POSITIVE SOLUTION CURVES OF AN INFINITE SEMIPOSITONE PROBLEM

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Abstract. In this article we consider the infinite semipositone problem $-\Delta u = \lambda f(u)$ in $\Omega$, a smooth bounded domain in $\mathbb{R}^N$, and $u = 0$ on $\partial \Omega$, where $f(t) = t^q - t^{-\beta}$ and $0 < q, \beta < 1$. Using stability analysis we prove the existence of a connected branch of maximal solutions emanating from infinity. Under certain additional hypothesis on the extremal solution at $\lambda = \Lambda$ we prove a version of Crandall-Rabinowitz bifurcation theorem which provides a multiplicity result for $\lambda \in (\Lambda, \Lambda + \epsilon)$.

1. Introduction

Consider the infinite semi-positone problem

\[\begin{align*}
-\Delta u &= \lambda f(u) \quad \text{in } \Omega \\
0 &= \frac{\partial u}{\partial \nu} \quad \text{on } \partial \Omega,
\end{align*}\]

where $f(t) = t^q - t^{-\beta}$, $0 < q < 1$ and $\beta \in (0, 1)$ and $\lambda$ a positive parameter. Here $\Omega$ is assumed to be a bounded domain with smooth boundary in $\mathbb{R}^N$. Note that $f(0) = -\infty$ (hence the name infinite semipositone problem) and $f$ is an increasing concave function in $\mathbb{R}^+$. Finding a positive solution for semipositone problems are always challenging and in fact proving the existence of multiple positive solutions are even more difficult. The existence of a positive solution for (1.1) when $\lambda$ large is studied using sub-super solutions technique in [18]. Later in [10], it was additionally shown that when $\lambda$ is large there exists a maximal positive solution for (1.1) which is in fact bounded below by the distance function $d(x, \partial \Omega) = \inf \{ |x - y| : y \in \partial \Omega \}$. The aim of this work is to further understand this maximal branch of solution of (1.1) which emanates from $\infty$.

Definition 1.1. We say $u$ is a solution of (1.1), if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u(x) \geq c d(x, \partial \Omega)$ for some positive constant $c = c(\lambda)$.

Suppose that $\partial \Omega$ is smooth and $u$ is a solution of $(P_\lambda)$, then the outward normal derivative $\frac{\partial u}{\partial \nu}(x_0) < 0$ for all $x_0 \in \partial \Omega$. Conversely if we assume that $\frac{\partial u}{\partial \nu}|_{\partial \Omega} < 0$ then by the tubular neighbourhood lemma $u(x) \geq c, d(x, \partial \Omega)$ for some $c > 0$.

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Definition 1.2. Let \( S \) = \{ (\lambda, u_{\lambda}) : u_{\lambda} \) is a solution to \((1.1)\) and let \( \lambda = \inf \{ \lambda > 0 : (1.1) \) admits at least one solution}.

Definition 1.3. We say \( \lambda_{\infty} = \infty \) is a bifurcation point at infinity for \((1.1)\) if there exists a sequence \( (\lambda_n, u_{\lambda_n}) \in S \) such that \( \lambda_n \rightarrow \lambda_{\infty} \) and \( \| u_{\lambda_n} \| \rightarrow \infty \).

The principal eigenvalue of the linearized operator associated to \((1.1)\) is denoted by \( \Lambda_1(\lambda) \) and defined as
\[
\Lambda_1(\lambda) = \inf_{\varphi \in H_0^1(\Omega), \| \varphi \|_2 = 1} \left( \int_{\Omega} |\nabla \varphi|^2 - \lambda \int_{\Omega} f'(u) \varphi^2 \right).
\]
where \( u \) solves \((1.1)\) as in definition 1.1. Since the solution \( u(x) \) behaves like \( d(x) \) near \( \partial \Omega \), by Hardy’s inequality the term \( \int_{\Omega} f'(u) \varphi^2 \) make sense. The functional \( \int_{\Omega} |\nabla \varphi|^2 - \lambda \int_{\Omega} f'(u) \varphi^2 \) is bounded below and coercive on the set \( \{ \varphi \in H_0^1(\Omega) : \| \varphi \|_2 = 1 \} \) and hence a minimizer exists. Also one can show that \( \Lambda_1(\lambda) \) satisfies the differential equation \(-\Delta \psi - \lambda f'(u) \psi = \Lambda_1(\lambda) \psi \) for some non-negative \( \psi \in H_0^1(\Omega) \).

We say that a solution \( u \) of \((1.1)\) is stable if \( \Lambda_1(\lambda) \) is strictly positive. Our main result is the following theorem.

Theorem 1.4. Assume that \( \Omega \) is a bounded open set in \( \mathbb{R}^N \) with smooth boundary and consider the infinite semipositone problem \((1.1)\) \(-\Delta u = \lambda (u^q - u^\beta) \) in \( \Omega \) for \( 0 < q, \beta < 1 \) and \( u = 0 \) on \( \partial \Omega \).

(a) There exists a \( \lambda \in (0, \infty) \) and for all \( \lambda > \Lambda \), there exists a maximal positive solution \( u_{\lambda} \) solving \((1.1)\). And \( \| u_{\lambda} \|_\infty \rightarrow \infty \) as \( \lambda \rightarrow \infty \), i.e. \( \lambda_{\infty} \) is a bifurcation point at infinity. Also if \( \lambda \in (0, \Lambda) \), the problem \((1.1)\) does not admit any positive solution.

(b) The maximal solution \( u_{\lambda} \) is stable for all \( \lambda > \Lambda \).

(c) There exists an unbounded connected branch \( \mathcal{C} \) of solutions of \((1.1)\) emanating from \((\infty, \infty)\) consisting of the maximal solution \( u_{\lambda} \). The map \( (\Lambda, \infty) \ni \lambda \rightarrow u_{\lambda} \) is of class \( C^2 \) in \( \mathbb{R} \times C^0(\Omega) \).

We prove results (a) and (b) in Section 2 (see Theorems 2.1 and 2.5). We introduce the operator \( A \) and the space \( C_0(\Omega) \) in section 3 and prove the differentiability of the map \( A \) (in fact we prove \( A \) is a \( C^2 \) map) in the Appendix. Using the stability analysis and smoothness of the map \( A \) we prove (c) in Theorem 3.3. Existence of a positive solution for large \( \lambda \) for similar problems are well studied in literature. For example Shi-Yao\[21\] and Hernández et al. \[10\] consider the semipositone problem of the type \(-\Delta u = \lambda u^q - u^\beta \) with Dirichlet boundary condition in an arbitrary smooth domain \( \Omega \) and establish the existence of positive solution bounded below by the distance function using sub-super solution techniques. We also use similar techniques to prove the existence of solution for large \( \lambda \), but here in this work we additionally show that the maximal solution curve \( \lambda \rightarrow u_{\lambda} \) is in fact smooth. Also see \[13\] \[9\] \[14\] for related problems where they prove stability results for infinite semipositone problems. In \[5\] the authors discuss a bifurcation phenomenon for semipositone problems \((f(0) \in (-\infty, 0))\) depending on the behaviour of \( f(t) \) at infinity, i.e. depending on if \( f \) is sublinear, superlinear or asymptotically linear at infinity. Positive solutions curves of concave semipositone problems are also studied in \[8\] and \[7\].

In Section 4, existence of a non-negative weak solution at \( \lambda = \Lambda \) is proved using a limiting argument (see Proposition 4.1). We conclude our paper by proving the following result.
Thus the solution is bounded below by \( \psi \). There exists a maximal solution \( v \) in the ordered interval \( [-\epsilon, \epsilon) \) with \( \lambda(0) = \Lambda, \lambda'(0) = 0, \lambda''(0) < 0 \) and \( u(0) = u_\Lambda \).

To the best of our knowledge a complete bifurcation diagram for semipositone problems is understood in either of the following two situations: (a) in case of \( f(0) = -\infty \) and dimension \( N = 1 \) (see [17] or (b) in case of strictly semipositone problems, i.e. \( -\infty < f(0) < 0 \) in a ball (see [6]). In the latter work the results were obtained by using shooting methods for ODE as any positive solution for a semipositone problem in a ball is known to be radially symmetric. In Theorem 1.5 we make an attempt to understand the bifurcation curve in arbitrary domain \( \Omega \) under certain additional hypothesis on extremal function \( u_\Lambda \). The second alternative gives a precise description of the bifurcation branch at \( \lambda = \Lambda \). At least in dimension \( N = 1 \) and \( \beta \in (0, \frac{1}{2}) \), it is clear from [17, Theorem 2] that the first case does not arise. The second alternative also suggests the existence of multiple positive solutions for (1.1) when \( \lambda \in (\Lambda, \Lambda + \epsilon) \) for some \( \epsilon > 0 \). In fact the solution in the lower branch (the non-maximal solution) is also bounded below by \( c(\lambda)d(x, \partial \Omega) \). It is expected that the solutions exhibit a “free boundary” condition (i.e. a non negative solution becomes zero in a set of positive measure) beyond \( \Lambda + \epsilon \).

2. Stability analysis

**Theorem 2.1.** There exists a \( \Lambda \in (0, \infty) \) and for all \( \lambda > \Lambda \), there exists a positive function \( u_\lambda \) solving (1.1) as defined in (1.7). In fact, the function \( u_\lambda \) is the maximal solution for (1.1).

**Proof.** For \( \lambda \) large enough the existence of a positive solution bounded below by \( d(x, \partial \Omega) \) is obtained in Section 5 of [10] for more general nonlinear function \( f \). Here we briefly explain the sub and supersolution to be chosen for our particular nonlinearity \( f(t) = t^q - t^{-\beta} \). Following the lines of proof of [10, Example 5.6] we define \( \psi = \lambda^r (\phi_1 + \phi_1^{1+r}) \), where \( \phi_1 \) is the first eigenfunction of \( -\Delta \), and \( 1 < r < \frac{1}{1+q} \) is chosen so that \( -\Delta \psi \leq \lambda (\psi^q - \psi^{-\beta}) \). We define a super-solution \( \phi = v_\lambda \) where

\[
-\Delta v_\lambda = \lambda v_\lambda^q \text{ in } \Omega, \quad v_\lambda = 0 \text{ on } \partial \Omega. \tag{2.1}
\]

Then we know that \( v_\lambda = \lambda^{\frac{1}{1+q}} v_1 \) and hence for large \( \lambda \) we have \( \psi \leq \phi \). Now by [10, Theorem 5.5] there exists a maximal solution \( u_\lambda \) in the ordered interval \([\psi, \phi]\). Thus the solution is bounded below by \( \psi \) and hence

\[
u_\lambda(x) \geq \psi = \lambda^r (\phi_1 + \phi_1^{1+r}), \quad \text{i.e. } \|u_\lambda\|_\infty \rightarrow \infty \text{ as } \lambda \rightarrow \infty. \tag{2.2}
\]

Suppose \( u \) is a solution of (1.1). Then, \( -\Delta u \leq \lambda u^q \) and by comparison [20, Lemma 2.2] \( u \leq v_\lambda \). Thus the \( \bar{u}_\lambda \) that we constructed via sub-super solution is in fact the maximal positive solution of (1.1). Now define \( \Lambda = \inf \{\lambda > 0 : (P_\lambda) \text{ admits at least one solution} \} \). Next we claim that

\[
0 < \Lambda < \infty. \tag{2.3}
\]

Clearly from our previous discussion \( \Lambda < \infty \). We shall now prove that \( \Lambda > 0 \). Suppose on the contrary that \( \Lambda = 0 \), then there exists a sequence \( (\lambda_m, u_{\lambda_m}) \in \mathcal{S} \) and \( \lambda_m \rightarrow 0 \). By comparison Lemma we have \( 0 < u_{\lambda_m} \leq v_{\lambda_m} \). Therefore for large
Proposition 2.2. The maximal solution following proposition.

Proof. For a fixed solution of (1.1).

Thanks to the lower and upper bound of the sequence \( \{u_n\} \), bounded say in \( W^{2,p}(\Omega) \) and if we define \( u_{\lambda} = \lim_{n \to \infty} u_n \), then we claim that \( u_{\lambda} > \psi \) solving (1.1).

Clearly, \( \psi < v_{\lambda} < v_{\lambda} \). We claim that \( \psi < v_{\lambda} \). Let \( \psi < v_{\lambda} \). We claim that \( \psi < v_{\lambda} \). Let \( \psi < v_{\lambda} \).

By mathematical induction we can easily prove that \( 0 < u_{\lambda} < 1 \) and \( -\Delta u_{\lambda} = \lambda u_{\lambda} - u_{\lambda}^{-\beta} < 0 \).

This leads to a contradiction, since by maximum principle any such solution \( u_{\lambda} \) has to be necessarily negative and hence \( \lambda > 0 \).

Next we claim that for any \( \lambda > \Lambda \) there exists at least one solution for (1.1). Fix \( \lambda > \Lambda \), then by definition there exists a \( \lambda' \in (\Lambda, \lambda) \) such that (1.1) with \( \lambda = \lambda' \) admits at least one solution which we call \( \psi \). Note that we do not claim \( \psi \) is a sub-solution for (1.1), but still we prove that there exists a \( u_{\lambda} > \psi \) solving (1.1).

Clearly, \( \psi < v_{\lambda} < v_{\lambda} \). We claim that \( \psi < v_{\lambda} \). Let \( \psi < v_{\lambda} \). We claim that \( \psi < v_{\lambda} \). Let \( \psi < v_{\lambda} \).

By the standard weak comparison principle for the functions in \( W^{2,p}(\Omega) \) we obtain \( u_1 < u_0 \). We claim that \( \psi < u_1 < u_0 \). In fact,

\[
\begin{align*}
-\Delta(u_1 - \psi) &= \lambda f(u_0) - \lambda' f(\psi) \\
&\geq \lambda f(\psi) - \lambda' f(\psi) \\
&= \left( \frac{\lambda - \lambda'}{\lambda'} \right) \lambda' f(\psi) = -\Delta(\delta \psi)
\end{align*}
\]

where \( \delta = (\lambda - \lambda')/\lambda' > 0 \). Thus once again by comparison method we prove the claim. Iteratively if we define the sequence

\[
\begin{align*}
-\Delta u_{n+1} &= \lambda (u_n^q - u_n^{-\beta}) \quad \text{in } \Omega \\
u_{n+1} &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

by mathematical induction we can easily prove that

\[
\psi < \cdots < u_{n+1} \leq u_n \leq \cdots u_1 < u_0.
\]

Thanks to the lower and upper bound of the sequence \( \{u_n\} \), we have have \( u_n \in C^{1,\gamma}(\Omega) \cap C^2(\Omega) \) (see [10, Theorem 5.2] and [13]). Hence the sequence \( \{u_n\} \) is bounded say in \( H_0^1(\Omega) \) and if we define \( u_{\lambda} = \lim_{n \to \infty} u_n \), then \( u_{\lambda} \) is the maximal solution of (1.1).

Our next aim is to prove that the principal eigenvalue of the linearized operator about the maximal solution \( u_{\lambda} \) is positive. As a first step towards it we prove the following proposition.

Proposition 2.2. The maximal solution \( u_{\lambda} \) is semi-stable or the principal eigenvalue of the linearized operator

\[
\Lambda_1(\lambda) = \inf_{\varphi \in H_0^1(\Omega), \|\varphi\|_2 = 1} \left( \int_{\Omega} |\nabla \varphi|^2 - \lambda \int_{\Omega} f'(u_{\lambda}) \varphi^2 \right) \geq 0.
\]

Proof. For a fixed \( \lambda > \Lambda \) we consider the \( \epsilon \)-approximate regular problem

\[
\begin{align*}
-\Delta w &= \lambda \left( (w + \epsilon)^q - (w + \epsilon)^{-\beta} \right) \quad \text{in } \Omega, \\
w &> 0 \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

Let

\[
-\Delta v_{\lambda}^\epsilon = \lambda (v_{\lambda}^\epsilon + \epsilon)^q \quad \text{in } \Omega \quad v_{\lambda}^\epsilon > 0 \quad \text{in } \Omega; \quad v_{\lambda}^\epsilon = 0 \quad \text{on } \partial\Omega.
\]

It is easy to check that \( v_{\lambda}^\epsilon \) exists and \( v_{\lambda}^\epsilon < v_{\lambda} \). Note that \( u_{\lambda} \) and \( v_{\lambda}^\epsilon \) are respectively sub and super solutions of (2.4) and by standard monotone iteration there exists a \( w_\epsilon \in [u_{\lambda}, v_{\lambda}] \) solving (2.4). In fact \( w_\epsilon \) is the maximal solution of (2.4). By Hopf’s
maximum principle for some $\theta_1 > 0$ we have $w_\epsilon(x) + \theta_1 d(x, \partial \Omega) \leq v_\lambda^\epsilon$. Next we observe that the sequence $\{w_\epsilon\}$ is bounded independent of $\epsilon$ since

$$
\int_\Omega |\nabla w_\epsilon|^2 \leq \lambda \int_\Omega (w_\epsilon + 1)^{q+1} \leq \lambda \int_\Omega (v_\lambda + 1)^{q+1} < \infty.
$$

Clearly $w_\epsilon$ converges to some function $\tilde{w}$ which is a weak solution of (1.1) and $u_\lambda \leq \tilde{w} \leq v_\lambda^\epsilon$. Since $u_\lambda$ is the maximal solution of (1.1) we must have

$$
\lim_{\epsilon \to 0} w_\epsilon = u_\lambda.
$$

Let us write $f_\epsilon(t) = (t + \epsilon)^q - (t + \epsilon)^{-\beta}$.

Claim: $\Lambda_1'(\lambda) = \inf_{\varphi \in H_0^1(\Omega), \|\varphi\|_2 = 1} \left( \int_\Omega |\nabla \varphi|^2 - \lambda \int_\Omega f_\epsilon'(w_\epsilon)\varphi^2 \right) \geq 0$. On the contrary suppose that $\Lambda_1'(\lambda) < 0$ and $\varphi_\epsilon \in H_0^1(\Omega)$ be the associated non-negative eigenfunction of

$$
-\Delta \varphi_\epsilon - \lambda f_\epsilon'(w_\epsilon)\varphi_\epsilon = \Lambda_1'(\lambda)\varphi_\epsilon.
$$

We will show that $(w_\epsilon + \theta \varphi_\epsilon)$ is a sub solution of (2.4). For a non-negative $\varphi \in H_0^1(\Omega)$,

\[
\int_\Omega \nabla (w_\epsilon + \theta \varphi_\epsilon) \nabla \varphi - \lambda \int_\Omega f_\epsilon(w_\epsilon + \theta \varphi_\epsilon) \varphi \\
= \lambda \int_\Omega f_\epsilon'(w_\epsilon) \varphi - f_\epsilon(w_\epsilon + \theta \varphi_\epsilon) \varphi + \theta f_\epsilon'(w_\epsilon) \varphi_\epsilon \varphi + \theta \Lambda_1'(\lambda) \int_\Omega \varphi \varphi_\epsilon \\
= o(\theta) + \theta \Lambda_1'(\lambda) \int_\Omega \varphi \varphi_\epsilon.
\]

Choosing $\theta > 0$ small enough we have $(w_\epsilon + \theta \varphi_\epsilon)$ is a sub-solution of (2.4). If required we may choose $\theta$ smaller so that $w_\epsilon(x) + \theta \varphi_\epsilon \leq v_\lambda^\epsilon$. Thus $w_\epsilon + \theta \varphi_\epsilon$ and $v_\lambda^\epsilon$ forms an ordered pair of sub and super solution of (2.4) and we obtain a solution $\tilde{w}_\epsilon \in [w_\epsilon + \theta \varphi_\epsilon, v_\lambda^\epsilon]$ of (2.4). This contradicts the fact that $w_\epsilon$ is the maximal solution of (2.4) and hence the claim is verified. Thus for every $\varphi \in H_0^1(\Omega)$ such that $\|\varphi\|_2 = 1$,

$$
\int_\Omega |\nabla \varphi|^2 - \lambda \int_\Omega f_\epsilon'(w_\epsilon)\varphi^2 \geq 0.
$$

Now passing through the limit using (2.5) and Hardy’s inequality we obtain that $\Lambda_1'(\lambda) \geq 0$.

\begin{proposition}
The semi-stable solution of (1.1) is unique.
\end{proposition}

\begin{proof}
Let $u_\lambda$ be the maximal solution of (1.1) and $v_\lambda$ be any other solution of (1.1). We know that $u_\lambda$ is semi-stable by Proposition 2.2 and assume that $v_\lambda$ is also semi-stable. Then

$$
\int_\Omega |\nabla w|^2 \geq \lambda \int_\Omega f'(v_\lambda)w^2
$$

for all $w \in H_0^1(\Omega)$. In particular,

$$
\int_\Omega |\nabla (u_\lambda - v_\lambda)|^2 \geq \lambda \int_\Omega f'(v_\lambda)(u_\lambda - v_\lambda)^2.
$$

(2.6)

Since $v_\lambda$ and $u_\lambda$ are both the solutions of (1.1)

$$
\int_\Omega |\nabla (u_\lambda - v_\lambda)|^2 = \lambda \int_\Omega (f(u_\lambda) - f(v_\lambda))(u_\lambda - v_\lambda).
$$

(2.7)
Combining the above two equations we have
\[ \int_{\Omega} \{ f(u_\lambda) - f(v_\lambda) - f'(v_\lambda)(u_\lambda - v_\lambda)\} (u_\lambda - v_\lambda) \geq 0. \]
Since \( u_\lambda \) is the maximal solution this implies
\[ \int_{\{u_\lambda > v_\lambda\}} \{ f(u_\lambda) - f(v_\lambda) - f'(v_\lambda)(u_\lambda - v_\lambda)\} (u_\lambda - v_\lambda) \geq 0. \]
Since \( f \) is strictly concave the above integral is strictly negative if the Lebesgue measure of the set \( \{ x : u_\lambda(x) > v_\lambda(x) \} \) is non-zero. Thus \( u_\lambda \equiv v_\lambda \), or the semi-stable solution is unique. \( \square \)

Next we shall prove our main result of this section, the maximal \( u_\lambda \) is stable. We consider here a different approximate problem (2.8) for a parameter \( \theta < 0 \).

\[ -\Delta z = \lambda(z^{q} - z^{-\beta} + \theta) \quad \text{ in } \Omega, \]
\[ z > 0 \quad \text{ in } \Omega, \]
\[ z = 0 \quad \text{ on } \partial \Omega. \]

**Lemma 2.4.** For each \( \theta \in (\theta_0, 0) \) there exists a function \( z_\theta \) which is a maximal solution of (2.8). If \( \theta < \theta' \) then \( z_\theta \leq z_{\theta'} \) and \( z_\theta \neq z_{\theta'} \).

**Proof.** Fix a \( \lambda \in (\Lambda, \infty) \) and choose \( \lambda' \in (\Lambda, \lambda) \). Let

\[ -\Delta V_\lambda = \lambda \quad \text{in } \Omega, \quad V_\lambda = 0 \quad \text{in } \partial \Omega \]

and \( u_\lambda \) be the maximal solution of (1.1). Define \( z_\varepsilon = \frac{\lambda}{\lambda'} u_{\lambda'} - \varepsilon V_\lambda \). Then for some positive constants \( C_1, C_2 \)

\[ z_\varepsilon - u_{\lambda'} = \left( \frac{\lambda - \lambda'}{\lambda'} \right) u_{\lambda'} - \varepsilon V_\lambda \geq (C_1 - C_2) d(x, \partial \Omega). \]

If we choose \( 0 < \varepsilon < |\theta_0| \) for some small \( \theta_0 < 0 \), we have \( z_\varepsilon > u_{\lambda'} \). For all \( \theta \in (\theta_0, 0) \) define

\[ z_\theta = \frac{\lambda}{\lambda'} u_{\lambda'} + \theta V_\lambda. \]

Then
\[ -\Delta z_\theta = \lambda(u_\lambda^{q} - u_{\lambda'}^{-\beta} + \theta) \leq \lambda(z_\theta^{q} - z_{\theta'}^{-\beta} + \theta) \]

and hence a sub solution of (2.8). It is easy to check that \( z_\theta = v_\lambda \) is a super solution of (2.8) for all \( \theta < 0 \). Since \( u_{\lambda'} < v_\lambda \)

\[ z_\theta - z_\theta = \frac{\lambda}{\lambda'} u_{\lambda'} + \theta V_\lambda - v_\lambda < \frac{\lambda}{\lambda'} u_{\lambda'} - v_\lambda = \left( \frac{\lambda}{\lambda'} (\lambda')^{\frac{1}{1-\beta}} - \lambda^{\frac{1}{1-\beta}} \right) v_1 < 0. \]

Thus there exists a solution \( z_\theta \) of (2.8) in between the ordered pair \( [z_\varepsilon, z_\theta] \). As before using comparison lemma one can easily observe that \( z_\theta \) is the maximal solution of (2.8). Now let \( \theta < \theta' \) and \( z_\theta, z_{\theta'} \) be the maximal solutions of (2.8) and (2.8) with \( \theta = \theta' \) respectively. Then

\[ -\Delta z_\theta \leq \lambda(z_\theta^{q} - z_{\theta'}^{-\beta} + \theta') \quad \text{and} \quad z_\theta \leq z_{\theta'}. \]

Since \( z_{\theta'} \) is the maximal solution of (2.8) with \( \theta = \theta' \) we conclude that \( z_\theta \leq z_{\theta'} \). \( \square \)

**Theorem 2.5.** The maximal solution \( u_\lambda \) of (1.1) is stable.
Proof. Let $\Lambda^\theta_1(\lambda)$ denote the principal eigenvalue of (2.8). Repeating the calculations of Proposition 2.2 we can show that $\Lambda^\theta_1(\lambda) \geq 0$. If $\theta_1 < \theta_2$ using the strict concavity of $f$ and Lemma 2.4 we have for all $\varphi \in H^1_0(\Omega)$, $\|\varphi\|_2 = 1$,
\[
\int_\Omega |\nabla \varphi|^2 - \lambda \int_\Omega f'(z_{\theta_1})\varphi^2 < \int_\Omega |\nabla \varphi|^2 - \lambda \int_\Omega f'(z_{\theta_2})\varphi^2.
\]
Since $\inf f \varphi \in H^1_0(\Omega) \int_\Omega |\nabla \varphi|^2 - \lambda \int_\Omega f'(z_{\theta})\varphi^2$ is attained, we have $\Lambda^\theta_1(\lambda) < \Lambda^\theta_2(\lambda)$. Observe that $z_{\theta} \rightarrow u_{\lambda}$ as $\theta \rightarrow 0^-$ and $\lim_{\theta \rightarrow 0^{-}} \Lambda^\theta_1(\lambda) = \Lambda_1(\lambda)$. Thus
\[
\Lambda_1(\lambda) > \Lambda^\theta_1(\lambda) \geq 0
\]
which is the main result. \hfill \Box

3. Bifurcation analysis

In the previous section we have shown that for each $\lambda > \Lambda$ there exists a maximal solution for (1.1). In this section we try to understand this maximal branch of solution using bifurcation theory. For $\lambda > \Lambda$, consider the function $u_{\lambda'}$ which is a solution of (1.1) with $\lambda = \lambda'$ for some $\lambda' \in [\Lambda, \lambda)$ and $v_{\lambda}$ as in (2.1). To ease notation we omit the subscript $\lambda$ and denote $\psi = \psi_{\lambda} = u_{\lambda'}$ and $\phi = \phi_{\lambda} = v_{\lambda}$, then clearly $\psi < \phi$. Let
\[
C_\lambda = \{ u \in C_0(\Omega) : \psi \leq u \leq \phi \}.
\]
For each $u \in C_\lambda$ there exists $w \in C^0_1(\Omega) \cap C^2(\Omega)$ which is a solution of
\[
-\Delta w = \lambda f(u) \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega.
\]
The existence of $w \in W^{2,p}(\Omega)$ easily follows from the lower estimate on $u$ and the regularity of $w$ by [13] (see Section 5 of [10] for the details). Since we would repeatedly use the regularity result of Gui-Lin [13], for the sake of completeness we quote the result below.

**Theorem 3.1** (Gui-Lin [13] Prop. 3.4). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies
\[
|\Delta u(x)| \leq M d(x)^{-\beta} \quad \text{and} \quad |u(x)| \leq M d(x)\alpha
\]
for some positive constants $M, \alpha$. Then there exists some $\gamma \in (0, 1)$ depending upon $\beta$ and $\alpha$ such that $\|u\|_{C^{1,\gamma}(\overline{\Omega})} \leq C(M, \alpha, \beta)$.

We can in fact prove that the solution $w$ of (3.2) belongs to $C_\lambda$. One can observe that $w \leq \phi$ since $\phi$ is a supersolution of (3.2). It is not clear if $\psi$ is a subsolution of (3.2) or not. But still by the specific choice of $\psi$ we can show that
\[
-\Delta (w - \psi) = \lambda f(u) - \lambda' g(\psi) \geq \frac{\lambda - \lambda'}{\lambda'} (-\Delta \psi)
\]
Since $\lambda' < \lambda$ it follows that $w > \psi$ and hence $w \in C_\lambda$. For a fixed $\lambda \in (\Lambda, \infty)$ we define the map
\[
\mathcal{A}: C_\lambda \rightarrow C_\lambda \text{ is defined as } \mathcal{A}(u) = w \text{ if } w \text{ is a solution of (3.2)}.
\]

We aim to employ the well known abstract setting of bifurcation theory to prove the existence of a connected branch of solutions. If we consider the map $\mathcal{A}: C_\lambda \rightarrow C_\lambda$ it is not possible to use the implicit function theorem since the set $C_\lambda \subset C_0(\overline{\Omega})$ has empty interior. Hence we introduce the space $C_e(\overline{\Omega})$ as in [1] and consider the set $C_\lambda$ with the topology induced from $C_e(\overline{\Omega})$ in which $C_\lambda$ has nonempty interior.
Let \( e \in C^2(\overline{\Omega}) \) denote the unique positive solution of
\[
-\Delta e = 1 \quad \text{in } \Omega \\
\quad \quad e = 0 \quad \text{on } \partial \Omega.
\]

Then \( e(x) > 0 \) in \( \Omega \), \( \frac{\partial e}{\partial n} < 0 \) on \( \partial \Omega \) and thus \( e(x) \geq kd(x, \partial \Omega) \) for some constant \( k > 0 \). \( C_c(\overline{\Omega}) \) is the set of functions in \( u \in C(\overline{\Omega}) \) such that \( -te \leq u \leq te \) for some \( t \geq 0 \). \( C_c(\overline{\Omega}) \) equipped with \( \|u\|_e = \inf \{ t > 0 : -te \leq u \leq te \} \) is a Banach space.

Also the following continuous embedding holds:
\[
C_0^1(\overline{\Omega}) \hookrightarrow C_c(\overline{\Omega}) \hookrightarrow C_0(\overline{\Omega}).
\]

Further \( C_c(\overline{\Omega}) \) is an ordered Banach space(OBS) whose positive cone \( P_e = \{ u \in C_c(\overline{\Omega}) : u(x) \geq 0 \} \) is normal and has non empty interior. In particular the interior of \( P_e \) consists of all those functions \( u \in C(\overline{\Omega}) \) with \( t_1e \leq u \leq t_2e \) for some \( t_1, t_2 > 0 \). Define
\[
\mathcal{M}_\lambda = \{ u \in C_c(\overline{\Omega}) : \psi \leq u \leq \phi \}
\]
(3.5)

Using the lower and upper bounds for \( \psi \) and \( \phi \) in terms of \( d(x, \Omega) \) we find that set theoretically \( C_\lambda \) is same as \( \mathcal{M}_\lambda \). But topologically they are different and in fact \( \mathcal{M}_\lambda \) has non empty interior which we denote by \( U_\lambda \) where
\[
U_\lambda = \{ u \in \mathcal{M}_\lambda : \psi + t_1e \leq u \leq \phi - t_2e \text{ for some } t_1, t_2 > 0 \}. \quad (3.6)
\]

By definition the set \( U_\lambda \) is open and we denote the restriction of the map \( A \) to \( U_\lambda \) as \( A \) itself. From \( (3.3) \) \( A \) maps \( U_\lambda \) to \( C_\lambda \). In the next proposition we prove that \( A \) maps \( U_\lambda \) to itself and it is a \( C^2 \) map.

**Proposition 3.2.** The map \( A:U_\lambda \to U_\lambda \) is twice continuously differentiable. The map \( A'(u):C_c(\overline{\Omega}) \to C_c(\overline{\Omega}) \) is continuous linear and compact.

**Proof.** Let \( u \in U_\lambda \), i.e there exists some \( t_1, t_2 > 0 \) such that \( \psi + t_1e \leq u \leq \phi - t_2e \) and let \( A(u) = w \). Then \( -\Delta (w - \phi) < 0 \) in \( \Omega \) and \( w - \phi = 0 \) on \( \partial \Omega \), and by Hopf Maximum principle there exists \( t_2 > 0 \) for which \( w \leq \phi - t_2e \). From our previous discussion \( (3.3) \) if we take \( t_1 = \frac{\lambda - \lambda'}{\lambda} \) we find \( w \geq \psi + t_1e \). Thus \( A \) maps \( U_\lambda \) into itself. Proof of the smoothness of the map \( A \) and the compactness of \( A'(u) \) is much technical and we shall give the details in the Appendix. \( \square \)

Next we shall treat \( \lambda \) as a variable and define the map \( A: (\Lambda, \infty) \times U_\lambda \to U_\lambda \) as \( A(\lambda, u) = w \) if \( w \) is a solution of
\[
-\Delta w = \lambda f(u) \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial \Omega. \quad (3.7)
\]

Fix \( \lambda_1, \lambda_2 \) such that \( \Lambda < \lambda_1 < \lambda_2 < \infty \). Then for all \( \lambda \in [\lambda_1, \lambda_2] \) we can in fact fix the indexed set \( U_\lambda \) independent of \( \lambda \) in the following way. By the definition of \( \Lambda \) there exists a \( \lambda' \in [\Lambda, \lambda_1) \) and \( (1.1) \) with \( \lambda = \lambda' \) is solvable. Let \( \psi = u_{\lambda'} \) and \( \phi = v_{\lambda_2} \) and let \( M_\lambda \) and \( U_\lambda \) defined as before in \( (3.5) \) and \( (3.6) \) for this choice of \( \psi \) and \( \phi \). Now \( U_\lambda \) is independent of \( \lambda \) for all \( \lambda \in [\lambda_1, \lambda_2] \). For this particular choice of \( U = U_\lambda \) we can prove that the map \( A \) is \( C^2 \) in \( \lambda \) and \( u \) variable in \( (\lambda_1, \lambda_2) \times U \).

**Theorem 3.3.** There exists a connected branch of positive maximal solutions of \( (1.1) \) bifurcating from \( \lambda_\infty = \infty \).

**Proof.** Fix an open interval \( I \subset (\Lambda, \infty) \) and \( \mathcal{I} \) compactly contained in \( (\Lambda, \infty) \). Let \( I = (\lambda_1, \lambda_2) \) and \( \psi = u_{\lambda_1} \) and \( \phi = v_{\lambda_2} \) as before. Thus for all \( \lambda \in I \) we define
\( \mathcal{M} = \{ u \in C_c(\Omega) : \psi \leq u \leq \phi \} \) and \( \mathcal{U} \) to be the interior of \( \mathcal{M} \). Consider the map \( F : I \times \mathcal{U} \to \mathcal{U} \) defined as

\[
F(\lambda, u) = u - A(\lambda, u). \tag{3.8}
\]

Clearly the zeroes of \( F \) are the solutions of \((1.1)\) and \( F(\lambda, u_{\lambda}) = 0 \) where \( u_{\lambda} \) is the maximal solution of \((1.1)\). Note that \( F : I \times \mathcal{U} \to \mathcal{U} \) is a \( C^2 \) map and \( \partial_u F(\lambda, u) = I - \partial_u A(\lambda, u) \) is a compact perturbation of identity. Fix \( \lambda_0 \in I \) and let \( u_0 = u_{\lambda_0} \) be the maximal solution of \((1.1)\) with \( \lambda = \lambda_0 \), then \( F(\lambda_0, u_0) = 0 \). From Theorem 2.5 we know that \( u_0 \) is a stable solution and hence \( \partial_u F(\lambda_0, u_0) \) is one-one. Now by Fredholm alternative it is onto as well. Thus the linear map \( \partial_u F(\lambda_0, u_0) \) is bijective and continuous, hence by open mapping theorem \( \partial_u F(\lambda_0, u_0) \) has a continuous inverse. Now we can apply implicit function theorem around \((\lambda_0, u_0)\) and deduce that there exists a \( C^2 \) curve \((\lambda, u(\lambda)) \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \times \mathcal{U} \) such that the set of all solutions of \( F(\lambda, u) = 0 \) in a neighbourhood of \((\lambda_0, u_0)\) is given by \((\lambda, u(\lambda))\). Note that this \( u(\lambda) \) may be different from the maximal solution \( u_{\lambda} \).

If we can show that \( \lambda \rightarrow u_{\lambda} \) (where \( u_{\lambda} \) is the maximal solution) is continuous then by the uniqueness of the solution near \((\lambda_0, u_0)\) we have a \( u(\lambda) = u_{\lambda} \). On the contrary suppose \( \lambda \rightarrow u_{\lambda} \) is not continuous at \( \lambda_0 \). i.e. there exists a sequence \( \lambda_n \to \lambda_0 \) such that \( u_{\lambda_n} \neq u_0 \). One can use Hardy’s inequality to prove that \( \{ u_{\lambda_n} \} \) is bounded in \( H_0^1(\Omega) \) and hence up to a subsequence \( u_{\lambda_n} \to \tilde{u} \) in \( H_0^1(\Omega) \). It is also easy to check that \( \tilde{u} \) is a solution of \((P_{\lambda_0})\). Since \( u_0 \) is the maximal solution of \((P_{\lambda_0})\) we have

\[
\tilde{u} \leq u_0 \quad \text{and} \quad \tilde{u} \neq u_0. \tag{3.9}
\]

On the other hand we have \( u(\lambda_n) \to u_0 \) and \( u(\lambda_n) \leq u_{\lambda_n} \). Taking limit as \( n \to \infty \) we find \( u_0 \leq \tilde{u} \) which contradicts (3.9). We have now \( u(\lambda) = u_{\lambda} \) and hence by implicit function theorem \( \lambda \to u_{\lambda} \) is a \( C^2 \) map which completes the proof of theorem. \( \square \)

**Remark 3.4.** The smoothness of the map \( \lambda \to u_{\lambda} \) for \( \lambda \in (\Lambda, \infty) \) is completely determined by the smoothness of the operator \( A \). We can in fact prove that the map is infinitely many times differentiable, hence \( \lambda \to u_{\lambda} \) is a \( C^\infty \) map.

The proof of our main result now follows from Theorem 2.1, equations (2.2), (2.3), Theorems 2.5, 3.3 and Remark 3.4.

### 4. Bifurcation Analysis at \( \lambda = \Lambda \)

**Proposition 4.1.** There exists a non-negative solution \( u_{\Lambda} \) solving \((1.1)\) with \( \lambda = \Lambda \) in the weak sense. The Lebesgue measure of the set \( \{ x : u_{\Lambda}(x) = 0 \} \) is zero.

**Proof.** Let \( \{ u_n \} \) denote the sequence of maximal solutions of \((P_{\lambda_n})\) where \( \lambda_n \downarrow \Lambda \) and \( \lambda_n < \bar{\Lambda} \). If \( \overline{\pi} \) denote the solution of \((2.1)\) for \( \lambda = \overline{\Lambda} \), we have \( 0 < u_{n+1} \leq u_n \leq \overline{\pi} \) and

\[
\int_\Omega |\nabla u_n|^2 = \lambda_n \int_\Omega (u_n^{\gamma+1} - u_n^{\gamma-1}) \leq \lambda_n \int_\Omega u_{n+1}^{\gamma+1} \leq \overline{\Lambda} \int_\Omega \overline{\pi}^{\gamma+1}. \tag{4.1}
\]

Thus the sequence \( \{ u_n \} \) is bounded in \( H_0^1(\Omega) \) and denote the weak limit of \( u_n \) as

\[
u_{\Lambda} := \lim_{n \to \infty} u_n. \tag{4.2}
\]

We will show that \( u_{\Lambda} \) is in fact a solution of \((1.1)\) with \( \lambda = \Lambda \) in the weak sense. As a first step we shall prove that \( \{ x \in \Omega : u_{\Lambda}(x) = 0 \} \) has Lebesgue measure zero. Let \( \phi_1 \) be the first eigenfunction of \(-\Delta \) and \( \gamma \in (0, 1), \epsilon > 0 \). Consider the function
\[ \psi = (\phi_1 + \epsilon)^\gamma - \epsilon^\gamma \in H^1_0(\Omega). \]
Then from a direct computation we find \(-\Delta \psi \geq 0\) and hence \(-\Delta u, \psi \geq H^1_0(\Omega) \times H^{-1}(\Omega) \geq 0\) which implies
\[ \lambda_n \int_\Omega (u_n^\beta - u_n^{-\beta}) \psi \geq 0. \] (4.3)
Thus
\[ \int_\Omega u_n^{-\beta} ((\phi_1 + \epsilon)^\gamma - \epsilon^\gamma) \leq \int_\Omega u_n^\beta ((\phi_1 + \epsilon)^\gamma - \epsilon^\gamma). \]
Now letting \(\epsilon \to 0\) and \(\gamma \to 0\) we have \(\int_\Omega u_n^{-\beta} \leq \int_\Omega u_n^\beta \leq \int_\Omega \psi^\beta < \infty\). Once again using Fatou’s lemma,
\[ \int_\Omega u_n^{-\beta} < \infty \] (4.4)
which in turn implies \(\{ x \in \Omega : u_\lambda(x) = 0 \}\) is of Lebesgue measure zero. Now we will prove that \(u_\lambda\) is a weak solution of (1.1) with \(\lambda = \Lambda\). We have
\[ \int_\Omega \nabla u_n \nabla \varphi = \lambda_n \int_\Omega (u_n^\beta - u_n^{-\beta}) \varphi \quad \text{for all} \quad \varphi \in C_c^\infty(\Omega). \]
The only difficulty arises while passing through the limit in the term involving \(u_n^{-\beta}\).
But note that \(u_n^{-\beta} |\varphi| \leq u_n^{-\beta} ||\varphi||_{\infty} \in L^1(\Omega)\) and by dominated convergence theorem \(u_\lambda\) is a weak solution of (1.1) with \(\lambda = \Lambda\).

Next we shall discuss a sufficient condition that ensures the existence of multiple solutions for (1.1). We make a crucial assumption that the non-negative solution \(u_\Lambda\) belongs to \(C_c(\Omega)\) and is bounded below by \(cd(x, \partial \Omega)\) for some \(c > 0\). By the above assumption \(u_\lambda\) is positive and it can be shown that the (1.1) with \(\lambda = \Lambda\) admits a unique positive solution. Indeed, if \(\tilde{u}_\Lambda\) is another positive solution of (1.1) with \(\lambda = \Lambda\) then we can show that a convex combination of \(u_\lambda\) and \(\tilde{u}_\Lambda\) is a positive solution of (1.1) with \(\lambda = \lambda'\) for some \(\lambda' < \Lambda\) which is impossible (see [19], Proposition 5) for details.

Now the uniqueness in the class of positive solutions imply that \(u_\lambda\) is maximal and by Proposition 2.2 \(\Lambda_1(\Lambda) = 0\). Indeed, since \(u_\Lambda\) is maximal it is clear that \(\Lambda_1(\Lambda) \geq 0\). Suppose \(\Lambda_1(\Lambda) > 0\), then implicit function theorem would guarantee the existence of a positive solution for some \(\lambda < \Lambda\) which would contradict the definition of \(\Lambda\). Next we shall prove a local bifurcation result of Crandall-Rabinowitz [3] for an infinite semipositive problem. Similar ideas of the proof were used in [4,11] when the authors studied a positone convex non-linearity.

**Lemma 4.2.** The solutions of \(F(\lambda, u) = 0\) near \((\Lambda, u_\lambda)\) are described by a curve \((\lambda(s), u(s)) = (\Lambda + \tau(s), u_\lambda + s\phi_\Lambda + x(s))\) where \(s \to (\tau(s), x(s)) \in \mathbb{R} \times C_c(\Omega)\) is a continuously differentiable function near \(s = 0\) with \(\tau(0) = \tau'(0) = 0\), \(\tau''(0) > 0\), and \(x(0) = x'(0) = 0\). Moreover \(\tau\) is of class \(C^2\) near 0.

**Proof.** Consider the map \(F(\lambda, u)\) and the Gateaux derivative of \(F\) at \((\Lambda, u_\lambda)\). Clearly \(\partial_\lambda F(\Lambda, u_\lambda) = -\partial_\lambda A(\Lambda, u_\lambda) = -\frac{u_\lambda}{\Lambda}\). Now consider the null space of the linear operator \(\partial_u F(\Lambda, u_\lambda)\). Since \(\Lambda_1(\Lambda) = 0\), there exists a \(\phi_\Lambda \in H^1_0(\Omega)\) such that
\[ -\Delta \phi_\Lambda = \Lambda f'(u_\lambda) \phi_\Lambda \quad \text{in} \quad \Omega, \]
\[ \phi_\Lambda = 0 \quad \text{on} \partial \Omega. \]
By the interior regularity results the eigenfunction \(\phi_\Lambda \in C^2(\Omega) \cap H^1_0(\Omega)\) itself. Now by [12], Theorem 8.16, the principal eigenvalue \(\Lambda_1(\Lambda)\) is simple and the corresponding eigenfunction \(\phi_\Lambda\) is positive. Hence \(\ker(\partial_u F(\Lambda, u_\lambda))\) is one dimensional and is
spanned by $\phi_\Lambda$. We claim that $\partial_\lambda F(\Lambda, u_\lambda) \notin \ker \partial_u F(\Lambda, u_\lambda)$. If so, then for some constant $k$ we have $u_\lambda = k\phi_\Lambda$. This implies $f(u_\lambda) = kf'(u_\lambda)\phi_\Lambda$ which is impossible since RHS is has a constant sign and LHS changes its sign inside $\Omega$ and hence that $X \rightarrow \ker \partial_u F(\Lambda, u_\lambda)$.

Let $X$ be any complement of the span of $\{\phi_\Lambda\}$ in $C_c(\mathbb{O})$ and the map $\theta: \mathbb{R} \times \mathbb{R} \times X \rightarrow C_c(\mathbb{O})$ be defined as

$$\theta(s, \tau, x) = F(\Lambda + \tau, u_\lambda + s\phi_\Lambda + x)$$

Then, we claim that $\partial_{\tau, x}\theta(0, 0, 0) = (\partial_\lambda F(\Lambda, u_\lambda), \partial_u F(\Lambda, u_\lambda))$ is an isomorphism from $\mathbb{R} \times X$ on to $X$. Since $\partial_\lambda F(\Lambda, u_\lambda) \notin \text{Range} \partial_u F(\Lambda, u_\lambda)$ the map $\partial_{\tau, x}\theta(0, 0, 0)$ is one-one in $\mathbb{R} \times X$. Now by Fredholm alternative $\partial_{\tau, x}\theta(0, 0, 0)$ is also onto. Now by implicit function theorem there exists an $\epsilon > 0$ and a $C^2$ function $p:(-\epsilon, \epsilon) \rightarrow \mathbb{R} \times X$ such that $p(s) = (\tau(s), x(s))$ and $\theta(s, p(s)) = 0$, $\tau(0) = 0$ and $x(0) = 0$, i.e.,

$$F(\Lambda + \tau(s), u_\lambda + s\phi_\Lambda + x(s)) = 0.$$  

Now differentiating with respect to $s$ variable and evaluating at $s = 0$, we obtain

$$\partial_\lambda F(\Lambda, u_\lambda)\tau'(0) + \partial_u F(\Lambda, u_\lambda)x'(0) = 0.$$  

Since $\partial_\lambda F(\Lambda, u_\lambda) \notin \text{Range}(\partial_u F(\Lambda, u_\lambda))$ we have $\tau'(0) = x'(0) = 0$. Once again differentiating $F(\Lambda + \tau(s), u_\lambda + s\phi_\Lambda + x(s))$ we obtain

$$\partial_\lambda F(\Lambda, u_\lambda)\tau''(0) + \partial_u F(\Lambda, u_\lambda)\phi_\Lambda^2 + \partial_u F(\Lambda, u_\lambda)x''(0) = 0.$$  

Let us write the middle term in the above expression as $W = \partial_{uu} F(\Lambda, u_\lambda)\phi_\Lambda^2$. Then one can easily check that

$$\Delta W = \Lambda f''(u_\lambda)\phi_\Lambda^2 \quad \text{in } \Omega,$$

$$W = 0 \quad \text{on } \partial\Omega.$$  

Since $f$ is concave, by maximum principle $W \geq 0$. Now call $w = \partial_u F(\Lambda, u_\lambda)x''(0)$ which by definition is equal to $x''(0) - \partial_u A(\Lambda, u_\lambda)x''(0)$. If $w_1 = \partial_u A(\Lambda, u_\lambda)x''(0)$ then $w_1$ solves

$$\begin{align*}
-\Delta w_1 &= \Lambda f''(u_\lambda)x''(0) \quad \text{in } \Omega, \\
 w_1 &= 0 \quad \text{on } \partial\Omega.
\end{align*}$$

Thus $\int_{\Omega} \nabla w_1 \nabla \phi_\Lambda = \int_{\Omega} \Lambda f''(u_\lambda)x''(0)\phi_\Lambda$. From the definition of $\phi_\Lambda$, we also have $\int_{\Omega} \nabla w_1 \nabla \phi_\Lambda = \int_{\Omega} \Lambda f''(u_\lambda)w_1\phi_\Lambda$. Thus

$$\int_{\Omega} \Lambda f''(u_\lambda)\phi_\Lambda w = 0.$$  

Now multiplying (4.5) by $\Lambda f''(u_\lambda)\phi_\Lambda$ and integrating over $\Omega$,

$$-\tau''(0) \int_{\Omega} u_\Lambda f''(u_\lambda)\phi_\Lambda + \int_{\Omega} W\Lambda f''(u_\lambda)\phi_\Lambda = 0.$$  

We know $f$ is monotonically increasing and $\phi_\Lambda$ is a non-negative function and $W \geq 0$. Thus $\tau'' \geq 0$ which completes the proof. \hfill \Box

**Proof of Theorem 1.3.** Suppose that alternative (a) does not hold. Then from the properties of $C_c(\mathbb{O})$ (see Section 3) there exists a constant $c_\lambda > 0$ such that $u_\lambda(x) \geq c_\lambda d(x, \partial\Omega)$. Thus $A_1(\Lambda)$ is well defined and is non-negative. Now by the definition of $\Lambda$ the principal eigenvalue $A_1(\Lambda)$ cannot be positive and hence the proof of Lemma 4.2 is applicable and which completes the Theorem. \hfill $\Box$
5. Appendix

Proposition 5.1. The map $\mathcal{A}U_\lambda \rightarrow U_\lambda$ is a $C^2$ map.

Proof. Let $u \in U_\lambda$, i.e. there exists some $t_1, t_2 > 0$ such that $\psi + t_1 e \leq u \leq \phi - t_2 e$ and let $\mathcal{A}(u) = w$. Then $-\Delta (w - \phi) < 0$ in $\Omega$ and $w - \phi = 0$ on $\partial \Omega$, and by Hopf Maximum principle there exists a $t_2 > 0$ for which $w \leq \phi - t_2 e$. From our previous discussion if we take $t_1 = \frac{\lambda - \phi}{\chi}$ we find $w \geq \psi + t_1 e$. Thus $\mathcal{A}$ maps $U_\lambda$ into itself.

Step I. $\mathcal{A}U_\lambda \rightarrow U_\lambda$ is continuous. Let $h \in C_\ell(\Omega)$ with $\|h\|_{C^{\gamma}(\Omega)}$ small so that $u + h \in U_\lambda$ and $\mathcal{A}(u + h) = w_h$. Then $(w_h - w)$ satisfies $-\Delta (w_h - w) = \lambda (f(u + h) - f(u))$ in $\Omega$ and $w_h - w = 0$ on $\partial \Omega$. For $p \in (1, \frac{1}{\beta})$ using $L^p$ estimate and dominated convergence theorem we find

$$\|w_h - w\|_{W^{2,p}(\Omega)} \leq C\|f(u + h) - f(u)\|_{L^p(\Omega)} \rightarrow 0 \quad \text{as } \|h\|_{C_\ell(\Omega)} \rightarrow 0. \quad (5.1)$$

Now since $w_h$ and $w$ belongs to $U_\lambda$ we have $|w_h - w| \leq C d(x, \partial \Omega)$. Now we can apply Theorem 3.1 and obtain $|w_h - w|_{C^{\gamma}(\Omega)}$ is bounded. Thanks to Ascoli-Arzelà theorem and (5.1) we have $w_h \rightarrow w$ in $C_\ell^0(\Omega)$. Finally using the continuity of the embedding $C_\ell^0(\Omega) \rightarrow C_\ell(\Omega)$ we conclude that $\mathcal{A}U_\lambda \rightarrow U_\lambda$ is continuous.

Step II. The map $\mathcal{A}U_\lambda \rightarrow U_\lambda$ is $C^1$. For a given $u \in U_\lambda$ and $h \in C_\ell(\Omega)$ consider the solution operator $z$ defined as

$$-\Delta z = \lambda f(u)h \quad \text{in } \Omega \quad \text{and } \quad u = 0 \quad \text{on } \partial \Omega. \quad (5.2)$$

Let us denote $\xi_\lambda \in C_\ell^0(\Omega) \cap C^2(\Omega)$ be the unique solution of

$$-\Delta \xi_\lambda = \lambda \xi_\lambda - \beta \quad \text{in } \Omega \quad \text{and } \quad \xi_\lambda = 0 \quad \text{on } \partial \Omega.$$ 

The existence and behaviour of the solution $\xi_\lambda$ near $\partial \Omega$ is studied in [5]. It is well known that $\xi_\lambda \sim d(x, \partial \Omega)$ and $d(x, \partial \Omega) \sim e(x)$ and thus $\xi_\lambda \sim e(x)$. We can estimate $f'(u)h$ in terms of $\xi_\lambda$ as

$$|f'(u)h| \leq C_{0,0} e(x)^{-(\beta+1)}|h(x)| \leq \frac{C_1 \|h\|_{C_\ell(\Omega)}}{\xi_\lambda^\beta}$$

for some positive constant $C_1$. Thus,

$$C_1 \|h\|\Delta \xi_\lambda \leq -\Delta z = \lambda f'(u)h \leq \lambda C_1 \|h\|\xi_\lambda - \beta = -C_1 \|h\|\Delta \xi_\lambda,$$

By the comparison principle and since $\xi_\lambda(x) \sim e(x)$ we have for some $C > 0$,

$$|z(x)| \leq C\|h\|_{C_\ell(\Omega)} e(x) \quad (5.3)$$

Now as in Step I, let $w_h = A(u + h)$ and $w = A(u)$, then using Taylor’s theorem

$$-\Delta (w_h - w - z) = \lambda f''(u + \theta h) \frac{h^2}{2} \quad \text{for some } \theta \in (0, 1).$$

Since $|f''(u + \theta h)h^2| \leq C \|h\|^2_{C^{\gamma}(\Omega)} e(x)^{-\beta}$ we have

$$\frac{|w_h - w - z|}{\|h\|_{C_\ell(\Omega)}} \|w_h - w - z\|_{W^{2,p}(\Omega)} \leq C\|h\|_{C_\ell(\Omega)}.$$

Up to a sub sequence $(w_h - w - z)/\|h\|_{C_\ell(\Omega)}$ converges to 0 as $\|h\|_{C_\ell(\Omega)} \rightarrow 0$. It can be shown that $|w_h - w - z|/\|h\|_{C_\ell(\Omega)} \leq C d(x, \partial \Omega)$ and thus $(w_h - w - z)/\|h\|_{C_\ell(\Omega)}$
satisfies the assumptions of theorem 3.1. Hence,
\[ \frac{w_h - w - z}{\|h\|_{C_e(\Omega)}} \text{ is bounded in } C^{1,\gamma}(\Omega) \] (5.4)

Now by using Ascoli-Arzela theorem and continuity of the embedding \( C^1_0(\Omega) \hookrightarrow C_\gamma(\Omega) \) we deduce that \( \frac{w_h - w - z}{\|h\|_{C_e(\Omega)}} \to 0 \) in \( C_\gamma(\Omega) \). If we call \( A'(u)h = z \) then
\[ \|A(u + h) - A(u) - A'(u)h\|_{C_e(\Omega)} = o(\|h\|). \]

Now from \( \ref{5.3} \) we note that \( A'(u):C_e(\Omega) \to C_e(\Omega) \) is a bounded linear map and hence the map \( A\mathcal{U}_\lambda \to \mathcal{U}_\lambda \) is continuous, i.e. \( u \to A'(u) \) is continuous. Let \( \tilde{u} \in C_e(\Omega) \) such that \( \|\tilde{u} - u\| < \delta \) and \( A'(\tilde{u})h = \tilde{z} \) for some \( h \in C_e(\Omega) \). Using Taylor’s theorem there exists some \( \theta(x) \in [u, \tilde{u}] \) and
\[ |f'(\tilde{u}) - f'(u)| = \lambda f''(\theta)|\tilde{u} - u|h| \leq C_0e(x)^2 \frac{d(x)^2}{\xi_1^2} \delta\|h\|_{C_e(\Omega)} \leq C_4\delta \|h\|_{C_e(\Omega)} \]
where the constant \( C_1 \) is independent of \( u \) and \( \tilde{u} \). As before estimating \(-\Delta(\tilde{z} - z)\) from above and below and using maximum principle we have \( |\tilde{z}(x) - z(x)| \leq C\delta\|h\|e(x) \). Now taking supremum over \( \|h\|_{C_e(\Omega)} \leq 1 \) we have
\[ \|A'(\tilde{u}) - A'(u)\| \leq C\|\tilde{u} - u\|_{C_e(\Omega)} \]
and thus \( A \) is continuously differentiable.

**Step III.** The map \( A \) is \( C^2 \). Now that we have proved \( \mathcal{A}\mathcal{U}_\lambda \to \mathcal{U}_\lambda \) is \( C^1 \), using the same idea we can prove that \( A \) is twice continuously differentiable. In order to avoid the repetition of the same arguments we skip the details of the proof of step III.

From \( \ref{5.4} \) of above proposition we know that \( \|\frac{w_h - w - z}{\|h\|}\|_{C_e(\Omega)} \) is bounded and similarly \( \|\frac{w_h - w}{\|h\|}\|_{C_e(\Omega)} \) is also bounded. So
\[ \|A'(u)h\|_{C_e(\Omega)} = \|z\|_{C_e(\Omega)} \leq \|w_h - w - z\|_{C_e(\Omega)} + \|w_h - w\|_{C_e(\Omega)} \]
\[ = \|h\| \|\frac{w_h - w - z}{\|h\|}\|_{C_e(\Omega)} + \|h\| \|\frac{w_h - w}{\|h\|}\|_{C_e(\Omega)} \]
\[ \leq M\|h\|_{C_e(\Omega)} \]

which implies \( A'(u) \in BL(C_e(\Omega), C_e(\Omega)) \) and hence \( A'(u):C_e(\Omega) \to C_e(\Omega) \) is compact.

**Corollary 5.2.** \( A'(u):C_e(\Omega) \to C_e(\Omega) \) is continuous linear and compact.

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