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SPECTRUM, GLOBAL BIFURCATION AND NODAL SOLUTIONS TO KIRCHHOFF-TYPE EQUATIONS

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Abstract. In this article, we consider a Dancer-type unilateral global bifurcation for the Kirchhoff-type problem

$$\begin{gathered} -\Big(a+b\int_0^1 |u'|^2\,dx\Big)u'' = \lambda u + h(x,u,\lambda) \quad \text{in } (0,1),\\ u(0)=u(1)=0. \end{gathered}$$

Under natural hypotheses on h, we show that $(a\lambda_k, 0)$ is a bifurcation point of the above problem. As applications we determine the interval of λ, in which there exist nodal solutions for the Kirchhoff-type problem

$$\begin{gathered} -\Big(a+b\int_0^1 |u'|^2\,dx\Big)u'' = \lambda f(x,u) \quad \text{in } (0,1),\\ u(0)=u(1)=0, \end{gathered}$$

where f is asymptotically linear at zero and is asymptotically 3-linear at infinity. To do this, we also establish a complete characterization of the spectrum of a nonlocal eigenvalue problem.

1. Introduction

We consider the unilateral global bifurcation phenomenon for the problem

$$\begin{gathered} -\Big(a+b\int_0^1 |u'|^2\,dx\Big)u'' = \lambda u + h(x,u,\lambda) \quad \text{in } (0,1),\\ u(0)=u(1)=0, \end{gathered} \tag{1.1}$$

where $a>0$, $b>0$ are real constants, λ is a parameter and $h:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function satisfying

$$\lim_{s\to 0}\frac{h(x,s,\lambda)}{s} = 0 \tag{1.2}$$

uniformly for all $x\in(0,1)$ and λ on bounded sets.

Problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff to describe the transversal oscillations of a stretched string [25]. Problem (1.1) received much attention only after Lions [27] proposed an abstract framework to it. Some important and interesting results can be found, for example, in [1, 5, 16, 17, 24]. Recently, many mathematicians have studied problem (1.1) by

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variational method, see [6, 7, 22, 26, 28, 29, 30, 31, 32, 35, 36, 37] and the references therein. The study of Kirchhoff-type equations has already been extended to the case involving the p-Laplacian and $p(x)$-Laplacian. We refer the readers to [2, 8, 10, 12, 13, 14, 18, 19, 21, 33] on this subject. A distinguishing feature of problem (1.1) is that the first equation contains a nonlocal coefficient $a + b\int_0^1 |u'|^2\,dx$, and hence the equation is no longer a pointwise equation. Moreover, the first equation of problem (1.1) with $h \equiv 0$ is not homogeneous. So problem (1.1) is a fully nonlinear problem which raises some essential difficulties to the study of this kind of problems.

It is well known that the problem

$$
\begin{gathered}
-u'' = \lambda u \quad \text{in } (0,1),\\
u(0) = u(1) = 0
\end{gathered}
$$

possesses infinitely many eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \lambda_k \to +\infty$, all of which are simple. The eigenfunction φ_k corresponding to λ_k has exactly $k-1$ simple zeros in $(0,1)$. Let S_k^+ denote the set of functions in $E := C_0^1[0,1]$ which have exactly $k-1$ interior nodal (i.e. non-degenerate) zeros in $(0,1)$ and are positive near $x = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. It is clear that S_k^+ and S_k^- are disjoint and open in E. Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$ under the product topology.

Theorem 1.1. *The pair $(a\lambda_k, 0)$ is a bifurcation point of problem (1.1). Moreover, there are two distinct unbounded continua in $\mathbb{R}\times H_0^1(0,1)$, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, consisting of the bifurcation branch $\mathscr{C}_k$ emanating from $(a\lambda_k, 0)$, such that $\mathscr{C}_k^\nu \subseteq (\{(a\lambda_k, 0)\} \cup \Phi_k^\nu)$, $\nu \in \{+,-\}$.*

We shall prove Theorem 1.1 in Section 2. If $b = 0$, the conclusions of Theorem 1.1 are well known. However, the case of $b > 0$ is nontrivial because problem (1.1) is nonlinear. So the standard bifurcation theory cannot be used directly here. In order to apply the Dancer unilateral global bifurcation theorem, we find a skillful transformation which can convert problem (1.1) to a desired form. This technique can also be used to deal with other similar problems.

To find more detailed information of $\mathscr{C}_k^\nu$, in Section 3 we study the eigenvalue problem

$$
\begin{gathered}
-\Big(\int_0^1 |u'|^2\,dx\Big)u'' = \mu u^3 \quad \text{in } (0,1),\\
u(0) = u(1) = 0.
\end{gathered} \tag{1.3}
$$

We shall establish a complete characterization of the spectrum of problem (1.3).

Theorem 1.2. *The set of all eigenvalues of problem (1.3) satisfy*

$$
0 < \mu_1 < \mu_2 < \dots < \mu_k < \dots \to +\infty.
$$

Every μ_k is simple and the corresponding one-dimensional space of solutions of problem (1.3) with $\mu = \mu_k$ is spanned by a function having precisely k bumps in $(0,1)$. Each k-bump solution is constructed by the reflection and compression of the eigenfunction φ_1 associated with μ_1.

Note that problem (1.3) is nonlinear and nonlocal. So the Prüfer-type transformation method cannot be used to get the desired results. We use variational method combined with reflection-compression technique to prove Theorem 1.2. We believe that the first trying of this way has significance to nonlinear eigenvalue problems.

In Section 4, we describe $\mathscr{C}_k^\nu$ more detailed for problem (1.1) with $h(x,s,\lambda) = \lambda f(x,s) - \lambda s$, i.e., the problem

$$\begin{gathered} -\Big(a + b\int_0^1 |u'|^2\,dx\Big)u'' = \lambda f(x,u) \quad \text{in } (0,1),\\ u(0) = u(1) = 0. \end{gathered} \tag{1.4}$$

We assume that f satisfies the following assumptions:

(A1) $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x,s)s > 0$ for all $x \in (0,1)$ and any $s \neq 0$.

(A2) there exist f_0, $f_\infty \in (0,+\infty)$ such that
$$\lim_{s\to 0} \frac{f(x,s)}{as} = f_0, \quad \lim_{|s|\to+\infty} \frac{f(x,s)}{bs^3} = f_\infty$$
uniformly with respect to $x \in (0,1)$.

Our last main theorem reads as follows.

Theorem 1.3. *Assume that* (A1)–(A2) *are satisfied. Then for*
$$\lambda \in \Big(\frac{\lambda_k}{f_0}, \frac{\mu_k}{f_\infty}\Big) \cup \Big(\frac{\mu_k}{f_\infty}, \frac{\lambda_k}{f_0}\Big),$$
problem (1.4) *possesses at least two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0, and u_k^- has exactly $k-1$ simple zeros in $(0,1)$ and is negative near 0.*

We end this section by introducing some notation convention which will be used later in this paper. Let X be the usual Sobolev space $H_0^1(0,1)$ with the norm $\|u\| = (\int_0^1 |u'|^2\,dx)^{1/2}$. For a measurable set A of $\mathbb{R}^N$, we denote its measure by $|A|$. Also, denote by c and c_i, $i \in \mathbb{N}$, some generic positive constants (the exact value may be different from line to line).

2. Global bifurcation

Firstly, we consider the auxiliary problem

$$\begin{gathered} -u'' = f(x) \quad \text{in } (0,1),\\ u(0) = u(1) = 0. \end{gathered} \tag{2.1}$$

It is well known that problem (2.1) possesses a unique weak solution for each $f \in L^1(0,1)$. Let us denote by $G(f)$ the unique solution of problem (2.1). Then $G : L^1(0,1) \to X$ is a linear continuous operator. Since the embedding of $X \hookrightarrow C[0,1]$ is compact, the restriction of G to X is a completely continuous operator.

Theorem 2.1. *The pair $(a\lambda_k, 0)$ is a bifurcation point of problem* (1.1)*. Moreover, there are two distinct continua in $\mathbb{R} \times X$, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, consisting of the bifurcation branch $\mathscr{C}_k$ emanating from $(a\lambda_k, 0)$, which contain $\{(a\lambda_k, 0)\}$ and satisfy either $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$ are both unbounded or $\mathscr{C}_k^+ \cap \mathscr{C}_k^- \neq \{(a\lambda_k, 0)\}$.*

Proof. Clearly, the pair (λ, u) is a solution of problem (1.1) if and only if (λ, u) satisfies

$$u = G\Big(\frac{1}{a + b\|u\|^2}(\lambda u + h(x,u,\lambda))\Big). \tag{2.2}$$

Let

$$Lu = \frac{1}{a}G(u), \quad \widetilde{H}(\lambda, u) = \frac{1}{a + b\|u\|^2}G(h(x, u, \lambda)) - \frac{\lambda b\|u\|^2}{a(a + b\|u\|^2)}G(u).$$

Clearly, $L{:}X \to X$ is linear completely continuous, $\widetilde{H}{:}\mathbb{R} \times X \to X$ is compact. Moreover, it is easy to see that $a\lambda_k$ is simple characteristic value of L. Then equation (2.2) is equivalent to

$$u = \lambda Lu + \widetilde{H}(\lambda, u).$$

Let $\widetilde{h}(x, u, \lambda) = \max_{0 \le |s| \le u} |h(x, s, \lambda)|$ for all $x \in (0, 1)$ and λ on bounded sets. Then $\widetilde{h}$ is nondecreasing with respect to u and

$$\lim_{u \to 0^+} \frac{\widetilde{h}(x, u, \lambda)}{u} = 0. \tag{2.3}$$

Further it follows from (2.3) that

$$\frac{h(x, u, \lambda)}{\|u\|} \le \frac{\widetilde{h}(x, |u|, \lambda)}{c\|u\|_\infty} \le \frac{\widetilde{h}(x, \|u\|_\infty, \lambda)}{c\|u\|_\infty} \to 0 \quad \text{as } \|u\| \to 0, \tag{2.4}$$

uniformly for $x \in (0, 1)$ and λ belonging to a bounded set, where $c > 0$ is the best embedding constant of $X \hookrightarrow C[0, 1]$, $\|u\|_\infty = \max_{x \in [0,1]} |u(x)|$. It follows that $\widetilde{H} = o(\|u\|)$ for u near 0 uniformly on bounded λ intervals. [15, Theorem 2] shows the desired conclusions. □

For the regularity of weak solution, we have the following result.

Proposition 2.2. *Any weak solution $u \in X$ of problem (1.1) is also a classical solution, i.e., $u \in C^2(0, 1) \cap C^{1,\alpha}[0, 1]$ satisfying (1.1) and $u(0) = u(1) = 0$.*

Proof. Let

$$f(x) = \frac{1}{a + b\|u\|^2}(\lambda u + h(x, u, \lambda)).$$

Then it is easy to see that $f \in L^2(0, 1)$. By [23, Theorem 8.12], we know that $u \in W^{2,2}(0, 1)$. Furthermore, by the general Sobolev embedding theorem [20, p. 270], we get $u \in C^{1,\alpha}[0, 1]$ for some $\alpha \in (0, 1)$. According to the definition of weak solution, we have

$$-\Big(a + b\int_0^1 |u'|^2\, dx\Big)u'' = \lambda u + h(x, u, \lambda)$$

in the sense of distribution, i.e.,

$$-\Big(a + b\int_0^1 |u'|^2\, dx\Big)u'' = \lambda u + h(x, u, \lambda) \quad \text{in } (0, 1) \setminus I_0 \tag{2.5}$$

for some $I_0 \subset (0, 1)$ which satisfies $|I_0| = 0$. Let $I := (0, 1)$. Then the equation (2.5) follows that $u \in C^2(I \setminus I_0)$ and

$$u''(x) = -f(x), \quad x \in I \setminus I_0. \tag{2.6}$$

For any $x_0 \in I_0$, it is easy to see that equation (2.6) implies the existence of $\lim_{x \to x_0} u''(x)$. [11, Proposition 1] follows that $u''(x_0) = \lim_{x \to x_0} u''(x)$. By the arbitrary property of x_0, we get that $u \in C^2(I)$ and satisfies (1.1). Clearly, one has $u(0) = u(1) = 0$. □

Lemma 2.3. *If (λ, u) is a solution of (1.1) and u has a double zero, then $u \equiv 0$.*

Proof. Let u be a solution of problem (1.1) and $x^* \in [0,1]$ be a double zero. We note that
$$u(x) = \frac{-1}{a+b\|u\|^2}\int_{x^*}^{x}\int_{x^*}^{s}(\lambda u + h(\tau,u,\lambda))\,d\tau\,ds.$$
Firstly, we consider $x \in [0, x^*]$. Then
$$\begin{aligned}|u(x)| &\le \frac{1}{a}\int_x^{x^*}|\lambda u + h(\tau,u,\lambda)|\,d\tau,\\ &\le \frac{1}{a}\int_x^{x^*}\Big(|\lambda| + |\frac{h(\tau,u(\tau),\lambda)}{u(\tau)}|\Big)|u(\tau)|\,d\tau.\end{aligned}$$
In view of (1.2), for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that
$$|h(x,s,\lambda)| \le \varepsilon|s|$$
uniformly with respect to all $x \in (0,1)$ and fixed λ when $|s| \in [0,\delta]$. Hence,
$$|u(x)| \le \int_x^{x^*}\frac{1}{a}\Big(|\lambda| + \varepsilon + \max_{s\in[\delta,\|u\|_\infty]}|\frac{h(\tau,s,\lambda)}{s}|\Big)|u(\tau)|\,d\tau.$$
By the Gronwall-Bellman inequality [4], we get $u \equiv 0$ on $[0, x^*]$. Similarly, we can get $u \equiv 0$ on $[x^*, 1]$ and the proof is complete. $\square$

Lemma 2.4. *We have $\mathscr{C}_k^\nu \cap (\mathbb{R}\times\{0\}) = \{(a\lambda_k, 0)\}$ if $\mathscr{C}_k^\nu \subseteq (\Phi_k^\nu \cup \{(a\lambda_k,0)\})$.*

Proof. By Proposition 2.2 follows that $\mathscr{C}_k^\nu \in \mathbb{R}\times E$. Suppose, on the contrary, if there exists $(\mu_m, u_m) \to (a\lambda_j, 0)$ when $m \to +\infty$ with $(\mu_m, u_m) \in \mathscr{C}_k^\nu$, $u_m \in S_k^\nu$, $u_m \not\equiv 0$ and $j \ne k$. Let $v_m := u_m/\|u_m\|$, then v_m should be a solution of the problem
$$v = G\Big(\frac{1}{a+b\|u_m\|^2}\Big(\mu_m v + \frac{h(x,u_m,\mu_m)}{\|u_m(x)\|}\Big)\Big). \tag{2.7}$$
By (2.4), (2.7) and the compactness of G we obtain that for some convenient subsequence $v_m \to v_0$ as $m \to +\infty$. Now v_0 verifies the equation
$$-v'' = \lambda_j v$$
and $\|v_0\| = 1$. Hence $v_0 \in S_j$ which is an open set in X, and as a consequence for some m large enough, $v_m \in S_j$, and this is a contradiction. $\square$

Proof of Theorem 1.1. By [34, Lemma 1.24] there exists a bounded open neighborhood $\mathscr{O}_k$ of $(a\lambda_k, 0)$ such that $(\mathscr{C}_k^\nu \cap \mathscr{O}_k) \subseteq (\Phi_k^\nu \cup \{(a\lambda_k,0)\})$ or $(\mathscr{C}_k^\nu \cap \mathscr{O}_k) \subseteq (\Phi_k^{-\nu} \cup \{(a\lambda_k,0)\})$. Without loss of generality, we assume that $(\mathscr{C}_k^\nu \cap \mathscr{O}_k) \subseteq (\Phi_k^\nu \cup \{(a\lambda_k,0)\})$.

Next, we show that $\mathscr{C}_k^\nu \subseteq (\Phi_k^\nu \cup \{(a\lambda_k,0)\})$. Suppose that $\mathscr{C}_k^\nu \not\subseteq (\Phi_k^\nu \cup \{(a\lambda_k,0)\})$. Then there exists $(\mu,u) \in \mathscr{C}_k^\nu \cap (\mathbb{R}\times\partial S_k^\nu)$ such that $(\mu,u) \ne (a\lambda_k,0)$ and $(\mu_n,u_n) \to (\mu,u)$ with $(\mu_n,u_n) \in \mathscr{C}_k^\nu \cap (\mathbb{R}\times S_k^\nu)$. Since $u \in \partial S_k^\nu$, by Lemma 2.3, $u \equiv 0$. Let $w_n := u_n/\|u_n\|$, then w_n should be a solution of the following problem
$$w = G\Big(\frac{1}{a+b\|u_n\|^2}\Big(\mu_n w + \frac{h(x,u_n,\mu_n)}{\|u_n(x)\|}\Big)\Big). \tag{2.8}$$
By (2.4), (2.8) and the compactness of G we obtain that for some convenient subsequence $w_n \to w_0 \ne 0$ as $n \to +\infty$. Now w_0 verifies the equation
$$-w'' = \frac{\mu}{a}w$$

and $\|w_0\| = 1$. Hence $\mu = a\lambda_j$, for some $j \neq k$. Therefore, $(\mu_n, u_n) \to (a\lambda_j, 0)$ with $(\mu_n, u_n) \in \mathscr{C}_k^\nu \cap (\mathbb{R} \times S_k^\nu)$. This contradicts Lemma 2.4. Thus, we have that

$$\mathscr{C}_k^\nu \subseteq (\Phi_k^\nu \cup \{(a\lambda_k, 0)\}).$$

We claim that both $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$ are unbounded. Without loss of generality, we may suppose that $\mathscr{C}_k^-$ is bounded. Therefore, there exists $(\lambda_*, u_*) \in \mathscr{C}_k^+ \cap \mathscr{C}_k^-$ such that $(\lambda_*, u_*) \neq (a\lambda_k, 0)$ and $u_* \in S_k^+ \cap S_k^-$. This contradicts the definitions of S_k^+ and S_k^-. □

3. Spectrum

By an argument similar to that of Proposition 2.2, we can get the following regularity result.

Proposition 3.1. *Any weak solution $u \in X$ of problem (1.3) is also a classical solution, i.e., $u \in C^2(0,1) \cap C^{1,\alpha}[0,1]$ satisfying (1.3) and $u(0) = u(1) = 0$.*

Lemma 3.2. *If (μ, u) is a solution of (1.3) and u has a double zero, then $u \equiv 0$.*

Proof. The homogeneity of problem (1.3) implies that it suffices to consider a solution such that $\|u\| = 1$, which therefore is solution of the ordinary differential equation

$$\begin{gathered} -u'' = \mu u^3 \quad \text{in } (0,1), \\ u(0) = u(1) = 0. \end{gathered}$$

Such a nontrivial solution necessarily cannot have a double zero by the uniqueness property of Cauchy problem for (1.3). □

Lemma 3.3. *Each nontrivial solution of (1.3) has a finite number of zeros.*

Proof. If a nontrivial solution has an infinite number of zeros, its accumulation point is a double zero, a contradiction. □

Lemma 3.4. *$\mu_1(I)$ with $I = (0,1)$ satisfies the strict monotonicity property with respect to the domain I, i.e. if J is a strict sub interval of I, then $\mu_1(I) < \mu_1(J)$.*

Proof. Let φ_1 with $\|\varphi_1\| = 1$ be the eigenfunction of (1.3) on J corresponding to $\mu_1(J)$, and denote by $\widetilde{\varphi}_1$ the extension by zero on I. Then we have that

$$\frac{1}{\mu_1(J)} = \int_J |\varphi_1|^4\,dx = \int_I |\widetilde{\varphi}_1|^4\,dx < \sup_{u \in X, \|u\|=1} \int_0^1 |u|^4\,dx = \frac{1}{\mu_1(I)}.$$

The last strict inequality holds from the fact that $\widetilde{\varphi}_1$ vanishes in $I \setminus J$ so cannot be an eigenfunction corresponding to the principal eigenvalue $\mu_1(I)$. □

Proof of Theorem 1.2. Let φ_1 be a positive eigenfunction corresponding to μ_1. It follows from the symmetry of (1.3) and [9, Theorem 1.2] that $\varphi_1(x) = \varphi_1(1-x)$ for $x \in [0,1]$, i.e. φ_1 is even with respect to $1/2$. For any $k \geq 2$, set

$$\varphi_k(x) = \begin{cases} \varphi_1(kx), & x \in [0, 1/k], \\ -\varphi_1(kx-1), & x \in [1/k, 2/k], \\ \dots \\ (-1)^k \varphi_1(kx-k+1), & x \in [\frac{k-1}{k}, 1]. \end{cases}$$

Then φ_k is an eigenfunction of (1.3) associated with the eigenvalue $\mu_k = k^4\mu_1$. On the other hand, let $u = u(x)$ be an eigenfunction of (1.3) associated with some eigenvalue $\mu_* > \mu_1$. According to [9, Theorem 1.2], u changes sign in $(0, 1)$. Lemmas 3.2 and 3.3 imply that $u \in S_k$ for some $k \geq 2$. Without loss of generality, we may assume that $u'(0) > 0$. Let

$$0 < \tau_1 < \tau_2 < \cdots < \tau_{k-1} < 1$$

denote the zeros of u in $(0, 1)$. Without loss of generality, we may assume that $\tau_1 \leq 1/k$. Applying Lemma 3.4 on $[0, 1/k]$, we have that $\mu_* \geq \mu_k$. By [3, Lemma 2], there exist integers p and q, $1 \leq p \leq k-1$, $1 \leq q \leq k-1$, such that

$$\tau_p \leq \frac{1}{q+1} < \frac{1}{q} \leq \tau_{p+1}.$$

Applying Lemma 3.4 on $[\tau_p, \tau_{p+1}]$, we have that $\mu_* \leq \mu_k$. So we have that $\mu_* = \mu_k$. Furthermore, if $\tau_1 < 1/k$, we have $\mu_* > \mu_k$; if $\tau_1 > 1/k$, we have $\mu_* < \mu_k$. Thus we have $\tau_1 = 1/k$ and $u = c_1\varphi_k(x)$ for $x \in [0, 1/k]$. Similarly, we can obtain that $\tau_i = i/k$ and $u = c_i\varphi_k(x)$ for $x \in [(i-1)/k, i/k]$, $2 \leq i \leq k-1$. Let us normalize u as $u'(0) = \varphi_k'(0)$. It follows that $c_1 = 1$. Hence $\varphi_k'(\frac{1}{k}) = c_2\varphi_k'(\frac{1}{k})$. So we have $c_2 = 1$. Similarly, one has $c_i = 1$ for all $3 \leq i \leq k-1$. Therefore, we have that $u(x) = \varphi_k(x)$, $x \in [0, 1]$. □

4. Nodal solutions

In this section, we apply Theorems 1.1 and 1.2 to study the existence of nodal solutions for (1.4).

Proof of Theorem 1.3. Let $g : (0, 1) \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x, s) = af_0 s + g(x, s)$$

with

$$\lim_{s\to 0} \frac{g(x, s)}{as} = 0 \quad \text{and} \quad \lim_{|s|\to +\infty} \frac{g(x, s)}{s^3} = bf_\infty \tag{4.1}$$

uniformly with respect to $x \in (0, 1)$.

From (4.1), we can see that λg satisfies the assumptions of (1.2). Now, by Theorem 1.1, there are two distinct unbounded continua, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$ emanating from $(\lambda_k/f_0, 0)$, such that

$$\mathscr{C}_k^\nu \subset (\{(\lambda_k/f_0, 0)\} \cup \Phi_k^\nu).$$

If $\mathscr{C}_k^\nu$ is unbounded in the parameter direction, the conclusion is done. Next we assume that $\mathscr{C}_k^\nu$ is bounded in the parameter direction. Then it is sufficient to show that $\mathscr{C}_k^\nu$ joins $(\lambda_k/f_0, 0)$ to $(\lambda_k/f_\infty, \infty)$. Let $(\xi_n, u_n) \in \mathscr{C}_k^\nu$ where $u_n \not\equiv 0$ satisfies $\xi_n + \|u_n\| \to +\infty$. Since (0,0) is the only solution of (1.4) for $\lambda = 0$, we have $\mathscr{C}_k^\nu \cap (\{0\} \times X) = \emptyset$. It follows that $\xi_n > 0$ for all $n \in \mathbb{N}$. Clearly, we have

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty.$$

Let h:$(0, 1) \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x, s) = bf_\infty s^3 + h(x, s)$$

with

$$\lim_{|s|\to +\infty} \frac{h(x, s)}{s^3} = 0 \quad \text{and} \quad \lim_{s\to 0} \frac{h(x, s)}{s} = af_0$$

uniformly with respect to $x \in (0,1)$. Then (ξ_n, u_n) satisfies

$$u_n = G\Big(\frac{\xi_n}{a+b\|u_n\|^2}(bf_\infty u_n^3 + h(x,u_n))\Big). \tag{4.2}$$

Dividing the above equation by $\|u_n\|$ and letting $\overline{u}_n = u_n/\|u_n\|$, we obtain

$$\overline{u}_n = G\Big(\frac{\xi_n\|u_n\|^2}{a+b\|u_n\|^2}(bf_\infty \overline{u}_n^3 + \frac{h(x,u_n)}{\|u_n\|^3})\Big).$$

Let

$$\widetilde{h}(x,u) = \max_{0\le|s|\le u} |h(x,s)| \quad \text{for } x \in (0,1).$$

Then $\widetilde{h}$ is nondecreasing with respect to u. Define

$$\overline{h}(x,u) = \max_{u/2\le|s|\le u} |h(x,s)| \quad \text{for } x \in (0,1).$$

Then we can see that

$$\lim_{u\to+\infty} \frac{\overline{h}(x,u)}{u^3} = 0 \quad \text{and} \quad \widetilde{h}(x,u) \le \widetilde{h}(x,\frac{u}{2}) + \overline{h}(x,u).$$

It follows that

$$\limsup_{u\to+\infty} \frac{\widetilde{h}(x,u)}{u^3} \le \limsup_{u\to+\infty} \frac{\widetilde{h}(x,\frac{u}{2})}{u^3} = \limsup_{u/2\to+\infty} \frac{\widetilde{h}(x,\frac{u}{2})}{8(\frac{u}{2})^3}.$$

So we have

$$\lim_{u\to+\infty} \frac{\widetilde{h}(x,u)}{u^3} = 0. \tag{4.3}$$

Further it follows from (4.3) that

$$\frac{h(x,u_n)}{\|u_n\|^3} \le \frac{\widetilde{h}(x,|u_n|)}{\|u_n\|^3} \le \frac{\widetilde{h}(x,\|u_n\|_\infty)}{\|u_n\|^3} \le c^3\frac{\widetilde{h}(x,c\|u_n\|)}{c^3\|u_n\|^3} \to 0$$

as $n \to +\infty$ uniformly for $x \in (0,1)$, where $c > 0$ is the best embedding constant of $X \hookrightarrow C[0,1]$.

From the compactness of G we obtain

$$-\|\overline{u}\|^2\overline{u}'' = \overline{\mu} f_\infty \overline{u}^3,$$

where $\overline{u} = \lim_{n\to+\infty} \overline{u}_n$ and $\overline{\mu} = \lim_{n\to+\infty} \xi_n$, again choosing a subsequence and relabeling it if necessary. It is clear that $\|\overline{u}\| = 1$ and $\overline{u} \in \mathscr{C}_k^\nu$. Theorem 1.2 shows that $\overline{\mu} = \mu_k/f_\infty$. Therefore, $\mathscr{C}_k^\nu$ joins $(\lambda_k/f_0, 0)$ to $(\mu_k/f_\infty, \infty)$. $\square$

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