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ELLIPTIC SECTORS AND EULER DISCRETIZATION

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Abstract. In this work we are interested in the elliptic sector of autonomous differential systems with a degenerate equilibrium point at the origin, and in their Euler discretization. When the linear part of the vector field at the origin has two zero eigenvalues, then the differential system has an elliptic sector, under some conditions. We describe this elliptic sector and we show that the associated Euler discretized system has an elliptic sector converging to that of the continuous system when the step size of the discretization tends to zero.

1. Introduction

In this work we consider the planar differential system

$$\begin{aligned}\dot{x} &= P(x,y)\\ \dot{y} &= Q(x,y)\end{aligned} \tag{1.1}$$

where P and Q are analytic functions from $\mathbb{R}^2$ to $\mathbb{R}$. Also we consider their Euler discretization

$$\begin{aligned}x_{n+1} &= x_n + hP(x_n,y_n)\\ y_{n+1} &= y_n + hQ(x_n,y_n)\end{aligned} \tag{1.2}$$

where $h > 0$ is the step size of the discretization. We assume that the origin is an isolated equilibrium point of (1.1) and that system (1.1) has an elliptic sector S_0.

The main aim of this work is to explore to what extent the discrete system (1.2) presents also an elliptic sector S_h and whether S_h tends to S_0 in the sense of the Hausdorff distance, when h tends to zero.

In order to define an elliptic sector of (1.1), we consider a circle C with center $(0,0)$ and radius r, containing no other equilibria than the origin. We will assume that there exist two solutions γ_1 and γ_2 of system (1.1) tending to the origin; we assume for example that γ_1 tends to the origin when t tends to $+\infty$ and γ_2 tends to the same point when t tends to $-\infty$. We denote by γ_1^* and γ_2^* the respective corresponding orbits to the solutions γ_1^* and γ_2^*. Let M_1 and M_2 be the respective intersection points of γ_1^* and γ_2^* with the circle C such that, taking into account the direction of the orbits, M_1 is the last intersection point of γ_1^* with the circle C and M_2 is the first intersection point of γ_2^* with C. Let σ be the closed curve

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made up of the two segments OM_1 and OM_2 parts of the two orbits γ_1^* and γ_2^*, the origin and the oriented arc of C joining M_1 to M_2 in the forward direction. The region R_σ delimited by σ is said to be a *sector*. An *elliptic sector* is a sector containing only nested homoclinic or parts of nested homoclinic orbits (Figure 1) [1]. We recall that a solution is said to be homoclinic if it is defined on $\mathbb{R}$ entirely and tends to the origin when t tends to $+\infty$ as well as when t tends to $-\infty$.

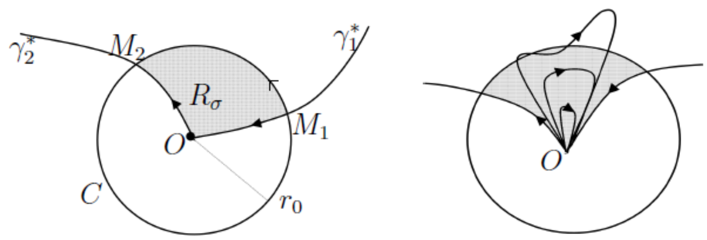

FIGURE 1. A sector and an elliptic sector

To pose the problem raised above correctly, we must define the notion of homoclinic orbits of system (1.2), even when the map

$$F_h:(x,y) \mapsto (x + hP(x,y), y + hQ(x,y))$$

is not invertible. This leads us to define the S-invertibility notion. The map F_h is said to be S-invertible if, for any m_0 in $\mathbb{R}^2$, there exist two reals $R > 0$ and $h_0 > 0$ such that for any h in $]0, h_0]$, there exists a unique $m_{-1,h}$ in $\mathbb{R}^2$ for which $\|m_{-1,h}\| \le R$ and $F_h(m_{-1,h}) = m_0$. In this case, we have necessarily

$$\lim_{h\to 0} m_{-1,h} = m_0.$$

In other words, F_h is S-invertible if any point of the plane has a unique "good predecessor" by the application F_h. The notation S comes from nonstandard analysis (NSA): in the context of NSA, a planar map is said to be S-invertible if any limited point of the plane has a unique limited predecessor; for the standard functions, this notion coincides with the usual invertibility.

Let U be an open set of the disc $D(0;r_0)$ and simply connected. We say that U is an elliptic sector of the discrete system (1.2) if any solution emanating from U is homoclinic.

The works of Beyn [2]. Fiedler and Scheurle [24] and that of Zou [40] deal with the problem of the persistence of the non-degenerate homoclinic orbits of an autonomous system after the discretization of the latter. They give an error estimate of order $O(h^d)$, for the difference between the homoclinic solution of the differential equation and that of the associated discrete equation, where h is the step size of the method of discretization and d its order. They also give the length $l(h)$ of the parameter interval over which the homoclinic orbit persists. On the other hand, given an autonomous differential system which has an equilibrium point at $(0,0)$ not necessarily hyperbolic and an unstable center manifold W_c^u, it is shown in [3] that, in a small neighborhood of the origin, the discrete system associated by a one step method has, under some conditions, an invariant manifold close to W_c^u. In the hyperbolic case, it is shown in [4] that, under some conditions, the phase portrait of the differential system is correctly reproduced in the associated discretization by a one step method, on an arbitrary time interval. In [19], the author examines the conditions which make the solution of the differential system close to that of the associated discretized one on an infinite time interval. A similar study has been made in [29] for the structurally stable systems without periodic solutions. On the other hand, it has been shown in [21] that in the hyperbolic case, the local stable manifold of the discrete system tends to that of the corresponding differential system when the step size tends to zero (see also [23, 28]). An extension of this result to nonautonomous differential systems is given in [27]. A Taylor expansion approximation of this manifold is given in [20]. In [22] it is shown that, in the hyperbolic case, the maps defining the vector fields and the associated discretized system by the one step method are uniformly topologically equivalent. Other results about numerical calculus of homoclinic, heteroclinic and periodic orbits are established in [5]-[11], [15]-[26], [39] and [40].

It is known that, when system (1.1) has an elliptic sector, then the associated Jacobian matrix M of the function $(x, y) \mapsto (P(x, y), Q(x, y))$ at point $(0, 0)$, has two zero eigenvalues (cf. [33, p. 241]); the origin is a non-hyperbolic equilibrium point. Two situations are possible. The first one is that where the matrix M is null. In this case, the behavior of the solutions near the origin is very complex; If the smallest degree of the nonlinear terms of (1.1) is m, then the neighborhood of the origin will be split into $2(m + 1)$ parabolic, hyperbolic or elliptic sectors. The number of elliptic sectors existing in system (1.1) depends on the index of the equilibrium point (cf. [36], p 151): let C be a Jordan curve containing $(0, 0)$ and no other critical point of (1.1) in its interior; then the index of the equilibrium point $(0, 0)$ with respect to (1.1) is given by

$$I_{(1.1)}(0,0) = I_{(1.1)}(C) = \frac{1}{2\pi}\oint_C \frac{PdQ - QdP}{P^2 + Q^2}.$$

The second situation is that where the matrix M can be reduced by a linear transformation to $M' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; it is in this situation that we are interested in this work. In this case, system (1.1) is reduced by a sequence of analytic changes of variables to the form (see [1])

$$\begin{gathered} \dot{x} = y \\ \dot{y} = ax^r\big(1 + k(x)\big) + bx^p y\big(1 + g(x)\big) + y^2 f(x, y) \end{gathered} \tag{1.3}$$

where f, g and k are analytic functions, such that $k(0) = g(0) = f(0, 0) = 0$, a and b are real parameters, r and p are integer parameters, satisfying $r = 2m + 1$, $m \geq 1$, $a < 0$, $b \neq 0$, $p \geq 1$, p odd and either

- $p = m$ and $\Delta = b^2 + 4(m + 1)a \geq 0$, or
- $p < m$.

Without loss of generality, by a linear change of variable, we assume in all the following that b is strictly positive. In all this work, we place ourselves in the situation $p = m = 1$ and $\Delta = b^2 + 8a > 0$. These assumptions are relatively natural. Indeed, the assumption $p = m = 1$ is generic in some sense. The assumption $\Delta \geq 0$ is necessary to get an elliptic sector, and the assumption $\Delta \neq 0$ is generic. The affine transformation $(x, y) \mapsto (x, \sqrt{-a}y)$ together with the time change $\tau = \sqrt{-a}t$ and the parameter change $b \mapsto b/\sqrt{-a}$, allow us to assume that $a = -1$. Thus we will be interested in the elliptic sector of system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^3(1+k(x)) + bxy\big(1+g(x)\big) + y^2 f(x,y) \end{aligned} \tag{1.4}$$

and in its persistence in the associated Euler discretized system

$$\begin{aligned} x_{n+1} &= x_n + hy_n \\ y_{n+1} &= y_n + h\Big(-x_n^3\big(1+k(x_n)\big) + bx_ny_n\big(1+g(x_n)\big) + y_n^2 f(x_n,y_n)\Big) \end{aligned} \tag{1.5}$$

The main result of this work is the following.

Theorem 1.1. *There exists $h_0 > 0$ such that for any h in $]0, h_0]$, there exists a subregion S_h of the elliptic sector S_0 of system* (1.4) *with the following properties:*

- *Any solution of system* (1.5) *starting from S_h is homoclinic.*
- *When h tends to zero, S_h tends to S_0 in the sense of the Hausdorff distance.*

We do not try to obtain a maximal region of homoclinic orbits of (1.5), whose structure may be highly complicated. In particular we do not study the behavior of orbits of (1.5) starting close to the boundary of S_0. A more precise study of these orbits remains to be done. The speed of convergence of S_h to S_0 as h tends to 0 is another interesting problem which is not discussed here.

The article is organized as follows. In section two, we give a local description of the elliptic sector of system (1.4) whose existence is stated above. To do so, we will use the results obtained during the study of the global behavior of the solutions of the model example

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^3 + bxy \end{aligned} \tag{1.6}$$

In section three, we give a complete proof of theorem 1.1. In the last section we deal with the elliptic sector of the family of differential systems which are diffeomorphic to system (1.4) and with its persistence in the corresponding discretized systems.

2. Description of the continuous system

We are first interested in the global behavior of the solutions of system (1.6) (Figure 2). The following proposition describes the phase portrait of (1.6). The symmetry with respect to the y-axis allows to consider only the solutions of (1.6) starting at points with positive abscissas. Let us denote $\Delta = b^2 - 8$ and $\alpha_1 = (b - \sqrt{\Delta})/4$ and $\alpha_2 = (b + \sqrt{\Delta})/4$ the solutions of the equation $-2\alpha^2 + b\alpha - 1 = 0$.

Proposition 2.1. *Let (x_0, y_0) be a point of the plane such that $x_0 > 0$ and let $\gamma(t, x_0, y_0)$ be the solution of system* (1.6) *emanating from this point.*

1. *If $y_0 < \alpha_1 x_0^2$, then the solution $\gamma(t, x_0, y_0)$ is homoclinic.*
2. *If $\alpha_1 x_0^2 < y_0 < \alpha_2 x_0^2$, then*

$$\begin{aligned} \lim_{t\to-\infty} x(t) &= \lim_{t\to-\infty} y(t) = 0, \\ \lim_{t\to+\infty} x(t) &= \lim_{t\to+\infty} y(t) = +\infty \end{aligned}$$

Moreover,

$$\lim_{t\to-\infty} \frac{y(t)}{x(t)^2} = \alpha_1 \quad \text{and} \quad \lim_{t\to+\infty} \frac{y(t)}{x(t)^2} = \alpha_2.$$

(3) *If $\alpha_1 x_0^2 < y_0$, then the trajectory γ crosses the y-axis at a point whose y-coordinate is positive. Also*
$$\lim_{t\to+\infty} x(t) = \lim_{t\to\pm\infty} y(t) = -\lim_{t\to-\infty} x(t) = +\infty.$$

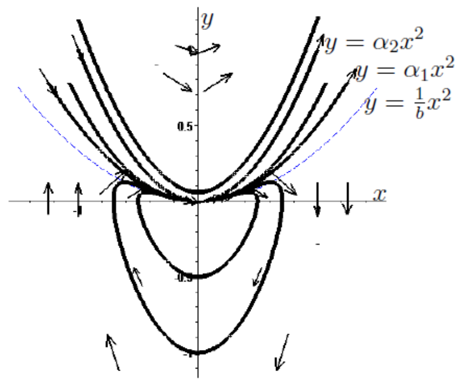

FIGURE 2. The phase portrait of system (1.6).

Hereafter, we give only a sketch of the proof since it is similar to proof of the proposition 2.2. Note however that it uses the traditional tools of trap trajectories, taking into account the fact that the curves $y = \alpha_1 x^2$ and $y = \alpha_2 x^2$ are solutions of system (1.6), the blow-up theory [35] and other tools of qualitative study of differential equations. Indeed, the application for system (1.6) of the two quasi-homogeneous blow-ups of degree $(1, 2)$
$$x = u, \quad y = u^2 v \quad \text{with } d\tau = u dt, \tag{2.1}$$
and
$$x = wz, \quad y = z^2 \quad \text{with } d\eta = z dt \tag{2.2}$$
produce respectively the systems
$$\begin{aligned} u' &= uv \\ v' &= -1 + bv - 2v^2 \end{aligned} \tag{2.3}$$
and
$$\begin{aligned} w' &= 1 + \frac{1}{2}w^4 - \frac{b}{2}w^2 \\ z' &= -\frac{1}{2}w^3 z + \frac{b}{2}wz \end{aligned} \tag{2.4}$$
The phase portraits of systems (2.3) and (2.4) are illustrated in Figure 3.

The interpretation of these phase portraits allows us to prove the stated result.

Now, we prove the following proposition giving a local description of the elliptic sector of system (1.4).

Proposition 2.2. *System* (1.4) *has an elliptic sector. Moreover, the behavior of the solutions of* (1.4) *near the origin is similar to that of system* (1.6)*. More precisely, in a neighborhood of the origin, solutions of* (1.4) *starting below the parabola $y = \alpha_1 x^2$ are homoclinic and solutions starting above the parabola $y = \alpha_2 x^2$ are not homoclinic.*

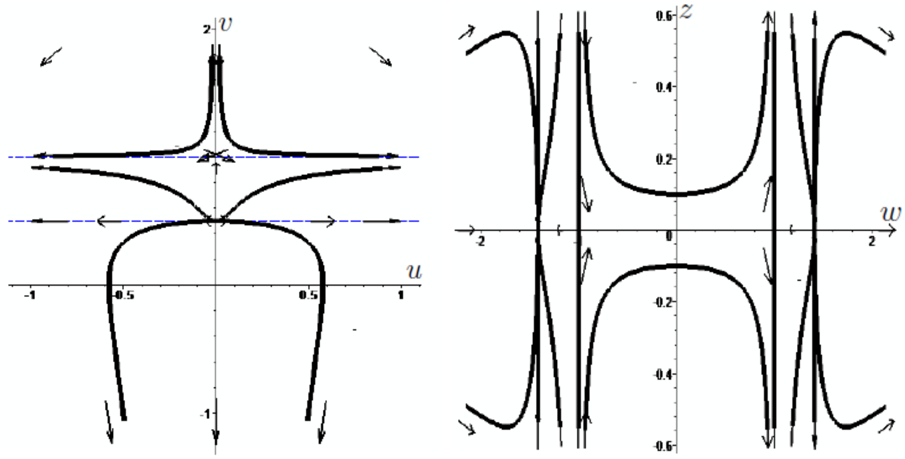

FIGURE 3. Phase portraits of systems (2.3) and (2.4) for $b = 3$.

Proof. The two previous blow-ups applied to (1.4) give respectively the two systems

$$\begin{aligned} u' &= uv \\ v' &= -1 + bv - 2v^2 - k(u) + bvg(u) + uv^2 f(u, u^2 v) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} w' &= 1 + \frac{w^4}{2}(1 + k(wz)) - \frac{bw^2}{2}(1 + g(wz)) - \frac{wz}{2} f(wz, z^2) \\ z' &= -\frac{w^3 z}{2}(1 + k(wz)) + \frac{bwz}{2}(1 + g(wz)) + \frac{z^2}{2} f(wz, z^2) \end{aligned} \tag{2.6}$$

The blow-up $x = XY$, $y = -Y^2$ together with $d\xi = Y dt$ applied to system (1.4) give

$$\begin{aligned} X' &= -1 - \frac{X^4}{2}(1 + k(XY)) - \frac{bX^2}{2}(1 + g(XY)) + \frac{XY}{2} f(XY, -Y^2) \\ Y' &= \frac{X^3 Y}{2}(1 + k(XY)) + \frac{bXY}{2}(1 + g(XY)) - \frac{Y^2}{2} f(XY, -Y^2) \end{aligned} \tag{2.7}$$

System (2.5) has an unstable node on $A_1 = (0, \alpha_1)$ and a saddle on $A_2 = (0, \alpha_2)$ (Fig. 4). The v-axis is invariant under system (2.5). The continuity of the solutions with respect to the initial conditions gives the existence of $\delta > 0$ such that for any u_0 in the interval $]-\delta, \delta]$, the solution of (2.5) starting at $(u_0, 0)$ tends to A_2 when τ tends to $-\infty$ and enters in the lower half-plane after some time $\tau_0 > 0$ (depending on u_0).

System (2.6) has as equilibria (Figure 4) two saddles on $B_1 = (-\sqrt{2\alpha_1}, 0)$ and on $B_2 = (\sqrt{2\alpha_1}, 0)$ and two nodes, stable on $B_3 = (-\sqrt{2\alpha_2}, 0)$ and unstable on $B_4 = (\sqrt{2\alpha_2}, 0)$. The solutions in the neighborhoods of B_1, B_2, B_3 and B_4 having the tangents $w = \sqrt{2\alpha_1}$, $w = -\sqrt{2\alpha_1}$, $w = \sqrt{2\alpha_2}$ or $w = -\sqrt{2\alpha_2}$, correspond in the (x, y) plane to the solutions of system (1.4) whose order two approximations near $(0, 0)$ are given by $y = \alpha_1 x^2$ and $y = \alpha_2 x^2$.

The w-axis is invariant under system (2.6). By continuity of the solutions of (2.6) with respect to the initial conditions, those that lie near B_1 and B_2 sufficiently close to the lines $w = -\sqrt{2\alpha_2}$ and $w = \sqrt{2\alpha_2}$, with the initial conditions (w_0, z_0)

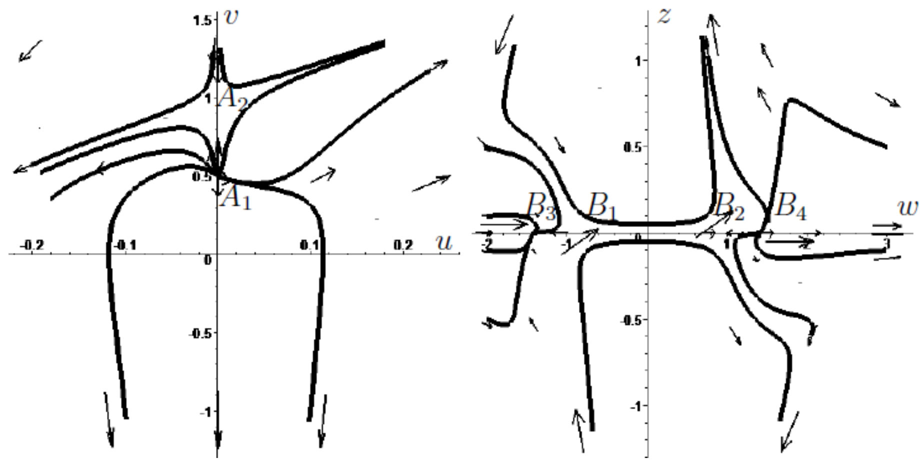

FIGURE 4. the phase portraits of system (2.5) near A_1 and A_2 and of system (2.6) near B_1, B_2, B_3 and B_4 for $f(x,y) = g(x) = k(x) = x$ and $b = 3$.

such that $-\sqrt{2\alpha_2} < w_0 < \sqrt{2\alpha_2}$, move close to the w-axis, then cross the z-axis on points whose y-coordinates are close to zero.

In the plane (x,y), this means that the corresponding solutions lie in the neighborhood of the origin close to the curves $y = \alpha_1 x^2$ and $y = \alpha_2 x^2$, then cross the y-axis near the origin.

System (2.7) has no equilibrium point and the X-axis is invariant under this system. When $Y = 0$, system (2.7) becomes

$$\begin{aligned} X' &= -1 - \frac{1}{2}X^4 - \frac{b}{2}X^2 \\ Y' &= 0 \end{aligned} \tag{2.8}$$

By continuity of solutions of (2.7) with respect to the initial conditions, there exist some solutions emanating from points close enough to the X-axis, that cross the Y-axis on points whose y-coordinates are sufficiently close to zero.

We deduce that, in the lower half-plane, there is no solution of system (1.4) which tends to $(0,0)$. A simultaneous interpretation of results of the three blow-ups applied to system (1.4) allows us to conclude. □

3. Homoclinic orbits of the discrete system (1.5)

In this part we give the proof of theorem 1.1. This proof is based on the properties of solutions of system (1.6).

For convenience, we have chosen to use the formalism of Nonstandard Analysis (NSA for short) [37] in the IST version due to E. Nelson [34]; In particular, we use the tools of stroboscopy [38] and of permanence principle [30]. In a classical framework, several statements can play the role of the permanence principle of NSA, e.g. the Kaplun's extension theorem [31] (see also [32, page 27]) or [18, Theorem 2.2.2].

From now on, the functions f, g, k of system (1.6) are assumed to be standard, and $h > 0$ is fixed infinitesimal. By the Transfer Principle of NSA, it is sufficient to prove that, for all limited $(x_0, y_0) \in S_0$ in the S-interior of S_0, the solution $(x_n, y_n)_{n\in\mathbb{Z}}$ of (1.5) starting from (x_0, y_0) is homoclinic. We recall that (x_0, y_0) is in the S-interior of S_0 if there exists a standard $r > 0$ such that the disk of center (x_0, y_0) and of radius r is in S_0. For $n < 0$, the points (x_n, y_n) are given recursively by Lemma 3.2.

The proof of Theorem 1.1 essentially uses the idea that for $\alpha_2 \lessapprox a \lessapprox \alpha_1$, the region $\{(x, y) \in \mathbb{R}^2; y < ax^2\}$ is positively invariant under system (1.6). By $a \lessapprox b$ we mean that a is less than b and not infinitly close to b.

To prove Theorem 1.1, we need a second lemma which is inspired by the idea above. The proof of this lemma requires to highlight an intermediate discrete system. Put $\ell(x, y) = -x^3(1 + k(x)) + bxy(1 + g(x)) + y^2 f(x, y)$ and $F_h(x, y) = (x + hy, y + h\ell(x))$.

Also there exists a function ψ defined near $x = 0$, satisfying $\ell(x, \psi(x)) = 0$ and

$$\psi(x) = \frac{1}{b}x^2 + o(x^2).$$

Then, the component $\dot{y}$ of the vector field (1.4) becomes zero on the curve $y = \psi(x)$.

Now, let (x_0, y_0) be a limited point in the elliptic sector of system (1.4), not infinitely close to its border. Several cases are possible, but only one is presented here: $x_0 > 0$ and $0 < y_0 < \psi(x_0)$. The other cases are similar.

The change of variables $x = \varepsilon X$, $y = \varepsilon^2 Y$ and the time change $\tau = \varepsilon t$ where $\varepsilon \simeq 0$, transform system (1.4) into

$$\begin{gathered} X' = Y \\ Y' = -X^3(1 + k(\varepsilon X)) + bXY(1 + g(\varepsilon X)) + \varepsilon Y^2 f(\varepsilon X, \varepsilon^2 Y) \end{gathered} \tag{3.1}$$

where $(') = d/d\tau$. By using the short shadow lemma [12], it appears that the solutions of system (3.1) starting at limited points (X_0, Y_0) have for shadows in a limited time interval, solutions of system (1.6). Let $(\bar{x}(t), \bar{y}(t))$ denote the solution of system (1.4) starting at point (x_0, y_0), and let $(x_n, y_n)_{n\in\mathbb{N}}$ be the solution of system (1.5) started from the same point.

For nh bounded, by using the stroboscopy lemma [38],

$$(x_n, y_n) \simeq (\bar{x}(nh), \bar{y}(nh)) \tag{3.2}$$

By permanence [12], there exists N in $\mathbb{N}$ such that Nh is infinitely large and for any $n < N$, approximation (3.2) remains true. Then

$$(\bar{x}(Nh), \bar{y}(Nh)) \simeq (0, 0)$$

Let $\varepsilon = -\bar{x}(Nh)$ and put $x_n = \varepsilon X_n$, $y_n = \varepsilon^2 Y_n$ and $\bar{h} = h\varepsilon$. System (1.5) becomes

$$\begin{aligned} X_{n+1} &= X_n + \bar{h}Y_n \\ Y_{n+1} &= Y_n + \bar{h}\Big(- X_n^3(1 + k(\varepsilon X_n)) + bX_nY_n(1 + g(\varepsilon X_n)) \\ &\quad + \varepsilon Y_n^2 f(\varepsilon X_n, \varepsilon^2 Y_n)\Big) \end{aligned} \tag{3.3}$$

The following lemmas are used in the proof of theorem 1.1.

Lemma 3.1. *Let $\alpha_2 \lessapprox a \lessapprox \alpha_1$. Any solution $(X_n, Y_n)_{n\in\mathbb{N}}$ of system (3.3) starting at a limited point (X_0, Y_0) in the region $\{(x, y) \in \mathbb{R}^2/x < 0,\ y < ax^2\}$ remains in it.*

Proof. We use induction on n:

$$X_{n+1} = YX_n + h_n \leq X_n + ha(X_n)^2 = X_n(1 + haX_n)$$

As X_n is assumed negative, then $X_{n+1} < 0$. Otherwise,

$$\begin{aligned} Y_{n+1} - aX_{n+1}^2 &= (Y_n - aX_n^2)(1 + \bar{h}(b - 2a + bg(\varepsilon X_n))X_n) \\ &\quad + \bar{h}(-2a^2 + ba - 1 + bag(\varepsilon X_n) - k(\varepsilon X_n))X_n^3 \\ &\quad + \bar{h}\varepsilon Y_n^2 f(\varepsilon X_n, \varepsilon^2 Y_n) - a\bar{h}^2 Y_n^2 \end{aligned}$$

On the other hand, $-2a^2 + ba - 1 \gtrsim 0$. Since g and k are standard, continuous and satisfy $g(0) = k(0) = 0$, we deduce that

$$-2a^2 + ba - 1 + bag(\varepsilon X_n) - k(\varepsilon X_n) \gtrsim 0.$$

Hence

$$\Big(-2a^2 + ba - 1 + bag(\varepsilon X_n) - k(\varepsilon X_n)\Big)X_n^3 + \varepsilon Y_n^2 f(\varepsilon X_n, \varepsilon^2 Y_n) < 0 \tag{3.4}$$

Thus $Y_{n+1} - aX_{n+1}^2 < 0$, when $Y_n - aX_n^2 < 0$. □

Lemma 3.2. *Any limited point in the plane has a unique limited predecessor by system* (1.5).

Proof. With the previous notation ℓ and F_h, we must show that for any limited point (u, v) of the plane, there exists a unique limited point (x, y) such that

$$F_h(x, y) = (u, v). \tag{3.5}$$

We mention that such a point, if it exists, is necessarily infinitely close to (x, y). Moreover, x is given by $x = u - hy$. Then, it suffices to show that there exists a unique $y \in]v - 1, v + 1[$ satisfying

$$v = y + hl(u - hy, v). \tag{3.6}$$

For a fixed (u, v), we denote

$$L(y) = v - y - hl(u - hy, v)$$

Since $l(u - h(v + 1), v)$ is limited,

$$L(v + 1) = -1 - hl(u - h(v + 1), v) < 0$$

In the same way,

$$L(v - 1) = 1 - hl(u - h(v - 1), v) > 0$$

Also, for any limited y,

$$L'(y) = -1 - h(-h\frac{\partial l}{\partial x}(u - hy, v) + \frac{\partial l}{\partial y}(u - hy, v)) < 0$$

Since L is continuous, the equation $L(x) = 0$ has a unique limited solution y in $]v - 1, v + 1[$. □

We fix a limited point (x_0, y_0). Let $(x_{-n}, y_{-n})_{n\in\mathbb{N}^*}$ be the predecessors sequence of system (1.5) defined by lemma 3.2. Then, this sequence is uniquely defined, as long as (x_{-n}, y_{-n}) is limited, by

$$\begin{gathered} x_{-n} = x_{-n-1} + hy_{-n-1} \\ y_{-n} = y_{-n-1} + h\ell(x_{-n-1}, y_{-n-1}) \end{gathered}$$

satisfying $(x_{-1}, y_{-1}) \simeq (x_0, y_0)$ and $(x_{-n-1}, y_{-n-1}) \simeq (x_{-n}, y_{-n})$.

Lemma 3.3. *Let $a \in \mathbb{R}$ such that $\alpha_2 \lessapprox a \lessapprox \alpha_1$ and (X_0, Y_0) a limited point in the plane such that $0 < Y_0 < aX_0^2$ and $X_0 > 0$. The solution $(X_{-n}, Y_{-n})_{n\in\mathbb{N}^*}$ of system* (3.3) *starting at (X_0, Y_0), does not leave the region of the plane*

$$\{(X, Y) \in \mathbb{R}^2 : X > 0,\ 0 < Y < aX^2\}$$

Proof. We use induction. We will show that $Y_{-1} < aX_{-1}^2$; The same reasoning allows us to show that $Y_{-n} < aX_{-n}^2$ for any $n > 1$. We denote $\delta = -2a^2 + ba - 1 + bag(\varepsilon X_{-1}) - k(\varepsilon X_{-1})$ and $\beta = \varepsilon f(\varepsilon X_{-1}, \varepsilon^2 Y_{-1}) - a\bar{h}$. We have

$$Y_0 - aX_0^2 = (Y_{-1} - aX_{-1}^2)\Big(1 + \bar{h}\big(b - 2a + bg(\varepsilon X_{-1})\big)X_{-1}\Big) + \bar{h}\delta X_{-1}^3 + \bar{h}\beta$$

We will distinguish two cases: If $\beta \geq 0$, since $\delta > 0$ we have

$$0 \geq Y_0 - aX_0^2 \geq (Y_{-1} - aX_{-1}^2)(1 + \bar{h}(b - 2a + bg(\varepsilon X_{-1}))X_{-1}).$$

This means that $Y_{-1} - aX_{-1}^2 < 0$.

On the other hand,

$$\delta X_{-1}^3 + \beta Y_{-1}^2 = \beta Y_{-1}(Y_{-1} - aX_{-1}^2) + a\beta X_{-1}^2\Big(Y_{-1} + \frac{\delta}{a\beta}X_{-1}\Big).$$

Since $\delta \not\simeq 0$, $\delta/(a\beta)$ is infinitely large, then if $\beta < 0$,

$$\frac{\delta}{a\beta}X_{-1} + aX_{-1}^2 = \Big(\frac{\delta}{a\beta} + aX_{-1}\Big)X_{-1} < 0.$$

Thus,

$$\delta X_{-1}^3 + \beta Y_{-1}^2 \geq \beta(Y_{-1} - aX_{-1}^2)(Y_{-1} + aX_{-1}^2),$$

hence

$$\begin{aligned} &Y_0 - aX_0^2 \\ &\geq (Y_{-1} - aX_{-1}^2)\Big(1 + \bar{h}(b - 2a + bg(\varepsilon X_{-1}))X_{-1} + \beta(Y_{-1} + aX_{-1}^2)\Big) \end{aligned}$$

It results in this case that $Y_{-1} < aX_{-1}^2$. □

Proof of theorem 1.1. By applying the stroboscopy lemma, the solution $(x_n, y_n)_{n\in\mathbb{N}}$ reaches a point of the region

$$\{(x, y) \in \mathbb{R}^2 : x < 0,\ 0 < y < \psi(x)\}$$

and as long as $y_n < \alpha_2 x_n^2$ then $x_{n+1} < 0$. Furthermore, this solution reaches a point (x_q, y_q) infinitely close to the origin, in the region

$$\{(x, y) \in \mathbb{R}^2 : x < 0, \psi(x) < y < ax^2\}.$$

On the other hand, by lemma 3.1,

$$\forall n > q, x_n < 0, \psi(x_n) < y_n < ax_n^2$$

and the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ is convergent; its limit is the stationary point $(0, 0)$ of (1.5).

Likewise, by using lemmas 3.2 and 3.3, it results that the sequence $(x_{-n}, y_{-n})_{n\in\mathbb{N}}$ converges to $(0, 0)$ when n tends to $+\infty$. □

4. Discretization of the family of planar differential systems diffeomorphic to (1.4)

In this section we consider a system of the form (1.1) which is diffeomorphic to system (1.4) in the following sense: there exists a diffeomorphism $\varphi:\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\varphi = (\varphi_1, \varphi_2) : (x, y) \mapsto (u, v) = (\varphi_1(x, y), \varphi_2(x, y))$$

such that a function from $\mathbb{R}$ to $\mathbb{R}^2$, $t \mapsto (x(t), y(t))$ is a solution of (1.1) if and only if the function $t \mapsto (u(t), v(t)) = \varphi(x(t), y(t))$ is a solution of (1.4) (with the variables denoted u and v instead of x and y).

It will also be possible to make a time change, but the question which interests us concerns the elliptic sectors and these do not depend on the parametrization of the integral curves. We then assume immediately that the two systems are diffeomorphic without time change. We mention that it is not the same to discretize a differential system before or after applying diffeomorphisms.

System (1.1) can be written as

$$\begin{aligned} \dot{x} &= ax + by + P_1(x, y) \\ \dot{y} &= cx + dy + Q_1(x, y) \end{aligned} \tag{4.1}$$

where a, b, c and d are reals and P_1 and Q_1 are analytic functions whose Taylor expansion near the origin begins with terms of total degree on x and y greater than or equal to two.

We know that when system (4.1) has an elliptic sector, then the two eigenvalues of the associated matrix of its linear part are zero [33, 36]. This means that

$$\begin{gathered} a = -d, \\ a^2 + bc = 0 \\ |a| + |b| + |c| \neq 0 \end{gathered}$$

The most generic case is when $a \neq 0$, $b \neq 0$ and $c \neq 0$. In this case, the change of variables

$$\chi:(u, v) \mapsto (x, y) = \chi(u, v)$$

defined by

$$\begin{gathered} x = v \\ y = cu - av + \bar{P}(v, cu - av) \end{gathered}$$

transforms system (4.1) into system (1.4). The Jacobian of χ at point $(0, 0)$ is $(-c) \neq 0$. This means that in some neighborhood of $(0, 0)$, the map χ is bijective. It suffices to put $\varphi = \chi^{-1}$. We write

$$\begin{gathered} \varphi_1(x, y) = \bar{f}(x, y) \\ \varphi_2(x, y) = x \end{gathered}$$

where $\bar{f}$ is an analytic function such that $\bar{f}(0, 0) = 0$. It follows that the diffeomorphism φ transforms system (1.2) into

$$\begin{aligned} u_{n+1} &= u_n + hv_n \\ v_{n+1} &= v_n + hl(u_n, v_n) + h^2 G(x_n, y_n), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} G(u_n, v_n) = \frac{1}{d_n}\Big(&\frac{(u_{n+1}-u_n)^2}{2}\frac{\partial^2\varphi_1}{\partial u^2}(u_n, v_n) + \frac{(v_{n+1}-v_n)^2}{2}\frac{\partial^2\varphi_1}{\partial v^2}(u_n, v_n) \\ &+ (u_{n+1}-u_n)(v_{n+1}-v_n)\frac{\partial^2\varphi_1}{\partial u\partial v}\Big)\frac{\partial\varphi_2}{\partial u}(u_n, v_n) \\ &+ o\Big(\|(u_{n+1}, v_{n+1}) - (u_n, v_n)\|^3\Big) \end{aligned}$$

with

$$d_n = \frac{\partial\varphi_1}{\partial u}(u_n, v_n)\frac{\partial\varphi_2}{\partial v}(u_n, v_n) - \frac{\partial\varphi_2}{\partial u}(u_n, v_n)\frac{\partial\varphi_1}{\partial v}(u_n, v_n)$$

We can check that G can be written for any $(x, y) \in \mathbb{R}^2$ as

$$G(x, y) = xyG_1(x, y) + y^2G_2(x, y)$$

where G_1 and G_2 are analytic functions in $\mathbb{R}^2$.

In the same way as system (1.2), system (1.4) has an elliptic sector. Hence, system (1.1) has also an elliptic sector. We will show by passing through system (4.2) obtained by diffeomorphism from (1.4) and by adapting the same proof as for theorem 1.1, that system (1.2) has an elliptic sector tending to that of system (1.1) when h tends to zero.

We remark that the study of elliptic sectors in system (1.4) and in its discretized one (1.5) is similarly generalized to system (1.3) and to its discretized one obtained by Euler in the case where $p = m$ and $\Delta > 0$. The case $\Delta = 0$ is treated separately.

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