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# EXISTENCE OF SOLUTIONS FOR SUBLINEAR EQUATIONS ON EXTERIOR DOMAINS

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Abstract. In this article we consider the radial solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius $R > 0$, $B_R$, centered at the origin in $\mathbb{R}^N$ with $u = 0$ on $\partial B_R$ and $\lim_{r\to\infty} u(r) = 0$ where $N > 2$, $f$ is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on $(\beta, \infty)$, $f(u) \sim u^p$ with $0 < p < 1$ for large $u$ and $K(r) \sim r^{-\alpha}$ with $\frac{(N+2)-p(N-2)}{2} \le \alpha < N - p(N-2)$ for large $r$. We prove existence of $n$ solutions - one with exactly $n$ zeros on $[R, \infty)$ - if $R > 0$ is sufficiently small. If $R > 0$ is sufficiently large then there are no solutions with $\lim_{r\to\infty} u(r) = 0$.

## 1. Introduction

In this article we study radial solutions of

$$\Delta u + K(r)f(u) = 0 \quad \text{in } \mathbb{R}^N \backslash B_R, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial B_R, \tag{1.2}$$

$$u \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

where $B_R$ is the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ and $K(r) > 0$. We assume:

- (H1) $f$ is odd and locally Lipschitz, $f < 0$ on $(0, \beta)$, $f > 0$ on $(\beta, \infty)$, and $f'(0) < 0$.

Let $F(u) = \int_0^u f(s)\, ds$. Since $f$ is odd it follows that $F$ is even and from (H1) it follows that $F$ is bounded below by $-F_0 < 0$, $F$ has a unique positive zero, $\gamma$, with $0 < \beta < \gamma$, and

- (H2) $-F_0 < F < 0$ on $(0, \gamma)$, $F > 0$ on $(\gamma, \infty)$.

We also assume:

- (H3) There exists $p$ with $0 < p < 1$ such that $f(u) = |u|^{p-1}u + g(u)$ where $\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0$ and $g(u) \ge 0$ for $u \ge \gamma$.

In addition, we assume $K$ is differentiable on $(R, \infty)$ and that there exist constants $k_2 > k_1 > 0$, and $\alpha$ with $\frac{(N+2)-p(N-2)}{2} \le \alpha < N - p(N-2)$ such that

- (H4) $k_1 r^{-\alpha} < K(r) < k_2 r^{-\alpha}$ on $[R, \infty)$, and:
- (H5) $\lim_{r\to\infty} \frac{rK'}{K} = -\alpha$ for $\frac{(N+2)-p(N-2)}{2} \le \alpha < N - p(N-2)$, and $\frac{rK'}{K} > -2(N-1)$ on $[R, \infty)$.

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Note that (H5) implies $r^{2(N-1)}K(r)$ is increasing.

Interest in the topic for this article comes from recent papers [5, 6, 11, 12, 14] about solutions of differential equations on exterior domains. When $f$ grows superlinearly at infinity - i.e. $\lim_{u\to\infty} f(u)/u = \infty$, $\Omega = \mathbb{R}^N$, and $K(r) \equiv 1$ then the problem (1.1), (1.3) has been extensively studied [1, 2, 3, 9, 13, 15].

In this article we consider the case where $f$ grows sublinearly at infinity - i.e. $\lim_{u\to\infty} \frac{f(u)}{u^p} = c_0 > 0$ with $0 < p < 1$ and $K(r) \sim r^{-\alpha}$ with $\frac{(N+2)-p(N-2)}{2} \le \alpha < N - p(N-2)$. In earlier papers [7, 8, 10] the cases where $f$ is sublinear with $0 < \alpha < 2$, $N - p(N-2) < \alpha < 2(N-1)$, and $\alpha > 2(N-1)$ were investigated.

In [14] existence of a positive solution if $R > 0$ is sufficiently small was proved using sub and super solutions when $f$ is semipositone. Here we prove the existence of $n$ solutions if $R > 0$ is sufficiently small.

Since we are interested in radial solutions of (1.1)-(1.3) we assume that $u(x) = u(|x|) = u(r)$ where $x \in \mathbb{R}^N$ and $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ so that $u$ solves

$$u''(r) + \frac{N-1}{r} u'(r) + K(r) f(u(r)) = 0 \quad \text{on } (R,\infty) \text{ where } R > 0, \tag{1.4}$$

$$u(R) = 0, \quad u'(R) = b \in \mathbb{R}. \tag{1.5}$$

In this article we prove the following result.

**Theorem 1.1.** *Assume* (H1)–(H5) *hold. Then given a nonnegative integer $n$ there exists a solution of* (1.4)-(1.5) *with $n$ zeros on $(R,\infty)$ and $\lim_{r\to\infty} u(r) = 0$ if $R > 0$ is sufficiently small.*

We note that if $0 < \alpha < 2$ then it was shown in [7] that there are solutions for *all* $R > 0$ whereas when $\alpha > 2$ and $R$ is sufficiently large then it was shown in [8] that there are *no* solutions of (1.1)-(1.3) with $\lim_{r\to\infty} u(r) = 0$. In this paper we prove existence of solutions of (1.1)-(1.3) for $\frac{(N+2)-p(N-2)}{2} \le \alpha < N - p(N-2)$ and $R > 0$ sufficiently small. A similar result was proved in [8], [10] for $N - p(N-2) < \alpha < 2(N-1)$ and for $\alpha > 2(N-1)$.

## 2. Behavior for small $b > 0$

We suppose that $U(r)$ solves (1.4) and then make the change of variables:

$$U(r) = u(r^{2-N}).$$

Then for $0 < t < \infty$ we see that $u$ satisfies

$$u'' + h(t) f(u) = 0, \tag{2.1}$$

where

$$h(t) = \frac{t^{\frac{2(N-1)}{2-N}} K(t^{\frac{1}{2-N}})}{(N-2)^2}.$$

It follows from (H4), (H5) that

$$h(t) > 0, \quad h'(t) < 0, \quad h_1 t^{-q} < h(t) < h_2 t^{-q} \tag{2.2}$$

for $t > 0$ where $q = \frac{2(N-1)-\alpha}{N-2}$, $h_i = \frac{k_i}{(N-2)^2}$. In addition it follows from (H3), (H5), and (2.2) that

$$1 < p + 1 < q < 2. \tag{2.3}$$

We also assume that

$$u(0) = 0, \quad u'(0) = b > 0. \tag{2.4}$$

For the rest of this article we will focus on finding solutions of (2.1), (2.4) such that $u(R^{2-N}) = 0$. If such a solution exists then $U(r) = u(r^{2-N})$ will be a solution of (1.1)-(1.3).

We first examine (2.1), (2.4) assuming (H1)–(H5). It is straightforward to show that there is a unique solution $u \in C^1([0,\epsilon)) \cap C^2(0,\epsilon)$ of (2.1) for some $\epsilon > 0$ such that (2.4) holds. A proof is provided in the appendix. Then from (H3) since $|f(u)| \le C_1|u|$ for some constant $C_1 > 0$ it follows that the solution of (2.1), (2.4) exists on $(0,\infty)$.

Since $u > 0$ for small positive $t$ (by (2.4)) and since $f(u) < 0$ for $0 < u < \beta$ by (H1) it follows from (2.1) that $u'' > 0$ when $0 < u < \beta$. Therefore $u' > u'(0) = b$ when $0 < u < \beta$. Integrating this on $(0,t)$ and using (2.4) we obtain

$$u > bt \quad \text{when } 0 < u < \beta. \tag{2.5}$$

It then follows that there is a smallest positive value of $t, t_b$, such that $u(t_b) = \beta$ and $0 < u < \beta$ on $(0, t_b)$.

We next show that $u$ gets larger than $\gamma$. So suppose not. That is suppose $0 < u < \gamma$ for all $t > 0$ and consider

$$E = \frac{1}{2}\frac{u'^2}{h(t)} + F(u) \quad \text{for } t > 0. \tag{2.6}$$

Then from (2.1) and (2.2) we have

$$E' = \Big(\frac{1}{2}\frac{u'^2}{h(t)} + F(u)\Big)' = -\frac{h'u'^2}{h^2} \ge 0 \tag{2.7}$$

and so by the initial conditions (2.4) we see that

$$E = \frac{1}{2}\frac{u'^2}{h(t)} + F(u) > 0 \quad \text{for } t > 0. \tag{2.8}$$

Now while $0 < u < \gamma$ we have $F(u) < 0$ and since $F(\gamma) = 0$ with $F'(\gamma) = f(\gamma) > 0$ (since $\gamma > \beta$) then we find that $-F(u) > a_0(\gamma - u)$ for $\beta < u < \gamma$ for some $a_0 > 0$. Using this in (2.8) we obtain

$$u'^2 \ge -2hF(u) > 2a_0 h(\gamma - u) \quad \text{for } t > t_b. \tag{2.9}$$

Then by (2.2) and (2.9) we have

$$\frac{u'}{\sqrt{\gamma - u}} > \sqrt{2a_0 h_1}\, t^{-\frac{q}{2}} \quad \text{for } t > t_b. \tag{2.10}$$

Integrating this on $(t_b, t)$ gives

$$\sqrt{\gamma - u(t)} < \sqrt{\gamma - \beta} - \sqrt{2a_0 h_1}\Big(\frac{t^{1-\frac{q}{2}} - t_b^{1-\frac{q}{2}}}{1 - \frac{q}{2}}\Big) \quad \text{for } t > t_b. \tag{2.11}$$

Since $1 < q < 2$ it follows that the right-hand side of (2.11) goes to $-\infty$ as $t \to \infty$ contradicting that $0 < u < \gamma$. Thus $u$ eventually gets larger than $\gamma$ and so there exists $t_{b,\gamma} > t_b$ such that $u(t_{b,\gamma}) = \gamma$ and $0 < u < \gamma$ on $(0, t_{b,\gamma})$.

Next we denote $t_{b,2}$ as the smallest positive value of $t$ such that $u(t_{b,2}) = \frac{\beta}{2}$. Note that $t_{b,2} < t_b < t_{b,\gamma}$.

**Lemma 2.1.** *Assume* (H1)–(H5) *and let $u$ solve* (2.1), (2.4). *Then* $\lim_{b\to 0^+} t_{b,2} = \infty$.

*Proof.* From the above arguments we know $u > 0$ and $u' > 0$ on $(0, t_{b,\gamma})$. Also it follows from (H1) that

$$f(u) \geq -a_1 u \quad \text{for } u > 0 \text{ for some } a_1 > 0 \tag{2.12}$$

so that integrating (2.1) twice on $(0, t)$ gives

$$0 < u \leq bt + a_1 \int_0^t \int_0^s h(x)u(x)\,dx\,ds \leq bt + a_1 t \int_0^t h(x)u(x)\,dx. \tag{2.13}$$

Now let

$$y = b + a_1 \int_0^t h(x)u(x)\,dx. \tag{2.14}$$

Then from (2.13)-(2.14) we have

$$0 < u \leq ty \quad \text{and} \quad y' = a_1 hu \tag{2.15}$$

and from (2.2), (2.15) we obtain

$$y' = a_1 hu \leq a_1 thy \leq a_1 h_2 t^{1-q} y. \tag{2.16}$$

Dividing (2.16) by $y$, integrating on $(0, t)$, exponentiating, and recalling that $q < 2$ gives

$$u \leq bt \exp\Big(\frac{a_1 h_2 t^{2-q}}{2-q}\Big) \quad \text{for } t > 0 \tag{2.17}$$

so that evaluating (2.17) at $t_{b,2}$ gives

$$\frac{\beta}{2} \leq bt_{b,2} \exp\Big(\frac{a_1 h_2 t_{b,2}^{2-q}}{2-q}\Big). \tag{2.18}$$

Since $q < 2$ it then follows from (2.18) that $t_{b,2} \to \infty$ as $b \to 0^+$. This completes the proof. □

Now let $c = \frac{2-q}{1-p}$ and note from (2.3) and (H5) that $\frac{1}{2} \leq c < 1$. We also note that $y = c_0 t^c$ with $c_0 = c(1-c)^{-\frac{1}{1-p}} > 0$ is a solution of $y'' + \frac{1}{t^q} y^p = 0$ for $t > 0$.

**Lemma 2.2.** *Assume* (H1)–(H5) *and let $u$ solve* (2.1), (2.4). *Then*

$$\lim_{b \to 0^+} t_{b,2}^{1-c} u'(t_{b,2}) = 0.$$

*Proof.* Using (2.2) and (2.12) in (2.1) gives

$$u'' - \frac{a_1 h_2}{t^q} u \leq 0. \tag{2.19}$$

Letting

$$u(t) = \sqrt{t}\, w\Big(\frac{2\sqrt{a_1 h_2}}{2-q} t^{\frac{2-q}{2}}\Big) \tag{2.20}$$

then using (2.4) and (2.19) it follows that $w$ satisfies

$$w'' + \frac{1}{t} w' - \Big(1 + \frac{1}{(2-q)^2 t^2}\Big) w \leq 0, \quad \lim_{t \to 0^+} \frac{w(t)}{t^{\frac{1}{2-q}}} = \Big(\frac{2-q}{2\sqrt{a_1 h_2}}\Big)^{\frac{1}{2-q}} b. \tag{2.21}$$

Now let $w_1$ satisfy

$$w_1'' + \frac{1}{t} w_1' - \Big(1 + \frac{1}{(2-q)^2 t^2}\Big) w_1 = 0, \quad \lim_{t \to 0^+} \frac{w_1(t)}{t^{\frac{1}{2-q}}} = \Big(\frac{2-q}{2\sqrt{a_1 h_2}}\Big)^{\frac{1}{2-q}} b. \tag{2.22}$$

That is, $w_1$ satisfies the modified Bessel equation and in fact

$$w_1(t) = \Big(\frac{2-q}{2\sqrt{a_1 h_2}}\Big)^{\frac{1}{2-q}} \frac{b}{a} I_{\frac{1}{2-q}}(t) \tag{2.23}$$

where $I_{\frac{1}{2-q}}(t)$ is the modified Bessel function of order $\frac{1}{2-q}$ that is finite at the origin. It is well-known [4] that

$$\begin{gathered} I_{\frac{1}{2-q}}(t) > 0, \quad I'_{\frac{1}{2-q}}(t) > 0 \quad \text{for } t > 0, \lim_{t\to\infty} I_{\frac{1}{2-q}}(t) = \infty, \\ \text{there exists } a > 0 \text{ such that } \lim_{t\to 0^+} \frac{I_{\frac{1}{2-q}}(t)}{t^{\frac{1}{2-q}}} = a. \end{gathered} \tag{2.24}$$

It is also true that

$$\Big|\frac{I'_{\frac{1}{2-q}}(t)}{I_{\frac{1}{2-q}}(t)}\Big| \le a_2 \quad \text{for } t > 1 \text{ for some } a_2 > 0. \tag{2.25}$$

A proof of (2.25) is provided in the appendix.

It then follows from (2.23), (2.25) that

$$\frac{w_1'}{w_1} = \frac{I'_{\frac{1}{2-q}}}{I_{\frac{1}{2-q}}} \le a_2 \quad \text{for } t > 1. \tag{2.26}$$

Next multiplying (2.21) by $tw_1$, (2.22) by $tw$, and subtracting gives

$$[t(w'w_1 - w_1'w)]' \le 0 \quad \text{for } t > 0. \tag{2.27}$$

Since

$$\lim_{t\to 0^+} \frac{w(t)}{t^{\frac{1}{2-q}}} = \lim_{t\to 0^+} \frac{w_1(t)}{t^{\frac{1}{2-q}}} = \Big(\frac{2-q}{2\sqrt{a_1 h_2}}\Big)^{\frac{1}{2-q}} b$$

and

$$\lim_{t\to 0^+} \frac{w'(t)}{t^{\frac{q-1}{2-q}}} = \lim_{t\to 0^+} \frac{w_1'(t)}{t^{\frac{q-1}{2-q}}} = \frac{1}{2-q}\Big(\frac{2-q}{2\sqrt{a_1 h_2}}\Big)^{\frac{1}{2-q}} b,$$

then integrating (2.27) on $(0,t)$ gives

$$w'w_1 - ww_1' \le 0 \quad \text{for } t > 0. \tag{2.28}$$

Since $u > 0$ on $(0, t_{b,2})$ it follows from (2.20) that $w > 0$ on $(0, t_{b,2})$. It also follows that $w_1 > 0$ since $I_{\frac{1}{2-q}} > 0$ by (2.24). Now dividing (2.28) by $ww_1$, using (2.26), and recalling $t_{b,2} \to \infty$ as $b \to 0^+$ from Lemma 2.1 gives

$$\frac{w'}{w} \le \frac{w_1'}{w_1} \le a_2 \quad \text{for } t > t_{b,2} \text{ if } b > 0 \text{ is sufficiently small.} \tag{2.29}$$

Next we return to (2.20) and differentiate to obtain

$$u' = \frac{t^{-1}u}{2} + \sqrt{a_1 h_2}\, t^{-\frac{q}{2}} u \frac{w'}{w}. \tag{2.30}$$

Using (2.29)-(2.30), Lemma 2.1, and assuming $b > 0$ is sufficiently small we obtain

$$t^{1-c}u' = \frac{t^{-c}u}{2} + \sqrt{a_1 h_2}\, t^{1-c-\frac{q}{2}} u \frac{w'}{w} \le \frac{t^{-c}u}{2} + a_2\sqrt{a_1 h_2}\, t^{1-c-\frac{q}{2}} u \tag{2.31}$$

for $t > t_{b,2}$. Since $1-c-\frac{q}{2} = -\frac{(2-q)(1+p)}{2(1-p)} < 0$ and $\frac{1}{2} \le c < 1$, evaluating (2.31) at $t_{b,2}$ along with Lemma 2.1 gives

$$0 < t_{b,2}^{1-c} u'(t_{b,2}) \le \frac{\beta t_{b,2}^{-c}}{4} + \frac{\beta a_2}{2} \sqrt{a_1 h_2}\, t_{b,2}^{1-c-\frac{q}{2}} \to 0 \quad \text{as } b \to 0^+.$$

This completes the proof. □

**Lemma 2.3.** *Assume* (H1)-(H5) *and let $u$ solve* (2.1), (2.4). *Then*

$$\lim_{b\to 0^+} t_{b,\gamma}^{1-c} u'(t_{b,\gamma}) = 0.$$

*Proof.* Rewriting (2.1) we see that

$$(t^{1-c}u')' = (1-c)t^{-c}u' - t^{1-c}h(t)f(u). \tag{2.32}$$

Integrating this on $(t_{b,2}, t_{b,\gamma})$ gives

$$t_{b,\gamma}^{1-c}u'(t_{b,\gamma}) = t_{b,2}^{1-c}u'(t_{b,2}) + \int_{t_{b,2}}^{t_{b,\gamma}} (1-c)s^{-c}u'\,ds - \int_{t_{b,2}}^{t_{b,\gamma}} s^{1-c}h(s)f(u)\,ds. \tag{2.33}$$

The first term on the right-hand side of (2.33) goes to 0 as $b \to 0^+$ by Lemma 2.2. Integrating the first integral in (2.33) by parts gives

$$\int_{t_{b,2}}^{t_{b,\gamma}} (1-c)s^{-c}u'\,ds = (1-c)t_{b,\gamma}^{-c}\gamma - (1-c)t_{b,2}^{-c}\frac{\beta}{2} + \int_{t_{b,2}}^{t_{b,\gamma}} c(1-c)s^{-c-1}u\,ds. \tag{2.34}$$

The first two terms on the right-hand side of this go to 0 as $b \to 0^+$ by Lemma 2.1. Since $\frac{\beta}{2} < u < \gamma$ on $(t_{b,2}, t_{b,\gamma})$ estimating the third term on the right-hand side of (2.34) and using Lemma 2.1 we obtain

$$\Big|\int_{t_{b,2}}^{t_{b,\gamma}} c(1-c)s^{-c-1}u\,ds\Big| \le (1-c)\gamma(t_{b,2}^{-c} - t_{b,\gamma}^{-c}) \to 0 \quad \text{as } b \to 0^+. \tag{2.35}$$

Finally, on $[\beta/2, \gamma]$ we have $|f| \le a_3$ for some constant $a_3 > 0$ and thus by (2.2),

$$\Big|\int_{t_{b,2}}^{t_{b,\gamma}} s^{1-c}h(s)f(u)\,ds\Big| \le \frac{a_3 h_2 (t_{b,\gamma}^{2-q-c} - t_{b,2}^{2-q-c})}{2-q-c} \to 0 \quad \text{as } b \to 0^+$$

since $2-q-c = -\frac{p(2-q)}{1-p} < 0$. Thus it follows from Lemma 2.2 and (2.34)-(2.35) that the right-hand side of (2.33) goes to 0 as $b \to 0^+$ and so the lemma follows. This completes the proof. □

**Lemma 2.4.** *Assume* (H1)–(H5) *and let $u$ solve* (2.1), (2.4). *Let $u = t^c v$. Then* $\lim_{b\to 0^+} t_{b,\gamma} v'(t_{b,\gamma}) = 0$.

*Proof.* We make the change of variables $u = t^c v$ and after differentiating we obtain

$$t^{1-c}u' = cv + tv'. \tag{2.36}$$

Evaluating (2.36) at $t_{b,\gamma}$ we see by Lemma 2.3 that the left-hand side of (2.36) goes to 0 as $b \to 0^+$. Also by Lemma 2.1 since $t_{b,\gamma} > t_{b,2} \to \infty$ as $b \to 0^+$ and since $v(t_{b,\gamma}) = \frac{u(t_{b,\gamma})}{t_{b,\gamma}^c} = \frac{\gamma}{t_{b,\gamma}^c} \to 0$ as $b \to 0^+$ we see that the first term on the right-hand side of (2.36) goes to 0 as $b \to 0^+$ when evaluated at $t_{b,\gamma}$ and thus it follows that

$$t_{b,\gamma} v'(t_{b,\gamma}) \to 0 \text{ as } b \to 0^+.$$

This completes the proof. □

Next substituting $u = t^c v$ into (2.1) and using (H3) gives

$$t^2 v'' + 2ctv' + c(c-1)v + t^q h(t) v^p + t^{2-c} h(t) g(t^c v) = 0 \quad \text{for } t > t_{b,\gamma}. \tag{2.37}$$

Now for $t > t_{b,\gamma}$ we have $u > \gamma > 0$ (and hence $v > 0$) and then by (H3) we have $g(t^c v) = g(u) \ge 0$. Then by (2.2) we have

$$t^2 v'' + 2ctv' + d(v) < 0 \quad \text{for } t > t_{b,\gamma} \tag{2.38}$$

where $d(v) = c(c-1)v + h_1 |v|^{p-1} v$. Next we let $v_1$ be the solution of

$$t^2 v_1'' + 2ctv_1' + d(v_1) = 0, \tag{2.39}$$

$$v_1(t_{b,\gamma}) = v(t_{b,\gamma}), \quad v_1'(t_{b,\gamma}) = v'(t_{b,\gamma}). \tag{2.40}$$

Now let

$$E_1 = \frac{1}{2} t^2 v_1'^2 + D(v_1) \tag{2.41}$$

where $D(v_1) \equiv \int_0^{v_1} d_1(s)\, ds = \frac{c(c-1)}{2} v_1^2 + \frac{h_1}{p+1} |v_1|^{p+1}$. It follows from (2.39) that

$$E_1' = (1-2c) t v_1'^2 \le 0 \quad \text{since } \frac{1}{2} \le c < 1. \tag{2.42}$$

Note $d(0) = 0$ and that $d(v_1)$ is increasing near $v_1 = 0$ since $0 < p < 1$. Also notice $d(v_1)$ has a unique positive zero at $\alpha_1 > 0$ and $d(v_1) < 0$ for $v > \alpha_1$. Similarly $D(0) = 0$ and $D(v_1)$ is increasing near $v_1 = 0$. In addition, $D$ has a local maximum at $\alpha_1$ and a unique positive zero at $\alpha_2 > \alpha_1 > 0$. Also $D(v_1) < 0$ for $v_1 > \alpha_2$.

Next it follows from Lemma 2.4 and since $v_1(t_{b,\gamma}) = v(t_{b,\gamma}) = \frac{\gamma}{t_{b,\gamma}^c} \to 0$ as $b \to 0^+$ (by Lemma 2.1) that

$$E_1(t_{b,\gamma}) = \frac{1}{2} t_{b,\gamma}^2 v_1'^2(t_{b,\gamma}) + D(v_1(t_{b,\gamma})) = \frac{1}{2} t_{b,\gamma}^2 v'^2(t_{b,\gamma}) + D(v(t_{b,\gamma})) \to 0 \tag{2.43}$$

as $b \to 0^+$, and since $E_1$ is non-increasing for $\frac{1}{2} \le c < 1$ (by (2.42)) it follows that

$$\frac{1}{2} t^2 v_1'^2 + D(v_1) = E_1(t) \le E_1(t_{b,\gamma}) < \epsilon \tag{2.44}$$

for $t > t_{b,\gamma}$ and $b$ sufficiently small.

**Lemma 2.5.** *Assume* (H1)–(H5) *and let $u$ solve* (2.1), (2.4). *If $b$ is sufficiently small then $v_1$ has a local maximum, $M_{1,b}$, and $v_1' > 0$ on $(t_{b,\gamma}, M_{1,b})$.*

*Proof.* We suppose by way of contradiction that $v_1' > 0$ for all $t > t_{b,\gamma}$. Then $v_1(t) > v_1(t_{b,\gamma}) = \frac{\gamma}{t_{b,\gamma}^c} > 0$ and by (2.44) if $b$ is sufficiently small then $D(v_1) \le E_1 < \epsilon$ if $t > t_{b,\gamma}$ and so it follows that $0 < v_1 < \frac{1}{2}\alpha_1$ if $b$ is sufficiently small. In particular, $v_1$ is bounded and since $v_1$ is increasing then $v_1 \to L$ as $t \to \infty$ where $0 < L < \alpha_1$. Since $D(v_1) \to D(L)$ as $t \to \infty$ and since $E_1' \le 0$ it follows that $E_1$ has a finite limit as $t \to \infty$. Thus from (2.41) we have $t^2 v_1'^2 \to A \ge 0$ for some $A \ge 0$. If $A > 0$ then $v_1' > \frac{A}{2t}$ for large $t$ implying $v_1(t) > v_1(t_0) + A \ln(t/t_0) \to \infty$ as $t \to \infty$ contradicting that $v_1$ is bounded. It follows therefore that $A = 0$ and thus $tv_1' \to 0$ as $t \to \infty$. Using this and taking limits in (2.39) we see that

$$\lim_{t \to \infty} t^2 v_1'' = -d(L).$$

If $d(L) \ne 0$ then using a similar argument as we just showed with $v_1$ would imply that $v_1'$ is unbounded contradicting $tv_1' \to 0$ so it must be that $d(L) = 0$ but this is impossible since $0 < L < \alpha_1$. Therefore from this contradiction we see that $v_1$ has a first local maximum, $M_{1,b}$, and $v_1' > 0$ on $(t_{b,\gamma}, M_{1,b})$ if $b > 0$ is sufficiently small. This completes the proof. $\square$

**Lemma 2.6.** *Assume* (H1)–(H5) *and let $u$ solve* (2.1), (2.4). *Then $v$ has a local maximum on $(t_{b,\gamma}, M_{1,b})$ if $b$ is sufficiently small.*

*Proof.* We assume $v' > 0$ on $(t_{b,\gamma}, M_{1,b})$ otherwise we are done. Since $v' > 0$ it follows that $v > v(t_{b,\gamma}) = \gamma > 0$ on $(t_{b,\gamma}, M_{1,b})$. Multiplying (2.38) by $v_1$, (2.39) by $v$, and subtracting gives

$$t^2(v_1v' - vv_1')' + 2ct(v_1v' - vv_1') + h_1(v^pv_1 - v_1^pv) \le 0.$$

Multiplying this by $t^{2c-2}$ gives

$$\big(t^{2c}(v_1v' - vv_1')\big)' + h_1t^{2c-2}vv_1(v^{p-1} - v_1^{p-1}) \le 0 \quad \text{for } t > t_{b,\gamma}. \tag{2.45}$$

Notice it follows from (2.38)–(2.40) that

$$t_{b,\gamma}^2(v''(t_{b,\gamma}) - v_1''(t_{b,\gamma})) < 0$$

and so $v < v_1$ on $(t_{b,\gamma}, t_{b,\gamma}+\epsilon_0)$ for some $\epsilon_0 > 0$. We next show that $v$ and $v_1$ do not intersect on $(t_{b,\gamma}, M_{1,b})$. Suppose they did and so there is a $t_0$ with $t_{b,\gamma} < t_0 < M_{1,b}$ such that $v < v_1$ on $(t_{b,\gamma}, t_0)$ and $v(t_0) = v_1(t_0)$. Integrating (2.45) on $(t_{b,\gamma}, t_0)$ and using (2.40) gives

$$t_0^{2c}\left(v_1(t_0)v'(t_0) - v(t_0)v_1'(t_0)\right) + h_1\int_{t_{b,\gamma}}^{t_0} t^{2c-2}vv_1(v^{p-1} - v_1^{p-1})\,dt \le 0. \tag{2.46}$$

Since $v < v_1$ on $(t_{b,\gamma}, t_0)$ and $0 < p < 1$ it follows that $v^{p-1} > v_1^{p-1}$ on $(t_{b,\gamma}, t_0)$ and so the integral in (2.46) is positive. So from (2.46) we see $(v_1(t_0)v'(t_0) - v(t_0)v_1'(t_0)) = v(t_0)(v'(t_0) - v_1'(t_0)) < 0$ and since $v(t_0) > 0$ we see that

$$v'(t_0) < v_1'(t_0). \tag{2.47}$$

On the other hand, since $v < v_1$ on $(t_{b,\gamma}, t_0)$ and $v(t_0) = v_1(t_0)$ then $v(t) - v(t_0) < v_1(t) - v_1(t_0)$ and so $\frac{v(t)-v(t_0)}{t-t_0} > \frac{v_1(t)-v_1(t_0)}{t-t_0}$ for $t_{b,\gamma} < t < t_0$. Therefore taking limits as $t \to t_0^-$ gives

$$v'(t_0) \ge v_1'(t_0) \tag{2.48}$$

which contradicts (2.47). Thus we see that $v < v_1$ on $(t_{b,\gamma}, M_{1,b})$. Now we integrate (2.45) on $(t_{b,\gamma}, M_{1,b})$ and obtain

$$M_{1,b}^{2c}v_1(M_{1,b})v'(M_{1,b}) + h_1\int_{t_{b,\gamma}}^{M_{1,b}} t^{2c-2}vv_1(v^{p-1} - v_1^{p-1})\,dt \le 0. \tag{2.49}$$

Then as earlier it follows that $v^{p-1} > v_1^{p-1}$ on $(t_{b,\gamma}, M_{1,b})$ thus the integral term in (2.49) is positive. But also $v_1(M_{1,b})v'(M_{1,b}) \ge 0$ since $v' > 0$ on $(t_{b,\gamma}, M_{1,b})$ and so we get a contradiction to (2.49). Thus $v$ must have a local maximum, $M_{v,b}$, on $(t_{b,\gamma}, M_{1,b})$ if $b > 0$ is sufficiently small. This completes the proof. $\square$

**Lemma 2.7.** *Assume* (H1)–(H5) *and let $u$ solve* (2.1)-(2.4). *Then $u$ has a local max if $b$ is sufficiently small.*

*Proof.* Since $u = t^cv$, it follows that $u' = t^{c-1}(tv' + cv)$ and so in order to show $u' < 0$ somewhere we want to show that $tv' + cv$ gets negative for some $t > t_{b,\gamma}$. So by the way of contradiction let us suppose $u' > 0$ for $t > t_{b,\gamma}$. Thus it follows that $u > 0$ and hence $v > 0$ for $t > t_{b,\gamma}$. We next show that $v < v_1$ for $t > t_{b,\gamma}$. Integrating (2.45) on $(t_{b,\gamma}, t)$ gives

$$t^{2c}(v_1 v' - v v_1') + h_1 \int_{t_{b,\gamma}}^{t} t^{2c-2} v v_1 (v^{p-1} - v_1^{p-1})\, dt \le 0 \quad \text{for } t > t_{b,\gamma}. \tag{2.50}$$

Now using an identical argument as in Lemma 2.6 it follows that $v$ and $v_1$ do not intersect for $t > t_{b,\gamma}$ and thus $0 < v < v_1$ for $t > t_{b,\gamma}$. It follows then from (2.50) that $v_1 v' - v v_1' < 0$ for $t > t_{b,\gamma}$. And since $v_1 > v > 0$ we then have $\frac{v'}{v} < \frac{v_1'}{v_1}$. From this it follows that $tv' + cv < t\frac{v_1'}{v_1} v + cv = \frac{v}{v_1}(tv_1' + cv_1)$ for $t > t_{b,\gamma}$. Thus to show $u'$ gets negative it suffices to show $tv_1' + cv_1$ gets negative for some $t > t_{b,\gamma}$.

Now recall from Lemma 2.5 that $v_1$ has a local maximum at $M_{1,b}$. Thus $v_1'(M_{1,b}) = 0$ and $v_1''(M_{1,b}) \le 0$. In fact by the uniqueness of solutions of initial value problems $v_1''(M_{1,b}) < 0$ and so $d(v_1(M_{1,b})) > 0$. Thus

$$0 < v_1(M_{1,b}) < \alpha_1. \tag{2.51}$$

Now $v_1$ cannot have a positive local minimum at $m_{1,b} > M_{1,b}$ with $v_1' < 0$ on $(M_{1,b}, m_{1,b})$ for at such a point we would have $v_1'(m_{1,b}) = 0$ and $v_1''(m_{1,b}) \ge 0$ implying $d(v_1(m_{1,b})) \le 0$ which would imply $v_1(m_{1,b}) \le 0$ contradicting $0 < v < v_1$. Thus

$$v_1' < 0 \text{ for } t > M_{1,b}. \tag{2.52}$$

Next we observe that if $v_1'' < 0$ for $t > M_{1,b}$ then $v_1$ has a zero for $t > M_{1,b}$ which implies $v$ has a zero since $v < v_1$ but this contradicts that $v > 0$. Thus $v_1$ has an inflection point $t_2 > M_{1,b}$ with $v_1'' < 0$ on $(M_{1,b}, t_2)$. In addition, $v_1$ cannot have another inflection point $t_3 > t_2$ with $v'' > 0$ on $(t_2, t_3)$ for at such a point $t_3$ then $v_1'$ would have a local maximum so $v_1''(t_3) = 0$ and $v_1'''(t_3) \le 0$. Using this and then differentiating (2.39) this implies $(c^2 + c + ph_1 v_1^{p-1}(t_3))v_1'(t_3) \ge 0$ and so $v_1'(t_3) \ge 0$ but this contradicts (2.52). Thus $v_1'' > 0$ for $t > t_2$.

Now substituting that $v_1'' > 0$ for $t > t_2$ into (2.39) gives:

$$2ctv_1' + d(v_1) < 0 \text{ for } t > t_2$$

and so rewriting

$$tv_1' + cv_1 < \frac{1+c}{2} v_1 - \frac{h_1 v_1^p}{2c} = v_1^p \Big( \frac{1+c}{2} v_1^{1-p} - \frac{h_1}{2c} \Big) \quad \text{for } t > t_2. \tag{2.53}$$

Now for $b > 0$ sufficiently small we know $D(v_1) < \epsilon$ for $t > t_{b,\gamma}$ by (2.44). In particular, $D(v_1(M_{1,b})) < \epsilon$ and since we also know (2.51) this implies that for $b > 0$ sufficiently small then $v_1$ is also arbitrarily small. In particular, for sufficiently small $b > 0$ the right-hand side of (2.53) is negative which implies $tv_1' + cv_1$ and hence $u'$ gets negative. Therefore $u$ has a local maximum. This completes the proof. $\square$

**Lemma 2.8.** *Assume* (H1)–(H5) *and let $u$ solve* (2.1), (2.4). *Then $u$ has a zero if $b$ sufficiently small.*

*Proof.* From Lemma 2.8 we know that $u$ has a local maximum, $M_{u,b}$, with $M_{u,b} > t_b$, if $b > 0$ is sufficiently small and so from (2.1) we see that $u'' < 0$ while $f(u) > 0$. It follows then that $u$ must become less than or equal to $\beta$ and so there exists $s_b > M_{u,b}$ such that $u(s_b) = \beta$. Returning to (2.8) we see then that $\frac{1}{2}\frac{u'^2}{h} + F(u) = E \ge E_0 > 0$ for $t > s_b$ and since $F(u) < 0$ for $0 < u < \beta$ it follows from (2.2) that $-u' \ge \sqrt{2E_0} h_1 t^{-\frac{q}{2}}$ for $t > s_b$. Integrating this on $(s_b, t)$ it then follows that

$u(t) \le u(s_b) - \sqrt{2E_0h_1}\big(\frac{t^{1-\frac{q}{2}} - s_b^{1-\frac{q}{2}}}{1-\frac{q}{2}}\big) \to -\infty$ as $t \to \infty$ since $1 < q < 2$ and so $u$ has a zero. This completes the proof. □

## 3. Behavior for large $b > 0$

We now examine (2.1), (2.4) for large $b > 0$. We want to show that either $u_b > 0$ on $(0, \infty)$ or if there exists $z_{u,b} > 0$ such that $u_b(z_{u,b}) = 0$ with $u > 0$ on $(0, z_{u,b})$ then $z_{u,b} \to \infty$ as $b \to \infty$. So let us suppose that $z_{u,b}$ is finite and suppose there exists $S$ such that $0 < z_{u,b} \le S$ for all large $b$.

Integrating (2.1) twice on $(0, t)$ gives

$$u_b = bt - \int_0^t \int_0^s h(x) f(u_b)\, dx\, ds.$$

Letting $\psi_b = \frac{u_b}{b}$ gives

$$\psi_b = t - \int_0^t \int_0^s \frac{h(x) f(b\psi_b)}{b}\, dx\, ds \tag{3.1}$$

and then letting $\psi_b = t y_b$ gives

$$y_b = 1 - \frac{1}{t} \int_0^t \int_0^s \frac{h(x) f(bx y_b)}{b}\, dx\, ds.$$

It follows from (H1) and (H3) that $|f(u)| \le C_1 |u|$ for some $C_1 > 0$. Thus

$$|y_b| \le 1 + C_1 \int_0^t x h(x) |y_b(x)|\, dx. \tag{3.2}$$

We now denote the right-hand side of (3.2) as $\phi_b$. Then we see $\phi_b' = C_1 t h(t) |y_b|$ and so (3.2) reads $\frac{\phi_b'}{C_1 t h} = |y_b| \le \phi_b$. Rewriting and integrating this on $(0, t)$ gives

$$\frac{|\psi_b|}{t} = |y_b| \le \phi_b \le \exp\Big(\frac{C_1 h_2}{2-q} t^{2-q}\Big).$$

Thus

$$|\psi_b| \le t \exp\Big(\frac{C_1 h_2}{2-q} t^{2-q}\Big) \le C_2 \quad \text{on } [0, S] \text{ for some constant } C_2 > 0. \tag{3.3}$$

Also from (3.1) we have

$$\psi_b' = 1 - \int_0^t \frac{h(x) f(b\psi_b)}{b}\, dx. \tag{3.4}$$

It follows then from (3.3)-(3.4), since $|f(u)| \le C_1 |u|$, and since $0 < t h(t) < h_2 t^{1-q}$ is integrable near $t = 0$ (because $1 < q < 2$) that there is a constant $C_3 > 0$ such that

$$|\psi_b'| \le 1 + C_1 \int_0^t h(x) |\psi_b|\, dx \le 1 + C_1 \int_0^t x h(x) \exp\Big(\frac{C_1 h_2}{2-q} x^{2-q}\Big)\, dx \le C_3 \tag{3.5}$$

on $[0, S]$. Thus from (3.3) and (3.5) we have $|\psi_b|$ and $|\psi_b'|$ are uniformly bounded on $[0, S]$ and so the $\{\psi_b\}$ are equicontinuous on $[0, S]$. In addition, differentiating (3.4) and using (3.3) we see there is some constant $C_4 > 0$ such that

$$|\psi_b''| \le \Big|\frac{h(t) f(b\psi_b)}{b}\Big| \le C_1 h(t) |\psi_b| \le C_1 t h(t) \exp\Big(\frac{C_1 h_2}{2-q} t^{2-q}\Big) \le C_4 t^{1-q}$$

on $[0,S]$. It then follows that

$$|\psi_b'(t_1)-\psi_b'(t_2)| \le \int_{t_1}^{t_2} |\psi_b''| \le \frac{C_4}{2-q}|t_2^{2-q}-t_1^{2-q}| \quad \text{on } [0,S]. \tag{3.6}$$

Thus since $q<2$ we see from (3.6) that $\{\psi_b'\}$ are also equicontinuous on $[0,S]$. It then follows by the Arzela-Ascoli theorem that there is a subsequence (still denoted $\psi_b$) such that

$$\psi_b \to \psi \text{ and } \psi_b' \to \psi' \text{ uniformly on } [0,S] \text{ as } b\to\infty. \tag{3.7}$$

Now since $u_b$ (and hence $\psi_b$) has a zero $z_{u,b}$ it follows that $u_b$ has a local maximum, $M_{u,b}$, with $0<M_{u,b}<z_{u,b}\le S$. Then since $E$ is nondecreasing (by (2.7)) we have

$$\frac{1}{2}\frac{u_b'^2}{h}+F(u_b)\le F(u_b(M_{u,b})) \quad \text{for } 0\le t\le M_{u,b}.$$

Rewriting and integrating this on $[0,M_{u,b}]$ using (2.2) gives

$$\begin{aligned}
\int_0^{u_b(M_{u,b})} \frac{dt}{\sqrt{2}\sqrt{F(u_b(M_{u,b}))-F(t)}}\,dt &= \int_0^{M_{u,b}} \frac{|u_b'|}{\sqrt{2}\sqrt{F(u_b(M_{u,b}))-F(u_b)}}\,dt\\
&\le \int_0^{M_{u,b}} \sqrt{h}\,dt\\
&\le \frac{\sqrt{h_2}M_{u,b}^{1-\frac{q}{2}}}{1-\frac{q}{2}}.
\end{aligned}$$

Since $F(t)\ge -F_0$ where $F_0>0$ it follows that the above inequality and (H3) imply

$$\frac{u_b(M_{u,b})}{\sqrt{2}\sqrt{C_5[u_b(M_{u,b})]^{p+1}+F_0}} \le \frac{\sqrt{h_2}M_{u,b}^{1-\frac{q}{2}}}{1-\frac{q}{2}} \le \frac{\sqrt{h_2}S^{1-\frac{q}{2}}}{1-\frac{q}{2}}$$

for some $C_5>0$. It follows from this and since $0<p<1$ that $u_b(M_{u,b})$ must be bounded and thus $u_b$ is bounded. Then since $\psi_b=\frac{u}{b}$ it follows that $\psi_b\to 0$ on $[0,S]$. Thus (3.7) implies $\psi\equiv 0$. In addition, (3.7) also implies $\psi_b'\to\psi'\equiv 0$ but from (3.4) it follows that $0=\psi'(0)\leftarrow\psi_b'(0)=1\ne 0$ yielding a contradiction. Thus the assumption that the $z_{u,b}$ are bounded must be false. Therefore $z_{u,b}\to\infty$ as $b\to\infty$.

## 4. Proof of Theorem 1.1

We now consider the set

$$S_0=\{b>0 : u_b>0 \text{ on } (0,R^{2-N})\}.$$

From section three it follows that if $b>0$ is sufficiently large then $u_b>0$ on $(0,R^{2-N})$ and so $S_0$ is nonempty. From section two we know that if $b>0$ is sufficiently small then $u_b$ will have a zero on $(0,\infty)$ and so then if $R>0$ is sufficiently small then since $N>2$ it follows that $R^{2-N}$ is sufficiently large and thus $u_b$ will have a zero on $(0,R^{2-N})$. Thus $S_0$ is bounded from below by a positive quantity. Then we let

$$b_0=\inf\{b>0 : u_b \text{ solves (2.1), (2.4) and } u_b>0 \text{ on } (0,R^{2-N})\}.$$

Then $b_0 > 0$ and a straightforward argument as in [13] shows $u_{b_0} > 0$ on $(0, R^{2-N})$ and $u_{b_0}(R^{2-N}) = 0$. We then define

$$S_1 = \{b > 0 | u_b \text{ solves (2.1), (2.4) and has exactly one zero on } (0, R^{2-N})\}.$$

Choosing $b$ slightly smaller than $b_0$ it follows then by continuity with respect to initial conditions that $u_b$ will have at least one zero on $(0, R^{2-N})$. And as in [13] it follows that if $b$ is sufficiently close to $b_0$ then $u_b$ has at most one zero on $(0, R^{2-N})$ and so $S_1$ is nonempty. Then letting

$$b_1 = \inf S_1$$

a similar argument shows $u_{b_1}$ has one zero on $(0, R^{2-N})$ and $u_{b_1}(R^{2-N}) = 0$. In a similar fashion we can show that given any $n$ then if $R > 0$ is sufficiently small then there exists $b_0, b_1, \cdots, b_n$ such that $u_{b_k}(t)$ has $k$ zeros on $(0, R^{2-N})$ and $u_{b_k}(R^{2-N}) = 0$. Finally, letting $U_{b_k}(t) = u_{b_k}(t^{2-N})$ it follows that $U_{b_k}(t)$ satisfies (1.1)-(1.3) and $U_{b_k}$ has $k$ zeros on $(R, \infty)$. This completes the proof of the main theorem.

## 5. Appendix

**Lemma 5.1.** *Assume* (H1)–(H5). *Then for any $b > 0$ there is a solution $u \in C^1[0, \epsilon) \cap C^2(0, \epsilon)$ of* (2.1) *on $(0, \epsilon)$ for some $\epsilon > 0$ and such that* (2.4) *holds.*

*Proof.* Integrating (2.1) twice on $(0, t)$ and using (2.4) gives

$$u = bt - \int_0^t \int_0^s h(x) f(u)\, dx\, ds.$$

Substituting $u = tv$ gives

$$v = b - \frac{1}{t} \int_0^t \int_0^s h(x) f(xv)\, dx\, ds. \tag{5.1}$$

Defining the right-hand side of (5.1) as $Tv$ then we see that $T : C[0, \epsilon] \to C[0, \epsilon]$ where $C[0, \epsilon]$ is the set of continuous functions on $[0, \epsilon]$ such that $v(0) = b$ with the supremum norm, $\| \cdot \|$. Then

$$Tv_1 - Tv_2 = -\frac{1}{t} \int_0^t \int_0^s h(x)[f(xv_1) - f(xv_2)]\, dx\, ds$$

and so

$$|Tv_1 - Tv_2| \le \frac{L_1}{t} \int_0^t \int_0^s x h(x) |v_1 - v_2|\, dx\, ds$$

where $L_1$ is the Lipschitz constant for $f$ near $v = b$. Then

$$|Tv_1 - Tv_2| \le \frac{L_1 h_2 \epsilon^{2-q}}{(2-q)(3-q)} \|v_1 - v_2\|$$

and so $T$ is a contraction if $\epsilon > 0$ is sufficiently small. It then follows from the contraction mapping principle that $v(t)$ is a solution of (5.1) and therefore $u(t) = tv(t)$ satisfies (2.1), (2.4) on $[0, \epsilon]$. This completes the proof. $\square$

**Lemma 5.2.** *Let $I$ be the solution of*

$$I'' + \frac{1}{t} I' - \Big(1 + \frac{1}{(2-q)^2 t^2}\Big) I = 0 \quad \textit{and} \quad \lim_{t \to 0^+} \frac{I(t)}{t^{\frac{1}{2-q}}} = a > 0. \tag{5.2}$$

*Then there exists a constant $C_6 > 0$ such that*

$$|\frac{I'}{I}| \leq C_6 \quad \text{for } t \geq 1\,.$$

*Proof.* A straightforward computation using (5.2) shows that

$$\Big(t^2 I'^2 - t^2 I - \frac{1}{(2-q)^2} I\Big)' = -tI^2 \leq 0\,.$$

Thus since $I(0) = 0$, it follows that

$$t^2 I'^2 - t^2 I^2 - \frac{1}{(2-q)^2} I^2 \leq 0 \quad \text{for } t > 0.$$

Therefore

$$\frac{I'^2}{I^2} \leq 1 + \frac{1}{(2-q)^2 t^2}\,.$$

So we see that there exists a $C_6 > 0$ such that

$$\Big|\frac{I'}{I}\Big| \leq C_6 \text{ for } t > 1\,.$$

This completes the proof. ∎

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