}$$

Consider (2.47) with $w = w_m'$, and then integrating in t, we obtain

$$
\begin{aligned}
&Z_m(t) \\
&= 2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle ds + 2\int_0^t \langle G_{m+1}(s) - G_m(s), w_{mx}'(s)\rangle ds \\
&\quad + \int_0^t ds \int_0^1 B_{m+1}'(x,s)(w_{mx}^2(x,s) + |w_{mx}'(x,s)|^2)dx \\
&\quad - 2\int_0^t \langle (B_{m+1}(s) - B_m(s))(u_{mx}(s) + \lambda_1 u_{mx}'(s) + u_{mx}''(s)), w_{mx}'(s)\rangle ds \\
&= J_1 + J_2 + J_3 + J_4,
\end{aligned}
\tag{2.48}
$$

with

$$
\begin{aligned}
Z_m(t) &= \|w_m'(t)\|^2 + \|\sqrt{B_{m+1}(t)}w_{mx}'(t)\|^2 + \|\sqrt{B_{m+1}(t)}w_{mx}(t)\|^2 \\
&\quad + 2\lambda_1 \int_0^t \|\sqrt{B_{m+1}(s)}w_{mx}'(s)\|^2 ds.
\end{aligned}
$$

From

$$
\begin{gathered}
\|F_{m+1}(s) - F_m(s)\| \le 2(1+2M)\bar{F}_M \|w_{m-1}\|_{W_1(T_*)}, \\
\|G_{m+1}(s) - G_m(s)\| \le 2(1+2M)\bar{G}_M \|w_{m-1}\|_{W_1(T_*)}, \\
|B_{m+1}'(x,s)| \le (1+M+8M^2)\bar{B}_M \equiv \tilde{B}_M, \\
|B_{m+1}(x,s) - B_m(x,s)| \le (1+4M)\bar{B}_M \|w_{m-1}\|_{W_1(T_*)}, \\
\|u_{mx}(s) + \lambda_1 u_{mx}'(s) + u_{mx}''(s)\| \le (2+\lambda_1)M,
\end{gathered}
$$

we obtain the estimates

$$
\begin{aligned}
J_1 + J_2 &= 2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle ds \\
&\quad + 2\int_0^t \langle G_{m+1}(s) - G_m(s), w_{mx}'(s)\rangle ds \\
&\le \frac{4}{b_0}(1+2M)^2(\bar{F}_M + \bar{G}_M)^2 T_* \|w_{m-1}\|_{W_1(T_*)}^2 + \int_0^t Z_m(s)ds;
\end{aligned}
\tag{2.49}
$$

$$
\begin{aligned}
J_3 &= \int_0^t ds \int_0^1 B_{m+1}'(x,s)(w_{mx}^2(x,s) + |w_{mx}'(x,s)|^2)dx \\
&\le \tilde{B}_M \int_0^t (\|w_{mx}(s)\|^2 + \|w_{mx}'(s)\|^2)ds \le \frac{\tilde{B}_M}{b_0}\int_0^t Z_m(s)ds;
\end{aligned}
$$

$$
\begin{aligned}
J_4 &= -2\int_0^t \langle (B_{m+1}(s) - B_m(s))(u_{mx}(s) + \lambda_1 u_{mx}'(s) + u_{mx}''(s)), w_{mx}'(s)\rangle ds \\
&\le 2(2+\lambda_1)(1+4M)M\bar{B}_M \|w_{m-1}\|_{W_1(T_*)} \int_0^t \|w_{mx}'(s)\| ds \\
&\le \frac{1}{b_0}(2+\lambda_1)^2(1+4M)^2 M^2 \bar{B}_M^2 T_* \|w_{m-1}\|_{W_1(T_*)}^2 + \int_0^t Z_m(s)ds.
\end{aligned}
$$

From (2.48) and (2.49) we have

$$Z_m(t) \le T_* \bar{D}_M \|w_{m-1}\|^2_{W_1(T_*)} + \big(2 + \frac{\tilde{B}_M}{b_0}\big) \int_0^t Z_m(s)ds, \tag{2.50}$$

with

$$\bar{D}_M = \frac{1}{b_0}[4(1+2M)^2(\bar{F}_M + \bar{G}_M)^2 + (2+\lambda_1)^2(1+4M)^2 M^2 \bar{B}_M^2]. \tag{2.51}$$

Using Gronwall's Lemma, (2.50) leads to

$$\|w_m\|_{W_1(T_*)} \le k_{T_*} \|w_{m-1}\|_{W_1(T_*)} \quad \forall m \in \mathbb{N}, \tag{2.52}$$

so

$$\|u_m - u_{m+p}\|_{W_1(T_*)} \le M(1-k_{T_*})^{-1} k_{T_*}^m, \quad \forall m, p \in \mathbb{N}. \tag{2.53}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T_*)$, so there exists $u \in W_1(T_*)$ such that

$$u_m \to u \text{ strongly in } W_1(T_*). \tag{2.54}$$

Note that $u_m \in W_1(M, T_*)$, so there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{gathered} u_{m_j} \to u \quad \text{in } L^\infty(0, T_*; H_0^1 \cap H^2) \text{ weakly*}, \\ u'_{m_j} \to u' \quad \text{in } L^\infty(0, T_*; H_0^1 \cap H^2) \text{ weakly*}, \\ u''_{m_j} \to u'' \quad \text{in } L^\infty(0, T_*; H_0^1) \text{ weakly*}, \\ u \in W(M, T_*). \end{gathered} \tag{2.55}$$

On the other hand, by (2.8), (2.10), (2.12) and $(2.55)_4$, we obtain

$$\begin{gathered} \|F_m(t) - F[u](t)\| \le 2(1+2M)\bar{F}_M \|u_{m-1} - u\|_{W_1(T_*)}, \\ \|G_m(t) - G[u](t)\| \le 2(1+2M)\bar{G}_M \|u_{m-1} - u\|_{W_1(T_*)}, \\ |B_{m+1}(x,t) - B[u](x,t)| \le (1+4M)\bar{B}_M \|u_{m-1} - u\|_{W_1(T_*)}. \end{gathered} \tag{2.56}$$

Then (2.54) and (2.56) imply

$$\begin{gathered} F_m \to F[u] \quad \text{strongly in } L^\infty(0, T_*; L^2), \\ G_m \to G[u] \quad \text{strongly in } L^\infty(0, T_*; L^2), \\ B_m \to B[u] \quad \text{strongly in } L^\infty(Q_{T_*}). \end{gathered} \tag{2.57}$$

Passing to limit in (2.11), (2.12) as $m = m_j \to \infty$, by (2.54), (2.55) and (2.57), there exists $u \in W(M, T_*)$ satisfying

$$\begin{aligned} &\langle u''(t), w\rangle + \langle B[u](t)(u''_x(t) + \lambda_1 u'_x(t) + u_x(t)), w_x\rangle \\ &= \langle f(t), w\rangle + \langle F[u](t), w\rangle + \langle G[u](t), w_x\rangle, \quad \forall w \in H_0^1, \end{aligned} \tag{2.58}$$

and satisfying the initial conditions

$$u(0) = \tilde{u}_0, \quad u'(0) = \tilde{u}_1. \tag{2.59}$$

Furthermore, assumption (H2) implies, from $(2.55)_4$ and (2.58) that

$$\begin{aligned} B[u]\Delta u'' &= -B[u](\Delta u + \lambda_1 \Delta u') - \frac{\partial}{\partial x}(B[u])(u_x + \lambda_1 u'_x + u''_x) \\ &\quad + u'' - F[u] + \frac{\partial}{\partial x} G[u] - f \equiv \Psi \in L^\infty(0, T_*; L^2). \end{aligned} \tag{2.60}$$

From

$$b_0 \|\Delta u''(t)\| \le \|B[u](t)\Delta u''(t)\| = \|\Psi(t)\| \le \|\Psi\|_{L^\infty(0,T_*;L^2)},$$

we obtain $u'' \in L^\infty(0,T_*;H_0^1 \cap H^2)$, and so $u \in W_1(M,T_*)$. The existence is proved.

(ii) Uniqueness. Let u_1, u_2 be two weak solutions of (1.1)–(1.3), such that

$$u_i \in W_1(M,T_*), \quad i=1,2. \tag{2.61}$$

Then $w=u_1-u_2$ satisfies

$$\begin{gathered}
\langle w''(t),w\rangle + \langle B_1(t)(w_x(t)+\lambda_1 w_x'(t)+w_x''(t)),w_x\rangle \\
= \langle F_1(t)-F_2(t),w\rangle + \langle G_1(t)-G_2(t),w_x\rangle \\
- \langle [B_1(t)-B_2(t)](u_{2x}(t)+\lambda_1 u_{2x}'(t)+u_{2x}''(t)),w_x\rangle, \quad \forall w \in H_0^1, \\
w(0)=w'(0)=0,
\end{gathered} \tag{2.62}$$

where

$$B_i = B[u_i], \quad F_i = F[u_i], \quad G_i = G[u_i], \quad i=1,2. \tag{2.63}$$

Taking $v=w=u_1-u_2$ in $(2.62)_1$ and integrating with respect to t, we obtain

$$\begin{aligned}
\rho(t) &= 2\int_0^t \langle F_1(s)-F_2(s),w'(s)\rangle ds + 2\int_0^t \langle G_1(s)-G_2(s),w_x'(s)\rangle ds \\
&\quad + \int_0^t ds \int_0^1 B_1'(x,s)(w_x^2(x,s)+|w_x'(x,s)|^2)dx \\
&\quad + 2\int_0^t \langle (B_1(s)-B_2(s))(u_{2x}(s)+\lambda_1 u_{2x}'(s)+u_{2x}''(s)),w_x'(s)\rangle ds,
\end{aligned} \tag{2.64}$$

where

$$\begin{aligned}
\rho(t) &= \|w'(t)\|^2 + \|\sqrt{B_1(t)}w_x'(t)\|^2 + \|\sqrt{B_1(t)}w_x(t)\|^2 \\
&\quad + 2\lambda_1 \int_0^t \|\sqrt{B_1(s)}w_x'(s)\|^2 ds.
\end{aligned} \tag{2.65}$$

On the other hand, by (H3)–(H5), we deduce from (2.8), (2.65), that

$$\begin{gathered}
|B_1'(x,s)| \le (1+M+8M^2)\bar{B}_M \equiv \tilde{B}_M, \\
|B_1(x,s)-B_2(x,s)| \le \sqrt{\frac{2}{b_0}}(1+4M)\bar{B}_M\sqrt{\rho(s)}, \\
\|F_1(s)-F_2(s)\| \le 2\sqrt{\frac{2}{b_0}}(1+2M)\bar{F}_M\sqrt{\rho(s)}, \\
\|G_1(s)-G_2(s)\| \le 2\sqrt{\frac{2}{b_0}}(1+2M)\bar{G}_M\sqrt{\rho(s)}, \\
\|u_{2x}(s)+\lambda_1 u_{2x}'(s)+u_{2x}''(s)\| \le (2+\lambda_1)M.
\end{gathered} \tag{2.66}$$

Combining (2.64) and (2.66) leads to

$$\rho(t) \le [4\sqrt{\frac{2}{b_0}}(1+2M)(\bar{F}_M+\bar{G}_M) + \frac{\tilde{B}_M}{b_0} + \frac{2\sqrt{2}}{b_0}(2+\lambda_1)(1+4M)M\bar{B}_M] \int_0^t \rho(s)ds.$$

By Gronwall's Lemma we have $\rho \equiv 0$, i.e., $u_1 \equiv u_2$. This completes the proof. $\square$

By proving Lemma 2.5, we complete the proof Theorem 2.2. $\square$

3. Blow up

In this section, we consider(1.1)–(1.3) with λ, $\lambda_1 > 0$, $B = B(x,t) \in C^1([0,1] \times [0,T])$, $B(x,t) \geq b_0 > 0$; $F = F(u,u_x) - \lambda u_t$, $G = G(u,u_x)$, F, $G \in C^1(\mathbb{R}^2;\mathbb{R})$ as follows

$$\begin{gathered} u_{tt} - \frac{\partial}{\partial x}[B(x,t)(u_x + \lambda_1 u_{xt} + u_{xtt})] + \lambda u_t \\ = F(u,u_x) - \frac{\partial}{\partial x}(G(u,u_x)) + f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x). \end{gathered} \tag{3.1}$$

Obviously, by the Theorem 2.2, (3.1) has a weak solution $u(x,t)$ such that

$$u \in C^1([0,T_*]; H^2 \cap H_0^1), \quad u'' \in L^\infty(0,T_*; H^2 \cap H_0^1), \tag{3.2}$$

for $T_* > 0$ small enough. Furthermore, if the following assumptions hold, then a blow up result is obtained.

- (H2') $f = 0$;
- (H3') $B \in C^1([0,1] \times [0,T])$ and there exist the positive constants b_0, $\bar{b}_0$, b_1 such that
  - (i) $b_0 \leq B(x,t) \leq \bar{b}_0$, for all $(x,t) \in [0,1] \times [0,T]$,
  - (ii) $-b_1 \leq B'(x,t) \leq 0$, for all $(x,t) \in [0,1] \times [0,T]$;
- (H4') There exist $\mathcal{F} \in C^2(\mathbb{R}^2;\mathbb{R})$ and the constants p, $q > 2$; d_1, $\bar{d}_1 > 0$, such that
  - (i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = F(u,v)$, $\frac{\partial \mathcal{F}}{\partial v}(u,v) = G(u,v)$,
  - (ii) $uF(u,v) + vG(u,v) \geq d_1 \mathcal{F}(u,v)$, for all $(u,v) \in \mathbb{R}^2$,
  - (iii) $\mathcal{F}(u,v) \geq \bar{d}_1(|v|^p + |u|^q)$, for all $(u,v) \in \mathbb{R}^2$;
- (H5') $0 < \lambda_1 < \frac{b_1}{2b_0}$;
- (H6') $d_1 > \max\big\{2 + \frac{2\lambda_1 b_1}{b_0}, \frac{b_1}{b_0\lambda_1} - 2\big\}$ with d_1 as in (H4').

Example 3.1. The following functions satisfy (H4'):

$$\begin{gathered} F(u,v) = \alpha\bar{\gamma}_2|u|^{\alpha-2}u|v|^\beta + q\bar{\gamma}_3|u|^{q-2}u, \\ G(u,v) = p\bar{\gamma}_1|v|^{p-2}v + \beta\bar{\gamma}_2|u|^\alpha|v|^{\beta-2}v, \end{gathered}$$

where α, β, p, $q > 2$; $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{\gamma}_3 > 0$ are the constants, with

$$\min\{p,q,\alpha+\beta\} > \max\{2 + \frac{2\lambda_1 b_1}{b_0}, \frac{b_1}{b_0\lambda_1} - 2\},$$

with b_0, b_1, λ_1 as in $(H3')$, $(H5')$. It is obvious that $(H4')$ holds, because there exists an $\mathcal{F} \in C^2(\mathbb{R}^2;\mathbb{R})$ defined by

$$\mathcal{F}(u,v) = \bar{\gamma}_1|v|^p + \bar{\gamma}_2|u|^\alpha|v|^\beta + \bar{\gamma}_3|u|^q,$$

such that

$$\begin{gathered} \frac{\partial \mathcal{F}}{\partial u}(u,v) = \alpha\bar{\gamma}_2|u|^{\alpha-2}u|v|^\beta + q\bar{\gamma}_3|u|^{q-2}u = F(u,v), \\ \frac{\partial \mathcal{F}}{\partial v}(u,v) = p\bar{\gamma}_1|v|^{p-2}v + \beta\bar{\gamma}_2|u|^\alpha|v|^{\beta-2}v = G(u,v), \\ uF(u,v) + vG(u,v) \geq d_1\mathcal{F}(u,v), \quad \text{for all } (u,v) \in \mathbb{R}^2, \end{gathered}$$

in which $d_1 = \min\{p, q, \alpha+\beta\} > \max\{2 + \frac{2\lambda_1 b_1}{b_0}, \frac{b_1}{b_0\lambda_1} - 2\}$,

$$\mathcal{F}(u,v) \geq \bar{d}_1(|v|^p + |u|^q) \quad \text{for all } (u,v) \in \mathbb{R}^2,$$

with $\bar{d}_1 = \min\{\bar{\gamma}_1, \bar{\gamma}_3\}$. Let us put

$$H(0) = -\frac{1}{2}\|\tilde{u}_1\|^2 - \frac{1}{2}\|\sqrt{B(0)}\tilde{u}_{1x}\|^2 - \frac{1}{2}\|\sqrt{B(0)}\tilde{u}_{0x}\|^2 + \int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{u}_{0x}(x))dx.$$

Theorem 3.2. *Let* (H2')–(H6') *hold. Then, for any* $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$, *such that* $H(0) > 0$, *the weak solution* $u = u(x,t)$ *of* (3.1) *blows up in finite time.*

Proof. It consists of two steps: the Lyapunov functional $L(t)$ is constructed in step1 and then the blow up is proved in step 2.

Step 1. We define the energy associated with (3.1) as

$$\begin{aligned} E(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\sqrt{B(t)}u'_x(t)\|^2 + \frac{1}{2}\|\sqrt{B(t)}u_x(t)\|^2 \\ - \int_0^1 \mathcal{F}(u(x,t), u_x(x,t))dx, \end{aligned} \tag{3.3}$$

and we put $H(t) = -E(t)$, for all $t \in [0, T_*)$. Multiplying $(3.1))_1$ by $u'(x,t)$ and integrating the resulting equation over $[0,1]$, we have

$$\begin{aligned} H'(t) &= \lambda\|u'(t)\|^2 + \lambda_1\|\sqrt{B(t)}u'_x(t)\|^2 \\ &\quad - \frac{1}{2}\int_0^1 B'(x,t)(u_x^2(x,t) + |u'_x(x,t)|^2)dx \geq 0. \end{aligned} \tag{3.4}$$

This implies

$$0 < H(0) \leq H(t), \quad \forall t \in [0, T_*), \tag{3.5}$$

so

$$\begin{aligned} &0 < H(0) \leq H(t) \leq \int_0^1 \mathcal{F}(u(x,t), u_x(x,t))dx; \\ &\|u'(t)\|^2 + \|\sqrt{B(t)}u'_x(t)\|^2 + \|\sqrt{B(t)}u_x(t)\|^2 \\ &\leq 2\int_0^1 \mathcal{F}(u(x,t), u_x(x,t))dx, \quad \forall t \in [0, T_*). \end{aligned} \tag{3.6}$$

Now, we define the functional

$$L(t) = H^{1-\eta}(t) + \varepsilon\Psi(t), \tag{3.7}$$

where

$$\Psi(t) = \langle u'(t), u(t)\rangle + \langle B(t)u'_x(t), u_x(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2 + \frac{\lambda_1}{2}\|\sqrt{B(t)}u_x(t)\|^2, \tag{3.8}$$

for ε small enough and

$$0 < \eta < 1, \ 2/(1-2\eta) \leq \min\{p, q\}. \tag{3.9}$$

Next we show that there exists a constant $\bar{L}_1 > 0$ such that

$$L'(t) \geq \bar{L}_1[H(t) + \|u_x(t)\|_{L^p}^p + \|u(t)\|_{L^q}^q + \|u'(t)\|^2 + \|u'_x(t)\|^2 + \|u_x(t)\|^2]. \tag{3.10}$$

Multiplying $(3.1)_1$ by $u(x,t)$ and integrating over $[0,1]$ leads to

$$\begin{aligned} \Psi'(t) &= \|u'(t)\|^2 + \|\sqrt{B(t)}u_x'(t)\|^2 - \|\sqrt{B(t)}u_x(t)\|^2 \\ &\quad + \langle B'(t)u_x'(t), u_x(t)\rangle + \frac{\lambda_1}{2}\int_0^1 B'(x,t)u_x^2(x,t)dx \\ &\quad + \langle F(u(t),u_x(t)),u(t)\rangle + \langle G(u(t),u_x(t)),u_x(t)\rangle. \end{aligned} \tag{3.11}$$

Therefore,

$$L'(t) = (1-\eta)H^{-\eta}(t)H'(t) + \varepsilon\Psi'(t) \geq \varepsilon\Psi'(t). \tag{3.12}$$

By $(H4')$, we obtain

$$\begin{gathered} \langle F(u(t),u_x(t)),u(t)\rangle + \langle G(u(t),u_x(t)),u_x(t)\rangle \\ \geq d_1\int_0^1 \mathcal{F}(u(x,t),u_x(x,t))dx, \\ \int_0^1 \mathcal{F}(u(x,t),u_x(x,t))dx \geq \bar{d}_1(\|u_x(t)\|_{L^p}^p + \|u(t)\|_{L^q}^q). \end{gathered} \tag{3.13}$$

On the other hand, by (H3'), we obtain

$$\begin{aligned} &\big|\langle B'(t)u_x'(t), u_x(t)\rangle + \frac{\lambda_1}{2}\int_0^1 B'(x,t)u_x^2(x,t)dx\big| \\ &\leq \frac{b_1}{b_0}\|\sqrt{B(t)}u_x'(t)\|\|\sqrt{B(t)}u_x(t)\| + \frac{\lambda_1 b_1}{2b_0}\|\sqrt{B(t)}u_x(t)\|^2 \\ &\leq \frac{b_1}{2b_0}\Big(\frac{1}{\lambda_1}\|\sqrt{B(t)}u_x'(t)\|^2 + \lambda_1\|\sqrt{B(t)}u_x(t)\|^2\Big) + \frac{\lambda_1 b_1}{2b_0}\|\sqrt{B(t)}u_x(t)\|^2 \\ &= \frac{b_1}{2b_0\lambda_1}\|\sqrt{B(t)}u_x'(t)\|^2 + \frac{\lambda_1 b_1}{b_0}\|\sqrt{B(t)}u_x(t)\|^2. \end{aligned} \tag{3.14}$$

From (3.11), (3.13), (3.14) it follows that

$$\begin{aligned} &\Psi'(t) \\ &\geq \|u'(t)\|^2 + \|\sqrt{B(t)}u_x'(t)\|^2 - \|\sqrt{B(t)}u_x(t)\|^2 \\ &\quad - [\frac{b_1}{2b_0\lambda_1}\|\sqrt{B(t)}u_x'(t)\|^2 + \frac{\lambda_1 b_1}{b_0}\|\sqrt{B(t)}u_x(t)\|^2] + d_1\int_0^1 \mathcal{F}(u(x,t),u_x(x,t))dx \\ &= \|u'(t)\|^2 + (1-\frac{b_1}{2b_0\lambda_1})\|\sqrt{B(t)}u_x'(t)\|^2 - \big(1+\frac{\lambda_1 b_1}{b_0}\big)\|\sqrt{B(t)}u_x(t)\|^2 \\ &\quad + d_1\delta_1\int_0^1 \mathcal{F}(u(x,t),u_x(x,t))dx \\ &\quad + d_1(1-\delta_1)[H(t) + \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\sqrt{B(t)}u_x'(t)\|^2 + \frac{1}{2}\|\sqrt{B(t)}u_x(t)\|^2] \\ &\geq d_1\delta_1\bar{d}_1(\|u_x(t)\|_{L^p}^p + \|u(t)\|_{L^q}^q) + d_1(1-\delta_1)H(t) + [1+\frac{d_1}{2}(1-\delta_1)]\|u'(t)\|^2 \\ &\quad + \frac{1}{2}[2+d_1-\frac{b_1}{b_0\lambda_1} - \delta_1 d_1]\|\sqrt{B(t)}u_x'(t)\|^2 \\ &\quad + \frac{1}{2}[d_1 - 2\big(1+\frac{\lambda_1 b_1}{b_0}\big) - \delta_1 d_1]\|\sqrt{B(t)}u_x(t)\|^2, \end{aligned}$$

for all $\delta_1 \in (0,1)$.

From $d_1 > \max\{2 + \frac{2\lambda_1 b_1}{b_0}, \frac{b_1}{b_0\lambda_1} - 2\}$, we have $d_1 - 2(1 + \frac{\lambda_1 b_1}{b_0}) > 0$ and $2 + d_1 - \frac{b_1}{b_0\lambda_1} > 0$, we can choose $\delta_1 > 0$ small enough such that

$$2 + d_1 - \frac{b_1}{b_0\lambda_1} - \delta_1 d_1 > 0 \text{ and } d_1 - 2\big(1 + \frac{\lambda_1 b_1}{b_0}\big) - \delta_1 d_1 > 0, \tag{3.15}$$

and then (3.10) holds.

From the formula of $L(t)$ and (3.10), we can choose $\varepsilon > 0$ small enough such that

$$L(t) \geq L(0) > 0, \quad \forall t \in [0, T_*). \tag{3.16}$$

Using the inequality

$$\Big(\sum_{i=1}^{5} x_i\Big)^r \leq 5^{r-1} \sum_{i=1}^{5} x_i^r, \quad \text{for all } r > 1, \text{ and } x_1, \dots, x_5 \geq 0, \tag{3.17}$$

we deduce from (3.7)–(3.9) that

$$\begin{aligned} L^{1/(1-\eta)}(t) \leq \text{const.}\big[&H(t) + |\langle u(t), u'(t)\rangle|^{1/(1-\eta)} + |\langle B(t)u_x'(t), u_x(t)\rangle|^{1/(1-\eta)} \\ &+ \|u(t)\|^{2/(1-\eta)} + \|\sqrt{B(t)}u_x(t)\|^{2/(1-\eta)}\big]. \end{aligned} \tag{3.18}$$

Using Young's inequality, we have

$$\begin{aligned} |\langle u(t), u'(t)\rangle|^{1/(1-\eta)} &\leq \|u(t)\|^{1/(1-\eta)}\|u'(t)\|^{1/(1-\eta)} \\ &\leq \frac{1-2\eta}{2(1-\eta)}\|u(t)\|^s + \frac{1}{2(1-\eta)}\|u'(t)\|^2 \\ &\leq \text{const.}(\|u_x(t)\|^s + \|u'(t)\|^2), \end{aligned} \tag{3.19}$$

where $s = 2/(1-2\eta) \leq \min\{p, q\}$ as in (3.9). Similarly, we obtain

$$\begin{aligned} |\langle B(t)u_x'(t), u_x(t)\rangle|^{1/(1-\eta)} &\leq \bar{b}_0^{1/(1-\eta)}\|u_x(t)\|^{1/(1-\eta)}\|u_x'(t)\|^{1/(1-\eta)} \\ &\leq \text{const.}\big\|u_x(t)\|^s + \|u_x'(t)\|^2\big). \end{aligned} \tag{3.20}$$

Combining (3.18)–(3.20), we obtain

$$\begin{aligned} L^{1/(1-\eta)}(t) \leq \text{const.}[&H(t) + \|u'(t)\|^2 + \|u_x'(t)\|^2 + \|u(t)\|^{2/(1-\eta)} \\ &+ \|u_x(t)\|^{2/(1-\eta)} + \|u_x(t)\|^s]. \end{aligned} \tag{3.21}$$

Step 2. We note that the following property for any $v \in H_0^1$.

Lemma 3.3. *Let $2 \leq r_1 \leq q$, $2 \leq r_2$, $r_3 \leq p$. Then, for any $v \in H_0^1$, we have*

$$\|v\|^{r_1} + \|v_x\|^{r_2} + \|v_x\|^{r_3} \leq 3\big(\|v\|_{L^q}^q + \|v_x\|_{L^p}^p + \|v_x\|^2\big). \tag{3.22}$$

The proof of the above lemma is not difficult, so we omit it. Using (3.21) and Lemma 3.3 with $r_1 = \frac{2}{1-\eta}$, $r_2 = 2/(1-\eta)$, $r_3 = s$, we obtain

$$\begin{aligned} L^{1/(1-\eta)}(t) \leq \text{const.}\big[&H(t) + \|u'(t)\|^2 + \|u_x'(t)\|^2 + \|u_x(t)\|^2 \\ &+ \|u(t)\|_{L^q}^q + \|u_x(t)\|_{L^p}^p\big], \quad \forall t \in [0, T_*). \end{aligned} \tag{3.23}$$

It follows from (3.10) and (3.23) that

$$L'(t) \geq \bar{L}_2 L^{1/(1-\eta)}(t), \quad \forall t \in [0, T_*), \tag{3.24}$$

where $\bar{L}_2$ is a positive constant. Integrating (3.24) over $(0,t)$ leads to

$$L^{\eta/(1-\eta)}(t) \geq \frac{1}{L^{-\eta/(1-\eta)}(0) - \frac{\bar{L}_2\eta}{1-\eta}t}, \quad 0 \leq t < \frac{1}{\bar{L}_2\eta}(1-\eta)L^{-\eta/(1-\eta)}(0). \tag{3.25}$$

Consequently, $L(t)$ blows up in a finite time given by $T_* = \frac{1}{\bar{L}_2\eta}(1-\eta)L^{-\eta/(1-\eta)}(0)$. The proof of Theorem 3.2 is complete. $\square$

4. Exponential decay

In this section, we consider Problem (3.1) under the following assumptions.

- (H2") $f \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2)$;
- (H3") $B \in C^1([0,1] \times \mathbb{R}_+)$ and there exist three positive constants b_0, $\bar{b}_0$, b_1 such that
  - (i) $b_0 \leq B(x,t) \leq \bar{b}_0$, for all $(x,t) \in [0,1] \times \mathbb{R}_+$,
  - (ii) $-b_1 \leq B'(x,t) \leq 0$, for all $(x,t) \in [0,1] \times \mathbb{R}_+$;
- (H4") There exist $\mathcal{F} \in C^2(\mathbb{R}^2;\mathbb{R})$ and the constants $p,q,\alpha,\beta > 2$; $2 < \alpha,\ \beta, q \leq p$; $d_2, \tilde{d}_1, \bar{d}_2 > 0$, such that
  - (i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = F(u,v)$, $\frac{\partial \mathcal{F}}{\partial v}(u,v) = G(u,v)$, for all $(u,v) \in \mathbb{R}^2$,
  - (ii) $\mathcal{F}_1(u,v) \equiv \mathcal{F}(u,v) + \tilde{d}_1|v|^p \leq \bar{d}_2(|u|^\alpha|v|^\beta + |u|^q)$, for all $(u,v) \in \mathbb{R}^2$,
  - (iii) $uF(u,v) + vG(u,v) \leq d_2\mathcal{F}(u,v)$, for all $(u,v) \in \mathbb{R}^2$;
- (H5") $d_2 < p$ with d_2 as in (H4").

Example 4.1. The functions satisfy (H4"):

$$\begin{gathered} F(u,v) = \alpha\tilde{\gamma}_2|u|^{\alpha-2}u|v|^\beta + q\tilde{\gamma}_3|u|^{q-2}u, \\ G(u,v) = -p\tilde{\gamma}_1|v|^{p-2}v + \beta\tilde{\gamma}_2|u|^\alpha|v|^{\beta-2}v, \end{gathered}$$

where $\alpha,\beta,p,q > 2$; $\tilde{\gamma}_1,\tilde{\gamma}_2,\tilde{\gamma}_3 > 0$ are the constants, with $2 < \alpha,\beta,q < p$ and $\alpha+\beta < p$. We see that (H4") holds. We consider $\mathcal{F} \in C^2(\mathbb{R}^2;\mathbb{R})$ defined by

$$\mathcal{F}(u,v) = -\tilde{\gamma}_1|v|^p + \tilde{\gamma}_2|u|^\alpha|v|^\beta + \bar{\gamma}_3|u|^q.$$

Then we have

$$\begin{gathered} \frac{\partial \mathcal{F}}{\partial u}(u,v) = \alpha\tilde{\gamma}_2|u|^{\alpha-2}u|v|^\beta + q\tilde{\gamma}_3|u|^{q-2}u = F(u,v), \\ \frac{\partial \mathcal{F}}{\partial v}(u,v) = -p\tilde{\gamma}_1|v|^{p-2}v + \beta\tilde{\gamma}_2|u|^\alpha|v|^{\beta-2}v = G(u,v), \\ \mathcal{F}_1(u,v) \equiv \mathcal{F}(u,v) + \tilde{\gamma}_1|v|^p \leq \bar{d}_2(|u|^\alpha|v|^\beta + |u|^q), \end{gathered}$$

for all $(u,v) \in \mathbb{R}^2$, where $\tilde{d}_1 = \tilde{\gamma}_1$, $\bar{d}_2 = \max\{\bar{\gamma}_2, \bar{\gamma}_3\}$.

On the other hand, $(H5'')$ holds, because

$$\begin{aligned} &uF(u,v) + vG(u,v) \\ &= (p-\varepsilon)\mathcal{F}(u,v) - \varepsilon\tilde{\gamma}_1|v|^p + \tilde{\gamma}_2(\alpha+\beta-p+\varepsilon)|u|^\alpha|v|^\beta + \bar{\gamma}_3(q-p+\varepsilon)|u|^q \\ &\leq d_2\mathcal{F}(u,v), \quad \text{for all } (u,v) \in \mathbb{R}^2, \end{aligned}$$

where $d_2 = p - \varepsilon < p$, with $\varepsilon > 0$ small enough such that

$$0 < \varepsilon < p, \quad \alpha+\beta-p+\varepsilon < 0, \quad q-p+\varepsilon < 0.$$

Now, we show the main result of this section. That is, the solution u of (3.1) is global and has exponential decay provided that $E(0)$ is small enough, and

$$I(0)=\|\sqrt{B(0)}\tilde{u}_{0x}\|^2-p\int_0^1\mathcal{F}_1(\tilde{u}_0(x),\tilde{u}_{0x}(x))dx>0,$$

where $p>\max\{2,d_2\}$ with d_2 given in (H4")(iii).

Let $u=u(x,t)$ be a weak solution of (3.1) satisfying (3.2) as note in section 3. To obtain the decay result, we construct the functional

$$\mathcal{L}(t)=E(t)+\delta\Psi(t), \tag{4.1}$$

with $\delta>0$; $E(t)$ and $\Psi(t)$ as definition in Section 3. We rewrite $E(t)$ as follows

$$\begin{aligned}
E(t)&=\frac{1}{2}\|u'(t)\|^2+\frac{1}{2}\|\sqrt{B(t)}u_x'(t)\|^2+\frac{1}{2}\|\sqrt{B(t)}u_x(t)\|^2+\tilde{d}_1\|u_x(t)\|_{L^p}^p\\
&\quad-\int_0^1\mathcal{F}_1(u(x,t),u_x(x,t))dx\\
&=\frac{1}{2}\|u'(t)\|^2+\frac{1}{2}\|\sqrt{B(t)}u_x'(t)\|^2+(\frac{1}{2}-\frac{1}{p})\|\sqrt{B(t)}u_x(t)\|^2\\
&\quad+\tilde{d}_1\|u_x(t)\|_{L^p}^p+\frac{1}{p}I(t),
\end{aligned}$$

where

$$I(t)=\|\sqrt{B(t)}u_x(t)\|^2-p\int_0^1\mathcal{F}_1(u(x,t),u_x(x,t))dx. \tag{4.2}$$

Theorem 4.2. *Assume that* (H2")-(H5") *hold. Let* $\tilde{u}_0,\tilde{u}_1\in H_0^1\cap H^2$ *such that* $I(0)>0$ *and the initial energy* $E(0)$ *satisfy*

$$\eta^*=b_0-p\bar{d}_2\Big[\Big(\frac{2p}{(p-2)b_0}E_*\Big)^{\frac{\alpha-2}{2}}\Big(\frac{E_*}{\tilde{d}_1}\Big)^{\beta/p}+\Big(\frac{2p}{(p-2)b_0}E_*\Big)^{\frac{q-2}{2}}\Big]>0, \tag{4.3}$$

where

$$E_*=\Big(E(0)+\frac{1}{2}\|f\|_{L^1(\mathbb{R}_+;L^2)}\Big)\exp\big(\|f\|_{L^1(\mathbb{R}_+;L^2)}\big). \tag{4.4}$$

Assume that

$$\|f(t)\|^2\le\bar{C}_1\exp(-\bar{\eta}_1t)\quad\text{for all } t\ge0, \tag{4.5}$$

where $\bar{C}_1$, $\bar{\eta}_1$ *are two positive constants. Then, there exist positive constants* $\bar{C}$, $\bar{\gamma}$ *such that*

$$\|u'(t)\|^2+\|u_x'(t)\|^2+\|u_x(t)\|^2+\|u_x(t)\|_{L^p}^p\le\bar{C}\exp(-\bar{\gamma}t),\quad\text{for all } t\ge0. \tag{4.6}$$

Proof. It consists of three steps.

Step 1. An estimate of $E'(t)$. We have

$$\begin{gathered}
E'(t)\le\frac{1}{2}\|f(t)\|+\|f(t)\|\|u'(t)\|^2,\\
E'(t)\le-(\lambda-\frac{\varepsilon_1}{2})\|u'(t)\|^2-\lambda_1\|\sqrt{B(t)}u_x'(t)\|^2+\frac{1}{2\varepsilon_1}\|f(t)\|^2,
\end{gathered} \tag{4.7}$$

for all $\varepsilon_1>0$. Indeed, multiplying $(3.1)_1$ by $u'(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}
E'(t)&=-\lambda\|u'(t)\|^2-\lambda_1\|\sqrt{B(t)}u_x'(t)\|^2\\
&\quad+\frac{1}{2}\int_0^1B'(x,t)(u_x^2(x,t)+|u_x'(x,t)|^2)dx+\langle f(t),u'(t)\rangle.
\end{aligned} \tag{4.8}$$

On the other hand
$$|\langle f(t), u'(t)\rangle| \leq \frac{1}{2}\|f(t)\| + \frac{1}{2}\|f(t)\|\|u'(t)\|^2. \tag{4.9}$$
From $B'(x,t) \leq 0$, by (4.8), (4.9), it is easy to see that $(4.7)_{(i)}$ holds. Similarly,
$$|\langle f(t), u'(t)\rangle| \leq \frac{1}{2\varepsilon_1}\|f(t)\|_0^2 + \frac{\varepsilon_1}{2}\|u'(t)\|^2, \quad \text{for all } \varepsilon_1 > 0. \tag{4.10}$$

By $B'(x,t) \leq 0$, (4.8) and (4.10), that $(4.7)_{(ii)}$ is valid.

Step 2. An estimate of $I(t)$. By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that
$$I(t) \geq 0, \quad \forall t \in [0, T_1], \tag{4.11}$$
this implies that
$$E(t) \geq \frac{1}{2}\|u'(t)\|^2 + (\frac{1}{2} - \frac{1}{p})\|\sqrt{B(t)}u_x(t)\|^2 + \tilde{d}_1\|u_x(t)\|_{L^p}^p, \ \forall t \in [0, T_1]. \tag{4.12}$$
Combining $(4.7)_i$ with (4.12) and using Gronwall's inequality we obtain
$$\frac{(p-2)b_0}{2p}\|u_x(t)\|^2 + \tilde{d}_1\|u_x(t)\|_{L^p}^p \leq E(t) \leq E_*, \quad \forall t \in [0, T_1]. \tag{4.13}$$
Hence, it follows from $(\bar{H}_4, (iii))$, (4.4), (4.13) that
$$\begin{aligned}
&p\int_0^1 \mathcal{F}_1(u(x,t), u_x(x,t))dx \\
&\leq p\bar{d}_2\Big(\int_0^1 |u(x,t)|^\alpha |u_x(x,t)|^\beta dx + \int_0^1 |u(x,t)|^q dx\Big) \\
&\leq p\bar{d}_2\Big(\|u_x(t)\|^\alpha \|u_x(t)\|_{L^\beta}^\beta + \|u_x(t)\|^q\Big) \\
&\leq p\bar{d}_2\Big(\|u_x(t)\|^\alpha \|u_x(t)\|_{L^p}^\beta + \|u_x(t)\|^q\Big) \\
&\leq p\bar{d}_2[(\frac{2p}{(p-2)b_0}E_*)^{\frac{\alpha-2}{2}}(\frac{E_*}{\tilde{d}_1})^{\beta/p} + (\frac{2p}{(p-2)b_0}E_*)^{\frac{q-2}{2}}]\|u_x(t)\|^2,
\end{aligned} \tag{4.14}$$
for all $t \in [0, T_1]$.

Consequently, $I(t) \geq \eta^*\|u_x(t)\|^2 > 0$, for all $t \in [0, T_1]$. Put $T_\infty = \sup\{T > 0 : I(t) > 0, \ t \in [0, T]\}$. If $T_\infty < +\infty$ then the continuity of $I(t)$ leads to $I(T_\infty) \geq 0$. By the same arguments, there exists $T'_\infty > T_\infty$ such that $I(t) > 0$, for all $t \in [0, T'_\infty]$. Hence, we conclude that $I(t) > 0$, for all $t \geq 0$.

Step 3. Decay result. First, we note that there exist the positive constants $\bar{\beta}_1$, $\bar{\beta}_2$ such that
$$\bar{\beta}_1 E_1(t) \leq \mathcal{L}(t) \leq \bar{\beta}_2 E_1(t), \quad \forall t \geq 0, \tag{4.15}$$
for δ small enough, where
$$E_1(t) = \|u'(t)\|^2 + \|\sqrt{B(t)}u'_x(t)\|^2 + \|\sqrt{B(t)}u_x(t)\|^2 + \|u_x(t)\|_{L^p}^p + I(t). \tag{4.16}$$
Indeed, we have
$$\begin{aligned}
\mathcal{L}(t) &= \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\sqrt{B(t)}u'_x(t)\|^2 + (\frac{1}{2} - \frac{1}{p})\|\sqrt{B(t)}u_x(t)\|^2 \\
&\quad + \tilde{d}_1\|u_x(t)\|_{L^p}^p + \frac{1}{p}I(t) + \delta\Big[\langle u'(t), u(t)\rangle \\
&\quad + \langle B(t)u'_x(t), u_x(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2 + \frac{\lambda_1}{2}\|\sqrt{B(t)}u_x(t)\|^2\Big].
\end{aligned} \tag{4.17}$$

On the other hand,

$$\begin{aligned} \langle u(t), u'(t)\rangle &\leq \frac{1}{2b_0}\|\sqrt{B(t)}u_x(t)\|^2 + \frac{1}{2}\|u'(t)\|^2, \\ \langle B(t)u'_x(t), u_x(t)\rangle &\leq \frac{1}{2}\|\sqrt{B(t)}u'_x(t)\|^2 + \frac{1}{2}\|\sqrt{B(t)}u_x(t)\|^2. \end{aligned} \tag{4.18}$$

Then

$$\begin{aligned} \mathcal{L}(t) &\geq \frac{1}{2}(1-\delta)\|u'(t)\|^2 + \frac{1}{2}(1-\delta)\|\sqrt{B(t)}u'_x(t)\|^2 \\ &\quad + (\frac{1}{2} - \frac{1}{p} - \frac{\delta}{2b_0})\|\sqrt{B(t)}u_x(t)\|^2 + \tilde{d}_1\|u_x(t)\|^p_{L^p} + \frac{1}{p}I(t) \\ &\geq \bar{\beta}_1 E_1(t), \end{aligned} \tag{4.19}$$

where δ is small enough, and

$$\bar{\beta}_1 = \min\big\{\frac{1-\delta}{2},\ \frac{1}{2} - \frac{1}{p} - \frac{\delta}{2b_0},\ \tilde{d}_1,\ \frac{1}{p}\big\} > 0, \quad 0 < \delta < \min\big\{1, \frac{(p-2)b_0}{p}\big\}. \tag{4.20}$$

Similarly,

$$\begin{aligned} \mathcal{F}(t) &\leq \frac{1}{2}(1+\delta)\|u'(t)\|^2 + \frac{1}{2}(1+\delta)\|\sqrt{B(t)}u'_x(t)\|^2 \\ &\quad + \tilde{d}_1\|u_x(t)\|^p_{L^p} + \frac{1}{p}I(t) \\ &\quad + [\frac{1}{2} - \frac{1}{p} + \frac{\delta}{2}(1 + \frac{1}{b_0} + \frac{\lambda}{b_0} + \lambda_1)]\|\sqrt{B(t)}u_x(t)\|^2 \\ &\leq \bar{\beta}_2 E_1(t), \end{aligned} \tag{4.21}$$

where

$$\bar{\beta}_2 = \max\big\{\frac{1+\delta}{2},\ \frac{1}{2} - \frac{1}{p} + \frac{\delta}{2}(1 + \frac{1}{b_0} + \frac{\lambda}{b_0} + \lambda_1),\ \tilde{d}_1\big\} > 0. \tag{4.22}$$

Next, we show that the functional $\Psi(t)$ satisfies

$$\begin{aligned} \Psi'(t) &\leq \|u'(t)\|^2 + \Big(1 + \frac{b_1^2}{2\varepsilon_2 b_0}\Big)\|\sqrt{B(t)}u'_x(t)\|^2 \\ &\quad - \Big(1 - \frac{d_2}{p} - \frac{\varepsilon_2}{b_0}\Big)\|\sqrt{B(t)}u_x(t)\|^2 \\ &\quad - \frac{d_2}{p}I(t) - d_2\tilde{d}_1\|u_x(t)\|^p_{L^p} + \frac{1}{2\varepsilon_2}\|f(t)\|^2, \end{aligned} \tag{4.23}$$

for all $\varepsilon_2 > 0$.

The proof is as follows. Multiplying $(3.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned} \Psi'(t) &= \|u'(t)\|^2 + \|\sqrt{B(t)}u'_x(t)\|^2 - \|\sqrt{B(t)}u_x(t)\|^2 \\ &\quad + \langle B'(t)u'_x(t), u_x(t)\rangle + \frac{\lambda_1}{2}\int_0^1 B'(x,t)u_x^2(x,t)dx \\ &\quad + \langle F(u(t), u_x(t)), u(t)\rangle + \langle G(u(t), u_x(t)), u_x(t)\rangle + \langle f(t), u(t)\rangle. \end{aligned} \tag{4.24}$$

Furthermore, by (H4”)$_{(iii)}$, we obtain

$$\begin{aligned} &\langle F(u(t),u_x(t)),u(t)\rangle+\langle G(u(t),u_x(t)),u_x(t)\rangle \\ &\leq d_2\int_0^1 \mathcal{F}(u(x,t),u_x(x,t))dx \\ &= d_2[\int_0^1 \mathcal{F}_1(u(x,t),u_x(x,t))dx-\tilde{d}_1\|u_x(t)\|_{L^p}^p] \\ &= \frac{d_2}{p}(\|\sqrt{B(t)}u_x(t)\|^2-I(t))-d_2\tilde{d}_1\|u_x(t)\|_{L^p}^p. \end{aligned} \tag{4.25}$$

We also have

$$\begin{gathered} \frac{\lambda_1}{2}\int_0^1 B'(x,t)u_x^2(x,t)dx\leq 0, \\ \langle B'(t)u_x'(t),u_x(t)\rangle\leq \frac{b_1^2}{2\varepsilon_2 b_0}\|\sqrt{B(t)}u_x'(t)\|^2+\frac{\varepsilon_2}{2b_0}\|\sqrt{B(t)}u_x(t)\|^2, \\ \langle f(t),u(t)\rangle\leq \frac{\varepsilon_2}{2b_0}\|\sqrt{B(t)}u_x(t)\|^2+\frac{1}{2\varepsilon_2}\|f(t)\|^2, \end{gathered} \tag{4.26}$$

for all $\varepsilon_2>0$. Combining (4.24)–(4.26), we obtain (4.23).

The estimates (4.7)$_{(ii)}$ and (4.23) give

$$\begin{aligned} \mathcal{L}'(t)\leq& -\big(\lambda-\frac{\varepsilon_1}{2}-\delta\big)\|u'(t)\|^2 \\ &-\big[\lambda_1-\delta\big(1+\frac{b_1^2}{2\varepsilon_2 b_0}\big)\big]\|\sqrt{B(t)}u_x'(t)\|^2 \\ &-\delta(1-\frac{d_2}{p}-\frac{\varepsilon_2}{b_0})\|\sqrt{B(t)}u_x(t)\|^2 \\ &-\frac{\delta d_2}{p}I(t)-\delta d_2\tilde{d}_1\|u_x(t)\|_{L^p}^p+\frac{1}{2}\big(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2}\big)\|f(t)\|^2, \end{aligned} \tag{4.27}$$

for all δ, ε_1, $\varepsilon_2>0$. Because $p>\max\{2,d_2\}\geq d_2$, we can choose $\varepsilon_2>0$ such that

$$\theta_1=1-\frac{d_2}{p}-\frac{\varepsilon_2}{b_0}>0. \tag{4.28}$$

Then, for ε_1 small enough such that $0<\frac{\varepsilon_1}{2}<\lambda$ and if $\delta>0$ such that

$$\begin{gathered} \theta_2=\lambda-\frac{\varepsilon_1}{2}-\delta>0,\quad \theta_3=\lambda_1-\delta(1+\frac{b_1^2}{2\varepsilon_2 b_0})>0, \\ 0<\delta<\min\{1,\frac{(p-2)b_0}{p}\}. \end{gathered} \tag{4.29}$$

By (4.27)-(4.29), we obtain

$$\mathcal{L}'(t)\leq -\bar{\beta}_3E_1(t)+\tilde{C}_1e^{-\bar{\eta}_1 t}\leq -\frac{\bar{\beta}_3}{\bar{\beta}_2}\mathcal{L}(t)+\tilde{C}_1e^{-\bar{\eta}_1 t}\leq -\bar{\gamma}\mathcal{L}(t)+\tilde{C}_1e^{-\bar{\eta}_1 t}, \tag{4.30}$$

where $\bar{\beta}_3=\min\{\delta\theta_1,\,\theta_2,\,\theta_3,\,\frac{\delta d_2}{p},\,\delta d_2\tilde{d}_1\}$, $0<\bar{\gamma}<\min\{\frac{\beta_*}{\bar{\beta}_2},\,\bar{\eta}_1\}$, $\tilde{C}_1=\frac{1}{2}(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2})\bar{C}_1$.

On the other hand, we have

$$\mathcal{L}(t)\geq \bar{\beta}_1\min\{1,b_0\}[\|u'(t)\|^2+b_0\|u_x'(t)\|^2+b_0\|u_x(t)\|^2+\|u(t)\|_{L^p}^p+\|u_x(t)\|_{L^p}^p].$$

This completes the proof. $\square$

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