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ON A FRACTIONAL PHASE TRANSITION MODEL IN FERROMAGNETISM

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Abstract. We consider a fractional model describing phase transition in ferromagnetic materials. This model includes the three-dimensional evolution of both thermodynamic and electromagnetic properties of the ferromagnetic material. We first prove existence of a global weak solution by using Faedo-Galerkin method. Then we establish uniqueness for the considered model.

1. Introduction

Modeling of many phenomena mostly rely on fractional calculus, and it has become a valuable tool in engineering applications, technological development, and industrial sciences for the description of the complex dynamics [2]. In this article, we are interested in a fractional version of a model arising in the theory of paramagnetic-ferromagnetic transition. Our investigation has its starting point in the paper [4] where the authors propose a three-dimensional evolutive model and establish the existence and uniqueness of weak solutions. The calculations combine phenomenological constitutive equations for magnetization vector $\mathbf{m}$ and the absolute temperature θ. To describe the model equations, we consider a rigid ferromagnetic conductor occupying a domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ and unit outward normal $\mathbf{n}$. According to Berti et al. [4], the system governing the evolution of the ferromagnetic material reads

$$\begin{gathered}\gamma\partial_t\mathbf{m} = \nu\Delta\mathbf{m} - \theta_c\big(|\mathbf{m}|^2 - 1\big)\mathbf{m} - \theta\mathbf{m} + \mathbf{H} = 0, \quad \text{in } Q \\ c_1\partial_t(\ln\theta) + c_2\partial_t\theta - \mathbf{m}\cdot\partial_t\mathbf{m} = k_0\Delta(\ln\theta) + k_1\Delta\theta + \hat{r}, \quad \text{in } Q,\end{gathered} \tag{1.1}$$

where $Q = (0,T)\times\Omega$, $T > 0$, γ, ν, c_1, c_2, k_0, k_1 are positive constants and θ_c is a certain temperature called Curie temperature. Here $\hat{r}$ is a known function of x, t. For simplicity we assume that $\hat{r} = 0$.

We shall neglect the displacement current $\partial_t\mathbf{E}$. This is a customary assumption in describing ferromagnetic phenomena. As a consequence, the magnetic field $\mathbf{H}$

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that appears in the Maxwell's equations verifies

$$\begin{gathered}
\mu\partial_t\mathbf{H} + \partial_t\mathbf{m} + \frac{1}{\sigma}\operatorname{curl}\operatorname{curl}\mathbf{H} = 0 \quad \text{in } Q, \\
\operatorname{div}(\mu\mathbf{H} + \mathbf{m}) = 0 \quad \text{in } Q, \\
(\mu\mathbf{H} + \mathbf{m})\cdot\mathbf{n} = 0, \quad \operatorname{curl}\mathbf{H}\times\mathbf{n} = 0, \quad \text{on } (0,T)\times\partial\Omega,
\end{gathered} \tag{1.2}$$

where σ is the conductivity and μ is the magnetic permeability.

Global existence and uniqueness for (1.1)-(1.2) are proved in [4] and some limiting problems for thin films are obtained in [11].

In this investigation we shall consider a fractional version of (1.1) where we replace the Laplacian operator by a fractional one of order α for the magnetization and β for the temperature, $\alpha, \beta \in (0,1)$. We also assume that $c_1 = k_0 = 0$. This assumption means that the heat conductivity and specific heat depend on the absolute temperature according to the laws: $k(\theta) = k_1\theta$ and $c(\theta) = \frac{c_2}{2}\theta^2$. Let us mention that a great variety of assumptions about heat conductivity and specific heat is depicted, see for instance [3]. We consider the spatial domain $\Omega = [0, 2\pi]^d$ where $d \geq 1$ with periodic boundary conditions. The model equations read

$$\begin{gathered}
\gamma\partial_t\mathbf{m} + \nu\Lambda^{2\alpha}\mathbf{m} + \theta_c\big(|\mathbf{m}|^2 - 1\big)\mathbf{m} + \theta\mathbf{m} - \mathbf{H} = 0, \\
c\partial_t\theta + k\Lambda^{2\beta}\theta - \mathbf{m}\cdot\partial_t\mathbf{m} = 0, \\
\mu\partial_t\mathbf{H} + \partial_t\mathbf{m} + \frac{1}{\sigma}\operatorname{curl}\operatorname{curl}\mathbf{H} = 0.
\end{gathered} \tag{1.3}$$

For the initial data let

$$\mathbf{m}(0,x) = \mathbf{m}(x), \quad \theta(0,x) = \theta(x), \quad \mathbf{H}(0,x) = \mathbf{H}(x), \tag{1.4}$$

be given functions in Ω.

The motivation behind our work is that fractional order calculus can represent systems with high-order dynamics and complex nonlinear phenomena using few coefficients, since the arbitrary order of the derivatives provides an additional degree of freedom to fit a specific behavior. Another important characteristic is that fractional order derivatives depend not only on local conditions but also on the entire history of the function. This nonlocal character is often useful when the system has a long-term "memory" and any evaluation point depends on the past values of the function. On the other hand, the freedom in the definition of fractional derivatives allows us to incorporate different types of information. At the same time, the fractional derivatives with noninteger exponents stress which algebraic scale properties are relevant to the data analysis. Inability of classical, integer order derivative models in explaining complex phenomena (especially in elastodynamics, material science, electrochemistry, chemical physics and rheology), propelled further research in field and demonstrated strength of fractional calculus in solving practical problems, in particular, any reduction in the order of initial differential equation produces a significant reduction in computation time. A non-exhaustive list of works that support the mentioned modern development of fractional calculus and its applications are for example in [6, 7].

The rest of this article is organized as follows. In the next section, we recall some definitions and properties of fractional laplacian. We also define the weak solution of the model (1.3). We prove in Section 3 a global existence result for the considered model by using Faedo-Galerkin method. Compared with classical system, the model with fractional Laplacian exhibits some less of regularity and lack of compactness.

The proof combines some compactness techniques in the framework of fractional Sobolev spaces and the available energy estimates are used in order to pass to the limit in the approximating models. In Section 4, we show that the weak solution of (1.3) is unique. The last section provides future directions for this work.

2. Preliminaries

We now review the notation in this paper. Let $\Omega = [0, 2\pi]^d$ denote the periodic box with period 2π in all the directions, and $\mathbb{Z}^d := \mathbb{Z} \times \cdots \times \mathbb{Z}$ by d-times denote the dual lattice associated to Ω. The Fourier transform for tempered distributions defined on the whole space $\mathbb{R}^d$ may be carried out to $\mathcal{S}^{'}(\Omega)$ with very few changes.

Indeed, $f \in \mathcal{S}^{'}(\Omega)$ can be decomposed into Fourier series

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) := \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi) e^{i\xi\cdot x}$$

with

$$\hat{f}(\xi) = \frac{1}{(2\pi)^d} \int_\Omega e^{-i\xi\cdot y} f(y)\,\mathrm{d}y.$$

The square root of the Laplacian $(-\Delta)^{1/2}$ will be denoted by Λ and obviously

$$\Lambda f(\xi) = \mathcal{F}^{-1}(|\xi|\hat{f}(\xi)).$$

More generally, $\Lambda^s f$ for $s \in \mathbb{R}$ can be identified with the Fourier transform

$$\Lambda^s f(\xi) = \mathcal{F}^{-1}(|\xi|^s \hat{f}(\xi)).$$

Let L^p denote the space of all the pth integrable functions f normed by

$$\|f\|_{L^p} = \Big(\int_\Omega |f(x)|^p\,\mathrm{d}x\Big)^{1/p}, \quad \|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\Omega} |f(x)|.$$

Finally, for any $s \in \mathbb{R}$, we define the homogeneous Sobolev space $\dot{H}^s$ of all tempered distribution f such that $\|f\|_{\dot{H}^s}$ is finite, where $\|f\|_{\dot{H}^s}$ is defined via the Fourier transform

$$\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2} = \Big(\sum_{\xi\in\mathbb{Z}^d} |\xi|^{2s} |\hat{f}(\xi)|^2\Big)^{1/2}.$$

For general $1 \le p \le \infty$ and $s \in \mathbb{R}$, the space $\dot{H}^{s,p}(\Omega)$ consists of all f which can be written in the form $f = \Lambda^{-s} g$ for some $g \in L^p(\Omega)$ and the $\dot{H}^{s,p}$-norm of f is defined by

$$\|f\|_{\dot{H}^{s,p}} = \|\mathcal{F}^{-1}(|\xi|^s \hat{f}(\xi))\|_{L^p}.$$

Instead of the homogeneous Sobolev spaces, one can define the inhomogeneous counterparts via the operator $\mathcal{J} = (I - \Delta)^{1/2}$. We define, for any $s \in \mathbb{R}$, the inhomogeneous Sobolev space H^s of any tempered distribution f on Ω such that

$$\|f\|_{H^s} = \|\mathcal{J}^s f\|_{L^2} = \Big(\sum_{\xi\in\mathbb{Z}^d} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\Big)^{1/2} < +\infty.$$

The inhomogeneous Sobolev space $H^{s,p}$ can be defined similarly for $p \in [1, +\infty]$ and we omit the details. For more details for the functional settings, the readers are referred to [10].

Throughout this article, for $k \in \mathbb{N}^*$, $\mathbb{L}^k(\Omega) = (L^k(\Omega))^3$ and $\mathbb{H}^k(\Omega) = (H^k(\Omega))^3$ are the usual Hilbert-type Lebesgue and Sobolev spaces, respectively. For $k = 2$, the norm in $\mathbb{L}^2(\Omega)$ is denoted by $\|\cdot\|$. The space $\dot{\mathbb{H}}^\alpha(\Omega)$ denotes the homogenous

Sobolev-Slobodetskii space and $\mathbb{H}^\alpha(\Omega)$ denotes the inhomogenous one. Let us now give the definition of weak solution for (1.3).

Definition 2.1. Let $\alpha,\beta\in(0,1)$, $\mathbf{m}_0\in\mathbb{H}^\alpha(\Omega)$, $\theta_0\in L^2(\Omega)$, and $\mathbf{H}_0\in\mathbb{L}^2(\Omega)$. We say that $(\mathbf{m},\theta,\mathbf{H})$ is a weak solution to (1.3)-(1.4) if

- For all $T>0$, $(\mathbf{m},\theta,\mathbf{H})$ satisfies
$$\begin{gathered}\mathbf{m}\in L^\infty(0,T,\mathbb{H}^\alpha(\Omega)),\quad \partial_t\mathbf{m}\in\mathbb{L}^2(Q),\\ \theta\in L^\infty(0,T,L^2(\Omega))\cap L^2(0,T,H^\beta(\Omega)),\\ \mathbf{H}\in L^\infty(0,T,\mathbb{L}^2(\Omega)).\end{gathered}$$
- For all $\mathbf{\Psi}\in\mathcal{C}^\infty(Q)$
$$\begin{aligned}&\gamma\int_Q\partial_t\mathbf{m}\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t+\nu\int_Q\Lambda^\alpha\mathbf{m}\cdot\Lambda^\alpha\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t+\int_Q\theta_c(|\mathbf{m}|^2-1)\mathbf{m}\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t\\&+\int_Q\theta\mathbf{m}\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t-\int_Q\mathbf{H}\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t=0,\end{aligned}\tag{2.1}$$
- For all $\psi\in\mathcal{C}^\infty(Q)$
$$c\int_Q\theta\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t-k\int_Q\Lambda^\beta\theta\,\Lambda^\beta\psi\,\mathrm{d}x\,\mathrm{d}t+\int_Q\mathbf{m}\cdot\partial_t\mathbf{m}\,\psi\,\mathrm{d}x\,\mathrm{d}t+c\int_\Omega\theta_0\psi(0,\cdot)\,\mathrm{d}x=0,\tag{2.2}$$
- For all $\mathbf{\Psi}\in\mathcal{C}^\infty(Q)$,
$$\begin{aligned}&\mu\int_Q\mathbf{H}\cdot\partial_t\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t+\int_Q\mathbf{m}\cdot\partial_t\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{\sigma}\int_Q\operatorname{curl}\mathbf{H}\cdot\operatorname{curl}\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t\\&+\mu\int_\Omega\mathbf{H}_0\cdot\mathbf{\Psi}(0,\cdot)\,\mathrm{d}x+\int_\Omega\mathbf{m}_0\cdot\mathbf{\Psi}(0,\cdot)\,\mathrm{d}x=0\,.\end{aligned}\tag{2.3}$$
- For all $t>0$,
$$\mathcal{E}(t)+\gamma\int_0^t\int_\Omega|\partial_t\mathbf{m}|^2\,\mathrm{d}x\,\mathrm{d}t+k\int_0^t\int_\Omega|\Lambda^\beta\theta|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\sigma}\int_0^t\int_\Omega|\operatorname{curl}\mathbf{H}|^2\,\mathrm{d}x\,\mathrm{d}t=\mathcal{E}(0),\tag{2.4}$$
where
$$\mathcal{E}(t)=\frac12\Big(\nu\int_\Omega|\Lambda^\alpha\mathbf{m}|^2\,\mathrm{d}x+\frac{\theta_c}{2}\int_\Omega(|\mathbf{m}|^2-1)^2\,\mathrm{d}x+c\int_\Omega|\theta|^2\,\mathrm{d}x+\mu\int_\Omega|\mathbf{H}|^2\,\mathrm{d}x\Big).$$

3. Existence of global weak solutions

This section we construct global weak solutions to (1.3)-(1.4) via Faedo-Galerkin method, by proceeding as in [1, 12]. Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be the eigenfunctions for the eigenvalue problem
$$\Lambda^{2\alpha}\varphi_i=\lambda_i\varphi_i,\quad i=1,2,\dots\tag{3.1}$$
under periodic boundary conditions with $\{\lambda_i\}_{i\in\mathbb{N}}$ being the corresponding eigenvalues. Then $\{\varphi_i\}_{i\in\mathbb{N}}$ constitutes an orthonormal basis for $L^2(\Omega)$ and an orthogonal basis in $H^\alpha(\Omega)$ for $\alpha\in\mathbb{R}$, and the inner product in $H^\alpha(\Omega)$ can be expressed as
$$\langle\varphi_i,\varphi_j\rangle_\alpha=\delta_{ij}\lambda_i^{\alpha/2}\lambda_j^{\alpha/2}$$
where δ_{ij} is the Kronecker symbol. for positive real α, H^α can be characterized as
$$H^\alpha=\big\{v\in L^2,\sum_{i=1}^\infty\lambda_i^\alpha(v,\varphi_i)^2<\infty\big\}$$

Similarly, we consider $\{\phi_i\}_{i\in\mathbb{N}}$ the eigenfunctions for the eigenvalue problem

$$\Lambda^{2\beta}\phi_i = \kappa_i\phi_i, \quad i = 1, 2, \dots \tag{3.2}$$

under periodic boundary conditions with $\{\kappa_i\}_{i\in\mathbb{N}}$ being the corresponding eigenvalues.

Now consider the approximating solutions $(\mathbf{m}^N, \theta^N, \mathbf{H}^N)$ of the form

$$\begin{aligned}
\mathbf{m}^N(t,x) &= \sum_{i=1}^N \mathbf{a}_i(t)\varphi_i(x),\\
\theta^N(t,x) &= \sum_{i=1}^N b_i(t)\phi_i(x),\\
\mathbf{H}^N(t,x) &= \sum_{i=1}^N \mathbf{c}_i(t)\varphi_i(x),
\end{aligned}$$

where $\mathbf{a}_i(t)$, $b_i(t)$ and $\mathbf{c}_i(t)$ are all three-dimensional vector valued functions of t, and are chosen such that for $1 \le i \le N$ it holds

$$\begin{aligned}
&\gamma\int_\Omega \partial_t\mathbf{m}^N\varphi_i \,\mathrm{d}x + \nu\int_\Omega \Lambda^{2\alpha}\mathbf{m}^N\varphi_i\,\mathrm{d}x + \int_\Omega \theta_c(|\mathbf{m}^N|^2-1)\mathbf{m}^N\varphi_i\,\mathrm{d}x\\
&+\int_\Omega \theta^N\mathbf{m}^N\varphi_i\,\mathrm{d}x - \int_\Omega \mathbf{H}^N\varphi_i\,\mathrm{d}x = 0,
\end{aligned}\tag{3.3}$$

$$c_2\int_\Omega \partial_t\theta^N\phi_i\,\mathrm{d}x + k\int_\Omega \Lambda^{2\beta}\theta^N\phi_i\,\mathrm{d}x - \int_\Omega \mathbf{m}^N\cdot\partial_t\mathbf{m}^N\phi_i\,\mathrm{d}x - \int_\Omega \hat{r}\phi_i\,\mathrm{d}x = 0, \tag{3.4}$$

$$\mu\int_\Omega \partial_t\mathbf{H}^N\varphi_i\,\mathrm{d}x + \int_\Omega \partial_t\mathbf{m}^N\varphi_i\,\mathrm{d}x + \frac{1}{\sigma}\int_\Omega \operatorname{curl}\operatorname{curl}\mathbf{H}^N\varphi_i\,\mathrm{d}x = 0. \tag{3.5}$$

The initial conditions are

$$\begin{aligned}
\int_\Omega \mathbf{m}^N(0,x)\varphi_i\,\mathrm{d}x &= \int_\Omega \mathbf{m}_0(x)\varphi_i\,\mathrm{d}x,\\
\int_\Omega \theta^N(0,x)\phi_i\,\mathrm{d}x &= \int_\Omega \theta_0(x)\phi_i\,\mathrm{d}x,\\
\int_\Omega \mathbf{H}^N(0,x)\varphi_i\,\mathrm{d}x &= \int_\Omega \mathbf{H}_0(x)\varphi_i\,\mathrm{d}x,
\end{aligned}\tag{3.6}$$

for all $1 \le i \le N$.

The existence of local (in time) solutions $(\mathbf{a}_{iN}, b_{iN}, \mathbf{c}_{iN})$ for $1 \le i \le N$ to (3.3)-(3.6) follows from the standard Picard's theorem, which can be found in a general ODE textbook. To take the limit $N \to \infty$, we need to make sure that all the functions are defined at least in a common interval $[0,T]$, and this is a consequence of Lemma 3.1 below.

3.1. **A priori estimates.** Define

$$\mathcal{E}_N(t) = \frac{1}{2}\Big(\nu\int_\Omega |\Lambda^\alpha\mathbf{m}^N|^2\,\mathrm{d}x + \frac{\theta_c}{2}\int_\Omega (|\mathbf{m}^N|^2-1)^2\,\mathrm{d}x + c\int_\Omega |\theta^N|^2\,\mathrm{d}x + \mu\int_\Omega |\mathbf{H}^N|^2\,\mathrm{d}x\Big).$$

Lemma 3.1. *Let $T > 0$, $\mathbf{m}_0 \in \mathbb{H}^\alpha(\Omega)$, $\theta_0 \in L^2(\Omega)$ and $\mathbf{H}_0 \in \mathbb{L}^2(\Omega)$. Then for the solutions $(\mathbf{m}^N, \theta^N, \mathbf{H}^N)$ to the approximating system* (3.3)-(3.6)*, the following* *estimates hold for all* $t \in (0,T)$*,*

$$\begin{aligned} &\mathcal{E}_N(t) + \gamma \int_0^t \int_\Omega |\partial_t \mathbf{m}^N|^2 \,\mathrm{d}x\,\mathrm{d}t + k \int_0^t \int_\Omega |\Lambda^\beta \theta^N|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \frac{1}{\sigma} \int_0^t \int_\Omega |\operatorname{curl} \mathbf{H}^N|^2 \,\mathrm{d}x\,\mathrm{d}t = \mathcal{E}_N(0), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\gamma \int_\Omega |\Lambda^\alpha \mathbf{m}^N|^2 \,\mathrm{d}x + \nu \int_0^t \int_\Omega |\Lambda^{2\alpha} \mathbf{m}^N|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le C \int_0^t \Big[\|\mathbf{m}^N\|^2_{\mathbb{H}^\alpha(\Omega)} \big(1 + \|\mathbf{m}^N\|^4_{\mathbb{H}^\alpha(\Omega)} + \|\theta\|^2_{H^\beta(\Omega)}\big) + \int_\Omega |\mathbf{H}^N|^2 \,\mathrm{d}x \Big] \,\mathrm{d}t, \end{aligned} \tag{3.8}$$

$$\|\partial_t \mathbf{H}^N\|_{L^2(0,T,\mathbb{H}^{-1}(\Omega))} \le C, \tag{3.9}$$

where C *is a positive constant independent of* N*.*

Proof. We multiply (3.3), (3.4) and (3.5) by $\partial_t \mathbf{a}_i$, b_i and $\mathbf{c}_i$, respectively, and add for $i = 1, \dots, N$ the resulting equations. We obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\nu \int_\Omega |\Lambda^\alpha \mathbf{m}^N|^2 \,\mathrm{d}x + \frac{\theta_c}{2} \int_\Omega (|\mathbf{m}^N|^2 - 1)^2 \,\mathrm{d}x + c \int_\Omega |\theta^N|^2 \,\mathrm{d}x \\ &+ \mu \int_\Omega |\mathbf{H}^N|^2 \,\mathrm{d}x \Big] + \gamma \int_\Omega |\partial_t \mathbf{m}^N|^2 \,\mathrm{d}x + k \int_\Omega |\Lambda^\beta \theta^N|^2 \,\mathrm{d}x \\ &+ \frac{1}{\sigma} \int_\Omega |\operatorname{curl} \mathbf{H}^N|^2 \,\mathrm{d}x = 0. \end{aligned} \tag{3.10}$$

Integrating (3.10) from 0 to t, we obtain (3.7).

Now, we test (3.3) by $\Lambda^{2\alpha}\mathbf{m}$ and using Young's inequality, we obtain

$$\begin{aligned} &\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \mathbf{m}^N|^2 \,\mathrm{d}x + \frac{\nu}{2} \int_\Omega |\Lambda^{2\alpha} \mathbf{m}^N|^2 \,\mathrm{d}x \\ &\le \frac{1}{2\nu} \int_\Omega \Big| -\theta_c(|\mathbf{m}^N|^2 - 1)\mathbf{m}^N - \theta^N \mathbf{m}^N + \mathbf{H}^N \Big|^2 \,\mathrm{d}x \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \mathbf{m}^N|^2 \,\mathrm{d}x + \frac{\nu}{2} \int_\Omega |\Lambda^{2\alpha} \mathbf{m}^N|^2 \,\mathrm{d}x \\ &\le C \int_\Omega \Big[(|\mathbf{m}^N|^4 + 1)|\mathbf{m}^N|^2 + |\theta^N|^2 |\mathbf{m}^N|^2 + |\mathbf{H}^N|^2 \Big] \,\mathrm{d}x. \end{aligned}$$

Now, for the term $\int_\Omega (|\mathbf{m}^N|^4 + 1)|\mathbf{m}^N|^2 \,\mathrm{d}x$, thanks to the Sobolev embedding $\mathbb{H}^\alpha(\Omega) \hookrightarrow \mathbb{L}^6(\Omega)$ for $\alpha \ge \frac{d}{3}$, we have

$$\begin{aligned} \int_\Omega (|\mathbf{m}^N|^4 + 1)|\mathbf{m}^N|^2 \,\mathrm{d}x &= \int_\Omega |\mathbf{m}^N|^2 \,\mathrm{d}x + \int_\Omega |\mathbf{m}^N|^6 \,\mathrm{d}x \\ &\le \|\mathbf{m}^N\|^2_{\mathbb{H}^\alpha(\Omega)} + C\|\mathbf{m}^N\|^6_{\mathbb{H}^\alpha(\Omega)} \\ &= C\|\mathbf{m}^N\|^2_{\mathbb{H}^\alpha(\Omega)}(1 + \|\mathbf{m}^N\|^4_{\mathbb{H}^\alpha(\Omega)}). \end{aligned}$$

On the other hand, using the fact that $\mathbb{H}^\beta(\Omega) \hookrightarrow \mathbb{L}^4(\Omega)$ for $\beta \ge \frac{d}{4}$, we obtain

$$\int_\Omega |\theta^N|^2 |\mathbf{m}^N|^2 \,\mathrm{d}x \le \|\theta^N\|^2_{L^4(\Omega)} \|\mathbf{m}^N\|^2_{\mathbb{L}^4(\Omega)} \le C\|\theta^N\|^2_{H^\beta(\Omega)} \|\mathbf{m}^N\|^2_{\mathbb{H}^\alpha(\Omega)}.$$

Then (3.10) implies (3.8).

Now, let $\mathbf{\Phi} \in L^2(0,T,\mathbb{H}^1(\Omega))$, from (3.5) and (3.7), we have

$$\begin{aligned}\Big|\int_Q \partial_t \mathbf{H}^N \cdot \mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t\Big| &\le \frac{1}{\mu}\|\partial_t \mathbf{m}^N\|_{\mathbb{L}^2(Q)}\|\mathbf{\Phi}\|_{\mathbb{L}^2(Q)} + \frac{1}{\mu\sigma}\|\operatorname{curl}\mathbf{H}^N\|_{\mathbb{L}^2(Q)}\|\operatorname{curl}\mathbf{\Phi}\|_{\mathbb{L}^2(Q)}\\ &\le C\|\mathbf{\Phi}\|_{L^2(0,T,\mathbb{H}^1(\Omega))}\end{aligned}$$

where C is a constant independent of N. The proof is complete. ☐

Lemma 3.2. *Let $(\mathbf{m}^N, \theta^N, \mathbf{H}^N)$ be solutions for the approximating system (3.3)-(3.6) then the following estimates hold*

$$\begin{aligned}\|\mathbf{m}^N(t_1,\cdot) - \mathbf{m}^N(t_2,\cdot)\|_{\mathbb{L}^2(\Omega)} &\le C|t_1-t_2|^{1/2},\\ \|\mathbf{H}^N(t_1,\cdot) - \mathbf{H}^N(t_2,\cdot)\|_{\mathbb{H}^{-1}(\Omega)} &\le C|t_1-t_2|^{1/2},\end{aligned} \tag{3.11}$$

where C is a constant independent of N.

Proof. By Young and Hölder inequalities we have

$$\begin{aligned}\|\mathbf{m}^N(t_1,\cdot) - \mathbf{m}^N(t_2,\cdot)\|_{\mathbb{L}^2(\Omega)} &= \Big\|\int_{t_2}^{t_1} \partial_t \mathbf{m}^N\,\mathrm{d}t\Big\|_{\mathbb{L}^2(\Omega)}\\ &\le \int_{t_2}^{t_1} \|\partial_t \mathbf{m}^N\|_{\mathbb{L}^2(\Omega)}\,\mathrm{d}t\\ &\le |t_1-t_2|^{1/2}\Big(\int_Q |\partial_t \mathbf{m}^N|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\\ &\le C|t_1-t_2|^{1/2}.\end{aligned}$$

By Lemma 3.1, we deduce that $(\partial_t \mathbf{H}^N)_N$ is bounded in $L^2(0,T,\mathbb{H}^{-1}(\Omega))$. Then

$$\begin{aligned}\|\mathbf{H}^N(t_1,\cdot) - \mathbf{H}^N(t_2,\cdot)\|_{\mathbb{H}^{-1}(\Omega)} &= \Big\|\int_{t_2}^{t_1} \partial_t \mathbf{H}^N\,\mathrm{d}t\Big\|_{\mathbb{H}^{-1}(\Omega)}\\ &\le \int_{t_2}^{t_1} \|\partial_t \mathbf{H}^N\|_{\mathbb{H}^{-1}(\Omega)}\,\mathrm{d}t\\ &\le |t_1-t_2|^{1/2}\Big(\int_0^T \|\partial_t \mathbf{H}^N\|^2_{\mathbb{H}^{-1}(\Omega)}\,\mathrm{d}t\Big)^{1/2}\\ &\le C|t_1-t_2|^{1/2},\end{aligned}$$

where the constant C is independent of N. The proof is complete. ☐

3.2. **Compactness argument and convergence.** In the following, we will take $N \to \infty$ to obtain a global weak solutions the problem (1.3)-(1.4). Before doing so, we give a compactness lemma first whose proof can be found in Lions [8], hence omitted.

Lemma 3.3. *Let B_0, B, B_1 be three Banach spaces such that $B_0 \hookrightarrow B \hookrightarrow B_1$, where the injections are continuous and B_0, B_1 are reflexive, and the injection $B_0 \hookrightarrow B$ is compact. Denote*

$$W = \big\{v \in L^{p_0}(0,T,B_0) : \frac{\mathrm{d}v}{\mathrm{d}t} \in L^{p_1}(0,T,B_1)\big\},$$

where T is finite and $1 < p_0, p_1 < \infty$. Then W equipped with the norm

$$\|v\|_W = \|v\|_{L^{p_0}(0,T,B_0)} + \Big\|\frac{\mathrm{d}v}{\mathrm{d}t}\Big\|_{L^{p_1}(0,T,B_1)}$$

is a Banach space and the embedding $W \hookrightarrow L^{p_0}(0,T,B)$ is compact. When $p_0=\infty$, $1<p_1\le\infty$, the embedding $W\hookrightarrow C([0,T],B)$ is compact.

Now, let $\mathbf{\Psi},\psi\in\mathcal{C}^\infty(Q)$ with $\mathbf{\Psi}(T,\cdot)=\psi(T,\cdot)=0$. Taking scalar product of (3.3), (3.5) with $\mathbf{\Psi}$ and (3.4) with ψ, summing up for $i=1,2,\dots,N$, integrating from over $[0,T]$ and using integration by parts formula, we obtain the following approximating equalities

$$
\begin{aligned}
&\gamma\int_Q \partial_t\mathbf{m}^N\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t+\nu\int_Q\Lambda^\alpha\mathbf{m}^N\cdot\Lambda^\alpha\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t+\int_Q\theta_c(|\mathbf{m}^N|^2-1)\mathbf{m}^N\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_Q\theta^N\mathbf{m}^N\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t-\int_Q\mathbf{H}^N\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t=0,\\
&\qquad c\int_Q\theta^N\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t-k\int_Q\Lambda^\beta\theta^N\Lambda^\beta\psi\,\mathrm{d}x\,\mathrm{d}t+\int_Q\mathbf{m}^N\cdot\partial_t\mathbf{m}^N\ \psi\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad +c\int_\Omega\theta^N(0,\cdot)\psi(0,\cdot)\,\mathrm{d}x=0,\\
&\quad\mu\int_Q\mathbf{H}^N\cdot\partial_t\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t-\int_Q\partial_t\mathbf{m}^N\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{\sigma}\int_Q\operatorname{curl}\mathbf{H}^N\cdot\operatorname{curl}\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mu\int_\Omega\mathbf{H}^N(0,\cdot)\cdot\mathbf{\Psi}(0,\cdot)\,\mathrm{d}x=0,
\end{aligned}
$$

Applying the compactness Lemma 3.3, we have the following compactness results. There is some $(\mathbf{m},\theta,\mathbf{H})$ such that up to a subsequence

$$
\begin{gathered}
\partial_t\mathbf{m}^N\rightharpoonup\partial_t\mathbf{m}\quad\text{weakly in } \mathbb{L}^2(Q),\\
\mathbf{m}^N\rightharpoonup\mathbf{m}\quad\text{weakly in } L^p(0,T,\mathbb{H}^\alpha(\Omega)),\ 1<p<\infty,\\
\mathbf{m}^N\to\mathbf{m}\quad\text{strongly in } C([0,T],\mathbb{H}^\rho(\Omega))\quad\text{and a.e. for } 0\le\rho<\alpha,\\
\theta^N\rightharpoonup\theta\quad\text{weakly in } L^2(0,T,H^\beta(\Omega)),\\
\mathbf{H}^N\rightharpoonup\mathbf{H}\quad\text{weak-}\star\text{ in } L^\infty(0,T,\mathbb{L}^2(\Omega)),\\
\operatorname{curl}\mathbf{H}^N\rightharpoonup\operatorname{curl}\mathbf{H}\quad\text{weakly in } L^2(0,T,\mathbb{L}^2(\Omega)).
\end{gathered}
$$

These compactness results enable us to prove the convergence of the above equalities. Indeed, it suffices to consider the convergence of the nonlinear terms. We will prove that

$$
\int_Q(|\mathbf{m}^N|^2-1)\mathbf{m}^N\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t\to\int_Q(|\mathbf{m}^N|^2-1)\mathbf{m}^N\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t,\quad\text{as } N\to\infty. \tag{3.12}
$$

Firstly, since $(|\mathbf{m}^N|^2-1)$ is bounded in $L^\infty(0,T,L^2(\Omega))$, by (3.7) we have $|\mathbf{m}^N|^2-1\rightharpoonup\chi$ weakly in $L^2(0,T,L^2(\Omega))$. On the other hand, $\mathbf{m}^N\to\mathbf{m}$ strongly in $L^2(0,T,\mathbb{L}^2(\Omega))$ and a.e. which implies that $\chi=|\mathbf{m}|^2-1$. Then $(|\mathbf{m}^N|^2-1)\mathbf{m}^N\rightharpoonup(|\mathbf{m}|^2-1)\mathbf{m}$ weakly in $L^1(0,T,\mathbb{L}^1(\Omega))$, therefore (3.12) is proved. Since $\mathbf{m}^N\to\mathbf{m}$ strongly in $L^2(0,T,\mathbb{L}^2(\Omega))$ and $\theta^N\rightharpoonup\theta$ weakly in $L^2(0,T,\mathbb{L}^2(\Omega))$, we now that $\theta^N\mathbf{m}^N\rightharpoonup\theta\mathbf{m}$ weakly in $\mathbb{L}^1(Q)$. Then

$$
\int_Q\theta^N\mathbf{m}^N\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t\to\int_Q\theta^N\mathbf{m}^N\cdot\mathbf{\Psi}\,\mathrm{d}x\,\mathrm{d}t,\quad\text{as } N\to\infty.
$$

We have that $\mathbf{m}^N \to \mathbf{m}$ strongly in $L^2(0,T,\mathbb{L}^2(\Omega))$ and $\partial_t\mathbf{m}^N \rightharpoonup \partial_t\mathbf{m}$ weakly in $L^2(0,T,\mathbb{L}^2(\Omega))$. Then

$$\int_Q \mathbf{m}^N\cdot\partial_t\mathbf{m}^N\psi\,\mathrm{d}x\,\mathrm{d}t \to \int_Q \mathbf{m}\cdot\partial_t\mathbf{m}\psi\,\mathrm{d}x\,\mathrm{d}t, \quad \text{as } N\to\infty.$$

Since the other terms are linear, their convergence is obvious. We have proved the following global existence result.

Theorem 3.4. Let $\alpha,\beta\in(0,1)$ such that $d\le\min(3\alpha,4\beta)$, $\mathbf{m}_0\in\mathbb{H}^\alpha(\Omega)$, $\theta_0\in L^2(\Omega)$, and $\mathbf{H}_0\in\mathbb{L}^2(\Omega)$. For all $T>0$, there exist a weak solution $(\mathbf{m},\theta,\mathbf{H})$ to the problem (1.3)-(1.4) in the sense of Definition 2.1. Furthermore, the solution satisfies

$$\begin{gathered}
\mathbf{m}\in L^\infty(0,T,\mathbb{H}^\alpha(\Omega))\cap C^{0,\frac12}(0,T,\mathbb{L}^2(\Omega)),\quad \partial_t\mathbf{m}\in\mathbb{L}^2(Q),\\
\theta\in L^\infty(0,T,L^2(\Omega))\cap L^2(0,T,H^\beta(\Omega)),\\
\mathbf{H}\in L^\infty(0,T,\mathbb{L}^2(\Omega))\cap C^{0,\frac12}(0,T,\mathbb{H}^{-1}(\Omega)).
\end{gathered}$$

4. Uniqueness of the weak solution

To prove uniqueness of weak solution to (1.3), let $(\mathbf{m}_i,\theta_i,\mathbf{H}_i)$, be two weak solutions corresponding to the data $\mathbf{m}_{0i},\theta_{0i}$ and $\mathbf{H}_{0i}$, $i=1,2$ respectively. We introduce the differences

$$\mathbf{m}=\mathbf{m}_1-\mathbf{m}_2,\quad \theta=\theta_1-\theta_2,\quad \mathbf{H}=\mathbf{H}_1-\mathbf{H}_2.$$

Then $(\mathbf{m},\theta,\mathbf{H})$ satisfies the following system in the weak sense

$$\begin{gathered}
\gamma\partial_t\mathbf{m}+\nu\Lambda^{2\alpha}\mathbf{m}+\theta_c(|\mathbf{m}_1|^2-1)\mathbf{m}_1-\theta_c(|\mathbf{m}_2|^2-1)\mathbf{m}_2\\
+\theta_1\mathbf{m}_1-\theta_2\mathbf{m}_2-\mathbf{H}=0,\\
c_2\partial_t\theta+k\Lambda^{2\beta}\theta-\mathbf{m}_1\cdot\partial_t\mathbf{m}_1+\mathbf{m}_2\cdot\partial_t\mathbf{m}_2=0,\\
\mu\partial_t\mathbf{H}+\partial_t\mathbf{m}+\frac{1}{\sigma}\operatorname{curl}\operatorname{curl}\mathbf{H}=0,
\end{gathered}\tag{4.1}$$

with initial data

$$\begin{gathered}
\mathbf{m}(0,x)=\mathbf{m}_{01}(x)-\mathbf{m}_{02}(x)=\mathbf{m}_0(x),\\
\theta(0,x)=\theta_{01}(x)-\theta_{02}(x)=\theta_0(x),\\
\mathbf{H}(0,x)=\mathbf{H}_{01}(x)-\mathbf{H}_{02}(x)=\mathbf{H}_0(x).
\end{gathered}$$

Integrate the second and the last equations of (4.1) over $(0,t)$, we obtain, respectively,

$$c\theta+k\int_0^t\Lambda^{2\beta}\theta\,\mathrm{d}s=\frac12(|\mathbf{m}_1|^2-|\mathbf{m}_2|^2)-\frac12(|\mathbf{m}_{01}|^2-|\mathbf{m}_{02}|^2)+c\theta_0,\tag{4.2}$$

$$\mu\mathbf{H}+\mathbf{m}+\frac{1}{\sigma}\int_0^t\operatorname{curl}\operatorname{curl}\mathbf{H}\,\mathrm{d}s=\mu\mathbf{H}_0+\mathbf{m}_0,\tag{4.3}$$

Multiplying the first equation of (4.1) by $\mathbf{m}$ and (4.3) by $\mathbf{H}$, integrating over Ω and adding the resulting equations, we obtain

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\Big[\gamma\|\mathbf{m}\|^2+\frac{1}{\sigma}\|\int_0^t\operatorname{curl}\mathbf{H}\,\mathrm{d}s\|^2\Big]+\nu\|\Lambda^\alpha\mathbf{m}\|^2+\mu\|\mathbf{H}\|^2:=I_1+I_2,\tag{4.4}$$

where

$$I_1 = \int_\Omega \big[\theta_c(|\mathbf{m}_2|^2 - 1)\mathbf{m}_2 - \theta_c(|\mathbf{m}_1|^2 - 1)\mathbf{m}_1 + \theta_2\mathbf{m}_2 - \theta_1\mathbf{m}_1\big] \cdot \mathbf{m}\,\mathrm{d}x,$$

$$I_2 = \int_\Omega (\mu\mathbf{H}_0 + \mathbf{m}_0) \cdot \mathbf{H}\,\mathrm{d}x.$$

Multiplying now (4.2) by θ, integrating over Ω, we obtain

$$c\int_\Omega |\theta|^2\,\mathrm{d}x + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\int_0^t \Lambda^\beta\theta\,\mathrm{d}s\|^2 := I_3, \tag{4.5}$$

where

$$I_3 = \int_\Omega \big[\frac{1}{2}(|\mathbf{m}_1|^2 - |\mathbf{m}_2|^2) - \frac{1}{2}(|\mathbf{m}_{01}|^2 - |\mathbf{m}_{02}|^2) + c\theta_0\big]\theta\,\mathrm{d}x.$$

- Estimate on I_1: Firstly, we rewrite

$$\begin{aligned} I_1 &= \int_\Omega \big[\theta_c(1 - |\mathbf{m}_1|^2) - \theta_1\big]|\mathbf{m}|^2\,\mathrm{d}x - \int_\Omega \theta_c[(\mathbf{m}_1 + \mathbf{m}_2)\cdot\mathbf{m}]\mathbf{m}_2\cdot\mathbf{m}\,\mathrm{d}x \\ &\quad - \int_\Omega \theta\mathbf{m}_2\cdot\mathbf{m}\,\mathrm{d}x = I_{11} + I_{12} + I_{13}, \end{aligned}$$

with

$$I_{11} = \int_\Omega \big[\theta_c(1 - |\mathbf{m}_1|^2) - \theta_1\big]|\mathbf{m}|^2\,\mathrm{d}x,$$

$$I_{12} = -\int_\Omega \theta_c[(\mathbf{m}_1 + \mathbf{m}_2)\cdot\mathbf{m}]\mathbf{m}_2\cdot\mathbf{m}\,\mathrm{d}x,$$

$$I_{13} = -\int_\Omega \theta\mathbf{m}_2\cdot\mathbf{m}\,\mathrm{d}x.$$

Next, we bound separately each term. Using the fact that, $H^\beta(\Omega) \hookrightarrow L^4(\Omega)$ for $\beta \ge d/4$, and $\mathbb{H}^{2\alpha}(\Omega) \hookrightarrow \mathbb{L}^\infty(\Omega)$ for $\alpha > d/4$, we have

$$\begin{aligned} |I_{11}| &\le \theta_c(1 + \|\mathbf{m}_2\|_\infty^2)\|\mathbf{m}\|^2 + \|\theta_1\|_{L^4(\Omega)}\|\mathbf{m}\|_{\mathbb{L}^4(\Omega)}\|\mathbf{m}\| \\ &\le \theta_c(1 + \|\mathbf{m}_2\|_{\mathbb{H}^{2\alpha}(\Omega)}^2)\|\mathbf{m}\|^2 + C\|\theta_1\|_{H^\beta(\Omega)}\|\mathbf{m}\|_{\mathbb{H}^\alpha(\Omega)}\|\mathbf{m}\| \\ &\le C\big[(1 + \|\mathbf{m}_2\|_{\mathbb{H}^{2\alpha}(\Omega)}^2)\|\mathbf{m}\|^2 + \|\theta_1\|_{H^\beta(\Omega)}\|\mathbf{m}\|_{\mathbb{H}^\alpha(\Omega)}\|\mathbf{m}\|\big]. \end{aligned}$$

Furthermore

$$\begin{aligned} |I_{12}| &\le \theta_c\|\mathbf{m}_1 + \mathbf{m}_2\|_\infty\|\mathbf{m}_2\|_\infty\|\mathbf{m}\|^2 \\ &\le 2\theta_c(\|\mathbf{m}_1\|_\infty^2 + \|\mathbf{m}_2\|_\infty^2)\|\mathbf{m}\|^2 \\ &\le C(\|\mathbf{m}_1\|_{\mathbb{H}^{2\alpha}(\Omega)}^2 + \|\mathbf{m}_2\|_{\mathbb{H}^{2\alpha}(\Omega)}^2)\|\mathbf{m}\|^2 \end{aligned}$$

and

$$|I_{13}| \le \|\mathbf{m}_2\|_\infty\|\theta\|\|\mathbf{m}\| \le C\|\mathbf{m}_2\|_{\mathbb{H}^{2\alpha}(\Omega)}\|\theta\|\|\mathbf{m}\|.$$

Then by Young's inequality, we obtain

$$\begin{aligned} |I_1| &\le C\Big[(1 + \|\mathbf{m}_1\|_{\mathbb{H}^{2\alpha}(\Omega)}^2 + \|\mathbf{m}_2\|_{\mathbb{H}^{2\alpha}(\Omega)}^2)\|\mathbf{m}\|^2 + \|\theta_1\|_{H^\beta(\Omega)}\|\mathbf{m}\|_{\mathbb{H}^\alpha(\Omega)}\|\mathbf{m}\| \\ &\quad + \|\mathbf{m}_2\|_{\mathbb{H}^{2\alpha}(\Omega)}\|\theta\|\|\mathbf{m}\|\Big] \\ &\le \varepsilon\|\mathbf{m}\|_{\mathbb{H}^\alpha(\Omega)}^2 + \varepsilon\|\theta\|^2 + C_\varepsilon\big(1 + \|\mathbf{m}_1\|_{\mathbb{H}^{2\alpha}(\Omega)}^2 + \|\mathbf{m}_2\|_{\mathbb{H}^{2\alpha}(\Omega)}^2 + \|\theta_1\|_{H^\beta(\Omega)}^2\big)\|\mathbf{m}\|^2 \end{aligned}$$

for $\varepsilon > 0$.

• Estimate on I_2: Young's inequality implies that

$$\begin{aligned}|I_2| &\leq \|\mu\mathbf{H}_0 + \mathbf{m}_0\|\|\mathbf{H}\|\\ &\leq (\mu\|\mathbf{H}_0\| + \|\mathbf{m}_0\|)\|\mathbf{H}\|\\ &\leq \frac{\mu}{2}\|\mathbf{H}\|^2 + C(\|\mathbf{H}_0\|^2 + \|\mathbf{m}_0\|^2).\end{aligned}$$

• Estimate on I_3: We rewrite

$$I_3 = \int_\Omega \big[\frac{1}{2}(\mathbf{m}_1 + \mathbf{m}_2)\cdot\mathbf{m} - \frac{1}{2}(\mathbf{m}_{01} + \mathbf{m}_{02})\cdot\mathbf{m}_0 + c\theta_0\Big]\theta\,\mathrm{d}x.$$

Then

$$\begin{aligned}|I_3| &\leq \big[\frac{1}{2}(\|\mathbf{m}_1\|_\infty + \|\mathbf{m}_2\|_\infty)\|\mathbf{m}\| + \frac{1}{2}(\|\mathbf{m}_{01}\|_\infty + \|\mathbf{m}_{02}\|_\infty)\|\mathbf{m}_0\| + c\|\theta_0\|\big]\|\theta\|\\ &\leq \frac{c}{2}\|\theta\|^2 + C\big[(\|\mathbf{m}_1\|^2_{\mathbb{H}^{2\alpha}(\Omega)} + \|\mathbf{m}_2\|^2_{\mathbb{H}^{2\alpha}(\Omega)})\|\mathbf{m}\|^2 + \|\mathbf{m}_0\|^2 + \|\theta_0\|^2\big]\end{aligned}$$

Adding (4.4) and (4.5), choosing ε such that $\varepsilon < \min(\nu, \frac{c}{2})$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\gamma\|\mathbf{m}\|^2 + \frac{1}{\sigma}\Big\|\int_0^t \operatorname{curl}\mathbf{H}\,\mathrm{d}s\Big\|^2 + k\Big\|\int_0^t \Lambda^\beta\theta\,\mathrm{d}s\Big\|^2\Big] + (\nu-\varepsilon)\|\Lambda^\alpha\mathbf{m}\|^2\\ &+ (\frac{c}{2} - \varepsilon)\|\theta\|^2 + \frac{\mu}{2}\|\mathbf{H}\|^2\\ &\leq C\big(\|\mathbf{m}_0\|^2 + \|\theta_0\|^2 + \|\mathbf{H}_0\|^2\big) + F(t)\|\mathbf{m}\|^2\end{aligned}\tag{4.6}$$

where $F \in L^1(0,T)$. Using Gronwall Lemma, there exists $C(T)$ such that

$$\|\mathbf{m}\|^2 \leq C(T)\big(\|\mathbf{m}_0\|^2 + \|\theta_0\|^2 + \|\mathbf{H}_0\|^2\big).\tag{4.7}$$

Integrating (4.6) over $(0,T)$ and using (4.7), we obtain

$$\int_0^T \big(\|\mathbf{m}\|^2_{\mathbb{H}^\alpha(\Omega)} + \|\theta\|^2 + \|\mathbf{H}\|^2\big)\,\mathrm{d}t \leq C_T\big(\|\mathbf{m}_0\|^2 + \|\theta_0\|^2 + \|\mathbf{H}_0\|^2\big).$$

We have proved the following uniqueness result.

Theorem 4.1. *Let* $(\mathbf{m}_1, \theta_1, \mathbf{H}_1)$ *and* $(\mathbf{m}_2, \theta_2, \mathbf{H}_2)$ *be two solutions of problem* (1.3)-(1.4)*, with initial data* $(\mathbf{m}_{01}, \theta_{01}, \mathbf{H}_{01})$*,* $(\mathbf{m}_{02}, \theta_{02}, \mathbf{H}_{02}) \in \mathbb{H}^\alpha(\Omega)\times L^2(\Omega)\times\mathbb{L}^2(\Omega)$*. Then, for each* $T > 0$*, there exists a positive constant* C_T *such that*

$$\begin{aligned}&\int_0^T (\|\mathbf{m}_1 - \mathbf{m}_2\|^2_{\mathbb{H}^\alpha(\Omega)} + \|\theta_1 - \theta_2\| + \|\mathbf{H}_1 - \mathbf{H}_2\|)\,\mathrm{d}t\\ &\leq C_T(\|\mathbf{m}_{01} - \mathbf{m}_{02}\|^2_{\mathbb{H}^\alpha(\Omega)} + \|\theta_{01} - \theta_{02}\| + \|\mathbf{H}_{01} - \mathbf{H}_{02}\|).\end{aligned}$$

In particular, the solution of problem (1.3)-(1.4) *is unique.*

5. Concluding remarks

In this paper, global existence and uniqueness of weak solution to a fractional model describing phase transition in ferromagnets are proved. The model couples thermodynamic and electromagnetic properties of the ferromagnetic material. Due to nonlocal nonlinearities in the model, special structures of the equations and some calculus inequalities of fractional order are exploited to get the convergence of the approximating solutions. There are a number of directions which are worth pursuing based on the developments presented here, we briefly mention some of them. We have assumed that $c_1 = k_0 = 0$ and we would like to extend our results to a more general assumptions on heat conductivity and specific heat as depicted in

[3]. Also, an interesting direction of future research is to design numerical scheme both for the model (1.1) and the fractional model studied in this paper. This will be helpful to give a strategy for efficient computer implementation which may address a comparative analysis of the models with integer and non-integer order derivatives. We finally note that these numerical issues may also give some help for studying periodic perforated media for which effective thermo-electromagnetic properties can be obtained by using the theory of periodic homogenization.

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