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INFINITE SEMIPOSITONE PROBLEMS WITH A FALLING ZERO AND NONLINEAR BOUNDARY CONDITIONS

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Abstract. We consider the problem

$$-u'' = h(t)\big(\frac{au-u^2-c}{u^\alpha}\big), \quad t \in (0,1),$$
$$u(0)=0, \quad u'(1)+g(u(1))=0,$$

where $a>0$, $c\geq 0$, $\alpha\in(0,1)$, h:$(0,1]\to(0,\infty)$ is a continuous function which may be singular at $t=0$, but belongs to $L^1(0,1)\cap C^1(0,1)$, and g:$[0,\infty)\to[0,\infty)$ is a continuous function. We discuss existence, uniqueness, and non existence results for positive solutions for certain values of a, b and c.

1. Introduction

In this article, we consider the boundary-value problem

$$\begin{gathered}-u'' = h(t)\big(\frac{au-u^2-c}{u^\alpha}\big), \quad t \in (0,1),\\ u(0)=0, u'(1)+g(u(1))=0,\end{gathered} \tag{1.1}$$

where $a>0$, $c\geq 0$, $\alpha\in(0,1)$, and g:$[0,\infty)\to[0,\infty)$ is a continuous function. The function h:$(0,1]\to(0,\infty)$ is a continuous function which satisfies:

- (H1) there exists $\epsilon_1>0, 0<\gamma<1-\alpha$, such that $h(s)\leq 1/s^\gamma$ for all $s\in(0,\epsilon_1)$,
- (H2) $\inf_{s\in(0,1)} h(s)=\hat{h}>0$.

Note that, for the nonlinear function $f(s)=(as-s^2-c)/s^\alpha$, $\lim_{s\to 0^+} f(s)=-\infty$. This singularity together with the fact that the solution needs to satisfy a Dirichlet boundary condition creates a challenge in establishing the existence of positive solutions. Such problems are referred in the literature as "infinite semipositone" problems. See [9, 11, 13, 17, 18], where infinite semipositone problems have been studied when the nonlinearity f only has a single zero beyond which it is positive and increasing to infinity. The analysis is more challenging when the reaction term f has a second zero (falling zero) beyond which it is negative. See [4, 14] where this study was achieved in the case when Dirichlet boundary conditions persisted on the entire boundary. In this paper, we extend this study to an even more challenging

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situation, namely when a nonlinear boundary condition is involved on part of the boundary.

Problems of the form (1.1) arise while studying radial solutions of

$$\begin{gathered}
-\Delta u = K(|x|)\big(\frac{au-u^2-c}{u^\alpha}\big), \quad x \in \Omega,\\
\frac{\partial u}{\partial \eta} + g(u) = 0, \quad \text{if } |x| = r_0,\\
u \to 0, \quad \text{as } |x| \to \infty,
\end{gathered} \tag{1.2}$$

where $\Omega = \{x \in \mathbb{R}^n : |x| > r_0\}$ is an exterior domain, $n > 2$, a, c, α are as before, and $K : [r_0, \infty) \to (0, \infty)$ belongs to a class of continuous functions such that $\lim_{r\to\infty} K(r) = 0$. By using the transformation: $r = |x|$ and $s = (\frac{r}{r_0})^{(2-n)}$, we can reduce (1.2) to (1.1), where $h(s) = \frac{r_0^2}{(2-n)^2} s^{\frac{-2(n-1)}{n-2}} K(r_0 s^{\frac{1}{2-n}})$ (see [2]). Note that if we assume $K \in C([r_0, \infty), (0, \infty))$ and satisfies $\frac{d_1}{r^{n+\sigma}} \le K(r) \le \frac{d_2}{r^{n+\sigma}}$ for some $d_1, d_2 > 0$, and for $\sigma \in ((n-2)\alpha, n-2)$, then h satisfies our assumptions (H1) and (H2).

When the boundary condition at $|x| = r_0$ is replaced by a Dirichlet's condition, i.e. $u = 0$, the same transformation reduces the problem to

$$\begin{gathered}
-u'' = h(t)\big(\frac{au-u^2-c}{u^\alpha}\big), \quad t \in (0,1),\\
u(0) = 0, \quad u(1) = 0.
\end{gathered} \tag{1.3}$$

The existence of positive solutions of this Dirichlet problem was studied in [4]. For given values of $a > 0, \alpha \in (0, 1)$, the authors established the existence of positive solution for small values of c. In this paper, we extend this study to the case when a nonlinear boundary condition is satisfied at $|x| = r_0$.

In particular, we will show that (1.1) has a positive solution with $u(1) > 0$, which clearly shows that it is not a solution of (1.3). Hence combining our result with the existence result obtained in [4], we also see that the problem

$$\begin{gathered}
-\Delta u = K(|x|)(\frac{au-u^2-c}{u^\alpha}), \quad x \in \Omega,\\
u\big[\frac{\partial u}{\partial \eta} + g(u)\big] = 0, \quad \text{if } |x| = r_0,\\
u \to 0, \quad \text{as } |x| \to \infty,
\end{gathered}$$

has at least two positive radial solutions for certain values of a and c. Existence of positive solutions to certain problems with such boundary conditions are discussed in [5, 8].

The study of such steady state reaction diffusion equations are of great importance in various applications. See in particular [16] for a problem arising in ecology. See also [1, 3, 5, 8]. Here we consider more challenging reaction diffusion models, namely, when nonlinear diffusion is involved (when the diffusion term is $u^\alpha \Delta u$ instead of Δu).

Below, we state our results for (1.1). We first establish a non existence result for (1.1). For this we assume

(H3) $h \in C^1\big((0, 1], (0, \infty)\big)$, and $h'(s) < 0$ for $s > 0$.

Note that if the weight function K in (1.2) is such that K is C^1 and $\frac{K(r^{-1})}{r^{2(n-1)}}$ is decreasing for $r > 0$, then the corresponding h satisfies (H3). A simple example of K which satisfies our assumptions is $K(r) = \frac{d_1}{r^{n+\sigma}}$, where $d_1 > 0$, and $\sigma \in ((n-2)\alpha, n-2)$.

Theorem 1.1. *Assume h satisfies* (H1), (H3), *and g:$[0,\infty) \to [0,\infty)$, is a continuous function. Then for given $a > 0$ and $\alpha \in (0,1)$, there exists $\hat{c}(a) = \frac{(3-\alpha)(1-\alpha)}{(2-\alpha)^2}\frac{a^2}{4}$ such that if $c > \hat{c}$,* (1.1) *has no nonnegative solution.*

Remark 1.2. Note that if $c > a^2/4$, then $f(s) = \frac{as-s^2-c}{s^\alpha} < 0$ for all $s > 0$ and this will immediately imply the non existence of nonnegative solution of (1.1). This follows from the fact that, since $u(0) = 0$ and $u'(1) \le 0$, there exists a $\tilde{t} \in (0,1)$ such that $u''(\tilde{t}) \le 0$.

Remark 1.3. From the proof of Theorem(1.1), we also see that, for a given $c > 0$ and $\alpha \in (0,1)$, there exists $\hat{a}(c)$ such that if $a < \hat{a}$, (1.1) has no nonnegative solution.

Next, we state an existence result for (1.1) for the case when $c = 0$.

Theorem 1.4. *Let $\alpha \in (0,1)$, $c = 0$, and g:$[0,\infty) \to [0,\infty)$ is a continuous function. Assume h:$(0,1] \to (0,\infty)$ is a continuous function which satisfies* (H1) *and* (H2)*. Then, there exists $\underline{a} > 0$ such that if $a \ge \underline{a}$,* (1.1) *has a positive solution u with $u(1) > 0$.*

Remark 1.5. If $\hat{g} = \inf_{s\in[0,\infty)} g(s) > 0$, then integrating (1.1) from 0 to 1 with $c = 0$, it is easy to see that for $a \le [\frac{(2-\alpha)^{2-\alpha}}{(1-\alpha)^{1-\alpha}}\frac{\hat{g}}{\|h\|_1}]^{\frac{1}{2-\alpha}}$, (1.1) has no positive solution.

Under an additional assumption on g, we also establish the uniqueness of the positive solution obtained in Theorem 1.4 for (1.1) when $c = 0$. For this we assume

(H4) $g(x)/x$ is nondecreasing for $x \in [0,\infty)$.

Then we have the following uniqueness result.

Theorem 1.6. *Let $a > 0$, $c = 0$, $\alpha \in (0,1)$, and h:$(0,1] \to (0,\infty)$ be a continuous function which satisfies* (H2)*. Assume also that g:$[0,\infty) \to [0,\infty)$ is a continuous function which satisfies* (H4)*. Then* (1.1) *has at most one positive solution.*

Finally, we state our main existence result in this paper for (1.1).

Theorem 1.7. *Let $\alpha \in (0,1)$ and g:$[0,\infty) \to [0,\infty)$ is a continuous function. Assume h:$(0,1] \to (0,\infty)$ is a continuous function which satisfies* (H1) *and* (H2)*. Then, there exists $\bar{a} > 0$, and for $a \ge \bar{a}$, $\bar{c}(a) > 0$ such that for $c \le \bar{c}$,* (1.1) *has a positive solution u with $u(1) > 0$. Further, this $\bar{c}$ is an increasing function of a such that $\bar{c}(a) \to \infty$ as $a \to \infty$.*

Remark 1.8. From the proof of Theorem(1.7), it is easy to see that, for any given $c \le \bar{c}(\bar{a})$, there exists $a_*(c)$ such that for $a \ge a_*$, (1.1) has a positive solution.

Figure 1 illustrates Theorem 1.7 and Remark 1.8. Here $\rho = \|u\|_\infty$.

In the next section we recall a method of sub and super solutions established in [12], which will be used to establish our existence results. We also provide some preliminary results about the existence of a positive eigenfunction for certain eigenvalue problems, which will be useful in the construction of our subsolution required in the proof of Theorem 1.7. The proofs of the theorems are provided in the later sections. In the last section, we provide some exact bifurcation diagrams of positive solutions of (1.1) when $h(t) \equiv 1$.

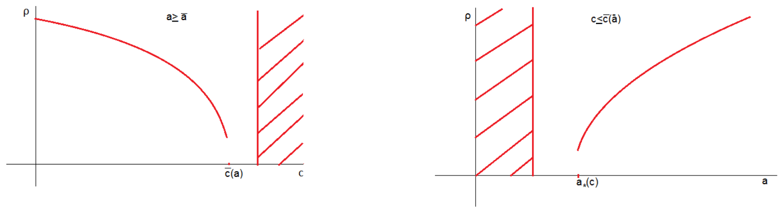

Figure 1. Bifurcation diagram of (1.1): left a versus ρ, right c versus ρ

2. Preliminary results

We first discuss the method of sub and super solutions. By a subsolution of (1.1), we mean a function $\psi \in C^2(0,1) \cap C^1[0,1]$ which satisfies

$$\begin{gathered}
-\psi''(t) \le h(t)\Big(\frac{a\psi(t) - \psi^2(t) - c}{\psi^\alpha(t)}\Big), \quad t \in (0,1), \\
\psi(t) > 0, \quad t \in (0,1], \\
\psi'(1) + g(\psi(1)) \le 0, \\
\psi(0) = 0,
\end{gathered} \tag{2.1}$$

and by a supersolution of (1.1), we mean a function $\phi \in C^2(0,1) \cap C^1[0,1]$ which satisfies

$$\begin{gathered}
-\phi''(t) \ge h(t)\Big(\frac{a\phi(t) - \phi^2(t) - c}{\phi^\alpha(t)}\Big), \quad t \in (0,1), \\
\phi(t) > 0, \quad t \in (0,1], \\
\phi'(1) + g(\phi(1)) \ge 0, \\
\phi(0) = 0.
\end{gathered} \tag{2.2}$$

Lemma 2.1 (See [12]). *If there exist a subsolution ψ and a supersolution ϕ of (1.1) such that $\psi \le \phi$, then (1.1) has at least one solution $u \in C^2(0,1) \cap C^1[0,1]$ satisfying $\psi \le u \le \phi$ in $[0,1]$.*

We note here that, in our case, the difficulty lies in the construction of a positive subsolution, as the subsolution, ψ, needs to satisfy $\lim_{t \to 0^+} -\psi''(t) = -\infty$, and $-\psi'' > 0$ in a large part of the interior.

Next, we discuss the Sturm-Liouville problem

$$\begin{gathered}
y''(t) + \lambda y(t) = 0, \quad t \in (0,1), \\
y(0) = 0, \\
y'(1) + l y(1) = 0,
\end{gathered} \tag{2.3}$$

where $l > 0$, and λ is a real parameter. We first observe (see also [15]) that the following result holds.

Lemma 2.2. *For a given $l > 0$, the first eigenvalue of (2.3), $\lambda_1 \in (\frac{\pi^2}{4}, \pi^2)$, and the corresponding eigenfunction ϕ_1 is positive, and is given by $\phi_1(t) = \sin\sqrt{\lambda_1}t$. Moreover, as $l \to 0$, $\lambda_1 \to \frac{\pi^2}{4}$, and as $l \to \infty$, $\lambda_1 \to \pi^2$.*

Proof. The solution of the equation $y'' + \lambda y = 0$ is given by $\phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. Using the boundary conditions, we reduce that $\tan\eta = \frac{-1}{l}\eta$, where $\eta = \sqrt{\lambda}$. This equation does not possess an explicit solution. But the graphical solutions of this equation can be determined by plotting functions $y = \tan\eta$ and $y = -\frac{1}{l}\eta$ (see Figure 2).

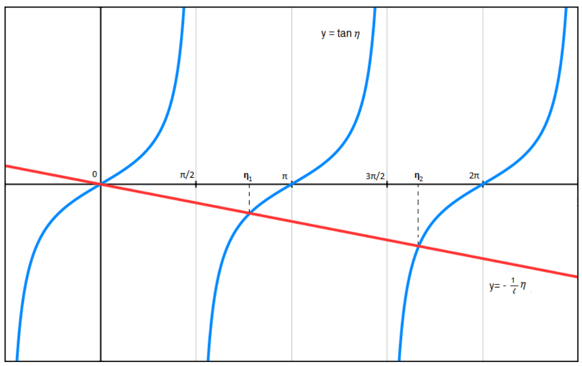

FIGURE 2. Graph of $\tan\eta$ vs $-1/(l\eta)$

From Figure 2, it is clear that, there are infinitely many roots η_n for $n = 1, 2, \dots$. To each root η_n, there corresponds an eigenvalue $\lambda_n = \eta_n^2$, $n = 1, 2, 3, \dots$. Thus there exists a sequence of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ and the corresponding eigenfunctions are $\phi_n = \sin\sqrt{\lambda_n}x$. From the graph, we observe that the first eigenvalue $\lambda_1 = \eta_1^2 \in (\pi^2/4, \pi^2)$, and hence ϕ_1 is positive. Also note that as $l \to \infty$, $\eta_1 \to \pi$ and, as $l \to 0$, $\eta_1 \to \pi/2$. $\square$

3. Proof of Theorem 1.1

We will first prove the following lemma.

Lemma 3.1. *Let $a > 0$, $c \geq 0$, $\alpha \in (0, 1)$, and $F(s) = \int_0^s f(t)\,dt$, where $f(s) = \frac{as-s^2-c}{s^\alpha}$. Let $h \in C((0, 1), (0, \infty))$ satisfy* (H1) *and* (H3)*. If $F(s) < 0$ for all $s > 0$, then* (1.1) *has no nonnegative solution.*

Proof. Let us assume that (1.1) has a nonnegative solution $u(t)$. Since $u(0) = 0$ and $u'(1) \leq 0$, there exists a $t_0 > 0$ such that $u'(t_0) = 0$. Now define $E(t) := F(u(t))h(t) + \frac{[u'(t)]^2}{2}$. From (H1), there exists a $d > 0$ such that $h(t) \leq \frac{d}{t^\gamma}$ for $t \in (0, 1)$. Integrating (1.1) from t to t_0 and using the fact for $s > 0$, $f(s) \leq R$ for some $R > 0$, we obtain

$$u'(t) = \int_t^{t_0} h(s)f(u(s))\,ds \leq \frac{dR}{1-\gamma}(t_0^{1-\gamma} - t^{1-\gamma}) \leq \frac{dR}{1-\gamma} = R_0. \tag{3.1}$$

Again integrating (3.1) from 0 to t, $t < t_0$, we have $u(t) < R_0 t$, for $t \in (0, t_0)$. Since f is integrable, there exist $k > 0$ and $\epsilon > 0$ such that $|F(u)| \leq ku$ for $u \in (0, \epsilon)$. Hence

$$\lim_{t\to 0^+} |F(u(t))|h(t) \leq \lim_{t\to 0^+} kR_0 d t^{1-\gamma} = 0.$$

This implies that $\lim_{t\to 0^+} E(t) \geq 0$. Now note that $E'(t) = F(u)h'(t)$. From (H3), $h'(t) < 0$ for $t \in (0,1)$, and $F(s) < 0$ for all $s > 0$, $E'(t) > 0$ for all $t > 0$. Therefore $E(t) > 0$ for all $t > 0$. But $E(t_0) < 0$, which is a contradiction. $\square$

Proof of Theorem 1.1. We have

$$F(s) = \int_0^s f(t)\,\mathrm{d}t = \int_0^s \frac{at - t^2 - c}{t^\alpha}\,\mathrm{d}t = s^{1-\alpha}\Big(\frac{a}{2-\alpha}s - \frac{1}{3-\alpha}s^2 - \frac{c}{1-\alpha}\Big).$$

The zeros of $F(s)$ are $s = 0$ and

$$s = \frac{\frac{a}{2-\alpha} \pm \sqrt{\frac{a^2}{(2-\alpha)^2} - \frac{4c}{(3-\alpha)(1-\alpha)}}}{\frac{2}{3-\alpha}}.$$

If $c > \hat{c}(a)$ then $\frac{a^2}{(2-\alpha)^2} - \frac{4c}{(3-\alpha)(1-\alpha)} < 0$. This implies $F(s)$ has only one zero at $s = 0$. Since $\lim_{s\to 0^+} F'(s) = -\infty$ and $F(0) = 0$, $F(s) < 0$ for all $s > 0$. Hence by Lemma 3.1, (1.1) has no nonnegative solution. $\square$

4. Proof of Theorem 1.4

We first construct a subsolution for (1.1) (when $c = 0$). Let ϕ_1 be the eigen function corresponding to the first eigenvalue λ_1 of the problem $-\phi''(t) = \lambda\phi(t), t \in (0,1)$, $\phi(0) = \phi(1) = 0$. Note that, $\phi_1(t) = \sin \pi t$, and $\lambda_1 = \pi^2$. Fix $k > 0$ such that $k \geq \frac{-(g(1)+1)}{\phi_1'(1)}$. We now define our subsolution to be $\psi(t) = k\phi_1(t) + t$. Let $\underline{a} = \frac{\lambda_1(k+1)^\alpha}{\hat{h}} + (k+1)$. For $a \geq \underline{a}$, we will show that ψ is a subsolution of (1.1). To prove this, we need to show that $-\psi'' = \lambda_1 k\phi_1 \leq h(t)(a\psi^{1-\alpha} - \psi^{2-\alpha})$, $\psi(0) \leq 0$ and $\psi'(1) + g(\psi(1)) \leq 0$. We will first show that

$$\lambda_1(k\phi_1(t) + t) \leq \hat{h}(a(k\phi_1(t) + t)^{1-\alpha} - (k\phi_1(t) + t)^{2-\alpha}), \tag{4.1}$$

where $\hat{h} = \inf_{s\in(0,1)} h(s)$. This clearly implies $-\psi'' \leq h(t)(a\psi^{1-\alpha} - \psi^{2-\alpha})$ (since $\psi(t) \leq k+1$ for all t, $a\psi^{1-\alpha} - \psi^{2-\alpha} > 0$). From the choice of $\underline{a}$,

$$\lambda_1(k+1)^\alpha \leq \hat{h}(a - (k+1)).$$

From this, we obtain

$$\lambda_1(k\phi_1 + t)^\alpha \leq \hat{h}(a - (k\phi_1 + t)),$$

and (4.1) follows. Clearly $\psi(0) = 0$. Also $\psi'(1) + g(\psi(1)) = k\phi_1'(1) + 1 + g(1) \leq 0$, by the choice of k. Hence ψ is a subsolution of (1.1). Next we construct a supersolution of (1.1). Let e be the solution of

$$\begin{gathered} -e''(t) = h(t), \quad t \in (0,1), \\ e(0) = e'(1) = 0. \end{gathered}$$

Integrating the above equation from t to 1, we see that $e'(t) = \int_t^1 h(s)\,\mathrm{d}s > 0$, and hence e is an increasing function for $t \in [0,1]$. Choose a constant $M > 0$ such that $\frac{as-s^2}{s^\alpha} < M$, for all $s \geq 0$. Then clearly $\phi = Me$ is a supersolution of (1.1). Also since $e'(0) > 0$ if we choose M large enough then, $\psi(t) \leq \phi(t)$ for all $t \in [0,1]$. Hence, by Lemma 2.1, there exist a solution u of (1.1) such that $\psi(t) \leq u(t) \leq \phi(t)$ for all $t \in [0,1]$. Clearly $u(1) > 0$ since $\psi(1) > 0$.

5. Proof of Theorem 1.6

Let u and v be two positive solutions of (1.1) with $c = 0$ such that $u \not\equiv v$. Without loss of generality let $t_1 \in [0,1)$ be such that $v(t_1) - u(t_1) = 0$, $v(t) - u(t) \geq 0$ in $[t_1, 1]$, and $v(t) - u(t) > 0$ for some $(s_1, s_2) \subset [t_1, 1]$. For $t \in (s_1, s_2)$, we have

$$
\begin{aligned}
-(uv'' - vu'') &= h(t)\Big(u\frac{av - v^2}{v^\alpha} - v\frac{au - u^2}{u^\alpha}\Big) \\
&= h(t)\frac{(av - v^2)(au - u^2)}{u^\alpha v^\alpha}\Big(\frac{u^{1+\alpha}}{au - u^2} - \frac{v^{1+\alpha}}{av - v^2}\Big).
\end{aligned}
$$

Since for any positive solution u, $\|u\|_\infty < a$, and $\tilde{f}(s) = \frac{s^{1+\alpha}}{as - s^2}$ is a strictly increasing function for $s \in (0, a)$, we see that $\int_{t_1}^1 -(uv'' - vu'')\,\mathrm{d}t < 0$. Using $v(t_1) = u(t_1)$, $v'(t_1) \geq u'(t_1)$, and (H4), we obtain

$$
\begin{aligned}
&\int_{t_1}^1 -(uv'' - vu'')(t)\,\mathrm{d}t \\
&= [-uv' + vu']_{t_1}^1 \\
&= v(1)u'(1) - u(1)v'(1) - (v(t_1)u'(t_1) + u(t_1)v'(t_1)) \\
&= -v(1)g(u(1)) + u(1)g(v(1)) + u(t_1)v'(t_1) - u(t_1)u'(t_1) \\
&\geq -v(1)g(u(1)) + u(1)g(v(1)) \\
&\geq u(1)v(1)\left(\frac{g(v(1))}{v(1)} - \frac{g(u(1))}{u(1)}\right) \geq 0,
\end{aligned}
$$

which a contradiction, and hence $u \equiv v$.

6. Proof of Theorem 1.7

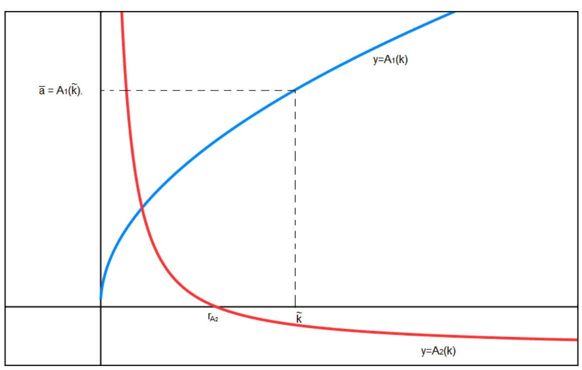

Figure 3. Graph of $A_1(k)$ vs $A_2(k)$

We first construct a subsolution. For this, we fix a $\beta \in (1, \frac{2-\gamma}{1+\alpha})$. From (H1), it is clear that this interval is nonempty. Now, for $k \geq 0$, we define

$$
A_1(k) := 2k + \frac{2\beta\pi^2 k^\alpha}{\hat{h}}, \tag{6.1}
$$

$$
A_2(k) := -\frac{3\pi}{4\sqrt{2}} + \frac{g(\frac{1}{k^{\beta-1}})}{\beta k^{2-\beta}}. \tag{6.2}
$$

It is easy to see that $A_1(k)$ is an increasing function of k and $A_2(k)$ is negative for k large (see Figure (3)). Let r_{A_2} be the least nonnegative number such that $A_2(k) \leq 0$ for all $k \geq r_{A_2}$. Choose $\bar{k} = \max\{\sqrt{2}, r_{A_2}\}$. Let $\bar{a} = A_2(\bar{k})$. Now, for given $a \geq \bar{a}$, there exists $\tilde{k}(a) \geq \bar{k}$ such that $a = A_1(\tilde{k})$. From Lemma 2.2, note that there exist $\tilde{l} > 0$ such that $\tilde{k} = \frac{1}{\phi_1(1)}$, where ϕ_1 is the eigenfunction corresponding to the first eigenvalue λ_1 of

$$
\begin{gathered}
y''(t) + \lambda y(t) = 0, \quad t \in (0,1), \\
y(0) = 0, \\
y'(1) + \tilde{l} y(1) = 0.
\end{gathered}
$$

We now define our subsolution ψ to be $\psi := \tilde{k}\phi_1^{\beta}$. Since $\phi_1(t) = \sin\sqrt{\lambda_1}t$, it is easy to see that ϕ_1 has the following properties. There exist $\epsilon < \epsilon_1$ (ϵ_1 as in H_1) and $\mu > 0$ such that $|\phi_1'| \geq \eta_1/2$ on $(0,\epsilon]$, where $\eta_1 = \sqrt{\lambda_1}$, $\phi_1 \geq \mu$ on $(\epsilon, 1)$, and $0 \leq \phi_1(t) \leq \eta_1 t$ for all $t \in (0,1)$. For $a \geq \bar{a}$, define

$$
\bar{c}(a) = \min\Big\{\tilde{k}^{1+\alpha}\beta(\beta-1)\frac{\eta_1^{2-\gamma}}{4}, \frac{1}{2}\tilde{k}\mu^{\beta}\Big(a - \frac{\beta\lambda_1\tilde{k}^{\alpha}}{\hat{h}}\Big)\Big\}. \tag{6.3}
$$

Note that $\bar{c} > 0$ by the choice of $\tilde{k}$ and β. Next, we calculate

$$
-\psi'' = \tilde{k}\lambda_1\beta\phi_1^{\beta} - \tilde{k}\beta(\beta-1)\frac{\phi_1'^2}{\phi_1^{2-\beta}}.
$$

To prove ψ is a subsolution, we need to establish

$$
\tilde{k}\lambda_1\beta\phi_1^{\beta} - \tilde{k}\beta(\beta-1)\frac{\phi_1'^2}{\phi_1^{2-\beta}} \leq h(t)\Big(a\tilde{k}^{1-\alpha}\phi_1^{\beta(1-\alpha)} - \tilde{k}^{2-\alpha}\phi_1^{\beta(2-\alpha)} - \frac{c}{\tilde{k}^{\alpha}\phi_1^{\alpha\beta}}\Big) \tag{6.4}
$$

and $\psi'(1) + g(\psi(1)) \leq 0$ (Clearly $\psi(0) = 0$).

First we show that (6.4) satisfied. Note that

$$
\begin{aligned}
\tilde{k}\lambda_1\beta\phi_1^{\beta} &= \frac{\hat{h}\tilde{k}\lambda_1\beta\phi_1^{\beta}}{\hat{h}} \\
&\leq h(t)\Big[a\tilde{k}^{1-\alpha}\phi_1^{\beta(1-\alpha)} - \frac{1}{2}\tilde{k}^{1-\alpha}\phi_1^{\beta(1-\alpha)}\Big(a - \frac{\tilde{k}^{\alpha}\lambda_1\beta\phi_1^{\alpha\beta}}{\hat{h}}\Big) \\
&\quad - \frac{1}{2}\tilde{k}^{1-\alpha}\phi_1^{\beta(1-\alpha)}\Big(a - \frac{\tilde{k}^{\alpha}\lambda_1\beta\phi_1^{\alpha\beta}}{\hat{h}}\Big)\Big].
\end{aligned}
$$

To prove (6.4) holds in $(0,1)$, it is sufficient to show the following three inequalities hold:

$$
-\frac{1}{2}\tilde{k}^{1-\alpha}\phi_1^{\beta(1-\alpha)}\Big(a - \frac{\tilde{k}^{\alpha}\lambda_1\beta\phi_1^{\alpha\beta}}{\hat{h}}\Big) \leq -\tilde{k}^{2-\alpha}\phi_1^{\beta(2-\alpha)} \quad \text{in } (0,1), \tag{6.5}
$$

$$
-\frac{1}{2}\tilde{k}^{1-\alpha}\phi_1^{\beta(1-\alpha)}\Big(a - \frac{\tilde{k}^{\alpha}\lambda_1\beta\phi_1^{\alpha\beta}}{\hat{h}}\Big) \leq -\frac{c}{\tilde{k}^{\alpha}\phi_1^{\alpha\beta}} \quad \text{in } (\epsilon,1), \tag{6.6}
$$

$$
-\tilde{k}\beta(\beta-1)\frac{\phi_1'^2}{\phi_1^{2-\beta}} \leq -h(t)\frac{c}{\tilde{k}^{\alpha}\phi_1^{\alpha\beta}} \quad \text{in } (0,\epsilon]. \tag{6.7}
$$

From the definition of a, we have $2\tilde{k} + \frac{\tilde{k}^{\alpha}\lambda_1\beta}{\hat{h}} < a$. Then

$$
-\Big(a - \frac{\tilde{k}^{\alpha}\lambda_1\beta\phi_1^{\alpha\beta}}{\hat{h}}\Big) < -2\tilde{k}.
$$

Hence

$$
\begin{aligned}
-\frac{1}{2}\tilde{k}^{1-\alpha}\phi_1^{\beta(1-\alpha)}\Big(a-\frac{\tilde{k}^{\alpha}\lambda_1\beta\phi_1^{\alpha\beta}}{\hat{h}}\Big) &< -\tilde{k}^{2-\alpha}\phi_1^{\beta(1-\alpha)} \\
&< -\tilde{k}^{2-\alpha}\phi_1^{\beta(2-\alpha)} \quad \text{in } (0,1).
\end{aligned} \tag{6.8}
$$

Using $\phi_1 \geq \mu$ in $(\epsilon, 1)$, and $c \leq \frac{1}{2}\tilde{k}\mu^{\beta}(a - \frac{\beta\lambda_1\tilde{k}^{\alpha}}{\hat{h}})$,

$$
\begin{aligned}
-\frac{1}{2}\tilde{k}^{1-\alpha}\phi_1^{\beta(1-\alpha)}\Big(a-\frac{\tilde{k}^{\alpha}\lambda_1\beta\phi_1^{\alpha\beta}}{\hat{h}}\Big) &\leq \frac{1}{\tilde{k}^{\alpha}\phi_1^{\alpha\beta}}\Big(\frac{-1}{2}\tilde{k}\phi_1^{\beta}\Big(a-\frac{\tilde{k}^{\alpha}\lambda_1\beta}{\hat{h}}\Big)\Big) \\
&\leq -\frac{c}{\tilde{k}^{\alpha}\phi_1^{\alpha\beta}} \quad \text{in } (\epsilon,1).
\end{aligned} \tag{6.9}
$$

Next, we prove that (6.7) holds in $(0, \epsilon]$. Since $|\phi_1'| \geq \eta_1/2$ and $2 - \beta > \alpha\beta + \gamma$ we have

$$
-\tilde{k}\beta(\beta-1)\frac{\phi_1'^2}{\phi_1^{2-\beta}} \leq -\frac{\tilde{k}^{1+\alpha}\beta(\beta-1)\eta_1^2}{4\tilde{k}^{\alpha}\phi_1^{\alpha\beta}\phi_1^{\gamma}} \leq -\frac{\tilde{k}^{1+\alpha}\beta(\beta-1)\eta_1^2}{4\tilde{k}^{\alpha}\phi_1^{\alpha\beta}\eta_1^{\gamma}t^{\gamma}}.
$$

Since $h(t) \leq \frac{1}{t^{\gamma}}$ in $(0, \epsilon]$, and $c \leq \tilde{k}^{1+\alpha}\beta(\beta-1)\eta_1^{2-\gamma}/4$, it follows that

$$
-\tilde{k}\beta(\beta-1)\frac{|\phi_1'|^2}{\phi_1^{2-\beta}} \leq -h(t)\frac{c}{\tilde{k}^{\alpha}\phi_1^{\alpha\beta}} \quad \text{in } (0,\epsilon]. \tag{6.10}
$$

Thus from (6.8), (6.9) and (6.10) we see that (6.4) holds in $(0, 1)$.

Next we will show that $\psi'(1) + g(\psi(1)) \leq 0$ and

$$
\psi'(1) + g(\psi(1)) = \tilde{k}\beta\phi_1^{\beta-1}(1)\phi_1'(1) + g(\tilde{k}\phi_1^{\beta}(1)).
$$

Since $\tilde{k} = \frac{1}{\phi_1(1)}$, it follows that

$$
\psi'(1) + g(\psi(1)) = \beta\tilde{k}^{2-\beta}\phi_1'(1) + g(\tilde{k}^{1-\beta}) = \beta\tilde{k}^{2-\beta}\Big(\phi'(1) + \frac{g(\tilde{k}^{1-\beta})}{\beta\tilde{k}^{2-\beta}}\Big).
$$

Now note that, since $\tilde{k} > \sqrt{2}$, $\phi_1(1) = \sin\sqrt{\lambda_1} < \frac{1}{\sqrt{2}}$, which implies $\sqrt{\lambda_1} \in (\frac{3\pi}{4}, \pi)$. Hence $\phi_1'(1) < -3\pi/(4\sqrt{2})$ and thus

$$
\psi'(1) + g(\psi(1)) \leq \beta\tilde{k}^{2-\beta}\Big(-\frac{3\pi}{4\sqrt{2}} + \frac{g(\frac{1}{\tilde{k}^{\beta-1}})}{\beta\tilde{k}^{2-\beta}}\Big) \leq 0
$$

since $A_2(\tilde{k}) \leq 0$. Therefore $\psi = \tilde{k}\phi_1^{\beta}$ is a subsolution of (1.1). Next we will construct a supersolution of (1.1). For this, we proceed as in the proof of Theorem 1.4. Let e be the solution of

$$
\begin{gathered}
-e''(t) = h(t), \quad t \in (0,1), \\
e(0) = e'(1) = 0.
\end{gathered}
$$

As discussed earlier, e is an increasing function for $t \in [0, 1]$. Choose a constant $M > 0$ such that $\frac{as-s^2-c}{s^{\alpha}} < M$, for all $s \geq 0$. Then clearly $\phi = Me$ is a supersolution of (1.1). Also if we choose M large enough then, $\psi(t) \leq \phi(t)$ for all $t \in [0, 1]$. Hence, by Lemma 2.1, there exist a solution u of (1.1) such that $\psi(t) \leq u(t) \leq \phi(t)$ for all $t \in [0, 1]$. Clearly $u(1) > 0$ since $\psi(1) > 0$.

We now show that $\bar{c}$, given by (6.3), is an increasing function of a. By definition, $\tilde{k}$ increases as a increases and hence $\tilde{k}^{1+\alpha}\beta(\beta-1)\frac{\eta_1^{2-\gamma}}{4}$ is an increasing function of a. Also,

$$\begin{aligned}
\frac{d}{da}\Big(\frac{1}{2}\tilde{k}\mu^\beta(a-\frac{\beta\lambda_1\tilde{k}^\alpha}{\hat{h}})\Big) &= \frac{1}{2}\frac{d\tilde{k}}{da}\mu^\beta\Big(a-\frac{\beta\lambda_1\tilde{k}^\alpha}{\hat{h}}\Big)+\frac{1}{2}\tilde{k}\mu^\beta\Big(1-\frac{\beta\lambda_1\alpha\tilde{k}^{\alpha-1}}{\hat{h}}\frac{d\tilde{k}}{da}\Big)\\
&= \frac{1}{2}\tilde{k}\mu^\beta+\frac{\mu^\beta}{2}\frac{d\tilde{k}}{da}\Big(a-\frac{(\alpha+1)\beta\lambda_1\tilde{k}^\alpha}{\hat{h}}\Big)\\
&> \frac{1}{2}\tilde{k}\mu^\beta+\frac{\mu^\beta}{2}\frac{d\tilde{k}}{da}\Big(a-\frac{2\beta\pi^2\tilde{k}^\alpha}{\hat{h}}\Big)>0.
\end{aligned}$$

Hence $\bar{c}(a)$ is an increasing function of a and $\bar{c}(a)\to\infty$ as $a\to\infty$.

7. Numerical results

In this section, we consider the boundary-value problem

$$\begin{gathered}
-u''=\Big(\frac{au-u^2-c}{u^\alpha}\Big),\quad t\in(0,1),\\
u(0)=0,\quad u'(1)+g(u(1))=0,
\end{gathered}\tag{7.1}$$

where $a>0$, $c\ge 0$, $\alpha\in(0,1)$, and g:$[0,\infty)\to[0,\infty)$, is a continuous function. We plot the exact bifurcation diagram of positive solutions of (7.1) (c versus $\|u\|_\infty$ and a versus $\|u\|_\infty$) using Mathematica. For this, we adapt the quadrature method discussed in [6, 7, 10]. Let $u(t)$ be a positive solution of (7.1). Let $F(z)=\int_0^z f(s)ds$, where $f(s)=\frac{as-s^2-c}{s^\alpha}$, $\rho:=\|u\|_\infty$, and $q=u(1)$. Following the arguments in [6], u is a solution of (7.1) if and only if ρ,q satisfy:

$$2\int_0^\rho\frac{ds}{\sqrt{F(\rho)-F(s)}}-\int_0^q\frac{ds}{\sqrt{F(\rho)-F(s)}}=\sqrt{2},\tag{7.2}$$

$$F(\rho)-F(q)=\frac{(g(q))^2}{2}.\tag{7.3}$$

Let θ_1 be the positive zero of F (see figure 4) and r_2 be the falling zero of f (see figure 4).

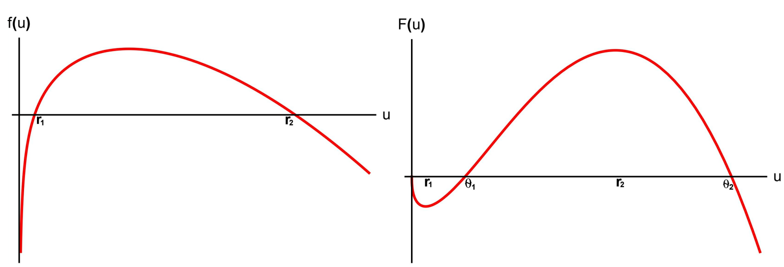

FIGURE 4. Graph of $f(u)$ (left). Graph of $F(u)$ (right)

We note that if $\rho\in(\theta_1,r_2)$ then the integrals in (7.2) are well defined (see [6] for details). Now, using (7.2) and (7.3), we are able to plot exact bifurcation diagram of positive solutions of (7.1) by implementing a numerical root finding algorithm in Mathematica. Figures 5, 6 are bifurcation diagrams c versus ρ for the cases $g(t)\equiv 1$

and $g(t) = t^2$ when $a = 10$ and $a = 15$. Figures 7 has bifurcation diagrams a versus ρ for the cases $g(t) = t^2$ when $c = 0.1$ and $c = 1$.

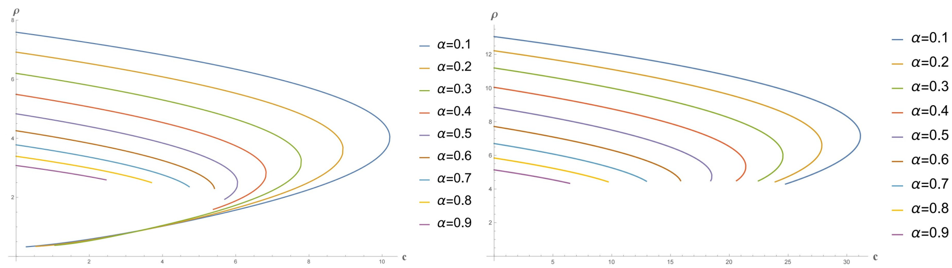

FIGURE 5. Bifurcation of (7.1) when $g(t) \equiv 1$, $a = 10$ (left); when $g(t) \equiv 1$, $a = 15$ (right)

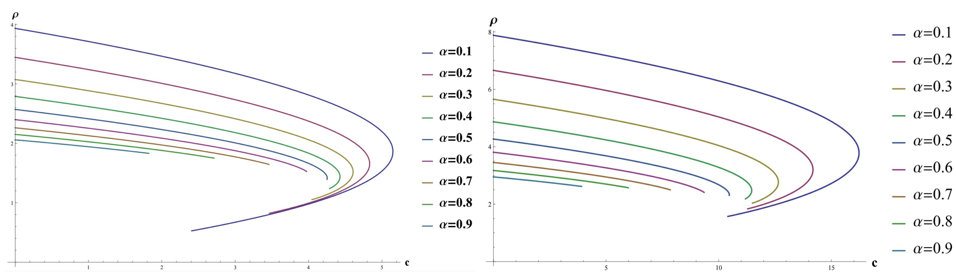

FIGURE 6. Bifurcation of (7.1) when $g(t) = t^2$, $a = 10$ (left); when $g(t) = t^2$, $a = 15$ (right)

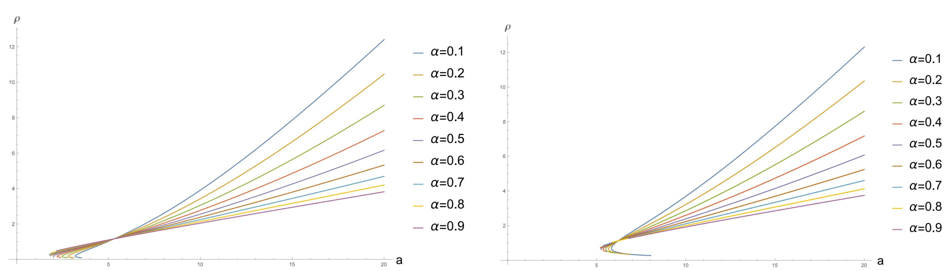

FIGURE 7. Bifurcation of (7.1) when $g(t) = t^2$, $c = 0.1$ (left); when $g(t) = t^2$, $c = 1$ (right)

Our bifurcation diagrams illustrate the existence result in Theorem 1.7 for the case $h(t) \equiv 1$, $g(t) \equiv 1$ or $g(t) = t^2$, and $a = 10$ or 15. We see that for each $\alpha \in (0, 1)$, there exists a $\bar{c} > 0$ such that for $c < \bar{c}$, (7.1) has a positive solution. Also from the bifurcation diagrams (Figure 7) we can see that for given $c \leq \bar{c}(\bar{a})$, there exists $a_*(c)$ such that for $a > a_*$, (7.1) has a positive solution. For $c = 0$, the bifurcation diagrams show that the positive solution is unique which illustrates Theorem 1.6. The following observations can also be made from the bifurcation diagrams for the special cases considered. For $c \approx 0$, it appears that (7.1) has unique positive solution and for a certain range of c, (7.1) has multiple positive solutions. Also, for a fixed $c \leq \bar{c}(\bar{a})$ we observe that for large values of a, (7.1) has unique positive solution and for a certain range of a, (7.1) has multiple positive solutions. Proving these results for (1.1) (at least for certain cases of g) remains an open question.

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