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CARLEMAN ESTIMATE AND NULL CONTROLLABILITY OF A CASCADE DEGENERATE PARABOLIC SYSTEM WITH GENERAL CONVECTION TERMS

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Abstract. This article shows Carleman estimate and null controllability of a cascade control system governed by the semilinear degenerate parabolic equations with the general convection terms. The semilinear parabolic equations are weakly degenerate on the boundary and the convection terms cannot be controlled by the diffusion terms. We establish the Carleman estimate and the observability inequality for the linear conjugate system, and prove that the control system is null controllable.

1. Introduction

In this article, we study the Carleman estimate and the null controllability of the cascade semilinear degenerate parabolic system with convection terms

$$u_t - (x^\alpha u_x)_x + (P_1(x,t,u))_x + F_1(x,t,u) = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \tag{1.1}$$

$$v_t - (x^\alpha v_x)_x + (P_2(x,t,v))_x + F_2(x,t,u,v) = 0, \quad (x,t) \in Q_T, \tag{1.2}$$

$$u(0,t) = v(0,t) = 0, \quad u(1,t) = v(1,t) = 0, \quad t \in (0,T), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in (0,1), \tag{1.4}$$

where $0 < \alpha < 1/2$, $Q_T = (0,1) \times (0,T)$, h is the control function, $\omega \subset (0,1)$ is an interval, χ_ω is the characteristic function of ω, $u_0, v_0 \in L^2(0,1)$, and P_1, P_2, F_1, F_2 are measurable functions. It is noted that (1.1) and (1.2) are degenerate at the boundary $x = 0$. The cascade system (1.1) and (1.2) arises in some models from mathematical biology and physics, such as the Keller-Segel model [6] and the Lotka-Volterra model [25].

Controllability theory has been widely investigated for nondegenerate parabolic equations and systems over the last forty years and the known results are almost complete (see [3, 4, 5, 16, 18, 19, 20, 21, 22] and the references therein). Recently, controllability theory for degenerate ones has been studied and there have been many results (see [1, 2, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 23, 24, 26, 27, 28, 29, 31]). But it is far from being solved. The null controllability of the following system, governed by a degenerate diffusion equation, is already investigated

$$u_t - (x^\alpha u_x)_x + c(x,t)u = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \tag{1.5}$$

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$$\begin{gathered} u(0,t)=u(1,t)=0 \quad \text{if } 0<\alpha<1,\ t\in(0,T), \\ (x^\alpha u_x)(0,t)=u(1,t)=0 \quad \text{if } \alpha\geq 1,\ t\in(0,T), \end{gathered} \tag{1.6}$$

$$u(x,0)=u_0(x), \quad x\in(0,1), \tag{1.7}$$

where $\alpha>0$, $c\in L^\infty(Q_T)$. It was shown that the system (1.5)–(1.7) is null controllable if $0<\alpha<2$ [1, 10, 11, 24], while not if $\alpha\geq 2$ [9]. Although system (1.5)–(1.7) is not null controllable for $\alpha\geq 2$, it was proved in [14, 26, 27, 28] and [7, 8, 9] that it is approximately controllable in $L^2(0,1)$ and regional null controllable for each $\alpha>0$, respectively. Flores and Teresa [17] studied the degenerate convection-diffusion equation

$$u_t-(x^\alpha u_x)_x+x^{\alpha/2}b(x,t)u_x+c(x,t)u=h(x,t)\chi_\omega, \quad (x,t)\in Q_T \tag{1.8}$$

with $b\in L^\infty(Q_T)$ and proved that the system (1.8), (1.6) and (1.7) is null controllable for $0<\alpha<2$. Clearly, the convection term can be controlled by the diffusion term in (1.8). Wang and Du [29] considered the degenerate convection-diffusion equation

$$u_t-(x^\alpha u_x)_x+(b(x,t)u)_x+c(x,t)u=h(x,t)\chi_\omega, \quad (x,t)\in Q_T, \tag{1.9}$$

and proved that system (1.9), (1.6) and (1.7) is null controllable if $0<\alpha<1/2$. Here $0<\alpha<1/2$ is optimal when one establishes the Carleman estimate in such a way as in [29]. Since (1.9) is degenerate, the convection term can cause essential differences. For example, problem (1.9), (1.6) and (1.7) is well-posed in the weakly degenerate case ($0<\alpha<1$), while may be ill-posed in the strongly degenerate case ($\alpha\geq 1$) (see [30, 32]). Moreover, the system

$$u_t-(x^\alpha u_x)_x+b(x,t)u_x+c(x,t)u=h(x,t)\chi_\omega, \quad (x,t)\in Q_T$$

with (1.6) and (1.7) was shown to be null controllable for $0<\alpha<1$ if $b,b_x,b_{xx},b_t\in L^\infty(Q_T)$ in [31]. As for systems, the authors in [2] studied the null controllability of system (1.1)–(1.4) without convection term, i.e. the case that $P_1=P_2=0$. It was shown in [2] that the system is null controllable if $0<\alpha<2$ (a different boundary condition at $x=0$ is prescribed if $1\leq\alpha<2$). Du and Xu [15] considered the linear case of system (1.1)–(1.4) when the convection terms can be controlled by the diffusion terms. Furthermore, the Carleman estimate in [15] depends on the derivatives of the coefficients of the convection terms and cannot be used to treat the semilinear case.

In this article, we consider the more general system (1.1)–(1.4) where the convection terms are independent of the diffusion terms. More precisely, we assume that P_1,P_2,F_1,F_2 are measurable functions satisfying

$$\begin{gathered} P_1(x,t,0)=P_2(x,t,0)=F_1(x,t,0)=F_2(x,t,0,0)=0, \\ F_2(x,t,\cdot,\cdot)\in C^1(\mathbb{R}^2), \quad (x,t)\in Q_T, \end{gathered} \tag{1.10}$$

$$\begin{gathered} |P_1(x,t,y_1)-P_1(x,t,y_2)|\leq K|y_1-y_2|, \\ |P_2(x,t,z_1)-P_2(x,t,z_2)|\leq K|z_1-z_2|, \\ |F_1(x,t,y_1)-F_1(x,t,y_2)|\leq K|y_1-y_2|, \\ (x,t)\in Q_T,\ y_1,y_2,z_1,z_2\in\mathbb{R}, \end{gathered} \tag{1.11}$$

$$\Big|\frac{\partial F_2(x,t,y,z)}{\partial y}\Big|\leq K, \quad \Big|\frac{\partial F_2(x,t,y,z)}{\partial z}\Big|\leq K, \quad (x,t)\in Q_T,\ y,z\in\mathbb{R}, \tag{1.12}$$

$$\frac{\partial F_2}{\partial y}\Big|_{(x_0,x_1)\times(0,T)\times\mathbb{R}^2} \ge c_0 \quad \text{or} \quad \frac{\partial F_2}{\partial y}\Big|_{(x_0,x_1)\times(0,T)\times\mathbb{R}^2} \le -c_0, \tag{1.13}$$

where $K > 0$, $c_0 > 0$ and $0 < x_0 < x_1 < 1$ such that $(x_0, x_1) \subset \omega$. Since (1.1) and (1.2) are degenerate at the boundary $x = 0$, the classical solution may not exist and weak solution should be considered. The key to prove the null controllability of system (1.1)–(1.4) is the Carleman estimate for its linear conjugate problem. The degeneracy of the equations and the existence of the general convection terms cause essential difficulties for the Carleman estimate. In order to establish the needed Carleman estimate for the degenerate parabolic system, we use the Carleman estimate for a single degenerate parabolic equation in [29], where the auxiliary functions for the Carleman estimate were chosen by the method of undetermined coefficients and it was turned out that only the case $0 < \alpha < 1/2$ can be treated in such a way. Using the Carleman estimate in [29] and the classical Carleman estimate, together with some other energy estimates, we establish the Carleman estimate for the linear conjugate problem of problem (1.1)–(1.4). Some technical difficulties caused by the degeneracy of the equations and the existence of the general convection terms have to be overcome. After the Carleman estimate, we establish the observability inequality and further prove the null controllability of system (1.1)–(1.4).

This article is organized as follows. In § 2, we introduce the well-posedness of problem (1.1)–(1.4) and some a priori estimates. The Carleman estimate and the observability inequality are established in § 3. Finally, the null controllability of system (1.1)–(1.4) is shown in § 4.

2. Well-posedness and some a priori estimates

Consider the linear nondegenerate parabolic problem

$$\begin{gathered} u^\eta_t - ((x+\eta)^\alpha u^\eta_x)_x + (c_1(x,t)u^\eta)_x + c_3(x,t)u^\eta + c_4(x,t)v^\eta = f_1(x,t), \\ (x,t) \in Q_T, \end{gathered} \tag{2.1}$$

$$\begin{gathered} v^\eta_t - ((x+\eta)^\alpha v^\eta_x)_x + (c_2(x,t)v^\eta)_x + c_5(x,t)u^\eta + c_6(x,t)v^\eta = f_2(x,t), \\ (x,t) \in Q_T, \end{gathered} \tag{2.2}$$

$$u^\eta(0,t) = v^\eta(0,t) = 0, \quad u^\eta(1,t) = v^\eta(1,t) = 0, \quad t \in (0,T), \tag{2.3}$$

$$u^\eta(x,0) = u_0(x), \quad v^\eta(x,0) = v_0(x), \quad x \in (0,1), \tag{2.4}$$

where $0 < \alpha < 1$, $0 < \eta < 1$, $c_i \in L^\infty(Q_T)$ $(1 \le i \le 6)$, $f_1, f_2 \in L^2(Q_T)$, and $u_0, v_0 \in L^2(0,1)$. The problem (2.1)–(2.4) admits a unique solution (u^η, v^η) with $u^\eta, v^\eta \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1))$. Moreover, (u^η, v^η) satisfies the following a priori estimates.

Lemma 2.1. *Assume that* $0 < \alpha < 1$, $0 < \eta < 1$, $c_i \in L^\infty(Q_T)$ *with* $\|c_i\|_{L^\infty(Q_T)} \le K$ $(1 \le i \le 6)$, $f_1, f_2 \in L^2(Q_T)$, *and* $u_0, v_0 \in L^2(0,1)$. *Then the solution* (u^η, v^η) *to problem* (2.1)–(2.4) *satisfies*

$$\begin{aligned} &\|u^\eta\|_{L^\infty(0,T;L^2(0,1))} + \|(x+\eta)^{\alpha/2}u^\eta_x\|_{L^2(Q_T)} + \|v^\eta\|_{L^\infty(0,T;L^2(0,1))} \\ &+ \|(x+\eta)^{\alpha/2}v^\eta_x\|_{L^2(Q_T)} \\ &\le N(\|f_1\|_{L^2(Q_T)} + \|f_2\|_{L^2(Q_T)} + \|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)}), \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&\Big|\int_0^1 (u^\eta(x,t_2)-u^\eta(x,t_1))\varsigma(x)\,\mathrm{d}x\Big| + \Big|\int_0^1 (v^\eta(x,t_2)-v^\eta(x,t_1))\varsigma(x)\,\mathrm{d}x\Big| \\
&\le N(t_2-t_1)^{1/2}(\|f_1\|_{L^2(Q_T)}+\|f_2\|_{L^2(Q_T)}+\|u_0\|_{L^2(0,1)} \\
&\quad +\|v_0\|_{L^2(0,1)})\|\varsigma\|_{H^1(0,1)}, \quad 0\le t_1<t_2\le T, \quad \varsigma\in H^1(0,1),
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
&\int_0^{T-\delta}\int_0^1 (u^\eta(x,\tau+\delta)-u^\eta(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&+\int_0^{T-\delta}\int_0^1 (v^\eta(x,\tau+\delta)-v^\eta(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le N\delta^{1/2}(\|f_1\|^2_{L^2(Q_T)}+\|f_2\|^2_{L^2(Q_T)}+\|u_0\|^2_{L^2(0,1)}+\|v_0\|^2_{L^2(0,1)}), \\
&\quad 0<\delta<T,
\end{aligned} \tag{2.7}
$$

where $N>0$ depends only on K, T, and α.

Proof. Without loss of generality, it is assumed that (u^η, v^η) is a smooth solution. Otherwise, one can mollify c_i $(1\le i\le 6)$, f_1, f_2, u_0, v_0, and prove the lemma by a standard limit process. For convenience, u^η and v^η are abbreviated as u and v, respectively, in this proof.

For $0<s<T$, multiplying (2.1) and (2.2) by u and v, respectively, then integrating over $(0,1)\times(0,s)$ by parts and summing up, we obtain

$$
\begin{aligned}
&\frac12\int_0^s\int_0^1 ((u^2)_t+(v^2)_t)\,\mathrm{d}x\,\mathrm{d}t+\int_0^s\int_0^1 (x+\eta)^\alpha(u_x^2+v_x^2)\,\mathrm{d}x\,\mathrm{d}t \\
&=\int_0^s\int_0^1 (c_1uu_x+c_2vv_x)\,\mathrm{d}x\,\mathrm{d}t-\int_0^s\int_0^1 (c_3u^2+(c_4+c_5)uv+c_6v^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\int_0^s\int_0^1 (f_1u+f_2v)\,\mathrm{d}x\,\mathrm{d}t \\
&\le K\int_0^s\int_0^1 (|uu_x|+|vv_x|)\,\mathrm{d}x\,\mathrm{d}t+2K\int_0^s\int_0^1 (u^2+v^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\int_0^s\int_0^1 (|f_1u|+|f_2v|)\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac12\int_0^s\int_0^1 (x+\eta)^\alpha(u_x^2+v_x^2)\,\mathrm{d}x\,\mathrm{d}t+\frac12K^2\int_0^s\int_0^1 (x+\eta)^{-\alpha}(u^2+v^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\Big(2K+\frac12\Big)\int_0^s\int_0^1 (u^2+v^2)\,\mathrm{d}x\,\mathrm{d}t+\frac12\int_0^s\int_0^1 (f_1^2+f_2^2)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^1 (u^2(x,s)+v^2(x,s))\,\mathrm{d}x+\int_0^s\int_0^1 (x+\eta)^\alpha(u_x^2+v_x^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\le\int_0^1 (u_0^2(x)+v_0^2(x))\,\mathrm{d}x+K^2\int_0^s\int_0^\kappa (x+\eta)^{-\alpha}(u^2+v^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +(\kappa^{-\alpha}K^2+4K+1)\int_0^s\int_0^1 (u^2+v^2)\,\mathrm{d}x\,\mathrm{d}t+\int_0^s\int_0^1 (f_1^2+f_2^2)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{2.8}
$$

where $0<\kappa<1$ will be determined. It follows from (2.3) and $0<\alpha<1$ that

$$\begin{aligned}
&\int_0^s\int_0^\kappa (x+\eta)^{-\alpha}u^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^s\int_0^\kappa (x+\eta)^{-\alpha}\Big(\int_0^x u_x(\tilde{x},t)\,\mathrm{d}\tilde{x}\Big)^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^s\int_0^\kappa (x+\eta)^{-\alpha}\Big(\int_0^x (\tilde{x}+\eta)^{-\alpha}\,\mathrm{d}\tilde{x}\Big)\Big(\int_0^x (\tilde{x}+\eta)^{\alpha}u_x^2(\tilde{x},t)\,\mathrm{d}\tilde{x}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{1}{1-\alpha}\int_0^\kappa (x+\eta)^{1-2\alpha}\,\mathrm{d}x\int_0^s\int_0^\kappa (x+\eta)^{\alpha}u_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{(\kappa+\eta)^{2-2\alpha}-\eta^{2-2\alpha}}{2(1-\alpha)^2}\int_0^s\int_0^1 (x+\eta)^{\alpha}u_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{2.9}$$

Similarly, it holds that

$$\begin{aligned}
&\int_0^s\int_0^\kappa (x+\eta)^{-\alpha}v^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{(\kappa+\eta)^{2-2\alpha}-\eta^{2-2\alpha}}{2(1-\alpha)^2}\int_0^s\int_0^1 (x+\eta)^{\alpha}v_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{2.10}$$

Take $0<\kappa<1$ so small that

$$K^2\max_{0\le\gamma\le1}((\kappa+\gamma)^{2-2\alpha}-\gamma^{2-2\alpha})\le(1-\alpha)^2. \tag{2.11}$$

Substituting (2.9)–(2.11) into (2.8) yields

$$\begin{aligned}
&\int_0^1 (u^2(x,s)+v^2(x,s))\,\mathrm{d}x+\frac12\int_0^s\int_0^1 (x+\eta)^{\alpha}(u_x^2+v_x^2)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^1 (u_0^2(x)+v_0^2(x))\,\mathrm{d}x+(\kappa^{-\alpha}K^2+4K+1)\int_0^s\int_0^1 (u^2+v^2)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^s\int_0^1 (f_1^2+f_2^2)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{2.12}$$

Then, (2.5) follows from (2.12) and the Gronwall inequality. For $0\le t_1<t_2\le T$ and $\varsigma\in H^1(0,1)$, multiplying (2.1) and (2.2) by ς, and then integrating over $(0,1)\times(t_1,t_2)$ by parts, we obtain

$$\begin{aligned}
&\int_0^1 (u(x,t_2)-u(x,t_1))\varsigma(x)\,\mathrm{d}x\\
&=\int_{t_1}^{t_2}\int_0^1 (-(x+\eta)^{\alpha}u_x\varsigma_x+c_1u\varsigma_x-c_3u\varsigma-c_4v\varsigma+f_1\varsigma)\,\mathrm{d}x\,\mathrm{d}t,\\
&\int_0^1 (v(x,t_2)-v(x,t_1))\varsigma(x)\,\mathrm{d}x\\
&=\int_{t_1}^{t_2}\int_0^1 (-(x+\eta)^{\alpha}v_x\varsigma_x+c_2v\varsigma_x-c_5u\varsigma-c_6v\varsigma+f_2\varsigma)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

which, together with the Hölder inequality and (2.5), lead to (2.6). It remains to prove (2.7). For $0<\delta<T$ and $0<\tau<T-\delta$, multiplying (2.1) and (2.2) by $u(x,\tau+\delta)-u(x,\tau)$ and $v(x,\tau+\delta)-v(x,\tau)$, respectively, then integrating over

$(\tau,\tau+\delta)$ with respect to t and summing up, we obtain

$$
\begin{aligned}
&(u(x,\tau+\delta)-u(x,\tau))^2+(v(x,\tau+\delta)-v(x,\tau))^2\\
&=\int_\tau^{\tau+\delta}((x+\eta)^\alpha u_x)_x(u(x,\tau+\delta)-u(x,\tau))\,\mathrm{d}t\\
&\quad+\int_\tau^{\tau+\delta}((x+\eta)^\alpha v_x)_x(v(x,\tau+\delta)-v(x,\tau))\,\mathrm{d}t\\
&\quad-\int_\tau^{\tau+\delta}(c_1u)_x(u(x,\tau+\delta)-u(x,\tau))\,\mathrm{d}t-\int_\tau^{\tau+\delta}(c_2v)_x(v(x,\tau+\delta)-v(x,\tau))\,\mathrm{d}t\\
&\quad+\int_\tau^{\tau+\delta}(f_1-c_3u-c_4v)(u(x,\tau+\delta)-u(x,\tau))\,\mathrm{d}t\\
&\quad+\int_\tau^{\tau+\delta}(f_2-c_5u-c_6v)(v(x,\tau+\delta)-v(x,\tau))\,\mathrm{d}t.
\end{aligned}
$$

Integrating this equality by parts, over $(0,1)\times(0,T-\delta)$, yields

$$
\begin{aligned}
&\int_0^{T-\delta}\int_0^1(u(x,\tau+\delta)-u(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^{T-\delta}\int_0^1(v(x,\tau+\delta)-v(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&=-\int_0^{T-\delta}\int_0^1\int_\tau^{\tau+\delta}(x+\eta)^\alpha u_x(x,t)(u_x(x,\tau+\delta)-u_x(x,\tau))\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad-\int_0^{T-\delta}\int_0^1\int_\tau^{\tau+\delta}(x+\eta)^\alpha v_x(x,t)(v_x(x,\tau+\delta)-v_x(x,\tau))\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\int_0^{T-\delta}\int_0^1\int_\tau^{\tau+\delta}c_1(x,t)u(x,t)(u_x(x,\tau+\delta)-u_x(x,\tau))\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\int_0^{T-\delta}\int_0^1\int_\tau^{\tau+\delta}c_2(x,t)v(x,t)(v_x(x,\tau+\delta)-v_x(x,\tau))\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\int_0^{T-\delta}\int_0^1\int_\tau^{\tau+\delta}(f_1(x,t)-c_3(x,t)u(x,t)-c_4(x,t)v(x,t))\\
&\quad\times(u(x,\tau+\delta)-u(x,\tau))\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\int_0^{T-\delta}\int_0^1\int_\tau^{\tau+\delta}(f_2(x,t)-c_5(x,t)u(x,t)-c_6(x,t)v(x,t))\\
&\quad\times(v(x,\tau+\delta)-v(x,\tau))\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\\
&\le\Big(T\int_\tau^{\tau+\delta}\int_0^1(x+\eta)^\alpha u_x^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\\
&\quad\times\Big(\delta\int_0^{T-\delta}\int_0^1(x+\eta)^\alpha(u_x(x,\tau+\delta)-u_x(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2}\\
&\quad+\Big(T\int_\tau^{\tau+\delta}\int_0^1(x+\eta)^\alpha v_x^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\\
&\quad\times\Big(\delta\int_0^{T-\delta}\int_0^1(x+\eta)^\alpha(v_x(x,\tau+\delta)-v_x(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2}\\
&\quad+K\Big(T\int_\tau^{\tau+\delta}\int_0^1(x+\eta)^{-\alpha}u^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(\delta \int_0^{T-\delta} \int_0^1 (x+\eta)^\alpha (u_x(x,\tau+\delta) - u_x(x,\tau))^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2} \\
&+ K\Big(T\int_\tau^{\tau+\delta} \int_0^1 (x+\eta)^{-\alpha} v^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&\times \Big(\delta \int_0^{T-\delta} \int_0^1 (x+\eta)^\alpha (v_x(x,\tau+\delta) - v_x(x,\tau))^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2} \\
&+ \Big(T\int_\tau^{\tau+\delta} \int_0^1 (|f_1| + K|u| + K|v|)^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&\times \Big(\delta \int_0^{T-\delta} \int_0^1 (u(x,\tau+\delta) - u(x,\tau))^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2} \\
&+ \Big(T\int_\tau^{\tau+\delta} \int_0^1 (|f_2| + K|u| + K|v|)^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&\times \Big(\delta \int_0^{T-\delta} \int_0^1 (v(x,\tau+\delta) - v(x,\tau))^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2} \\
\le{}& 2(T\delta)^{1/2} \int_0^T \int_0^1 (x+\eta)^\alpha u_x^2 \,\mathrm{d}x\,\mathrm{d}t + 2(T\delta)^{1/2} \int_0^T \int_0^1 (x+\eta)^\alpha v_x^2 \,\mathrm{d}x\,\mathrm{d}t \\
&+ 2K(T\delta)^{1/2}\Big(\int_0^T \int_0^1 (x+\eta)^{-\alpha} u^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^T \int_0^1 (x+\eta)^\alpha u_x^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&+ 2K(T\delta)^{1/2}\Big(\int_0^T \int_0^1 (x+\eta)^{-\alpha} v^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^T \int_0^1 (x+\eta)^\alpha v_x^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&+ 2(3T\delta)^{1/2}\Big(\int_0^T \int_0^1 (f_1^2 + K^2u^2 + K^2v^2) \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^T \int_0^1 u^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&+ 2(3T\delta)^{1/2}\Big(\int_0^T \int_0^1 (f_2^2 + K^2u^2 + K^2v^2) \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^T \int_0^1 v^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}.
\end{aligned}
\tag{2.13}
$$

For $0 < \kappa < 1$ satisfying (2.11), it follows from (2.9) and (2.10) that

$$
\begin{aligned}
&\int_0^T \int_0^1 (x+\eta)^{-\alpha} u^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{2K^2} \int_0^T \int_0^1 (x+\eta)^\alpha u_x^2 \,\mathrm{d}x\,\mathrm{d}t + \kappa^{-\alpha} \int_0^T \int_0^1 u^2 \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
&\int_0^T \int_0^1 (x+\eta)^{-\alpha} v^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{2K^2} \int_0^T \int_0^1 (x+\eta)^\alpha v_x^2 \,\mathrm{d}x\,\mathrm{d}t + \kappa^{-\alpha} \int_0^T \int_0^1 v^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{2.15}
$$

Then, (2.7) follows from (2.13)–(2.15) and (2.5). $\square$

Using the a priori estimates in Lemma 2.1, one can prove the well-posedness of problem (1.1)–(1.4).

Proposition 2.2. *Assume that $0 < \alpha < 1$, and P_1, P_2, F_1, F_2 satisfy (1.10)–(1.13). For $h \in L^2(Q_T)$ and $u_0, v_0 \in L^2(0,1)$, problem (1.1)–(1.4) admits a unique solution $(u,v) \in \mathscr{H}_\alpha(Q_T) \times \mathscr{H}_\alpha(Q_T)$. Furthermore, $u, v \in L^\infty(0,T; L^2(0,1)) \cap$*

$C_w([0,T];L^2(0,1))$. Here, $\mathscr{H}_\alpha(Q_T)=\{w\in L^2(Q_T): x^{\alpha/2}w_x\in L^2(Q_T)\}$ and a function $\xi\in C_w([0,T];L^2(0,1))$ means that $\int_0^1\xi(x,t)\gamma(x)\,\mathrm{d}x\in C([0,T])$ for each $\gamma\in L^2(0,1)$.

The proof of Proposition 2.2 is standard and is omitted here. We refer to [29, Theorem 2.1] for the single equation case.

3. Uniform Carleman estimate and observability inequality

In this section, we prove the uniform Carleman estimate and the observability inequality for the linear nondegenerate conjugate system

$$\begin{gathered} U_t^\eta+((x+\eta)^\alpha U_x^\eta)_x+c_1(x,t)U_x^\eta-c_3(x,t)U^\eta-c_4(x,t)V^\eta=0,\\ (x,t)\in Q_T, \end{gathered}\tag{3.1}$$

$$V_t^\eta+((x+\eta)^\alpha V_x^\eta)_x+c_2(x,t)V_x^\eta-c_5(x,t)V^\eta=0,\quad (x,t)\in Q_T,\tag{3.2}$$

$$U^\eta(0,t)=V^\eta(0,t)=0,\quad U^\eta(1,t)=V^\eta(1,t)=0,\quad t\in(0,T),\tag{3.3}$$

$$U^\eta(x,T)=U_T(x),\quad V^\eta(x,T)=V_T(x),\quad x\in(0,1),\tag{3.4}$$

where $0<\alpha<1/2$, $0<\eta<1$, $c_i\in L^\infty(Q_T)\,(1\le i\le 5)$, and $U_T,V_T\in L^2(0,1)$.

Let $\psi\in C^\infty([0,1])$ satisfy

$$\psi\begin{cases}=1, & x\in[0,(3x_0+2x_1)/5],\\ \in[0,1], & x\in\tilde\omega,\\ =0, & x\in[(2x_0+3x_1)/5,1],\end{cases}$$

where $\tilde\omega=((3x_0+2x_1)/5,(2x_0+3x_1)/5)$ and $\hat\omega=((4x_0+x_1)/5,(x_0+4x_1)/5)$. Set

$$\begin{gathered}\varphi(x,t)=\theta(t)g(x),\quad \Psi(x,t)=\theta(t)\big(\mathrm{e}^{2\zeta(0)}-\mathrm{e}^{\zeta(x)}\big),\quad (x,t)\in Q_T,\\ \Phi(x,t)=\psi(x)\varphi(x,t)-(1-\psi(x))\Psi(x,t),\quad (x,t)\in Q_T,\end{gathered}$$

where

$$\theta(t)=\frac{1}{(t(T-t))^4},\quad t\in(0,T),$$

$$g(x)=8((x+\eta)^{(4-2\alpha)/3}-8),\quad \zeta(x)=(1+\eta)^{1-\alpha/2}-(x+\eta)^{1-\alpha/2},\quad x\in(0,1).$$

The Carleman estimate for the solution to problem (3.1)–(3.4) is as follows.

Theorem 3.1 (Uniform Carleman estimate)**.** *Assume that $0<\alpha<1/2$, $0<\eta<1$, $c_i\in L^\infty(Q_T)$ with $\|c_i\|_{L^\infty(Q_T)}\le K\,(1\le i\le 5)$, and $c_4\big|_{(x_0,x_1)\times(0,T)}\ge c_0$ or $c_4\big|_{(x_0,x_1)\times(0,T)}\le -c_0$. There exist two constants $s_0>0$ and $M_0>0$ depending only on x_0, x_1, K, c_0, T, and α, such that for each $U_T,V_T\in L^2(0,1)$, the solution (U^η,V^η) to problem (3.1)–(3.4) satisfies*

$$\begin{aligned}&\int_0^T\int_0^1(s\theta(U_x^\eta)^2+s^3\theta^3(U^\eta)^2+s\theta(V_x^\eta)^2+s^3\theta^3(V^\eta)^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\ &\le M_0\int_0^T\int_\omega(U^\eta)^2\,\mathrm{d}x\,\mathrm{d}t,\quad s\ge s_0.\end{aligned}$$

Proof. For convenience, U^η and V^η are abbreviated by U and V, respectively, in the proof. Without loss of generality, it is assumed that $U,V\in C^2(\overline{Q}_T)$. Set

$$w(x,t)=\psi(x)U(x,t),\quad W(x,t)=\psi(x)V(x,t),\quad (x,t)\in Q_T.\tag{3.5}$$

Then, (w, W) solves

$$w_t + ((x+\eta)^\alpha w_x)_x = \rho_1, \quad W_t + ((x+\eta)^\alpha W_x)_x = \rho_2, \quad (x,t) \in Q_T, \tag{3.6}$$

where

$$\rho_1 = \varrho_1 - c_1 w_x + c_3 w, \quad \rho_2 = \varrho_2 - c_2 W_x + c_5 W, \quad (x,t) \in Q_T, \tag{3.7}$$

$$\varrho_1 = ((x+\eta)^\alpha \psi' U)_x + \psi'(x+\eta)^\alpha U_x + c_1\psi' U + c_4 \psi V, \quad (x,t) \in Q_T, \tag{3.8}$$

$$\varrho_2 = ((x+\eta)^\alpha \psi' V)_x + \psi'(x+\eta)^\alpha V_x + c_2 \psi' V, \quad (x,t) \in Q_T. \tag{3.9}$$

Set

$$Y(x,t) = \mathrm{e}^{s\varphi(x,t)} w(x,t), \quad Z(x,t) = \mathrm{e}^{s\varphi(x,t)} W(x,t), \quad (x,t) \in Q_T. \tag{3.10}$$

From (3.6) it follows that

$$\begin{aligned} &\mathrm{e}^{s\varphi}((\mathrm{e}^{-s\varphi}Y)_t + ((x+\eta)^\alpha(\mathrm{e}^{-s\varphi}Y)_x)_x) = \mathrm{e}^{s\varphi}\rho_1, \quad (x,t) \in Q_T, \\ &\mathrm{e}^{s\varphi}((\mathrm{e}^{-s\varphi}Z)_t + ((x+\eta)^\alpha(\mathrm{e}^{-s\varphi}Z)_x)_x) = \mathrm{e}^{s\varphi}\rho_2, \quad (x,t) \in Q_T. \end{aligned}$$

From [29, Proposition 3.1], there exist three positive constants M_1, M_2, and s_1 depending only on T and α, such that for each $s \ge s_1$,

$$\begin{aligned} &M_1 s \int_0^T \int_0^1 (x+\eta)^{(4\alpha-2)/3}\theta(Y_x^2 + Z_x^2)\,\mathrm{d}x\,\mathrm{d}t + M_2 s^3 \int_0^T \int_0^1 \theta^3(Y^2+Z^2)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^T \int_0^1 (\rho_1^2 + \rho_2^2)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

This formula, together with (3.7) and (3.10), leads to that for each $s \ge s_1$,

$$\begin{aligned} &M_1 s \int_0^T \int_0^1 (x+\eta)^{(4\alpha-2)/3}\theta(Y_x^2 + Z_x^2)\,\mathrm{d}x\,\mathrm{d}t + M_2 s^3 \int_0^T \int_0^1 \theta^3(Y^2+Z^2)\,\mathrm{d}x\,\mathrm{d}t \\ &\le 2\int_0^T \int_0^1 (\varrho_1^2 + \varrho_2^2)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + M_3 \int_0^T \int_0^1 (Y_x^2 + Z_x^2 + Y^2 + Z^2 + s^2(x+\eta)^{2(1-2\alpha)/3}\theta^2(Y^2+Z^2))\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

where $M_3 > 0$ depends only on K, T, and α. Therefore, there exist $s_2 \ge s_1$ and $M_4 > 0$ depend only on K, T, and α, such that for each $s \ge s_2$,

$$\begin{aligned} &s\int_0^T \int_0^1 (x+\eta)^{(4\alpha-2)/3}\theta((\mathrm{e}^{-s\varphi}Y)_x^2 + (\mathrm{e}^{-s\varphi}Z)_x^2)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t \\ &+ s^3 \int_0^T \int_0^1 \theta^3(Y^2+Z^2)\,\mathrm{d}x\,\mathrm{d}t \\ &\le M_4 \int_0^T \int_0^1 (\varrho_1^2 + \varrho_2^2)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.11}$$

From (3.11), $0 < \alpha < 1/2$, (3.8)–(3.10) and (3.5), we obtain

$$\begin{aligned} &\int_0^T \int_0^1 (s\theta w_x^2 + s^3\theta^3 w^2 + s\theta W_x^2 + s^3\theta^3 W^2)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t \\ &\le M_5\Big(\int_0^T \int_{\tilde\omega} (U^2 + U_x^2 + V^2 + V_x^2)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_0^1 W^2 \mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\Big), \quad s \ge s_2, \end{aligned}$$

where $M_5>0$ depends only on K, T, x_0, x_1, and α. Therefore, there exist $s_3\ge s_2$ and $M_6>0$ depend only on K, T, x_0, x_1, and α, such that for each $s\ge s_3$,

$$\begin{aligned}&\int_0^T\int_0^1(s\theta w_x^2+s^3\theta^3w^2+s\theta W_x^2+s^3\theta^3W^2)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\&\le M_6\int_0^T\int_{\tilde\omega}(U^2+U_x^2+V^2+V_x^2)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This formula, together with (3.5) and the definition of ψ, leads to that for each $s\ge s_3$,

$$\begin{aligned}&\int_0^T\int_0^{(3x_0+2x_1)/5}(s\theta U_x^2+s^3\theta^3U^2+s\theta V_x^2+s^3\theta^3V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\&\le M_6\int_0^T\int_{\tilde\omega}(U^2+U_x^2+V^2+V_x^2)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.12}$$

Set

$$q(x,t)=(1-\psi(x))U(x,t),\quad Q(x,t)=(1-\psi(x))V(x,t),\quad (x,t)\in Q_T.$$

From the classical Carleman estimate [1, Proposition 4.2], there exist $s_4>0$ and $M_7>0$ depend only on K, T, x_0, x_1, and α, such that for each $s\ge s_4$,

$$\begin{aligned}&\int_0^T\int_0^1(s\theta\mathrm{e}^{\zeta}q_x^2+s^3\theta^3\mathrm{e}^{3\zeta}q^2+s\theta\mathrm{e}^{\zeta}Q_x^2+s^3\theta^3\mathrm{e}^{3\zeta}Q^2)\mathrm{e}^{-2s\Psi}\,\mathrm{d}x\,\mathrm{d}t\\&\le M_7\int_0^T\int_{\tilde\omega}(U^2+U_x^2+V^2+V_x^2)\mathrm{e}^{-2s\Psi}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This formula, and the definition of ψ, leads to

$$\begin{aligned}&\int_0^T\int_{(2x_0+3x_1)/5}^1(s\theta\mathrm{e}^{\zeta}U_x^2+s^3\theta^3\mathrm{e}^{3\zeta}U^2+s\theta\mathrm{e}^{\zeta}V_x^2+s^3\theta^3\mathrm{e}^{3\zeta}V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\&\le M_7\int_0^T\int_{\tilde\omega}(U^2+U_x^2+V^2+V_x^2)\mathrm{e}^{-2s\Psi}\,\mathrm{d}x\,\mathrm{d}t,\quad s\ge s_4.\end{aligned}\tag{3.13}$$

It follows from (3.12), (3.13) and the definition of Φ,φ,Ψ,ψ that

$$\begin{aligned}&\int_0^T\int_0^1(s\theta U_x^2+s^3\theta^3U^2+s\theta V_x^2+s^3\theta^3V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\&\le M_8\int_0^T\int_{\tilde\omega}(U^2+U_x^2+V^2+V_x^2)(\mathrm{e}^{2s\varphi}+\mathrm{e}^{-2s\Psi}+\mathrm{e}^{2s\Phi})\,\mathrm{d}x\,\mathrm{d}t\\&\le 3M_8\int_0^T\int_{\tilde\omega}(U^2+U_x^2+V^2+V_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t,\quad s\ge\max\{s_3,s_4\},\end{aligned}\tag{3.14}$$

where $M_8>0$ depends only on K, T, x_0, x_1, and α. Let $\xi_1\in C^\infty([0,1])$ such that $\operatorname{supp}\xi_1\subset\hat\omega$, $0\le\xi_1\le1$ in $(0,1)$ and $\xi_1\equiv1$ in $\tilde\omega$. For $s>0$, (3.1)–(3.3) show

$$\begin{aligned}0&=\int_0^T\frac{d}{dt}\int_0^1\xi_1^2(U^2+V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\&=2s\int_0^T\int_0^1\xi_1^2\Phi_t(U^2+V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\&\quad+2\int_0^T\int_0^1\xi_1^2((x+\eta)^\alpha U_x^2+(x+\eta)^\alpha V_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$\begin{aligned}
&+ 4\int_0^T\int_0^1 \xi_1\xi_1'(U(x+\eta)^\alpha U_x + V(x+\eta)^\alpha V_x)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\
&- 2\int_0^T\int_0^1 \xi_1^2(c_1UU_x + c_2VV_x)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\
&+ 4s\int_0^T\int_0^1 \xi_1^2\Phi_x(U(x+\eta)^\alpha U_x + V(x+\eta)^\alpha V_x)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\
&+ 2\int_0^T\int_0^1 \xi_1^2(c_3U^2 + c_5V^2 + c_4UV)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

which leads to

$$\begin{aligned}
&\int_0^T\int_0^1 \xi_1^2((x+\eta)^\alpha U_x^2 + (x+\eta)^\alpha V_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{1}{2}\int_0^T\int_0^1 \xi_1^2((x+\eta)^\alpha U_x^2 + (x+\eta)^\alpha V_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + M_9(1+s^2)\int_0^T\int_{\hat\omega} \theta^2(U^2+V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

with $M_9 > 0$ depending only on K, T, x_0, x_1, and α. Hence, for $s > 0$,

$$\int_0^T\int_{\tilde\omega} (U_x^2 + V_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \le M_{10}(1+s^2)\int_0^T\int_{\hat\omega} \theta^2(U^2+V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t, \tag{3.15}$$

where $M_{10} > 0$ depends only on K, T, x_0, x_1, and α. From (3.14) and (3.15), there exist $s_5 \ge \max\{s_3, s_4\}$ and $M_{11} > 0$ depend only on K, T, x_0, x_1, and α, such that for each $s \ge s_5$,

$$\begin{aligned}
&\int_0^T\int_0^1 (s\theta U_x^2 + s^3\theta^3U^2 + s\theta V_x^2 + s^3\theta^3V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\
&\le M_{11}\int_0^T\int_{\hat\omega} (U^2+V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.16}$$

Let $\xi_2 \in C^\infty([0,1])$ such that $\operatorname{supp}\xi_2 \subset (x_0, x_1)$, $0 \le \xi_2 \le 1$ in $(0,1)$ and $\xi_2 \equiv 1$ in $\hat\omega$. Multiplying (3.1) by $\xi_2 V\mathrm{e}^{2s\Phi}$, then integrating by parts and using (3.2), one gets

$$\begin{aligned}
&\int_0^T\int_0^1 c_4\xi_2V^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\
&= -2\int_0^T\int_0^1 (x+\eta)^\alpha\xi_2U_xV_x\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_0^1 c_1\xi_2U_xV\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_0^T\int_0^1 c_2\xi_2UV_x\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_0^1 \Big((c_3+c_5+2s\Phi_t)\xi_2\\
&\quad - ((x+\eta)^\alpha(\xi_2\mathrm{e}^{2s\Phi})_x)_x\mathrm{e}^{-2s\Phi}\Big)UV\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.17}$$

The Hölder inequality gives

$$\begin{aligned}
&\Big|\int_0^T\int_0^1 (x+\eta)^\alpha\xi_2U_xV_x\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\le \frac{1}{2}\int_0^T\int_0^1 \theta U_x^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_0^1 \theta^{-1}V_x^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{3.18}$$

$$
\begin{aligned}
&\Big|\int_0^T\int_0^1 c_1\xi_2U_xV\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \frac{1}{2}\int_0^T\int_0^1 \theta U_x^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}K^2\int_0^T\int_0^1 \theta^{-1}V^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
&\Big|\int_0^T\int_0^1 c_2\xi_2UV_x\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \frac{1}{2}\int_0^T\int_0^1 \theta V_x^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}K^2\int_0^T\int_\omega \theta^{-1}U^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.20}
$$

and

$$
\begin{aligned}
&\Big|\int_0^T\int_0^1 ((c_3+c_5+2s\Phi_t)\xi_2-((x+\eta)^\alpha(\xi_2\mathrm{e}^{2s\Phi})_x)_x\mathrm{e}^{-2s\Phi})UV\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \frac{1}{2}(1+s^2)\int_0^T\int_0^1 \theta^3V^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + M_{12}(1+s^2)\int_0^T\int_\omega \theta^{-3}U^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.21}
$$

where $M_{12}>0$ depends only on K, T, x_0, x_1, and α. Since $c_4\big|_{(x_0,x_1)\times(0,T)}\ge c_0$ or $c_4\big|_{(x_0,x_1)\times(0,T)}\le -c_0$, it follows from (3.17)–(3.21) and the definition of ξ_2 that

$$
\begin{aligned}
&c_0\int_0^T\int_{\hat\omega} V^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{3}{2}\int_0^T\int_0^1 \theta U_x^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_0^T\int_0^1 \theta V_x^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +2\int_0^T\int_0^1 \theta^{-1}V_x^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}K^2\int_0^T\int_0^1 \theta^{-1}(U^2+V^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\frac{1}{2}(1+s^2)\int_0^T\int_0^1 \theta^3V^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + M_{12}(1+s^2)\int_0^T\int_\omega \theta^{-3}U^2\mathrm{e}^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.22}
$$

Then the theorem follows from (3.16) and (3.22). $\square$

Below we prove the observability inequality for the solution to problem (3.1)–(3.4).

Theorem 3.2 (Uniform observability inequality). *Assume that $0<\alpha<1/2$, $0<\eta<1$, $c_i\in L^\infty(Q_T)$ with $\|c_i\|_{L^\infty(Q_T)}\le K$ $(1\le i\le 5)$, and $c_4\big|_{(x_0,x_1)\times(0,T)}\ge c_0$ or $c_4\big|_{(x_0,x_1)\times(0,T)}\le -c_0$. There exists $M>0$ depending only on x_0, x_1, K, c_0, T, and α, but independent of η, such that for each $U_T,V_T\in L^2(0,1)$, the solution (U^η,V^η) to problem (3.1)–(3.4) satisfies*

$$
\int_0^1((U^\eta)^2(x,0)+(V^\eta)^2(x,0))dx\le M\int_0^T\int_\omega (U^\eta)^2\,\mathrm{d}x\,\mathrm{d}t.
$$

Proof. As in Theorem 3.1, U^η and V^η are abbreviated by U and V, respectively, and it is assumed that $U,V\in C^2(\overline{Q}_T)$. Multiplying (3.1) and (3.2) by U and V, respectively, and then integrating over $(0,1)$ with respect to x, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 U^2\,\mathrm{d}x - \int_0^1 (x+\eta)^\alpha U_x^2 \mathrm{d}x + \int_0^1 c_1 U U_x\,\mathrm{d}x - \int_0^1 c_3 U^2 \mathrm{d}x - \int_0^1 c_4 UV\mathrm{d}x = 0,$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 V^2\,\mathrm{d}x - \int_0^1 (x+\eta)^\alpha V_x^2 \mathrm{d}x + \int_0^1 c_2 V V_x\,\mathrm{d}x - \int_0^1 c_5 V^2\,\mathrm{d}x = 0,$$

for $t \in (0,T)$. Using the Hölder inequality and the Hardy inequality yields

$$-\frac{d}{dt}\int_0^1 (U^2+V^2)\,\mathrm{d}x \le \tilde{M}\int_0^1 (U^2+V^2)\,\mathrm{d}x, \quad t\in(0,T),$$

where $\tilde{M}>0$ depends only on K and α. Hence

$$\int_0^1 (U^2(x,0)+V^2(x,0))\,\mathrm{d}x \le \mathrm{e}^{\tilde{M}t}\int_0^1 (U^2(x,t)+V^2(x,t))\,\mathrm{d}x, \quad t\in(0,T). \tag{3.23}$$

Integrating (3.23) over $[T/4, 3T/4]$ leads to

$$\frac{T}{2}\int_0^1 (U^2(x,0)+V^2(x,0))\,\mathrm{d}x \le \mathrm{e}^{3\tilde{M}T/4}\int_{T/4}^{3T/4}\int_0^1 (U^2+V^2)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.24}$$

The theorem follows from (3.24), the Hardy inequality and Theorem 3.1. □

By a standard limit process, one can get the Carleman estimate and the observability inequality for the degenerate parabolic system

$$U_t + (x^\alpha U_x)_x + c_1(x,t)U_x - c_3(x,t)U - c_4(x,t)V = 0, \quad (x,t)\in Q_T, \tag{3.25}$$

$$V_t + (x^\alpha V_x)_x + c_2(x,t)V_x - c_5(x,t)V = 0, \quad (x,t)\in Q_T, \tag{3.26}$$

$$U(0,t)=V(0,t)=0, \quad U(1,t)=V(1,t)=0, \quad t\in(0,T), \tag{3.27}$$

$$U(x,T)=U_T(x), \quad V(x,T)=V_T(x), \quad x\in(0,1), \tag{3.28}$$

where $0<\alpha<1/2$, $c_i\in L^\infty(Q_T)$ $(1\le i\le 5)$, and $U_T, V_T \in L^2(0,1)$.

Theorem 3.3. *Assume that $0<\alpha<1/2$, $0<\eta<1$, $c_i\in L^\infty(Q_T)$ with $\|c_i\|_{L^\infty(Q_T)}\le K$ $(1\le i\le 5)$, and $c_4\big|_{(x_0,x_1)\times(0,T)}\ge c_0$ or $c_4\big|_{(x_0,x_1)\times(0,T)}\le -c_0$. There exist three positive constants s_0, M_0 and M depending only on x_0, x_1, K, c_0, T, and α, such that for each $U_T, V_T\in L^2(0,1)$, the solution (U,V) to problem* (3.25)–(3.28) *satisfies*

$$\begin{aligned}&\int_0^T\int_0^1 (s\theta U_x^2 + s^3\theta^3 U^2 + s\theta V_x^2 + s^3\theta^3 V^2)\mathrm{e}^{2s\tilde{\Phi}}\,\mathrm{d}x\,\mathrm{d}t\\ &\le M_0\int_0^T\int_\omega U^2\,\mathrm{d}x\,\mathrm{d}t, \quad s\ge s_0,\end{aligned}$$

$$\int_0^1 (U^2(x,0)+V^2(x,0))dx \le M\int_0^T\int_\omega U^2\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\tilde{\Phi}(x,t) = 8\theta(t)\psi(x)(x^{(4-2\alpha)/3}-8) - \theta(t)(1-\psi(x))\big(\mathrm{e}^2 - \mathrm{e}^{1-x^{1-\alpha/2}}\big),$$

for $(x,t)\in Q_T$.

4. Null controllability

In this section, we study the null controllability of system (1.1)–(1.4). First consider the nondegenerate parabolic system

$$u_t^\eta - ((x+\eta)^\alpha u_x^\eta)_x + (P_1(x,t,u^\eta))_x + F_1(x,t,u^\eta) = h^\eta \chi_\omega, \quad (x,t)\in Q_T, \tag{4.1}$$

$$v_t^\eta - ((x+\eta)^\alpha v_x^\eta)_x + (P_2(x,t,v^\eta))_x + F_2(x,t,u^\eta,v^\eta) = 0, \quad (x,t)\in Q_T, \tag{4.2}$$

$$u^\eta(0,t) = v^\eta(0,t) = 0, \quad u^\eta(1,t) = v^\eta(1,t) = 0, \quad t\in(0,T), \tag{4.3}$$

$$u^\eta(x,0) = u_0(x), \quad v^\eta(x,0) = v_0(x), \quad x\in(0,1), \tag{4.4}$$

where $0<\alpha<1/2$, $0<\eta<1$, and $u_0, v_0 \in L^2(0,1)$.

Lemma 4.1. *Assume that $0<\alpha<1/2$, $0<\eta<1$, and P_1,P_2,F_1,F_2 satisfy (1.10)–(1.13). For each $u_0,v_0\in L^2(0,1)$, there exists $h^\eta\in L^2(Q_T)$, such that the solution (u^η,v^η) to problem (4.1)–(4.4) satisfies*

$$u^\eta(x,T) = v^\eta(x,T) = 0, \quad x\in(0,1). \tag{4.5}$$

Furthermore, there exists $M>0$ depending only on x_0, x_1, K, c_0, T, and α, such that

$$\|h^\eta\|_{L^2(Q_T)} \le M(\|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)}). \tag{4.6}$$

Proof. For $(x,t,y,z)\in Q_T\times\mathbb{R}^2$, set

$$c_1(x,t,y) = \begin{cases} \frac{P_1(x,t,y)-P_1(x,t,0)}{y}, & y\neq 0,\\ 0, & y=0,\end{cases}$$

$$c_2(x,t,z) = \begin{cases} \frac{P_2(x,t,z)-P_2(x,t,0)}{z}, & z\neq 0,\\ 0, & z=0,\end{cases}$$

$$c_3(x,t,y) = \begin{cases} \frac{F_1(x,t,y)-F_1(x,t,0)}{y}, & y\neq 0,\\ 0, & y=0,\end{cases}$$

$$c_4(x,t,y,z) = \int_0^1 \frac{\partial F_2}{\partial y}(x,t,\lambda y,\lambda z)\,\mathrm{d}\lambda, \quad c_5(x,t,y,z) = \int_0^1 \frac{\partial F_2}{\partial z}(x,t,\lambda y,\lambda z)\,\mathrm{d}\lambda.$$

Then, (1.11)–(1.13) show that $c_1,c_2,c_3\in L^\infty(Q_T\times\mathbb{R})$ and $c_4,c_5\in L^\infty(Q_T\times\mathbb{R}^2)$ satisfy

$$\begin{gathered} \|c_i\|_{L^\infty(Q_T\times\mathbb{R})} \le K \quad (i=1,2,3)\\ \|c_j\|_{L^\infty(Q_T\times\mathbb{R}^2)} \le K \quad (j=4,5),\\ c_4\big|_{(x_0,x_1)\times(0,T)\times\mathbb{R}^2} \ge c_0 \quad \text{or} \quad c_4\big|_{(x_0,x_1)\times(0,T)\times\mathbb{R}^2} \le -c_0. \end{gathered}$$

Let y, $z\in L^2(Q_T)$. For $\varepsilon>0$, consider the problem

$$\min\Big\{\int_0^T\int_0^1 h^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^1 u^2(x,T)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^1 v^2(x,T)\,\mathrm{d}x : h\in L^2(Q_T)\Big\}, \tag{4.7}$$

where (u,v) is the solution to the problem

$$\begin{aligned} &u_t - ((x+\eta)^\alpha u_x)_x + (c_1(x,t,y(x,t))u)_x + c_3(x,t,y(x,t))u\\ &= h\chi_\omega, \quad (x,t)\in Q_T, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &v_t - ((x+\eta)^\alpha v_x)_x + (c_2(x,t,z(x,t))v)_x + c_4(x,t,y(x,t),z(x,t))u\\ &+ c_5(x,t,y(x,t),z(x,t))v = 0, \quad (x,t)\in Q_T, \end{aligned} \tag{4.9}$$

$$u(0,t)=v(0,t)=0,\quad u(1,t)=v(1,t)=0,\quad t\in(0,T), \tag{4.10}$$
$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad x\in(0,1). \tag{4.11}$$

As shown in [5], one can prove that problem (4.7)–(4.11) admits a unique solution $h^{\eta,\varepsilon,y,z}=U^{\eta,\varepsilon,y,z}\chi_\omega$, where $(U^{\eta,\varepsilon,y,z},V^{\eta,\varepsilon,y,z})$ is the solution to the problem

$$\begin{aligned}&U_t^{\eta,\varepsilon,y,z}+((x+\eta)^\alpha U_x^{\eta,\varepsilon,y,z})_x+c_1(x,t,y(x,t))U_x^{\eta,\varepsilon,y,z}\\&-c_3(x,t,y(x,t))U^{\eta,\varepsilon,y,z}-c_4(x,t,y(x,t),z(x,t))V^{\eta,\varepsilon,y,z}=0,\quad (x,t)\in Q_T,\end{aligned} \tag{4.12}$$

$$\begin{aligned}&V_t^{\eta,\varepsilon,y,z}+((x+\eta)^\alpha V_x^{\eta,\varepsilon,y,z})_x+c_2(x,t,z(x,t))V_x^{\eta,\varepsilon,y,z}\\&-c_5(x,t,y(x,t),z(x,t))V^{\eta,\varepsilon,y,z}=0,\quad (x,t)\in Q_T,\end{aligned} \tag{4.13}$$

$$U^{\eta,\varepsilon,y,z}(0,t)=V^{\eta,\varepsilon,y,z}(0,t)=0,\quad U^{\eta,\varepsilon,y,z}(1,t)=V^{\eta,\varepsilon,y,z}(1,t)=0,\quad t\in(0,T), \tag{4.14}$$

$$U^{\eta,\varepsilon,y,z}(x,T)=-\frac{1}{\varepsilon}u^{\eta,\varepsilon,y,z}(x,T),\quad V^{\eta,\varepsilon,y,z}(x,T)=-\frac{1}{\varepsilon}v^{\eta,\varepsilon,y,z}(x,T),\quad x\in(0,1), \tag{4.15}$$

with $(u^{\eta,\varepsilon,y,z},v^{\eta,\varepsilon,y,z})$ solving problem (4.8)–(4.11) for $h=h^{\eta,\varepsilon,y,z}$. Multiplying (4.8) with $h=h^{\eta,\varepsilon,y,z}$, (4.9), (4.12) and (4.13) by $U^{\eta,\varepsilon,y,z}$, $V^{\eta,\varepsilon,y,z}$, $u^{\eta,\varepsilon,y,z}$ and $v^{\eta,\varepsilon,y,z}$, respectively, and then integrating over Q_T by parts, we obtain

$$\begin{aligned}&\int_0^T\int_0^1 h^{\eta,\varepsilon,y,z}\chi_\omega U^{\eta,\varepsilon,y,z}\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\varepsilon}\int_0^1(u^{\eta,\varepsilon,y,z}(x,T))^2\,\mathrm{d}x\\&+\frac{1}{\varepsilon}\int_0^1(v^{\eta,\varepsilon,y,z}(x,T))^2\,\mathrm{d}x\\&=-\int_0^1U^{\eta,\varepsilon,y,z}(x,0)u_0(x)\,\mathrm{d}x-\int_0^1V^{\eta,\varepsilon,y,z}(x,0)v_0(x)\,\mathrm{d}x.\end{aligned} \tag{4.16}$$

It follows from (4.16), $h^{\eta,\varepsilon,y,z}=U^{\eta,\varepsilon,y,z}\chi_\omega$, the Hölder inequality and Theorem 3.2 that

$$\begin{aligned}&\int_0^T\int_0^1(h^{\eta,\varepsilon,y,z})^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\varepsilon}\int_0^1(u^{\eta,\varepsilon,y,z}(x,T))^2\,\mathrm{d}x+\frac{1}{\varepsilon}\int_0^1(v^{\eta,\varepsilon,y,z}(x,T))^2\,\mathrm{d}x\\&\le\int_0^1|U^{\eta,\varepsilon,y,z}(x,0)u_0(x)|\,\mathrm{d}x+\int_0^1|V^{\eta,\varepsilon,y,z}(x,0)v_0(x)|\,\mathrm{d}x\\&\le\frac{M}{2}\int_0^1(u_0^2(x)+v_0^2(x))\,\mathrm{d}x+\frac{1}{2M}\int_0^1((U^{\eta,\varepsilon,y,z}(x,0))^2+(V^{\eta,\varepsilon,y,z}(x,0))^2)\,\mathrm{d}x\\&\le\frac{M}{2}\int_0^1(u_0^2(x)+v_0^2(x))\,\mathrm{d}x+\frac{1}{2}\int_0^T\int_0^1(h^{\eta,\varepsilon,y,z})^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $M>0$ depending only on x_0, x_1, K, c_0, T and α, is given in Theorem 3.2. Hence

$$\begin{aligned}&\int_0^T\int_0^1(h^{\eta,\varepsilon,y,z})^2\,\mathrm{d}x\,\mathrm{d}t+\frac{2}{\varepsilon}\int_0^1(u^{\eta,\varepsilon,y,z}(x,T))^2\,\mathrm{d}x+\frac{2}{\varepsilon}\int_0^1(v^{\eta,\varepsilon,y,z}(x,T))^2\,\mathrm{d}x\\&\le M\int_0^1(u_0^2(x)+v_0^2(x))\,\mathrm{d}x,\end{aligned} \tag{4.17}$$

which, together with Lemma 2.1, leads to

$$\begin{aligned}
&\|u^{\eta,\varepsilon,y,z}\|_{L^\infty(0,T;L^2(0,1))} + \|(x+\eta)^{\alpha/2}u_x^{\eta,\varepsilon,y,z}\|_{L^2(Q_T)}\\
&+ \|v^{\eta,\varepsilon,y,z}\|_{L^\infty(0,T;L^2(0,1))} + \|(x+\eta)^{\alpha/2}v_x^{\eta,\varepsilon,y,z}\|_{L^2(Q_T)}\\
&\le N(\|h^{\eta,\varepsilon,y,z}\|_{L^2(Q_T)} + \|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)})\\
&\le N(M^{1/2}+1)(\|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)}),
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
&\int_0^{T-\delta}\int_0^1 (u^{\eta,\varepsilon,y,z}(x,\tau+\delta) - u^{\eta,\varepsilon,y,z}(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&+ \int_0^{T-\delta}\int_0^1 (v^{\eta,\varepsilon,y,z}(x,\tau+\delta) - v^{\eta,\varepsilon,y,z}(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\le N\delta^{1/2}(\|h^{\eta,\varepsilon,y,z}\|^2_{L^2(Q_T)} + \|u_0\|^2_{L^2(0,1)} + \|v_0\|^2_{L^2(0,1)})\\
&\le N\delta^{1/2}(M+1)(\|u_0\|^2_{L^2(0,1)} + \|v_0\|^2_{L^2(0,1)}), \quad 0<\delta<T,
\end{aligned} \tag{4.19}$$

where $N>0$ depending only on K, T and α, is given in Lemma 2.1. Define

$$\Lambda_\varepsilon : (y,z) \mapsto (u^{\eta,\varepsilon,y,z}, v^{\eta,\varepsilon,y,z}), \quad y,z \in B_R = \{w \in L^2(Q_T) : \|w\|_{L^2(Q_T)} \le R\}$$

with

$$R = NT^{1/2}(M^{1/2}+1)(\|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)}).$$

It follows from (4.18) and (4.19) that Λ_ε is a mapping from B_R to B_R, and Λ_ε is compact and continuous. Therefore, the Schauder fixed point theorem yields that Λ_ε admits a fixed point $(u^{\eta,\varepsilon}, v^{\eta,\varepsilon}) \in B_R \times B_R$, which solves problem (4.1)–(4.4) with $h = h^{\eta,\varepsilon} = h^{\eta,\varepsilon,u^{\eta,\varepsilon},v^{\eta,\varepsilon}}$. Moreover, (4.17)–(4.19) yield

$$\begin{aligned}
&\int_0^T\int_0^1 (h^{\eta,\varepsilon})^2\,\mathrm{d}x\,\mathrm{d}t + \frac{2}{\varepsilon}\int_0^1 (u^{\eta,\varepsilon}(x,T))^2\,\mathrm{d}x + \frac{2}{\varepsilon}\int_0^1 (v^{\eta,\varepsilon}(x,T))^2\,\mathrm{d}x\\
&\le M\int_0^1 (u_0^2(x) + v_0^2(x))\,\mathrm{d}x,\\
&\|u^{\eta,\varepsilon}\|_{L^\infty(0,T;L^2(0,1))} + \|(x+\eta)^{\alpha/2}u_x^{\eta,\varepsilon}\|_{L^2(Q_T)} + \|v^{\eta,\varepsilon}\|_{L^\infty(0,T;L^2(0,1))}\\
&+ \|(x+\eta)^{\alpha/2}v_x^{\eta,\varepsilon}\|_{L^2(Q_T)}\\
&\le N(M^{1/2}+1)(\|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)}),
\end{aligned}$$

$$\begin{aligned}
&\int_0^{T-\delta}\int_0^1 (u^{\eta,\varepsilon}(x,\tau+\delta) - u^{\eta,\varepsilon}(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&+ \int_0^{T-\delta}\int_0^1 (v^{\eta,\varepsilon}(x,\tau+\delta) - v^{\eta,\varepsilon}(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\le N\delta^{1/2}(M+1)(\|u_0\|^2_{L^2(0,1)} + \|v_0\|^2_{L^2(0,1)}).
\end{aligned}$$

Then there is $\varepsilon_n \in (0,1)$ with $\lim_{n\to\infty}\varepsilon_n = 0$, $E_m \subset Q_T$ with $\lim_{m\to\infty}\operatorname{meas} E_m = 0$, $h^\eta \in L^2(Q_T)$ and $u^\eta, v^\eta \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^1(0,1))$, such that

$$\begin{gathered}
h^{\eta,\varepsilon_n} \rightharpoonup h^\eta, \quad u^{\eta,\varepsilon_n} \rightharpoonup u^\eta, \quad v^{\eta,\varepsilon_n} \rightharpoonup v^\eta, \quad u_x^{\eta,\varepsilon_n} \rightharpoonup u_x^\eta, \quad v_x^{\eta,\varepsilon_n} \rightharpoonup v_x^\eta\\
\text{in } L^2(Q_T) \text{ as } n\to\infty,
\end{gathered} \tag{4.20}$$

$$\begin{gathered}
u^{\eta,\varepsilon_n} \to u^\eta, \quad v^{\eta,\varepsilon_n} \to v^\eta\\
\text{uniformly in } Q_T \backslash E_m \text{ as } n\to\infty \text{ for each positive integer } m,
\end{gathered} \tag{4.21}$$

$$u^\eta(x,T) = v^\eta(x,T) = 0, \quad x \in (0,1), \tag{4.22}$$

$$\int_0^T \int_0^1 (h^\eta)^2 \,\mathrm{d}x\,\mathrm{d}t \le M \int_0^1 (u_0^2(x) + v_0^2(x))\,\mathrm{d}x. \tag{4.23}$$

It follows from (1.11), (1.12) and (4.21) that

$$\begin{gathered} P_1(x,t,u^{\eta,\varepsilon_n}) \rightharpoonup P_1(x,t,u^\eta), \quad P_2(x,t,v^{\eta,\varepsilon_n}) \rightharpoonup P_2(x,t,v^\eta), \\ F_1(x,t,u^{\eta,\varepsilon_n}) \rightharpoonup F_1(x,t,u^\eta), \\ F_2(x,t,u^{\eta,\varepsilon_n}, v^{\eta,\varepsilon_n}) \rightharpoonup F_2(x,t,u^\eta,v^\eta) \quad \text{in } L^2(Q_T) \text{ as } n \to \infty. \end{gathered} \tag{4.24}$$

From (4.20) and (4.24), one can show that (u^η, v^η) is the solution to problem (4.1)–(4.4). Finally, (4.5) and (4.6) follow from (4.22) and (4.23). □

Now we are ready to prove the null controllability of the degenerate parabolic system (1.1)–(1.4).

Theorem 4.2. *Assume that $0 < \alpha < 1/2$, and P_1, P_2, F_1, F_2 satisfy (1.10)–(1.13). The system (1.1)–(1.4) is null controllable. More precisely, for each u_0, $v_0 \in L^2(0,1)$, there exists $h \in L^2(Q_T)$, such that the solution (u,v) to problem (1.1)–(1.4) satisfies*

$$u(x,T) = v(x,T) = 0, \quad x \in (0,1). \tag{4.25}$$

Furthermore, there exists $M > 0$ depending only on x_0, x_1, K, c_0, T, and α, such that

$$\|h\|_{L^2(Q_T)} \le M(\|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)}). \tag{4.26}$$

Proof. For each $0 < \eta < 1$, Lemma 4.1 shows that there exists $h^\eta \in L^2(Q_T)$ with

$$\|h^\eta\|_{L^2(Q_T)} \le M(\|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)}), \tag{4.27}$$

such that the solution (u^η, v^η) to problem (4.1)–(4.4) satisfies

$$u^\eta(x,T) = v^\eta(x,T) = 0, \quad x \in (0,1), \tag{4.28}$$

where $M > 0$ depends only on x_0, x_1, K, c_0, T, and α. Rewrite (4.1) and (4.2) into

$$\begin{gathered} u_t^\eta - ((x+\eta)^\alpha u_x^\eta)_x + (c_1(x,t)u^\eta)_x + c_3(x,t)u^\eta = h^\eta \chi_\omega, \quad (x,t) \in Q_T, \\ v_t^\eta - ((x+\eta)^\alpha v_x^\eta)_x + (c_2(x,t)v^\eta)_x + c_4(x,t)u^\eta + c_5(x,t)v^\eta = 0, \quad (x,t) \in Q_T, \end{gathered}$$

where for $(x,t) \in Q_T$,

$$\begin{gathered} c_1(x,t) = \begin{cases} \frac{P_1(x,t,u^\eta(x,t)) - P_1(x,t,0)}{u^\eta(x,t)}, & u^\eta(x,t) \ne 0, \\ 0, & u^\eta(x,t) = 0, \end{cases} \\ c_2(x,t) = \begin{cases} \frac{P_2(x,t,v^\eta(x,t)) - P_2(x,t,0)}{v^\eta(x,t)}, & v^\eta(x,t) \ne 0, \\ 0, & v^\eta(x,t) = 0, \end{cases} \\ c_3(x,t) = \begin{cases} \frac{F_1(x,t,u^\eta(x,t)) - F_1(x,t,0)}{u^\eta(x,t)}, & u^\eta(x,t) \ne 0, \\ 0, & u^\eta(x,t) = 0, \end{cases} \\ c_4(x,t) = \int_0^1 \frac{\partial F_2}{\partial y}(x,t,\lambda u^\eta(x,t), \lambda v^\eta(x,t))\,\mathrm{d}\lambda, \\ c_5(x,t) = \int_0^1 \frac{\partial F_2}{\partial z}(x,t,\lambda u^\eta(x,t), \lambda v^\eta(x,t))\,\mathrm{d}\lambda. \end{gathered}$$

Then, $c_i \in L^\infty(Q_T)$ with $\|c_i\|_{L^\infty(Q_T)} \le K$ $(1 \le i \le 5)$. Lemma 2.1 yields

$$\begin{aligned} &\|u^\eta\|_{L^\infty(0,T;L^2(0,1))} + \|(x+\eta)^{\alpha/2}u_x^\eta\|_{L^2(Q_T)} + \|v^\eta\|_{L^\infty(0,T;L^2(0,1))} \\ &+ \|(x+\eta)^{\alpha/2}v_x^\eta\|_{L^2(Q_T)} \\ &\le N(\|h^\eta\|_{L^2(Q_T)} + \|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)}), \end{aligned} \tag{4.29}$$

$$\begin{aligned} &\int_0^{T-\delta}\int_0^1 (u^\eta(x,\tau+\delta) - u^\eta(x,\tau))^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \int_0^{T-\delta}\int_0^1 (v^\eta(x,\tau+\delta) - v^\eta(x,\tau))^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &\le N\delta^{1/2}(\|h^\eta\|^2_{L^2(Q_T)} + \|u_0\|^2_{L^2(0,1)} + \|v_0\|^2_{L^2(0,1)}), \quad 0 < \delta < T, \end{aligned} \tag{4.30}$$

where $N > 0$ depending only on K, T and α, is given in Lemma 2.1. By (4.27)–(4.30), there exist $\eta_n \in (0,1)$ with $\lim_{n\to\infty}\eta_n = 0$, $E_m \subset Q_T$ with $\lim_{m\to\infty} \operatorname{meas} E_m = 0$, $h \in L^2(Q_T)$ and $u, v \in L^\infty(0,T;L^2(0,1)) \cap \mathscr{H}_\alpha$, such that

$$\begin{gathered} h^{\eta_n} \rightharpoonup h, \quad u^{\eta_n} \rightharpoonup u, \quad v^{\eta_n} \rightharpoonup v, \quad (x+\eta)^{\alpha/2}u_x^{\eta_n} \rightharpoonup x^{\alpha/2}u_x, \\ (x+\eta)^{\alpha/2}v_x^{\eta_n} \rightharpoonup x^{\alpha/2}v_x \quad \text{in } L^2(Q_T) \text{ as } n \to \infty, \end{gathered} \tag{4.31}$$

$$\begin{gathered} u^{\eta_n} \to u, \quad v^{\eta_n} \to v \\ \text{uniformly in } Q_T \backslash E_m \text{ as } n \to \infty \text{ for each positive integer } m, \end{gathered} \tag{4.32}$$

$$u(x,T) = v(x,T) = 0, \quad x \in (0,1), \tag{4.33}$$

$$\int_0^T\int_0^1 h^2 \,\mathrm{d}x\,\mathrm{d}t \le M\int_0^1 (u_0^2(x) + v_0^2(x))\,\mathrm{d}x. \tag{4.34}$$

It follows from (1.11), (1.12) and (4.32) that

$$\begin{gathered} P_1(x,t,u^{\eta_n}) \rightharpoonup P_1(x,t,u), \quad P_2(x,t,v^{\eta_n}) \rightharpoonup P_2(x,t,v), \\ F_1(x,t,u^{\eta_n}) \rightharpoonup F_1(x,t,u), \quad F_2(x,t,u^{\eta_n},v^{\eta_n}) \rightharpoonup F_2(x,t,u,v) \end{gathered} \tag{4.35}$$

in $L^2(Q_T)$ as $n \to \infty$. From (4.31) and (4.35), one can show that (u,v) solves problem (1.1)–(1.4). Finally, (4.25) and (4.26) follow from (4.33) and (4.34). $\square$

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References

[1] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli; *Carleman estimates for degenerate parabolic operators with applications to null controllability*, J. Evol. Equ., 6(2) (2006), 161–204.

[2] E. M. Ait, B. Hassi, F. Ammar Khodja, A. Hajjaj, L. Maniar; *Null controllability of degenerate parabolic cascade systems*, Portugal. Math., 68(3) (2011), 345–367.

[3] F. Ammar Khodja, A. Benabdallah, C. Dupaix, I. Kostin; *Controllability to the trajectories of phase-field models by one control force*, SIAM J. Control Optim., 42(5) (2003), 1661–1680.

[4] F. Ammar Khodja, A. Benabdallah, C. Dupaix, I. Kostin; *Null-controllability of some systems of parabolic type by one control force*, ESAIM Control Optim. Calc. Var., 11(3) (2005), 426–448.

[5] V. Barbu; *Controllability of parabolic and Navier-Stokes equations*, Sci. Math. Jpn., 56(1) (2002), 143–211.

[6] H. M. Byrne, M. R. Owen; *A new interpretation of the Keller-Segel model based on multiphase modelling*, J. Math. Biol., 49(6) (2004), 604–626.

[7] P. Cannarsa, G. Fragnelli; *Null controllability of semilinear degenerate parabolic equations in bounded domains*, Electron. J. Differential Equations, 2006(136) (2006), 1–20.

[8] P. Cannarsa, G. Fragnelli, J. Vancostenoble; *Regional controllability of semilinear degenerate parabolic equations in bounded domains*, J. Math. Anal. Appl., 320(2) (2006), 804–818.

[9] P. Cannarsa, P. Martinez, J. Vancostenoble; *Persistent regional null controllability for a class of degenerate parabolic equations*, Commun. Pure Appl. Anal., 3(4) (2004), 607–635.

[10] P. Cannarsa, P. Martinez, J. Vancostenoble; *Carleman estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim., 47(1) (2008), 1–19.

[11] P. Cannarsa, P. Martinez, J. Vancostenoble; *Null controllability of degenerate heat equations*, Adv. Differential Equations, 10(2) (2005), 153–190.

[12] P. Cannarsa, L. de Teresa; *Controllability of 1-d coupled degenerate parabolic equations*, Electron. J. Differential Equations, 2009(73) (2009), 1–21.

[13] R. M. Du, C. P. Wang; *Null controllability of a class of systems governed by coupled degenerate equations*, Appl. Math. Lett., 26 (2013), 113–119.

[14] R. M. Du, C. P. Wang, Q. Zhou; *Approximate controllability of a semilinear system involving a fully nonlinear gradient term*, Appl. Math. Optim., 70(1) (2014), 165–183.

[15] R. M. Du, F. D. Xu; *Null Controllability of a coupled degenerate system with the first order terms*, J. Dyn. Control. Syst., 24(1) (2018), 83–92.

[16] E. Fernández-Cara, E. Zuazua; *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Lineaire, 17(5) (2000), 583–616.

[17] C. Flores, L. Teresa; *Carleman estimates for degenerate parabolic equations with first order terms and applications*, C. R. Math. Acad. Sci. Paris, Ser. I, 348(7-8) (2010), 391–396.

[18] A. V. Fursikov, O. Y. Imanuvilov; *Controllability of evolution equations*, Lecture Notes Series 34, Seoul National University, Seoul, Korea, 1996.

[19] M. González-Burgos, R. Pérez-García; *Controllability of some coupled parabolic systems by one control force*, C. R. Math. Acad. Sci. Paris, 340(2)(2005), 125–130.

[20] M. González-Burgos, R. Pérez-García; *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal., 46(2)(2006), 123–162.

[21] M. González-Burgos, L. de Teresa; *Controllability results for cascade systems of m coupled parabolic PDEs by one control force*, Portugal. Math., 67(1) (2010), 91–113.

[22] S. Guerrero; *Null Controllability of some systems of two parabolic equations with one control force*, SIAM J. Control Optim., 46(2) (2007), 379–394.

[23] X. Liu, H. Gao, P. Lin; *Null controllability of a cascade system of degenerate parabolic equations*, Acta Math. Sci. A Chin. Ed., 28(6)(2008), 985–996.

[24] P. Martinez, J. Vancostenoble; *Carleman estimates for one-dimensional degenerate heat equations*, J. Evol. Equ., 6(2) (2006), 325–362.

[25] A. Schiaffino, A. Tesei; *Competition systems with Dirichlet boundary conditions*, J. Math. Biol., 15(1) (1982), 93–105.

[26] C. P. Wang; *Approximate controllability of a class of degenerate systems*, Appl. Math. Comput., 203(1) (2008), 447–456.

[27] C. P. Wang; *Approximate controllability of a class of semilinear systems with boundary degeneracy*, J. Evol. Equ., 10(1) (2010), 163–193.

[28] C. P. Wang and R. M. Du, Approximate controllability of a class of semilinear degenerate systems with convection term, J. Differential Equations, 254(9)(2013), 3665–3689.

[29] C. P. Wang, R. M. Du; *Carleman estimates and null controllability for a class of degenerate parabolic equations with convection terms*, SIAM J. Control Optim., 52(3) (2014), 1457–1480.

[30] C. P. Wang, L H. Wang, J. X. Yin, S. L.Zhou; *Hölder continuity of weak solutions of a class of linear equations with boundary degeneracy*, J. Differential Equations, 239(1) (2007), no. 1, 99–131.

[31] C. P. Wang, Y. N. Zhou, R. M. Du, Q. Liu; *Carleman estimate for solutions to a degenerate convection-diffusion equation*, Discrete Contin. Dyn. Syst. Ser. B, 23(10) (2018), 4207-4222.

[32] J. X. Yin, C. P. Wang; *Evolutionary weighted p-Laplacian with boundary degeneracy*, J. Differential Equations, 237(2) (2007), 421–445.

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