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UNIQUENESS OF SOLUTION IN A RECTANGULAR DOMAIN OF AN EVOLUTION DAM PROBLEM WITH HETEROGENEOUS COEFFICIENTS

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Abstract. In this article, we consider an evolution dam problem with heterogeneous coefficients of type $a(x_1)(u_{x_2}+\chi)_{x_2}-\chi_t=0$ in a bounded rectangular domain of $\mathbb{R}^2$. We establish uniqueness of the solution for this problem. Our proofs are based on the test function by using the method of doubling variables.

1. Introduction

Let $\Omega = (0,L)\times(0,l)$ a bounded rectangular domain in $\mathbb{R}^2$, Ω represents a porous medium, with Lipschitz boundary $\partial\Omega=\Gamma_1\cup\Gamma_2$ where $\Gamma_2=(\{0\}\times[0,l])\cup([0,L]\times\{l\})\cup(\{L\}\times[0,l])$ is the part in contact with air or covered by fluid and $\Gamma_1=[0,L]\times\{0\}$ is the impervious part of $\partial\Omega$. $Q=\Omega\times(0,T), T>0$, ϕ is a nonnegative Lipschitz function defined in $\overline{Q}$, $\Sigma_1=\Gamma_1\times(0,T)$, $\Sigma_2=\Gamma_2\times(0,T)$, $\Sigma_3=\Sigma_2\cap\{\phi>0\}$ and $\Sigma_4=\Sigma_2\cap\{\phi=0\}$. Moreover, let a be a function of the variable x_1 satisfying for two positive constants $0<\lambda\le\Lambda$:

$$\lambda\le a(x_1)\le\Lambda \quad \text{a.e. } x_1\in(0,L) \tag{1.1}$$

and χ_0 is a function of the variable x satisfying

$$0\le\chi_0(x)\le 1 \quad \text{a.e. } x\in\Omega. \tag{1.2}$$

Now, let us consider the following weak formulation of an evolution dam problem with heterogeneous coefficients [9, 5, 3, 7, 10, 11]: Find $(u,\chi)\in L^2(0,T;H^1(\Omega))\times L^\infty(Q)$ such that

$$\begin{gathered} u\ge 0,\quad 0\le\chi\le 1,\quad u.(1-\chi)=0 \quad \text{a.e. in } Q,\\ u=\phi \quad \text{on } \Sigma_2,\\ \int_Q [a(x_1)(u_{x_2}+\chi)\xi_{x_2}-\chi\xi_t]dx\,dt\le\int_\Omega \chi_0(x)\xi(x,0)\,dx \\ \forall\xi\in H^1(Q),\quad \xi=0 \text{ on } \Sigma_3,\quad \xi\ge 0 \text{ on } \Sigma_4,\quad \xi(x,T)=0 \text{ for a.e. } x\in\Omega. \end{gathered} \tag{1.3}$$

Note that the strong formulation corresponding to (1.3) is given by

$$u\ge 0,\quad 0\le\chi\le 1,\quad u(1-\chi)=0 \quad \text{in } Q$$

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$$\begin{gathered}
a(x_1)(u_{x_2}+\chi)_{x_2}-\chi_t=0 \quad \text{in } Q\\
u=\phi \quad \text{on } \Sigma_2\\
\chi(\cdot,0)=\chi_0 \quad \text{in } \Omega\\
a(x_1)(u_{x_2}+\chi)\cdot\nu=0 \quad \text{on } \Sigma_1\\
a(x_1)(u_{x_2}+\chi)\cdot\nu\le 0 \quad \text{on } \Sigma_4.
\end{gathered}$$

Regarding existence of a solution of the problem (1.3) we refer to [5] and [11] respectively for the evolutionary dam problem with homogeneous coefficients and for a class of free boundary problem in heterogeneous domain. The regularity of the solution of the problem (1.3) was discussed in [8], where it was proved that $\chi\in C^0([0,T];L^p(\Omega))$ for all $p\in[1,+\infty)$ in both the class of free boundary problem of types $\operatorname{div}(a(x)\nabla u+H(x)\chi)-\chi_t$ and $\operatorname{div}(a(x)\nabla u+H(x)\chi)-(u+\chi)_t$, and that $u\in C^0([0,T];L^p(\Omega))$ for all $p\in[1,2]$ in the second class.

Uniqueness of the solution for the evolutionary dam problem in the homogeneous case for both incompressible and compressible fluids was obtained in [2] by using the method of doubling variables. In the case of a rectangular dam wet at the bottom and dry near to the top, the uniqueness was obtained in [4] and [9] by a different method, respectively in homogeneous and heterogeneous porous media. For the evolution free boundary problem in theory of lubrication, we refer to [1].

In this paper, we consider the weak formulation of an evolution dam problem with heterogeneous coefficients (1.3) in a bounded rectangular domain Ω of $\mathbb{R}^2$. We establish uniqueness of the solution for this problem. Our proofs are based on the test function by using the method of doubling variables. This uniqueness result is new in the general framework of an heterogeneous and bounded rectangular domain.

2. Properties

We shall denote by (u,χ) a solution of (1.3).

Proposition 2.1 ([8])**.** *If $a\in C^{0,1}([0,L])$, then we have*

$$\chi\in C^0([0,T];L^p(\Omega))\quad \forall p\in[1,+\infty).$$

Proposition 2.2. *For $\epsilon>0$, $k\ge 0$ and $\xi\in\mathcal{D}(\mathbb{R}^2\times(0,T))$ such that $\xi\ge 0$, $\xi=0$ on Σ_3, we have*

$$\int_Q a(x_1)(u_{x_2}+\chi)\Big(\min\Big(\frac{(u-k)^+}{\epsilon},\xi\Big)\Big)_{x_2}\,dx\,dt=0 \tag{2.1}$$

and if $\xi=0$ on Σ_2,

$$\int_Q a(x_1)(u_{x_2}+\chi)\Big(\min\Big(\frac{(k-u)^+}{\epsilon},\xi\Big)-\min\Big(\frac{k}{\epsilon},\xi\Big)\Big)_{x_2}\,dx\,dt=0. \tag{2.2}$$

Proof. Let ζ be a smooth function such that $d(\operatorname{supp}(\zeta),\Sigma_2)>0$ and $\operatorname{supp}(\zeta)\subset\mathbb{R}^2\times(0,T)$. Then there exists $\tau_0>0$ such that for any $\tau\in(-\tau_0,\tau_0)$ the functions $(x,t)\mapsto\pm\zeta(x,t-\tau)$ vanishes on Σ_2 and in $\Omega\times\{0,T\}$. So, they are test functions for (1.3) and we have

$$\int_Q\big[a(x_1)(u_{x_2}+\chi)\zeta_{x_2}(x,t-\tau)+(1-\chi)\zeta_t(x,t-\tau))\big]\,dx\,dt=0$$

which can be written as

$$\int_Q a(x_1)(u_{x_2}+\chi)\zeta_{x_2}(x,t-\tau)\,dx\,dt = \frac{\partial}{\partial\tau}\Big(\int_Q (1-\chi(x,t+\tau))\zeta(x,t)\,dx\,dt\Big). \tag{2.3}$$

This identity still holds for any $\zeta \in L^2(0,T;H^1(\Omega))$ such that $\zeta = 0$ on Σ_2 and $\zeta = 0$ on $\Omega \times ((0,\tau_0) \cup (T-\tau_0,T))$. So, if we consider $\xi \in \mathcal{D}(\mathbb{R}^2 \times (\tau_0, T-\tau_0))$ such that $\xi \geq 0$, $\xi = 0$ on Σ_3, and set $\zeta = \min\big(\frac{(u-k)^+}{\epsilon}, \xi\big)$, we have from (2.3) for all $\tau \in (-\tau_0,\tau_0)$:

$$\begin{aligned} &\int_Q a(x_1)(u_{x_2}+\chi)\Big(\min\Big(\frac{(u-k)^+}{\epsilon},\xi\Big)\Big)_{x_2}(x,t-\tau)\,dx\,dt \\ &= \frac{\partial}{\partial\tau}\Big(\int_Q (1-\chi(x,t+\tau))\min\Big(\frac{(u-k)^+}{\epsilon},\xi\Big)(x,t)\,dx\,dt\Big) := G'(\tau) \end{aligned} \tag{2.4}$$

where

$$G(\tau) = \int_Q (1-\chi(x,t+\tau))\min\Big(\frac{(u-k)^+}{\epsilon},\xi\Big)(x,t)\,dx\,dt.$$

Since the integral on the left hand side of (2.4) is continuous on $(-\tau_0,\tau_0)$, the function G' belongs to $C^0(-\tau_0,\tau_0)$. So, G is in $C^1(-\tau_0,\tau_0)$. Moreover, for all $\tau \in (-\tau_0,\tau_0)$, $G(\tau) \geq 0 = G(0)$ since $u \geq 0$, $0 \leq \chi \leq 1$ and $u(1-\chi) = 0$ a.e. in Q. So, 0 is absolute minimum for G in $(-\tau_0,\tau_0)$ and $G'(0) = 0$. Using (2.4), we deduce that (2.1) holds for $\xi \in \mathcal{D}(\mathbb{R}^2 \times (0,T))$ such that $\xi \geq 0$, $\xi = 0$ on Σ_3.

Now if we consider $\xi = 0$ on Σ_2, and set $\zeta = \min\big(\frac{(k-u)^+}{\epsilon},\xi\big) - \min\big(\frac{k}{\epsilon},\xi\big)$, we have from (2.3) for all $\tau \in (-\tau_0,\tau_0)$:

$$\begin{aligned} &\int_Q a(x_1)(u_{x_2}+\chi)\Big(\min\Big(\frac{(k-u)^+}{\epsilon},\xi\Big) - \min\Big(\frac{k}{\epsilon},\xi\Big)\Big)_{x_2}(x,t-\tau)\,dx\,dt \\ &= \frac{\partial}{\partial\tau}\Big(\int_Q (1-\chi(x,t+\tau))\Big(\min\Big(\frac{(k-u)^+}{\epsilon},\xi\Big) - \min\Big(\frac{k}{\epsilon},\xi\Big)\Big)(x,t)\,dx\,dt\Big) \\ &:= K'(\tau) \end{aligned} \tag{2.5}$$

where

$$K(\tau) = \int_Q (1-\chi(x,t+\tau))\Big(\min\Big(\frac{(k-u)^+}{\epsilon},\xi\Big) - \min\Big(\frac{k}{\epsilon},\xi\Big)\Big)(x,t)\,dx\,dt.$$

Since $u \geq 0$, $0 \leq \chi \leq 1$ and $u(1-\chi) = 0$ a.e. in Q, we have for all $\tau \in (-\tau_0,\tau_0)$, $K(\tau) \leq 0 = K(0)$. So, 0 is absolute maximum for K in $(-\tau_0,\tau_0)$ and $K'(0) = 0$. Using (2.5), we deduce that (2.2) holds for $\xi \in \mathcal{D}(\mathbb{R}^2 \times (0,T))$ such that $\xi \geq 0$, $\xi = 0$ on Σ_2. □

3. Uniqueness of the solution

In this section, we state and prove our main result, that the solution of problem (1.3) is unique. Let us assume that

$$a \in C^1([0,L]). \tag{3.1}$$

We begin with the following theorem.

Theorem 3.1. *Let* (u_1,χ_1) *and* (u_2,χ_2) *be two solutions of* (1.3). *Then we have*

$$\begin{aligned} &\int_Q a(x_1)\big\{(u_1(x,t)-u_2(x,t))^+_{x_2}+(1-\chi_2(x,t))\chi_{\{u_1>u_2\}}\\ &+\big((1-\chi_2(x,t))+(1-u_{2x_2}(x,t))\big)\chi_{\{u_1>0\}}\big\}\eta\xi_{x_2}\,dx\,dt\le 0\\ &\quad\forall\xi\in\mathcal{D}(\Omega),\quad \xi\ge 0,\quad \forall\eta\in\mathcal{D}(0,T),\quad \eta\ge 0. \end{aligned} \tag{3.2}$$

Proof. Let us consider (u_1,χ_1) and (u_2,χ_2) related to the variables (x,t,y,s) in the following way

$$\begin{gathered} (u_1,\chi_1):(x,t,y,s)\mapsto(u_1(x,t),\chi_1(x,t))\\ (u_2,\chi_2):(x,t,y,s)\mapsto(u_2(y,s),\chi_2(y,s))\,. \end{gathered}$$

Let $\xi\in\mathcal{D}(\Omega),\eta\in\mathcal{D}(0,T)$ such that $\xi\ge 0$, $\eta\ge 0$. For all $(x,t,y,s)\in\overline{Q\times Q}$, we define

$$\zeta(x,t,y,s)=\xi(\frac{x_1+y_1}{2},\frac{x_2+y_2}{2})\eta(\frac{t+s}{2})\rho_{1,\delta}(\frac{x_1-y_1}{2})\rho_{2,\delta}(\frac{x_2-y_2}{2})\rho_{3,\delta}(\frac{t-s}{2}),$$

where $\rho_{1,\delta}(r)=\frac{1}{\delta}\rho_1(\frac{r}{\delta})$, $\rho_{2,\delta}(r)=\frac{1}{\delta}\rho_2(\frac{r}{\delta})$, $\rho_{3,\delta}(r)=\frac{1}{\delta}\rho_3(\frac{r}{\delta})$ with $\rho_1,\rho_2,\rho_3\in\mathcal{D}(\mathbb{R})$, $\rho_1,\rho_2,\rho_3\ge 0$, $\operatorname{supp}(\rho_1),\operatorname{supp}(\rho_2),\operatorname{supp}(\rho_3)\subset(-1,1)$. For δ small enough, we have

$$\zeta(\cdot,\cdot,y,s)\in\mathcal{D}(Q)\quad\forall(y,s)\in Q \tag{3.3}$$

$$\zeta(x,t,\cdot,\cdot)\in\mathcal{D}(Q)\quad\forall(x,t)\in Q. \tag{3.4}$$

Let ϵ be a positive real number. We define

$$\vartheta(x,t,y,s)=\min\Big(\frac{(u_1(x,t)-u_2(y,s))^+}{\epsilon},\zeta(x,t,y,s)\Big). \tag{3.5}$$

Now, for almost every $(y,s)\in Q$ we can apply (2.1) (of Proposition 2.2) to (u_1,χ_1) with $k=u_2(y,s),\xi(x,t)=\vartheta(x,t,y,s)$, from which it follows that

$$\int_Q a(x_1)(u_{1x_2}+\chi_1)\vartheta_{x_2}\,dx\,dt=0. \tag{3.6}$$

Since $u_1.(1-\chi_1)=0$ a.e. in Q, we have

$$\chi_1 a(x_1)\Big(\min\Big(\frac{(u_1-u_2)^+}{\epsilon},\zeta\Big)\Big)_{x_2}=a(x_1)\Big(\min\Big(\frac{(u_1-u_2)^+}{\epsilon},\zeta\Big)\Big)_{x_2}$$

a.e. in Q and (3.6) can be written as

$$\int_Q a(x_1)(u_{1x_2}+1)\vartheta_{x_2}\,dx\,dt=0.$$

By integrating over Q, we obtain

$$\int_{Q\times Q} a(x_1)(u_{1x_2}+1)\vartheta_{x_2}\,dx\,dt\,dy\,ds=0. \tag{3.7}$$

Similarly, for almost every $(x,t)\in Q$, we can apply (2.2) (of Proposition 2.2) to (u_2,χ_2) with $k=u_1(x,t)$, $\xi(y,s)=\vartheta(x,t,y,s)$ to get

$$\int_Q a(y_1)(u_{2y_2}+\chi_2)\Big(\vartheta-\min\Big(\frac{u_1}{\epsilon},\zeta\Big)\Big)_{y_2}\,dy\,ds=0.$$

By integrating over Q, we obtain

$$\int_{Q\times Q} a(y_1)(u_{2y_2}+\chi_2)\Big(\vartheta-\min\Big(\frac{u_1}{\epsilon},\zeta\Big)\Big)_{y_2}\,dy\,ds\,dx\,dt=0. \tag{3.8}$$

Then, subtracting (3.8) from (3.7), we obtain

$$\begin{aligned}
&\int_{Q\times Q} \big[a(x_1)u_{1x_2}\vartheta_{x_2} - a(y_1)u_{2y_2}\vartheta_{y_2}\\
&+ a(x_1)\vartheta_{x_2} - \chi_2 a(y_1)\vartheta_{y_2}\big]\,dx\,dt\,dy\,ds\\
&- \int_{Q\times Q} a(y_1)(u_{2y_2}+\chi_2)\min\Big(\frac{u_1}{\epsilon},\zeta\Big)_{y_2}\,dx\,dt\,dy\,ds = 0\,.
\end{aligned}\tag{3.9}$$

Moreover, from (3.3)-(3.5), we have

$$\int_{Q\times Q} a(x_1)u_{1x_2}\vartheta_{y_2}\,dx\,dt\,dy\,ds = 0 \tag{3.10}$$

$$\int_{Q\times Q} a(y_1)u_{2y_2}\vartheta_{x_2}\,dx\,dt\,dy\,ds = 0 \tag{3.11}$$

$$\int_{Q\times Q} a(x_1)(\partial_{x_2}+\partial_{y_2})\vartheta\,dx\,dt\,dy\,ds = 0 \tag{3.12}$$

$$\int_{Q\times Q} \chi_2 a(y_1)\vartheta_{x_2}\,dx\,dt\,dy\,ds = 0 \tag{3.13}$$

$$\int_{Q\times Q} a(y_1)(u_{2y_2}+\chi_2)\min\Big(\frac{u_1}{\epsilon},\zeta\Big)_{x_2}\,dx\,dt\,dy\,ds = 0 \tag{3.14}$$

$$\int_{Q\times Q} a(x_1)(\partial_{x_2}+\partial_{y_2})\min\Big(\frac{u_1}{\epsilon},\zeta\Big)\,dx\,dt\,dy\,ds = 0. \tag{3.15}$$

Then, by adding (3.10)-(3.15) and (3.9) we have

$$\begin{aligned}
&\int_{Q\times Q} \big[\big(a(x_1)(\partial_{x_2}+\partial_{y_2})u_1 - a(y_1)(\partial_{x_2}+\partial_{y_2})u_2\big)(\partial_{x_2}+\partial_{y_2})\vartheta\\
&+ \big(a(x_1)-\chi_2 a(y_1)\big)(\partial_{x_2}+\partial_{y_2})\vartheta\big]\,dx\,dt\,dy\,ds\\
&+ \int_{Q\times Q} \big(a(x_1)-a(y_1)(\partial_{x_2}+\partial_{y_2})u_2\big)(\partial_{x_2}+\partial_{y_2})\min\Big(\frac{u_1}{\epsilon},\zeta\Big)\,dx\,dt\,dy\,ds\\
&+ \int_{Q\times Q} \big(a(x_1)-\chi_2 a(y_1)\big)(\partial_{x_2}+\partial_{y_2})\min\Big(\frac{u_1}{\epsilon},\zeta\Big)\,dx\,dt\,dy\,ds = 0.
\end{aligned}\tag{3.16}$$

Now, let us introduce the change of variables

$$\frac{x+y}{2}=z,\quad \frac{x-y}{2}=\sigma,\quad \frac{t+s}{2}=\tau,\quad \frac{t-s}{2}=\theta. \tag{3.17}$$

Note that $(z,\tau)\in Q$ and $(\sigma,\theta)\in(-\frac{L}{2},\frac{L}{2})\times(-\frac{l}{2},\frac{l}{2})\times(-\frac{T}{2},\frac{T}{2}):=\Omega_1\times(-\frac{T}{2},\frac{T}{2}):=Q_1$. Then, from (3.16)-(3.17), we obtain

$$
\begin{aligned}
&\int_{Q\times Q_1}\big(a(z_1+\sigma_1)u_{1z_2}(z+\sigma,\tau+\theta)\\
&\quad -a(z_1-\sigma_1)u_{2z_2}(z-\sigma,\tau-\theta)\big)\vartheta_{z_2}\,dz\,d\tau\,d\sigma\,d\theta\\
&+\int_{Q\times Q_1}\big(a(z_1+\sigma_1)-\chi_2(z-\sigma,\tau-\theta)a(z_1-\sigma_1)\big)\vartheta_{z_2}dz\,d\tau\,d\sigma\,d\theta\\
&+\int_{Q\times Q_1}\big(a(z_1+\sigma_1)-\chi_2(z-\sigma,\tau-\theta)a(z_1-\sigma_1)\big)\\
&\times\min\Big(\frac{u_1}{\epsilon},\zeta\Big)_{z_2}dz\,d\tau\,d\sigma\,d\theta\\
&+\int_{Q\times Q_1}\big(a(z_1+\sigma_1)-u_{2z_2}(z-\sigma,\tau-\theta)a(z_1-\sigma_1)\big)\\
&\times\min\Big(\frac{u_1}{\epsilon},\zeta\Big)_{z_2}dz\,d\tau\,d\sigma\,d\theta=0.
\end{aligned}
\tag{3.18}
$$

Let us set

$$
\begin{gathered}
\begin{aligned}
I_{\epsilon,\delta}=\int_{Q\times Q_1}&\Big(a(z_1+\sigma_1)u_{1z_2}(z+\sigma,\tau+\theta)\\
&-a(z_1-\sigma_1)u_{2z_2}(z-\sigma,\tau-\theta)\Big)\vartheta_{z_2}\,dz\ d\tau\,d\sigma\,d\theta,
\end{aligned}\\
J_{\epsilon,\delta}=\int_{Q\times Q_1}(a(z_1+\sigma_1)-\chi_2(z-\sigma,\tau-\theta)a(z_1-\sigma_1))\vartheta_{z_2}dz\,d\tau\,d\sigma\,d\theta\\
\begin{aligned}
K^1_{\epsilon,\delta}=\int_{Q\times Q_1}&\big(a(z_1+\sigma_1)-\chi_2(z-\sigma,\tau-\theta)a(z_1-\sigma_1)\big)\\
&\times\min\Big(\frac{u_1}{\epsilon},\zeta\Big)_{z_2}dz\,d\tau\,d\sigma\,d\theta
\end{aligned}\\
\begin{aligned}
K^2_{\epsilon,\delta}=\int_{Q\times Q_1}&\big(a(z_1+\sigma_1)-u_{2z_2}(z-\sigma,\tau-\theta)a(z_1-\sigma_1)\big)\\
&\times\min\Big(\frac{u_1}{\epsilon},\zeta\Big)_{z_2}dz\,d\tau\,d\sigma\,d\theta.
\end{aligned}
\end{gathered}
$$

Thus, we have the following lemmas.

Lemma 3.2.

$$\lim_{\delta\to0}(\lim_{\epsilon\to0}J_{\epsilon,\delta})=\int_Q\eta\chi_{\{u_1>u_2\}}a(z_1)\big(1-\chi_2(z,\tau)\big)\xi_{z_2}dzd\tau. \tag{3.19}$$

$$\lim_{\delta\to0}(\lim_{\epsilon\to0}K^1_{\epsilon,\delta})=\int_Q\eta\chi_{\{u_1>0\}}a(z_1)\big(1-\chi_2(z,\tau)\big)\xi_{z_2}dzd\tau. \tag{3.20}$$

$$\lim_{\delta\to0}(\lim_{\epsilon\to0}K^2_{\epsilon,\delta})=\int_Q\eta\chi_{\{u_1>0\}}a(z_1)\big(1-u_{2z_2}(z,\tau)\big)\xi_{z_2}dzd\tau. \tag{3.21}$$

Proof. We have

$$\begin{aligned} J_{\epsilon,\delta} &= \int_{Q\times Q_1} (a(z_1+\sigma_1)-a(z_1-\sigma_1))\vartheta_{z_2}\,dz\,d\tau\,d\sigma\,d\theta \\ &\quad + \int_{Q\times Q_1} a(z_1-\sigma_1)(1-\chi_2(z-\sigma,\tau-\theta))\vartheta_{z_2}\,dz\,d\tau\,d\sigma\,d\theta \\ &:= J^1_{\epsilon,\delta}+J^2_{\epsilon,\delta}. \end{aligned} \tag{3.22}$$

We use integration by parts, (3.3)-(3.5), and the fact that $a(z_1+\sigma_1)-a(z_1-\sigma_1)$ does not depend on z_2, we obtain $J^1_{\epsilon,\delta}=0$. So,

$$\lim_{\delta\to 0}(\lim_{\epsilon\to 0} J^1_{\epsilon,\delta}) = 0. \tag{3.23}$$

Now, we will show that

$$\lim_{\delta\to 0}(\lim_{\epsilon\to 0} J^2_{\epsilon,\delta}) = \int_Q \eta a(z_1)\big(1-\chi_2(z,\tau)\big)\xi_{z_2}\,dz\,d\tau. \tag{3.24}$$

Let us set $A_\epsilon=\{(u_1-u_2)^+>\epsilon\zeta\}$ and $B_\epsilon=\{0<u_1-u_2\le\epsilon\zeta\}$. We have

$$\begin{aligned} J^2_{\epsilon,\delta} &= \int_{B_\epsilon} a(z_1-\sigma_1)\big(1-\chi_2(z-\sigma,\tau-\theta)\big)\Big(\frac{u_1-u_2}{\epsilon}\Big)_{z_2} dz\,d\tau\,d\sigma\,d\theta \\ &\quad + \int_{A_\epsilon} a(z_1-\sigma_1)\big(1-\chi_2(z-\sigma,\tau-\theta)\big)\zeta_{z_2} dz\,d\tau\,d\sigma\,d\theta \\ &:= J^{2,1}_{\epsilon,\delta}+J^{2,2}_{\epsilon,\delta}. \end{aligned} \tag{3.25}$$

Using (3.17) and that $(1-\chi_2)u_2=0$, $u_{1y_2}=0$ a.e. in Q, we obtain

$$J^{2,1}_{\epsilon,\delta} = \int_{B_\epsilon} a(y_1)\big(1-\chi_2(y,s)\big)\frac{u_{1x_2}}{\epsilon}\,dx\,dt\,dy\,ds.$$

Using (3.3) and that the function $(y,s)\mapsto a(y_1)(1-\chi_2(y,s))$ does not depend on x_2, integrating by parts we have

$$\begin{aligned} J^{2,1}_{\epsilon,\delta} &= \int_{Q\times Q} a(y_1)\big(1-\chi_2(y,s)\big)\Big(\min\Big(\frac{u_1}{\epsilon},\zeta\Big)\Big)_{x_2} dx\,dt\,dy\,ds \\ &\quad - \int_{A_\epsilon} a(y_1)\big(1-\chi_2(y,s)\big)\zeta_{x_2}\,dx\,dt\,dy\,ds \\ &= \int_{Q\times Q} a(y_1)\big(1-\chi_2(y,s)\big)\Big(\min\Big(\frac{u_1}{\epsilon},\zeta\Big)\Big)_{x_2} dx\,dt\,dy\,ds \\ &\quad - \int_{Q\times Q} a(y_1)\big(1-\chi_2(y,s)\big)\zeta_{x_2}\,dx\,dt\,dy\,ds \\ &\quad + \int_{B_\epsilon} a(y_1)\big(1-\chi_2(y,s)\big)\zeta_{x_2}\,dx\,dt\,dy\,ds \\ &= \int_{B_\epsilon} a(y_1)\big(1-\chi_2(y,s)\big)\zeta_{x_2}\,dx\,dt\,dy\,ds. \end{aligned}$$

Applying Hölder's inequality and taking into account that $\lim_{\epsilon\to 0}|B_\epsilon|=0$, we obtain $\lim_{\epsilon\to 0} J^{2,1}_{\epsilon,\delta}=0$. So,

$$\lim_{\delta\to 0}(\lim_{\epsilon\to 0} J^{2,1}_{\epsilon,\delta}) = 0. \tag{3.26}$$

For $J^{2,2}_{\epsilon,\delta}$, we pass to the limit as $\epsilon \to 0$ and $\delta \to 0$, respectively, we obtain

$$\lim_{\delta\to 0}(\lim_{\epsilon\to 0} J^{2,2}_{\epsilon,\delta}) = \int_Q a(z_1)\chi_{\{u_1>u_2\}}\big(1-\chi_2(z,\tau)\big)\xi_{z_2}\eta \,dzd\tau. \tag{3.27}$$

Hence, if we combine (3.26)-(3.27), we obtain (3.24) by letting $\epsilon \to 0$ and $\delta \to 0$ in (3.25). Now, we pass successively to the limit in (3.22), as $\epsilon \to 0$ and then as $\delta \to 0$ and using (3.23)-(3.24), we obtain (3.19). Finally, arguing as in the proof (3.19), we obtain (3.20) and (3.21). □

Lemma 3.3.

$$\lim_{\delta\to 0}(\lim_{\epsilon\to 0} I_{\epsilon,\delta}) \ge \int_Q \eta a(z_1)(u_1(z,\tau)-u_2(z,\tau))^+_{z_2}\xi_{z_2}dzd\tau. \tag{3.28}$$

Proof. We have

$$\begin{aligned} I_{\epsilon,\delta} = \int_{A_\epsilon} &\Big(a(z_1+\sigma_1)u_{1z_2}(z+\sigma,\tau+\theta)\\ &- a(z_1-\sigma_1)u_{2z_2}(z-\sigma,\tau-\theta)\Big)\zeta_{z_2}\,dz\,d\tau\,d\sigma\,d\theta\\ &+ \int_{B_\epsilon}\big(a(z_1+\sigma_1)u_{1z_2}(z+\sigma,\tau+\theta)\\ &- a(z_1-\sigma_1)u_{2z_2}(z-\sigma,\tau-\theta)\big)\big(\frac{u_1-u_2}{\epsilon}\big)_{z_2}dz\,d\tau\,d\sigma\,d\theta\\ :=& I^1_{\epsilon,\delta} + I^2_{\epsilon,\delta}. \end{aligned} \tag{3.29}$$

The integral $I^2_{\epsilon,\delta}$ can be decomposed as

$$\begin{aligned} &I^2_{\epsilon,\delta}\\ &= \frac{1}{\epsilon}\Big\{\int_{B_\epsilon}\big[a(z_1+\sigma_1)u_{1z_2}(z+\sigma,\tau+\theta)u_{1z_2}(z+\sigma,\tau+\theta)\\ &\quad + a(z_1-\sigma_1)u_{2z_2}(z-\sigma,\tau-\theta)u_{2z_2}(z-\sigma,\tau-\theta)\big]\,dz\,d\tau\,d\sigma\,d\theta\\ &\quad - \int_{B_\epsilon} a(z_1-\sigma_1)u_{2z_2}(z-\sigma,\tau-\theta)u_{1z_2}(z+\sigma,\tau+\theta)dz\,d\tau\,d\sigma\,d\theta\\ &\quad - \int_{B_\epsilon}\big(a(z_1+\sigma_1)u_{1z_2}(z+\sigma,\tau+\theta)u_{2z_2}(z-\sigma,\tau-\theta)\,dz\,d\tau\,d\sigma\,d\theta\Big\}\\ &:= I^{2,1}_{\epsilon,\delta} - I^{2,2}_{\epsilon,\delta} - I^{2,3}_{\epsilon,\delta}. \end{aligned}$$

From (1.1), the integral $I^{2,1}_{\epsilon,\delta}$ is positive. So,

$$I^2_{\epsilon,\delta} \ge -I^{2,2}_{\epsilon,\delta} - I^{2,3}_{\epsilon,\delta}. \tag{3.30}$$

For $I^{2,2}_{\epsilon,\delta}$, we use integration by parts, (3.3), (3.5), and the fact that the function $(y,s)\mapsto a(y_1)u_{2y_2}(y,s)$ does not depend on x_2, we obtain

$$\begin{aligned} I^{2,2}_{\epsilon,\delta} &= \int_{B_\epsilon} a(y_1)u_{2y_2}(y,s)\Big(\frac{u_1(x,t)-u_2(y,s)}{\epsilon}\Big)_{x_2}\,dx\,dtdyds\\ &= \int_{Q\times Q} a(y_1)u_{2y_2}(y,s)\vartheta_{x_2}\,\,dx\,dt\,dy\,ds\\ &\quad - \int_{A_\epsilon} a(y_1)u_{2y_2}(y,s)\zeta_{x_2}\,dx\,dt\,dy\,ds \end{aligned}$$

$$
\begin{aligned}
&= \int_{Q\times Q} a(y_1)u_{2y_2}(y,s)\vartheta_{x_2}\ dx\,dt\,dy\,ds \\
&\quad - \int_{Q\times Q} a(y_1)u_{2y_2}(y,s)\zeta_{x_2}\,dx\,dt\,dy\,ds \\
&\quad + \int_{B_\epsilon} a(y_1)u_{2y_2}(y,s)\zeta_{x_2}\,dx\,dt\,dy\,ds \\
&= \int_{B_\epsilon} a(y_1)u_{2y_2}(y,s)\zeta_{x_2}\,dx\,dt\,dy\,ds.
\end{aligned}
$$

Applying Hölder's inequality and taking into account that $\lim_{\epsilon\to 0}|B_\epsilon| = 0$, we obtain

$$\lim_{\epsilon\to 0}(I^{2,2}_{\epsilon,\delta}) = 0. \tag{3.31}$$

In the same way, we prove

$$\lim_{\epsilon\to 0}(I^{2,3}_{\epsilon,\delta}) = 0. \tag{3.32}$$

Combine (3.31)-(3.32), we obtain, by passing to the limit as $\epsilon\to 0$ in (3.30),

$$\lim_{\epsilon\to 0}(I^{2}_{\epsilon,\delta}) \geq 0. \tag{3.33}$$

Let us study $I^1_{\epsilon,\delta}$. Applying the Lebesgue theorem to $I^1_{\epsilon,\delta}$, we obtain

$$
\begin{aligned}
\lim_{\epsilon\to 0}(I^1_{\epsilon,\delta}) = \int_{Q\times Q_1} &\chi_{\{u_1>u_2\}}\big(a(z_1+\sigma_1)u_{1z_2}(z+\sigma,\tau+\theta) \\
&- a(z_1-\sigma_1)u_{2z_2}(z-\sigma,\tau-\theta)\big)\zeta_{z_2}\,dz\,d\tau\,d\sigma\,d\theta
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
\lim_{\epsilon\to 0}(I^1_{\epsilon,\delta}) &= \int_{Q\times Q_1} \chi_{\{u_1>u_2\}}a(z_1+\sigma_1)(u_1(z+\sigma,\tau+\theta) \\
&\quad - u_2(z-\sigma,\tau-\theta))_{z_2}\zeta_{z_2}dz\,d\tau\,d\sigma\,d\theta \\
&\quad + \int_{Q\times Q_1} \chi_{\{u_1>u_2\}}(a(z_1+\sigma_1) \\
&\quad - a(z_1-\sigma_1))u_{2z_2}(z-\sigma,\tau-\theta)\zeta_{z_2}dz\,d\tau\,d\sigma\,d\theta \\
&= I^{1,1}_\delta + I^{1,2}_\delta.
\end{aligned} \tag{3.34}
$$

Using (3.1) and taking into account $\operatorname{supp}(\rho_{1,\delta})\subset(-\delta,\delta)$, we obtain that for some positive constant C,

$$
\begin{aligned}
|I^{1,2}_\delta| &\leq C\int_{Q\times Q_1} |\sigma_1||u_{2z_2}||\xi_{z_2}|\eta\rho_{1,\delta}(\sigma_1)\rho_{2,\delta}(\sigma_1)\rho_{3,\delta}(\theta)dz\,d\tau\,d\sigma\,d\theta \\
&\leq \delta C\int_{Q\times Q_1} |u_{2z_2}||\xi_{z_2}|\eta\rho_{1,\delta}(\sigma_1)\rho_{2,\delta}(\sigma_1)\rho_{3,\delta}(\theta)dz\,d\tau\,d\sigma\,d\theta \\
&:= \delta C W_\delta.
\end{aligned}
$$

So, since $(W_\delta)_\delta$ is bounded, we obtain

$$\lim_{\delta\to 0} I^{1,2}_\delta = 0. \tag{3.35}$$

In $I^{1,1}_\delta$, we pass to the limit as $\delta\to 0$, to obtain

$$\lim_{\delta\to 0} I^{1,1}_\delta = \int_Q \eta a(z_1)(u_1(z,\tau)-u_2(z,\tau))^+_{z_2}\xi_{z_2}\,dz\,d\tau. \tag{3.36}$$

Hence, if we combine (3.35)-(3.36), we obtain by letting $\delta \to 0$ in (3.34):

$$\lim_{\delta\to 0}(\lim_{\epsilon\to 0} I^1_{\epsilon,\delta}) = \int_Q \eta a(z_1)(u_1(z,\tau) - u_2(z,\tau))^+{}_{z_2}\xi_{z_2}\, dz d\tau. \tag{3.37}$$

Finally, we pass successively to the limit in (3.29), as $\epsilon \to 0$ and then as $\delta \to 0$, and using (3.33) and (3.37), we obtain (3.28). □

Now, using Lemma 3.2 and Lemma 3.3, and letting successively $\epsilon \to 0$ and $\delta \to 0$ in (3.18), we obtain (3.2). This completes the proof of Theorem 3.1. □

Now, we can state our uniqueness theorem.

Theorem 3.4. *The solution of the problem* (1.3) *associated with the initial data* χ_0 *(see* (1.2)*) is unique .*

Proof. Let (u_1, χ_1) and (u_2, χ_2) be two solutions of the problem (1.3) such that $\chi_1(x, 0) = \chi_2(x, 0) = \chi_0(x)$ a.e. in Ω. Let us set $v = (u_1 - u_2)^+$ and $\gamma = (1 - \chi_2(x,t))\chi_{\{u_1>u_2\}} + \big((1 - \chi_2(x,t)) + (1 - u_{2x_2}(x,t))\big)\chi_{\{u_1>0\}}$. From Theorem 3.1, we have

$$\begin{gathered}\int_Q \eta a(x_1)(v_{x_2} + \gamma)\xi_{x_2}\, dx\, dt \le 0,\\ \forall \xi \in \mathcal{D}(\Omega),\ \xi \ge 0,\ \eta \in \mathcal{D}(0,T),\ \eta \ge 0.\end{gathered} \tag{3.38}$$

Let $\varepsilon_0 = d(\mathrm{supp}(\xi), \partial\Omega)$ and $A_{\varepsilon_0} = \{x \in \mathbb{R}^2 / d(x, \partial\Omega) > \varepsilon_0\}$. We extend v and γ outside Q by 0 and still denote by v (resp. γ) this function. Moreover, from (3.1), the function a admits an extension to $\mathbb{R}$, still denote by a, such that $a \in C^1(\mathbb{R}, \mathbb{R})$. For $\varepsilon \in (0, \frac{\varepsilon_0}{2})$, let $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^2)$ with $\mathrm{supp}(\rho_\varepsilon) \subset B(0, \varepsilon)$ be a regularizing sequence and let $f_\varepsilon = \rho_\varepsilon * f$, the regularized of a function f. We have by using Fubini's theorem and change of variables:

$$\begin{aligned}
&\int_{\mathbb{R}^2\times(0,T)} \eta a(x_1)(v_{\varepsilon x_2} + \gamma_\varepsilon)\xi_{x_2}\, dx\, dt\\
&= \int_{\mathbb{R}^2\times(0,T)} \eta\Big\{\int_{\mathbb{R}^2} (v_{x_2}(x-y,t) + \gamma(x-y,t))\rho_\varepsilon(y)dy\Big\} a(x_1)\xi_{x_2}(x_1,x_2)\, dx\, dt\\
&= \int_{\mathbb{R}^2} \rho_\varepsilon(y)\Big\{\int_{\mathbb{R}^2\times(0,T)} \eta(v_{x_2}(x-y,t) + \gamma(x-y,t))a(x_1)\xi_{x_2}(x_1,x_2)\, dx\, dt\Big\} dy\\
&= \int_{B(0,\varepsilon)} \rho_\varepsilon(y)\Big\{\int_Q \eta(v_{z_2}(z,t) + \gamma(z,t))(a(z_1+y_1)\xi(z+y))_{z_2} dzdt\Big\} dy\\
&= \int_{B(0,\varepsilon)} \rho_\varepsilon(y)\Big\{\int_Q \eta a(z_1)(v_{z_2}(z,t) + \gamma(z,t))\Big(\frac{a(z_1+y_1)\xi(z+y)}{a(z_1)}\Big)_{z_2} dzdt\Big\} dy.
\end{aligned}$$

For all $y \in B(0, \varepsilon)$, the function $z \mapsto \frac{a(z_1+y_1)\xi(z+y)}{a(z_1)}$ is nonnegative and belongs to $C^1_0(\Omega)$. Therefore, since (3.38) still holds for the functions $\varphi \in C^1_0(\Omega)$, $\varphi \ge 0$, we obtain by taking into account that $\rho_\varepsilon \ge 0$,

$$\begin{gathered}\int_{\mathbb{R}^2\times(0,T)} \eta a(x_1)(v_{\varepsilon x_2} + \gamma_\varepsilon)\xi_{x_2}\, dx\, dt \le 0\\ \forall \xi \in \mathcal{D}(\Omega),\ \xi \ge 0,\ d(\mathrm{supp}(\xi), \partial\Omega) = \varepsilon_0 > 0, \quad \forall \eta \in \mathcal{D}(0,T),\ \eta \ge 0\end{gathered}$$

which using integration by parts, can be written as

$$\begin{gathered} -\int_{A_{\varepsilon_0}\times(0,T)} \eta a(x_1) v_\varepsilon \xi_{x_2x_2}\,dx\,dt + \int_{A_{\varepsilon_0}\times(0,T)} \eta a(x_1)\gamma_\varepsilon \xi_{x_2}\,dx\,dt \le 0 \\ \xi\in\mathcal{D}(\Omega),\ \xi\ge 0,\ d(\operatorname{supp}(\xi),\ \partial\Omega)=\varepsilon_0>0,\quad \forall\eta\in\mathcal{D}(0,T),\ \eta\ge 0. \end{gathered} \tag{3.39}$$

We define

$$\alpha_\epsilon(x) = a(x_1)\int_0^T \eta v_\varepsilon dt,$$

and suppose there exists $x_0\in A_{\varepsilon_0}\cap\Omega$ and $\varepsilon_1\in(0,\frac{\varepsilon_0}{2})$ such that $\alpha_{\varepsilon_1}(x_0)>0$. Since α_{ε_1} is continuous and $A_{\varepsilon_0}\cap\Omega$ is open set, there exists $r>0$ such that $\overline{B(x_0,r)}\subset A_{\varepsilon_0}\cap\Omega$ and $\alpha_{\varepsilon_1}(x)>0$ for all $x\in\overline{B(x_0,r)}$. Therefore, we deduce from (1.1) that $\int_0^T \eta v_{\varepsilon_1}dt>0$ in $\overline{B(x_0,r)}$. Now, let us consider the following homogeneous Dirichlet problem for linear second order partial differential equation:

$$\begin{gathered} -\frac{1}{\alpha_{\varepsilon_1}(x)}\xi_{x_1x_1} - \xi_{x_2x_2} + \frac{\int_0^T \eta\gamma_{\epsilon_1}dt}{\int_0^T \eta v_{\varepsilon_1}dt}\xi_{x_2} = \frac{1}{\alpha_{\varepsilon_1}(x)} \quad \text{in } B(x_0,r) \\ \xi = 0 \quad \text{on } \partial(B(x_0,r)) \end{gathered} \tag{3.40}$$

which can be written as

$$\begin{gathered} -\sum_{i,j=1}^{2} a_{\varepsilon_1 ij}(x)\xi_{x_ix_j} + \beta_{\epsilon_1}(x)\xi_{x_2} = \frac{1}{\alpha_{\varepsilon_1}(x)} \quad \text{in } B(x_0,r) \\ \xi = 0 \quad \text{on } \partial(B(x_0,r)). \end{gathered}$$

where

$$a_{\epsilon_1 11}(x) = \frac{1}{\alpha_{\epsilon_1}(x)},\quad a_{\epsilon_1 12}(x) = a_{\epsilon_1 21}(x) = 0,\quad a_{\epsilon_1 22}(x) = 1,$$

$$\beta_{\epsilon_1}(x) = \frac{\int_0^T \eta\gamma_{\epsilon_1}dt}{\int_0^T \eta v_{\varepsilon_1}dt}.$$

Observe that the matrix $(a_{\varepsilon_1 ij}(x))_{ij}$ is strictly elliptic in $B(x_0,r)$ with constant

$$\min\Big(\frac{1}{\max_{x\in\overline{B(x_0,r)}}\alpha_{\varepsilon_1}(x)},1\Big) > 0$$

and the coefficients $\frac{1}{\alpha_{\varepsilon_1}}$, β_{ϵ_1} are in $C^1(\overline{B(x_0,r)})$. So, by the regularity theory (see [6] for example), the problem (3.40) has a unique solution $\hat\xi\in C^2(\overline{B(x_0,r)})$. Moreover, since the function in the right side of the first equation of (3.40) satisfies $\frac{1}{\alpha_{\varepsilon_1}}>0$ in $B(x_0,r)$, we have from the maximum principle, $\hat\xi\ge 0$ in $\overline{B(x_0,r)}$. Therefore, we see that $\hat\xi\in C_0^2(\Omega), \hat\xi\ge 0$ and $d(\operatorname{supp}(\hat\xi),\partial\Omega)\ge\varepsilon_0$. Then, we can choose $\xi=\hat\xi$ in (3.39) to obtain

$$\int_{A_{\varepsilon_0}\times(0,T)} \Big\{-\eta a(x_1)v_\varepsilon\hat\xi_{x_2x_2} + \eta a(x_1)\gamma_\varepsilon\hat\xi_{x_2}\Big\}\,dx\,dt \le 0 \quad \forall\varepsilon\in\Big(0,\frac{\varepsilon_0}{2}\Big).$$

When we write the first equation of (3.40) for $\hat\xi$ and multiplying by $\alpha_{\varepsilon_1}(x)$, we obtain

$$-\hat\xi_{x_1x_1} - \alpha_{\varepsilon_1}(x)\hat\xi_{x_2x_2} + a(x_1)\hat\xi_{x_2}\int_0^T \eta\gamma_{\varepsilon_1}dt = 1,$$

and by integrating over $\overline{B(x_0,r)}$, we obtain

$$\begin{aligned}
&\int_{\overline{B(x_0,r)}\times(0,T)} \big\{-\eta a(x_1)v_{\varepsilon_1}\hat{\xi}_{x_2x_2}+\eta a(x_1)\gamma_{\varepsilon_1}\hat{\xi}_{x_2}\big\}\,dx\,dt\\
&=\int_{\overline{B(x_0,r)}}\hat{\xi}_{x_1x_1}\,dx+\int_{\overline{B(x_0,r)}}dx\\
&=\int_{\Omega}\hat{\xi}_{x_1x_1}\,dx+\int_{\overline{B(x_0,r)}}dx
\end{aligned}$$

which leads, using integration by parts, to

$$\int_{A_{\varepsilon_0}\times(0,T)}\Big\{-\eta a(x_1)v_{\varepsilon_1}\hat{\xi}_{x_2x_2}+\eta a(x_1)\gamma_{\varepsilon_1}\hat{\xi}_{x_2}\Big\}\,dx\,dt=|\overline{B(x_0,r)}|>0.$$

So, we deduce that

$$\alpha_\varepsilon(x)=a(x_1)\int_0^T\eta(t)v_\varepsilon(x,t)dt\le 0\quad\forall\varepsilon\in\big(0,\frac{\varepsilon_0}{2}\big),\ \forall x\in A_{\varepsilon_0}\cap\Omega$$

from which, we obtain by taking into account that $a>0$ and integrating over $A_{\varepsilon_0}\cap\Omega$:

$$\int_{(A_{\varepsilon_0}\cap\Omega)\times(0,T)}\eta(t)v_\varepsilon(x,t)\,dx\,dt\le 0\quad\forall\varepsilon\in\big(0,\frac{\varepsilon_0}{2}\big).$$

Passing to the limit as $\varepsilon\to 0$, we obtain

$$0\le\int_{(A_{\varepsilon_0}\cap\Omega)\times(0,T)}\eta(t)(u_1-u_2)^+(x,t)\,dx\,dt\le 0$$

and since ε_0 is arbitrary, we have

$$\int_Q\eta(t)(u_1-u_2)^+(x,t)\,dx\,dt=0.$$

So, for all $\eta\in\mathcal{D}(0,T)$, $\eta\ge 0$, we have $\eta(u_1-u_2)^+=0$ a.e. in Q. This leads to $u_1\le u_2$ a.e. in Q. By exchanging the roles of u_1 and u_2, we obtain $u_2\le u_1$ a.e. in Q. We conclude that

$$u_1=u_2:=u\quad\text{a.e. in } Q. \tag{3.41}$$

Now, we are going to prove that

$$\chi_1=\chi_2\quad\text{a.e. in } Q. \tag{3.42}$$

Let us consider $s\in(0,T]$. For a positive real number δ we define the following function η on $[0,s]$ by

$$\eta(t)=\begin{cases}2(\frac{t}{\delta})^2 & \text{if } t\in[0,\frac{\delta}{2}]\\ 1-2(1-\frac{t}{\delta})^2 & \text{if } t\in(\frac{\delta}{2},\delta]\\ 1 & \text{if } t\in(\delta,s-\delta]\\ 1-2(1-\frac{s-t}{\delta})^2 & \text{if } t\in(s-\delta,s-\frac{\delta}{2}]\\ 2(\frac{s-t}{\delta})^2 & \text{if } t\in(s-\frac{\delta}{2},s].\end{cases}$$

Note that $\eta \in C^1([0,s])$ and

$$\eta'(t) = \begin{cases} 4\frac{t}{\delta^2} & \text{if } t \in [0, \frac{\delta}{2}] \\ \frac{4}{\delta}(1-\frac{t}{\delta}) & \text{if } t \in (\frac{\delta}{2}, \delta] \\ 0 & \text{if } t \in (\delta, s-\delta] \\ -\frac{4}{\delta}(1-\frac{s-t}{\delta}) & \text{if } t \in (s-\delta, s-\frac{\delta}{2}] \\ -\frac{4}{\delta}(\frac{s-t}{\delta}) & \text{if } t \in (s-\frac{\delta}{2}, s]. \end{cases}$$

We extend η outside $[0,s]$ by 0 and still denote by η this function and let us consider $\xi \in \mathcal{D}(\Omega)$. Note that $\xi\eta^2 \in H^1(Q)$, $\xi\eta^2 = 0$ on $\partial\Omega \times (0,T)$ and $(\xi\eta^2)(x,0) = (\xi\eta^2)(x,T) = 0$ a.e. in Ω. Choosing $\pm\xi\eta^2$ as test functions for (1.3), both for (u,χ_1) and (u,χ_2), we obtain

$$\int_{\Omega\times(0,s)} [a(x_1)(u_{x_2}+\chi_1)\xi_{x_2}\eta^2 - 2\chi_1\eta\eta'\xi]\,dx\,dt = 0 \tag{3.43}$$

$$\int_{\Omega\times(0,s)} [a(x_1)(u_{x_2}+\chi_2)\xi_{x_2}\eta^2 - 2\chi_2\eta\eta'\xi]\,dx\,dt = 0. \tag{3.44}$$

Subtracting (3.44) from and (3.43), we obtain

$$\begin{aligned} 0 = &\int_{\Omega\times(0,s)} a(x_1)(\chi_1-\chi_2)\xi_{x_2}\eta^2\,dx\,dt - \int_{\Omega\times(0,s)} 2(\chi_1-\chi_2)\eta\eta'\xi\,dx\,dt \\ :=& R_\delta^1 - R_\delta^2. \end{aligned} \tag{3.45}$$

Applying the Lebesgue theorem to R_δ^1, we obtain

$$\lim_{\delta\to 0} R_\delta^1 = \int_{\Omega\times(0,s)} a(x_1)(\chi_1-\chi_2)\xi_{x_2}\,dx\,dt. \tag{3.46}$$

Let us study R_δ^2. From the definition of η', we have

$$\begin{aligned} |R_\delta^2| =& 2\Big|\int_\Omega\int_0^\delta a(x_1)(\chi_1-\chi_2)\eta\eta'\xi\,dx\,dt + \int_\Omega\int_{s-\delta}^s a(x_1)(\chi_1-\chi_2)\eta\eta'\xi\,dx\,dt\Big| \\ \le& C\Big\{\int_0^\delta\Big(\int_\Omega|\chi_1-\chi_2|dx\Big)\eta|\eta'|dtdt + \int_{s-\delta}^s\Big(\int_\Omega|\chi_1-\chi_2|dx\Big)\eta|\eta'|dt\Big\} \\ :=& C(R_\delta^{2,1}+R_\delta^{2,2}) \end{aligned} \tag{3.47}$$

where $C = \sup_{(x_1,x_2)\in\Omega}|a(x_1)\xi(x_1,x_2)|$. We have $\chi_1-\chi_2 \in C^0([0,T];L^1(\Omega))$ (see Propositions 2.1), $\eta \in C^0([0,s])$, $\eta(0)=0$ and η is uniformly bounded independently of δ. In particular, the function $t \mapsto \Big(\int_\Omega|\chi_1-\chi_2|dx\Big)\eta$ is right-continuous and vanishes at 0, and uniformly bounded independently of δ. Therefore, since $|\eta'| \sim \frac{1}{\delta}$, we deduce that

$$\lim_{\delta\to 0} R_\delta^{2,1} = 0. \tag{3.48}$$

Similarly, since the function $t \mapsto \big(\int_\Omega|\chi_1-\chi_2|dx\big)\eta$ is left-continuous and vanishes at s, uniformly bounded independently of δ, and $|\eta'| \sim \frac{1}{\delta}$, we deduce that

$$\lim_{\delta\to 0} R_\delta^{2,2} = 0. \tag{3.49}$$

By letting $\delta \to 0$ in (3.47) and using (3.48)-(3.49), we obtain

$$\lim_{\delta\to 0} R_\delta^2 = 0. \tag{3.50}$$

Passing to the limit as $\delta \to 0$ in (3.45), and using (3.46) and (3.50), we obtain

$$0 = \int_0^s \int_\Omega a(x_1)(\chi_1 - \chi_2)\xi_{x_2}\,dx\,dt := F(s) \quad \forall s \in [0,T]. \tag{3.51}$$

Since $\int_\Omega a(x_1)(\chi_1 - \chi_2)\xi_{x_2}dx$ is continuous on $[0,T]$ and $\chi_1(x,0) - \chi_2(x,0) = \chi_0(x) - \chi_0(x) = 0$ a.e. in Ω, we deduce from (3.51) that $F'(s) = 0$ for all $s \in [0,T]$. So,

$$\int_\Omega a(x_1)(\chi_1 - \chi_2)(\cdot,t)\xi_{x_2}dx = 0 \quad \forall t \in [0,T],\ \forall \xi \in \mathcal{D}(\Omega). \tag{3.52}$$

Let $\varepsilon_0 = d(\text{supp}(\xi), \partial\Omega)$. Let us extend $\chi_1(\cdot,t)$ and $\chi_2(\cdot,t)$ outside Ω by 0 and still denote by $\chi_1(\cdot,t)$ (resp. $\chi_2(\cdot,t)$) this function. Moreover, since $a \in C^1([0,L],\mathbb{R})$, there exists an extension to $\mathbb{R}$, still denote by a, such that $a \in C^1(\mathbb{R},\mathbb{R})$. For $\varepsilon \in (0, \frac{\varepsilon_0}{2})$, let $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^2)$ with $\text{supp}(\rho_\varepsilon) \subset B(0,\varepsilon)$ be a regularizing sequence and let $f_\varepsilon = \rho_\varepsilon * f$, the regularized of a function f. Then, using (3.52), we obtain

$$\begin{gathered}\int_{\mathbb{R}^2} a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon \xi_{x_2}dx = 0 \\ \forall t \in [0,T],\ \forall \xi \in \mathcal{D}(\Omega),\ \xi \geq 0,\ d(\text{supp}(\xi), \partial\Omega) = \varepsilon_0.\end{gathered} \tag{3.53}$$

For positive real number δ, we choose $\min\Big(\frac{((\chi_1-\chi_2)(\cdot,t))_\varepsilon^+}{\delta}, 1\Big)\xi$ as test function in (3.53), we obtain

$$\begin{aligned}0 &= \int_{\mathbb{R}^2} a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon \xi_{x_2} \min\Big(\frac{((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+}{\delta}, 1\Big)dx \\ &\quad + \int_{\mathbb{R}^2} a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon \min\Big(\frac{((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+}{\delta}, 1\Big)_{x_2}\xi dx \\ &:= S_\delta^1 + S_\delta^2.\end{aligned} \tag{3.54}$$

Applying the Lebesgue theorem to S_δ^1, we obtain

$$\lim_{\delta \to 0} S_\delta^1 = \int_\Omega a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ \xi_{x_2}dx. \tag{3.55}$$

For S_δ^2, we have by using integration by parts

$$\begin{aligned}S_\delta^2 &= \frac{1}{2\delta}\int_{\mathbb{R}^2} a(x_1)\{\min(((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+, \delta)^2\}_{x_2}\xi dx \\ &= -\frac{1}{2\delta}\int_{\mathbb{R}^2} a(x_1)\xi_{x_2}\min(((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+, \delta)^2 dx.\end{aligned}$$

By letting $\delta \to 0$, we obtain

$$\lim_{\delta \to 0} S_\delta^2 = 0. \tag{3.56}$$

Passing to the limit as $\delta \to 0$ in (3.54) and using (3.55)-(3.56), we obtain

$$\begin{gathered}\int_\Omega a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ \xi_{x_2}dx = 0 \\ \forall t \in [0,T],\ \forall \xi \in \mathcal{D}(\Omega),\ \xi \geq 0,\ d(\text{supp}(\xi), \partial\Omega) = \varepsilon_0.\end{gathered} \tag{3.57}$$

Choosing $x_2\xi$ as a test function in (3.57), we obtain

$$\begin{gathered}\int_\Omega a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ \xi dx + \int_\Omega a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ x_2\xi_{x_2}dx = 0 \\ \forall t \in [0,T],\ \forall \xi \in \mathcal{D}(\Omega),\ \xi \geq 0,\ d(\text{supp}(\xi), \partial\Omega) = \varepsilon_0.\end{gathered} \tag{3.58}$$

Let $x_1^1, x_1^2 \in (0,L)$ and $x_2^1, x_2^2 \in (0,l)$ such that $x_1^1 < x_1^2$, $x_2^1 < x_2^2$ and $d(0,x_1^1) = d(0,x_2^1) = d(L,x_1^2) = d(l,x_2^2) = \varepsilon_0$. Now, for positive real number δ we define the following functions h and g in (x_1^1, x_1^2) (resp. (x_2^1, x_2^2)) by

$$h(x_1) = \begin{cases} 2(\frac{x_1 - x_1^1}{\delta})^2 & \text{if } x_1 \in [x_1^1, x_1^1 + \frac{\delta}{2}] \\ 1 - 2(1 - \frac{x_1 - x_1^1}{\delta})^2 & \text{if } x_1 \in (x_1^1 + \frac{\delta}{2}, x_1^1 + \delta] \\ 1 & \text{if } x_1 \in (x_1^1 + \delta, x_1^2 - \delta] \\ 1 - 2(1 - \frac{x_1^2 - x_1}{\delta})^2 & \text{if } x_1 \in (x_1^2 - \delta, x_1^2 - \frac{\delta}{2}] \\ 2(\frac{x_1^2 - x_1}{\delta})^2 & \text{if } x_1 \in (x_1^2 - \frac{\delta}{2}, x_1^2] \end{cases}$$

and

$$g(x_2) = \begin{cases} 2(\frac{x_2 - x_2^1}{\delta})^2 & \text{if } x_2 \in [x_2^1, x_2^1 + \frac{\delta}{2}] \\ 1 - 2(1 - \frac{x_2 - x_2^1}{\delta})^2 & \text{if } x_2 \in (x_2^1 + \frac{\delta}{2}, x_2^1 + \delta] \\ 1 & \text{if } x_2 \in (x_2^1 + \delta, x_2^2 - \delta] \\ 1 - 2(1 - \frac{x_2^2 - x_2}{\delta})^2 & \text{if } x_2 \in (x_2^2 - \delta, x_2^2 - \frac{\delta}{2}] \\ 2(\frac{x_2^2 - x_2}{\delta})^2 & \text{if } x_2 \in (x_2^2 - \frac{\delta}{2}, x_2^2]. \end{cases}$$

We have $h(x_1^1) = h(x_1^2) = g(x_2^1) = g(x_2^2) = 0$, $h \in C^2([x_1^1, x_1^2])$, $g \in C^2([x_2^1, x_2^2])$ and $h, g \geq 0$. If we set $\Omega_{\varepsilon_0} = (x_1^1, x_1^2) \times (x_2^1, x_2^2)$, we see that $hg^2 \in C^2(\overline{\Omega_{1,2}})$ and $hg^2 \geq 0$. Let us extend hg^2 outside Ω_{ε_0} by 0 and still denote by hg^2 this function. Since $d(\text{supp}(hg^2), \partial\Omega) = \varepsilon_0$, we can choose $\xi = hg^2$ as test function in (3.58) to obtain

$$\begin{aligned} 0 &= \int_{\Omega_{\varepsilon_0}} a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ hg^2 dx \\ &\quad + 2\int_{\Omega_{\varepsilon_0}} a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ x_2 hgg' dx \\ &:= N_\delta^1 + N_\delta^2. \end{aligned} \tag{3.59}$$

Applying the Lebesgue theorem to N_δ^1, we obtain

$$\lim_{\delta \to 0} N_\delta^1 = \int_{\Omega_{\varepsilon_0}} a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ dx. \tag{3.60}$$

Let us study N_δ^2. From the definition of g', we have

$$\begin{aligned} |N_\delta^2| &= 2\Big| \int_{x_1^1}^{x_1^2} \int_{x_2^1}^{x_2^1+\delta} a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ x_2 hgg'\, dx \\ &\quad + \int_{x_1^1}^{x_1^2} \int_{x_2^2-\delta}^{x_2^2} a(x_1)((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ x_2 hgg'\, dx \Big| \\ &\leq C\Big\{ \int_{x_1^1}^{x_1^2} \int_{x_2^1}^{x_2^1+\delta} ((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ g|g'| dx \\ &\quad + \int_{x_1^1}^{x_1^2} \int_{x_2^2-\delta}^{x_2^2} ((\chi_1 - \chi_2)(\cdot,t))_\varepsilon^+ g|g'| dx \Big\} \\ &:= C(N_\delta^{2,1} + N_\delta^{2,2}) \end{aligned} \tag{3.61}$$

where $C=\sup_{(x_1,x_2)\in\Omega}(a(x_1)x_2h(x_1))$ which is independent of δ. Since the function $x_2\mapsto\big(\int_{x_1^1}^{x_1^2}((\chi_1-\chi_2)(\cdot,t))^+_\varepsilon dx_1\big)g$ is right-continuous and vanishes at x_2^1, uniformly bounded independently of δ and $|g'|\sim\frac{1}{\delta}$, we deduce that

$$\lim_{\delta\to0}N_\delta^{2,1}=0. \tag{3.62}$$

Similarly, since the function $x_2\mapsto\big(\int_{x_1^1}^{x_1^2}((\chi_1-\chi_2)(\cdot,t))^+_\varepsilon dx_1\big)g$ is left-continuous and vanishes at x_2^2, uniformly bounded independently of δ, and $|g'|\sim\frac{1}{\delta}$, we deduce that

$$\lim_{\delta\to0}N_\delta^{2,2}=0. \tag{3.63}$$

By letting $\delta\to0$ in (3.61) and using (3.62)-(3.63), we obtain

$$\lim_{\delta\to0}N_\delta^{2}=0. \tag{3.64}$$

Now, passing to the limit as $\delta\to0$ in (3.59) and using (3.60) and (3.64) we obtain

$$\int_{\Omega_{\varepsilon_0}}a(x_1)((\chi_1-\chi_2)(\cdot,t))^+_\varepsilon dx=0\quad\forall t\in[0,T]. \tag{3.65}$$

Finally, by letting $\varepsilon\to0$ in (3.65), we obtain

$$\int_{\Omega_{\varepsilon_0}}a(x_1)((\chi_1-\chi_2)(\cdot,t))^+dx=0\quad\forall t\in[0,T]$$

and since ε_0 is arbitrary, we have

$$\int_{\Omega}a(x_1)((\chi_1-\chi_2)(\cdot,t))^+dx=0\quad\forall t\in[0,T].$$

This leads to $a(x_1)((\chi_1-\chi_2)(\cdot,t))^+=0$ a.e. in Ω for all $t\in[0,T]$. Thanks to (1.1), we deduce that $((\chi_1-\chi_2)(\cdot,t))^+=0$ a.e. in Ω for all $t\in[0,T]$. So, $\chi_1\le\chi_2$ a.e. in Q. By exchanging the roles of χ_1 and χ_2, we obtain $\chi_2\le\chi_1$ a.e. in Q. We conclude that $\chi_1=\chi_2$ a.e. in Q. Hence, (3.42) holds. If we combine (3.41) and (3.42), we see that the solution of problem (1.3) associated with the initial data χ_0 is unique. □

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