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GENERALIZED RIEMANN PROBLEM FOR A TOTALLY DEGENERATE HYPERBOLIC SYSTEM

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Abstract. We consider the generalized Riemann problem for the Suliciu relaxation system in Lagrangian coordinates and we calculate the first-order expansion given by LeFloch and Raviart to verify our results, then we show the explicit solution for the generalized Riemann problem in Eulerian coordinates, which has a similar structure as the classical Riemann problem.

1. Introduction

The aim of this article is to study the generalized Riemann problem associated with the Suliciu relaxation system [1, 14]

$$\begin{gathered}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + (\rho u^2 + s^2 v)_x = 0,\\ (\rho v)_t + (\rho u v + u)_x = 0,\end{gathered} \tag{1.1}$$

where s is a positive constant. The Suliciu relaxation system can be considered as a simplified viscoelastic shallow fluid model [11] where ρ denotes the layer depth of fluid, u is the horizontal velocity, s is a positive constant related to the stress tensor and v is the relaxed pressure. The existence of global weak solutions for the Suliciu relaxation system, including vacuum regions $\rho_0(x) \geq 0$, was obtained in [11] by using the vanishing viscosity method joint with a compensated compactness argument. The classical Riemann problem for the Suliciu relaxation system has been extensively studied in [1, 2, 3, 5]. The existence and uniqueness of delta shock solution for the Riemann problem were studied in [5, 6] and the generalized Riemann problem for the Suliciu relaxation system in Lagrangian coordinates was partially studied in [7]. In [5], the authors show uniqueness of global weak solutions for the classical Riemann and Cauchy problems for the Suliciu relaxation system. From [5, Theorem 2] with initial data $v_0(x) = -1/\rho_0(x)$ for $\rho_0(x) \geq \underline{\rho} > 0$, the

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system (1.1) is the relaxation for the isentropic Chaplygin gas dynamics system

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + (\rho u^2 - \frac{s^2}{\rho}) = 0. \end{aligned} \tag{1.2}$$

The system (1.2) was introduced by Chaplygin [4] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. The same model was rediscovered later by Tsien [15] and von Karman [16]. The negative pressure following from the equation of state could also be used for the description of certain effects in deformable solids [13]. The Chaplygin gas occurs in certain cosmology theories and has been announced as a possible model for dark energy [8, 12].

Although the uniqueness of global weak solutions for the Cauchy problem of the Suliciu relaxation system was studied in [5], in general the explicit solutions are difficult to construct. To understand better the explicit solutions we focus on the study of the generalized Riemann problem associated with the Suliciu relaxation system (1.1) in Eulerian coordinates with bounded initial data

$$\begin{gathered} (\rho, u, v)(x, 0) = (\rho_0, u_0, v_0)(x), \quad x \in \mathbb{R}, \\ \rho_0(x) \geq \underline{\rho} \end{gathered}$$

where $\underline{\rho}$ is a positive constant, the total variations of $u_0(x) \pm s v_0(x)$ are bounded and the functions ρ_0, u_0 and v_0 satisfy the generalized Lax shock condition

$$\sup_{x\in\mathbb{R}} \lambda_1(\rho_0, u_0, v_0) < \inf_{x\in\mathbb{R}} \lambda_3(\rho_0, u_0, v_0), \tag{1.3}$$

for the eigenvalues associated with the system

$$\lambda_1 = u - s/\rho, \quad \lambda_2 = u \quad \lambda_3 = u + s/\rho.$$

Observe that when the initial data is given by

$$(\rho, u, v)(x, 0) = \begin{cases} (\rho_l, u_l, v_l), & \text{if } x < 0, \\ (\rho_r, u_r, v_r), & \text{if } x > 0, \end{cases}$$

for the left and right constant states (ρ_l, u_l, v_l) and (ρ_r, u_r, v_r), respectively, the classical Lax shock condition [9, Definition 7.1] becomes

$$\lambda_1(\rho_l, u_l, v_l) < \lambda_3(\rho_r, u_r, v_r).$$

In this article we have expanded the results given in [7] for the case in Lagrangian coordinates, showing the interaction of elementary waves in Lagrangian coordinates. Additionally, we give an example of the interaction of elementary waves in Eulerian coordinates.

2. Generalized Riemann problem in Lagrangian coordinates

In this section, we show uniqueness of solutions for the Suliciu relaxation system in Lagrangian coordinates. Moreover, we study the interaction of elementary waves. Finally, we compare the solutions with the first-order asymptotic expansion of LeFloch-Raviart. Thereby, by the Euler-Lagrange (E-L) transformation $(x, t) \to (y, t) = (Y(x, t), t)$ defined by

$$dy = \rho dx - \rho u dt \quad \text{and} \quad Y(x, 0) = Y_0(x) \stackrel{def}{=} \int_0^x \rho_0(\xi)\, d\xi,$$

the Suliciu relaxation system (1.1) becomes

$$\begin{aligned} \omega_t - \nu_y &= 0,\\ \nu_t + s^2\kappa_y &= 0,\\ \kappa_t + \nu_y &= 0, \end{aligned} \tag{2.1}$$

where $\omega(y,t) = \frac{1}{\rho(x,t)}$, $\nu(y,t) = u(x,t)$ and $\kappa(y,t) = v(x,t)$. Now, we consider the Suliciu relaxation system in Lagrangian coordinates (2.1) with initial data

$$(\omega,\nu,\kappa)(y,0) = \begin{cases} (\omega_L^0, \nu_L^0, \kappa_L^0)(y), & \text{if } y < 0,\\ (\omega_R^0, \nu_R^0, \kappa_R^0)(y), & \text{if } y > 0, \end{cases} \tag{2.2}$$

where $\omega_i^0(y), \nu_i^0(y), \kappa_i^0(y)$, for $i = L$ or R, are piecewise smooth functions but discontinuous at $y = 0$. In this way, the solution of the generalized Riemann problem is

$$(\omega,\nu,\kappa)(y,t) = \begin{cases} (\omega_L, \nu_L, \kappa_L)(y,t), & \text{if } y < -st,\\ (\omega_*, \nu_*, \kappa_*)(y,t), & \text{if } -st < y < 0,\\ (\omega_{**}, \nu_{**}, \kappa_{**})(y,t), & \text{if } 0 < y < st,\\ (\omega_R, \nu_R, \kappa_R)(y,t), & \text{if } y > st, \end{cases} \tag{2.3}$$

where for $i = L$ or R,

$$\begin{aligned} \omega_i(y,t) &= \omega_i^0(y) + \kappa_i^0(y) - \kappa_i^0(y,t),\\ \nu_i(y,t) &= \Lambda^+_{\nu_i^0}(y,t) - s\Lambda^-_{\kappa_i^0}(y,t),\\ \kappa_i(y,t) &= \Lambda^+_{\kappa_i^0}(y,t) - \frac{1}{s}\Lambda^-_{\nu_i^0}(y,t) \end{aligned}$$

with $\Lambda_f^{\pm}(y,t) = \frac{1}{2}[f(y+st) \pm f(y-st)]$ and

$$\begin{aligned} \omega_*(y,t) &= \frac{\nu_R(y,t) - \nu_L(y,t)}{2s} - \frac{\kappa_R(y,t) - \kappa_L(y,t)}{2} + \omega_L(y,t),\\ \omega_{**}(y,t) &= \frac{\nu_R(y,t) - \nu_L(y,t)}{2s} + \frac{\kappa_R(y,t) - \kappa_L(y,t)}{2} + \omega_R(y,t),\\ \nu_*(y,t) &= \nu_{**}(y,t),\\ \kappa_*(y,t) &= \kappa_{**}(y,t). \end{aligned}$$

From the above, for the Suliciu relaxation system in Lagrangian coordinates we have the following result.

Theorem 2.1. *Given left and right states*

$$(\omega_L^0(y), \nu_L^0(y), \kappa_L^0(y)) \quad \textit{and} \quad (\omega_R^0(y), \nu_R^0(y), \kappa_R^0(y)),$$

respectively. The generalized Riemann problem for the Suliciu relaxation system in Lagrangian coordinates (2.1)–(2.2) *has an unique entropy solution.*

This result plays an important role in the study of the interaction of elementary waves for the Suliciu relaxation system in Lagrangian coordinates.

2.1. **Interaction of elementary waves.** For the interaction of elementary waves, we consider the Suliciu relaxation system in Lagrangian coordinates (2.1) with initial data

$$(\omega,\nu,\kappa)(y,0)=\begin{cases}(\omega_l^0,\nu_l^0,\kappa_l^0)(y), & \text{if } y<a,\\ (\omega_m^0,\nu_m^0,\kappa_m^0)(y), & \text{if } a<y<b,\\ (\omega_r^0,\nu_r^0,\kappa_r^0)(y), & \text{if } y>b,\end{cases} \tag{2.4}$$

with $a<b$.

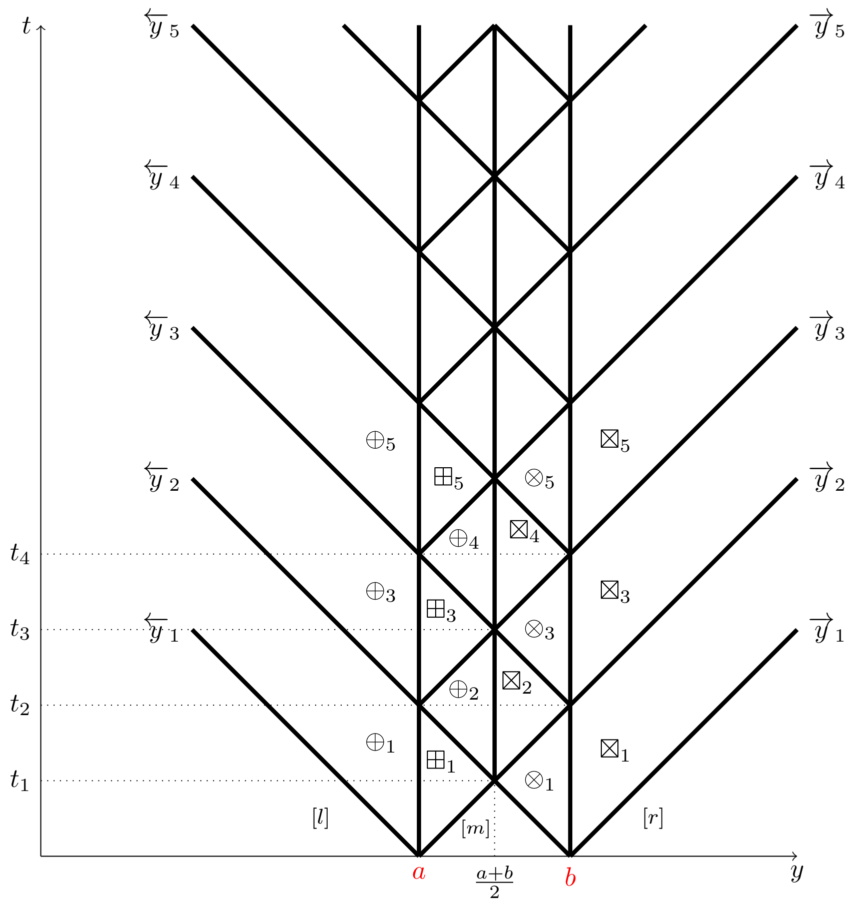

FIGURE 1. Interaction of elementary waves for the Suliciu relaxation system in Lagrangian coordinates.

In the Figure 1, the intermediate states are denoted by $\oplus_k$, $\boxplus_k$, $\otimes_k$ and $\boxtimes_k$. Here, the subscripts represent the k-interaction, $\oplus$ or $\otimes$ the first intermediate state while $\boxplus$ or $\boxtimes$ for the second one intermediate state in each Riemann problem. For example, for $0<t<t_1$ the 1-interaction of elementary waves is given by

$$\begin{gathered}(\omega_l,\nu_l,\kappa_l)(y,t)=[l], \quad \text{if } y<\overleftarrow{y_1}(t),\\ (\omega_*,\nu_*,\kappa_*)(y,t)=\oplus_1, \quad \text{if } \overleftarrow{y_1}(t)<y<a,\\ (\omega_{**},\nu_{**},\kappa_{**})(y,t)=\boxplus_1, \quad \text{if } a<y<\overrightarrow{y_2}(t),\end{gathered}$$

$$(\omega_m, \nu_m, \kappa_m)(y,t) = [m], \quad \text{if } y > \overrightarrow{y_2}(t),$$

and

$$\begin{gathered}
(\omega_m, \nu_m, \kappa_m)(y,t) = [m], \quad \text{if } y < \overleftarrow{y_2}(t),\\
(\widetilde{\omega}_*, \widetilde{\nu}_*, \widetilde{\kappa}_*)(y,t) = \otimes_1, \quad \text{if } \overleftarrow{y_2}(t) < y < b,\\
(\widetilde{\omega}_{**}, \nu_{**}, \widetilde{\kappa}_{**})(y,t) = \boxtimes_1, \quad \text{if } b < y < \overrightarrow{y_1}(t),\\
(\omega_r, \nu_r, \kappa_r)(y,t) = [r], \quad \text{if } y > \overrightarrow{y_1}(t),
\end{gathered}$$

where $\overleftarrow{y_1}(t) = -st + a$, $\overleftarrow{y_2}(t) = -st + b$, $\overrightarrow{y_1}(t) = st + b$ and $\overrightarrow{y_2}(t) = st + a$. Moreover, for $k = 1, 2, \dots,$

$$\begin{gathered}
t_k = k\Big(\frac{b-a}{2s}\Big),\\
\overleftarrow{y_k}(t) = -st + (2-k)a + (k-1)b,\\
\overrightarrow{y_k}(t) = st + (k-1)a + (2-k)b.
\end{gathered}$$

Let $(\overline{y}, \overline{t})$ be a point in $\mathbb{R} \times \mathbb{R}^+$. Consider the Riemann problem for the Suliciu relaxation system (2.1) with initial data

$$(\omega, \nu, \kappa)(y, \overline{t}) = \begin{cases} V_- = (\omega_-, \nu_-, \kappa_-)(y, \overline{t}), & \text{if } y < \overline{y},\\ V_+ = (\omega_+, \nu_+, \kappa_+)(y, \overline{t}), & \text{if } y > \overline{y}. \end{cases} \tag{2.5}$$

For $t > \overline{t}$, the solution for the Riemann problem (2.1)–(2.5) is given by

$$(\omega, \nu, \kappa)(y,t) = \begin{cases} V_- = (\omega_-, \nu_-, \kappa_-)(y,t), & \text{if } y < -s(t-\overline{t}) + \overline{y},\\ V_{\overline{*}} = (\omega_{\overline{*}}, \nu_{\overline{*}}, \kappa_{\overline{*}})(y,t), & \text{if } -s(t-\overline{t}) + \overline{y} < y < \overline{y},\\ V_{\overline{**}} = (\omega_{\overline{**}}, \nu_{\overline{**}}, \kappa_{\overline{**}})(y,t), & \text{if } \overline{y} < y < s(t-\overline{t}) + \overline{y},\\ V_+ = (\omega_+, \nu_+, \kappa_+)(y,t), & \text{if } y > s(t-\overline{t}) + \overline{y}. \end{cases} \tag{2.6}$$

To solve the Riemann problem (2.1)–(2.4), we consider two problems. In the first problem, we choose $V_- = (\omega_l, \nu_l, \kappa_l)$, $V_+ = (\omega_m, \nu_m, \kappa_m)$ and using (2.6) is obtained the first solution. In the second one, we choose $V_- = (\omega_m, \nu_m, \kappa_m)$, $V_+ = (\omega_r, \nu_r, \kappa_r)$ and once again by (2.6) is obtained the other solution. Observe that the states of the first Riemann problem are separated by the lines $\overleftarrow{y_1} = -st + a$, $y = a$ and $\overrightarrow{y_2} = st + a$, while the states of second problem by $\overleftarrow{y_2} = -st + b$, $y = b$ and $\overrightarrow{y_1} = st + b$. But the lines $\overrightarrow{y_2}$ and $\overleftarrow{y_2}$ intersect at $t_1 = \frac{b-a}{2s}$ and $y_1 = \frac{a+b}{2}$.

A new Riemann problem appears here with a second intermediate state of first Riemann problem and a first intermediate state of the second Riemann problem. Now, we choose $V_- = \boxplus_1$, $V_+ = \otimes_1$ and once again by (2.6) is obtained the solution for $t > t_1$.

In general, for $t_i = i\left(\frac{b-a}{2s}\right)$, $i = 1, 2, \dots,$ we have the following two situations:

(1) The Riemann problem with initial data $V_- = \boxplus_{2i-1}$ and $V_+ = \otimes_{2i-1}$, $i = 1, 2, \dots$. In this case, for $t_{2i-1} < t < t_{2i}$ the solution is given by

$$(\omega, \nu, \kappa)(y,t) = \begin{cases} \boxplus_{2i-1}, & \text{if } y < -s(t - t_{2i-1}) + \frac{a+b}{2},\\ \oplus_{2i}, & \text{if } -s(t - t_{2i-1}) + \frac{a+b}{2} < y < \frac{a+b}{2},\\ \boxtimes_{2i}, & \text{if } \frac{a+b}{2} < y < s(t - t_{2i-1}) + \frac{a+b}{2},\\ \otimes_{2i-1}, & \text{if } y > s(t - t_{2i-1}) + \frac{a+b}{2}. \end{cases}$$

(2a) The Riemann problem with initial data $V_- = \oplus_{2i-1}$ and $V_+ = \oplus_{2i}$, $i = 1, 2, \dots$. For $t_{2i} < t < t_{2i+1}$ the solution is given by

$$(\omega,\nu,\kappa)(y,t) = \begin{cases} \oplus_{2i-1}, & \text{if } y < -s(t-t_{2i}) + a, \\ \oplus_{2i+1}, & \text{if } -s(t-t_{2i}) + a < y < a, \\ \boxplus_{2i+1}, & \text{if } a < y < s(t-t_{2i}) + a, \\ \oplus_{2i}, & \text{if } y > s(t-t_{2i}) + a. \end{cases}$$

(2b) The Riemann problem with initial data $V_- = \boxtimes_{2i}$ and $V_+ = \boxtimes_{2i-1}$, $i = 1, 2, \dots$. For $t_{2i} < t < t_{2i+1}$ the solution is given by

$$(\omega,\nu,\kappa)(y,t) = \begin{cases} \boxtimes_{2i}, & \text{if } y < -s(t-t_{2i}) + b, \\ \otimes_{2i+1}, & \text{if } -s(t-t_{2i}) + b < y < b, \\ \boxtimes_{2i+1}, & \text{if } b < y < s(t-t_{2i}) + b, \\ \boxtimes_{2i-1}, & \text{if } y > s(t-t_{2i}) + b. \end{cases}$$

Now, for smooth solutions we compare the solution (2.3) with the asymptotic expansion of LeFloch-Raviart.

2.2. **Asymptotic expansion of LeFloch-Raviart.** For smooth solutions of the generalized Riemann problem, we consider the Taylor expansions of the initial data (2.2), $\omega_i^0(y) = \omega_i^0 + \sum_{j=1}^{\infty} \omega_i^j y^j$, $\nu_i^0(y) = \nu_i^0 + \sum_{j=1}^{\infty} \nu_i^j y^j$ and $\kappa_i^0(y) = \kappa_i^0 + \sum_{j=1}^{\infty} \kappa_i^j y^j$, $i = L$ or R. Then, by the asymptotic expansion of LeFloch-Raviart [10], for the first-order, we obtain that

$$\begin{aligned} \omega_i(y,t) &\approx \omega_i^0 + (y\omega_i^1 + t\nu_i^1), \\ \nu_i(y,t) &\approx \nu_i^0 + (y\nu_i^1 - s^2 t\kappa_i^1), \\ \kappa_i(y,t) &\approx \kappa_i^0 + (y\kappa_i^1 - t\nu_i^1), \quad \text{for } i = L \text{ or } R, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \omega_*(y,t) &\approx \omega_*^0 + y(\omega_L^1 + \kappa_L^1) - \Phi^-(y,t)/s, \\ \omega_{**}(y,t) &\approx \omega_{**}^0 + y(\omega_R^1 + \kappa_R^1) - \Phi^-(y,t)/s, \\ \nu_*(y,t) = \nu_{**}(y,t) &\approx \nu_*^0 + \Phi^+(y,t), \\ \kappa_*(y,t) = \kappa_{**}(y,t) &\approx \kappa_*^0 + \Phi^-(y,t)/s, \end{aligned} \tag{2.8}$$

where

$$\Phi^{\pm}(y,t) = \frac{1}{2}[(y-st)(\nu_L^1 + s\kappa_L^1) \pm (y+st)(\nu_R^1 - s\kappa_R^1)].$$

Note that for smooth solutions, the first-order of the Taylor expansion of the exact solution evaluated in $y = 0$, $(\omega,\nu,\kappa)(0,t)$, coincides with the expansion of Lefloch-Raviart (2.7)–(2.8).

Example 2.2. For $s > 1$, consider the generalized Riemann problem for (2.1) with initial data

$$(\omega,\nu,\kappa)(y,0) := \begin{cases} (2,0,1), & \text{if } y < 0, \\ (1,\cos(y),\sin(y)), & \text{if } y > 0. \end{cases}$$

By the first-order LeFloch-Raviart expansion, we obtain

$$\begin{gathered} \omega_R(y,t) \equiv 1, \quad \omega_L(y,t) \equiv 2, \\ \nu_R(y,t) \approx 1 - s^2 t, \quad \nu_L(y,t) \equiv 0, \\ \kappa_R(y,t) \approx y, \quad \kappa_L(y,t) \equiv 1, \end{gathered}$$

$$\begin{gathered}
\omega_*(y,t) \approx (5s+1)/2s - (y+st)/2, \\
\omega_{**}(y,t) \approx (s+1)/2 + (y-st)/2, \\
\nu_*(y,t) = \nu_{**}(y,t) \approx (s+1)/2 - s(y+st)/2, \\
\kappa_*(y,t) = \kappa_{**}(y,t) \approx \frac{(s-1)+s(y+st)}{2s}.
\end{gathered}$$

The exact solution of the generalized Riemann problem satisfies

$$\begin{gathered}
\omega_L(0,t) \equiv 2, \quad \nu_L(0,t) \equiv 0, \quad \kappa_L(0,t) \equiv 1, \\
\omega_R(0,t) \equiv 1, \quad \kappa_R(0,t) \equiv 0, \\
\nu_R(0,t) = \cos(st) - s\sin(st) = 1 - s^2 t + \mathcal{O}((st)^2), \\
\omega_*(0,t) = \frac{5s+1}{2s} - \frac{s}{2}t + \mathcal{O}(st^2), \quad \omega_{**}(0,t) = \frac{s+1}{2} - \frac{s}{2}t + \mathcal{O}(st^2), \\
\nu_*(0,t) = \frac{s+1}{2} - \frac{s^2}{2}t + \mathcal{O}((st)^2), \quad \kappa_*(0,t) = \frac{s-1}{2s} + \frac{s}{2}t + \mathcal{O}(st^2).
\end{gathered}$$

3. Generalized Riemann problem in Eulerian coordinates

Now, we consider the Suliciu relaxation system in Eulerian coordinates (1.1) with initial data

$$(\rho,u,v)(x,0) = \begin{cases} (\rho_l^0, u_l^0, v_l^0)(x), & \text{if } x<0, \\ (\rho_r^0, u_r^0, v_r^0)(x), & \text{if } x>0, \end{cases} \tag{3.1}$$

where $\rho_i^0(x), u_i^0(x), v_i^0(x)$, for $i = l, r$, are piecewise smooth functions but discontinuities at $x = 0$. The solution of generalized Riemann problem is of the form

$$(\rho,u,v)(x,t) = \begin{cases} (\rho_l, u_l, v_l)(x,t), & \text{if } x < x_1(t), \\ (\rho_*, u_*, v_*)(x,t), & \text{if } x_1(t) < x < x_2(t), \\ (\rho_{**}, u_{**}, v_{**})(x,t), & \text{if } x_2(t) < x < x_3(t), \\ (\rho_r, u_r, v_r)(x,t), & \text{if } x > x_3(t). \end{cases} \tag{3.2}$$

Each component in (3.2) is given by

$$\begin{gathered}
\rho_i(x,t) = \frac{1}{\frac{1}{\rho_i^0(X_0(Y(x,t)))} + v_i^0(X_0(Y(x,t))) - v_l(x,t)}, \\
u_i(x,t) = \Gamma^+_{u_i^0}(x,t) - s\Gamma^-_{v_i^0}(x,t), \\
v_i(x,t) = \Gamma^+_{v_i^0}(x,t) - \frac{1}{s}\Gamma^-_{u_i^0}(x,t),
\end{gathered}$$

where the functions X_0, Y_0, X and Y are defined by the E-L transformation,

$$\Gamma_g^\pm(x,t) = \frac{1}{2}[g(X_0(Y(x,t)+st)) \pm g(X_0(Y(x,t)-st))]$$

and the intermediate states are

$$\begin{gathered}
\frac{1}{\rho_*(x,t)} = \frac{1}{\rho_l(x,t)} + \frac{u_r(x,t)-u_l(x,t)}{2s} - \frac{v_r(x,t)-v_l(x,t)}{2}, \\
\frac{1}{\rho_{**}(x,t)} = \frac{1}{\rho_r(x,t)} + \frac{u_r(x,t)-u_l(x,t)}{2s} + \frac{v_r(x,t)-v_l(x,t)}{2}, \\
u_*(x,t) = \frac{u_r(x,t)+u_l(x,t)}{2} - s\frac{v_r(x,t)-v_l(x,t)}{2} = u_{**}(x,t),
\end{gathered}$$

$$v_*(x,t) = \frac{v_r(x,t)+v_l(x,t)}{2} - \frac{u_r(x,t)-u_l(x,t)}{2s} = v_{**}(x,t).$$

Moreover, the k-th contact discontinuity $x = x_k(t)$, $k = 1,2,3$, satisfies

$$\begin{gathered}\frac{dx_1(t)}{dt} = u_l(x_1(t),t) - \frac{s}{\rho_l(x_1(t),t)} = u_*(x_1(t),t) - \frac{s}{\rho_*(x_1(t),t)},\\ x_1(0) = 0,\\ x_1'(0) = u_l(0,0) - \frac{s}{\rho_l(0,0)},\end{gathered}$$

$$\begin{gathered}\frac{dx_2(t)}{dt} = u_*(x_2(t),t) = u_{**}(x_2(t),t),\\ x_2(0) = 0,\\ x_2'(0) = u_*(0,0),\end{gathered}$$

and

$$\begin{gathered}\frac{dx_3(t)}{dt} = u_r(x_3(t),t) + \frac{s}{\rho_r(x_3(t),t)} = u_{**}(x_3(t),t) + \frac{s}{\rho_{**}(x_3(t),t)},\\ x_3(0) = 0,\\ x_3'(0) = u_r(0,0) + \frac{s}{\rho_r(0,0)},\end{gathered}$$

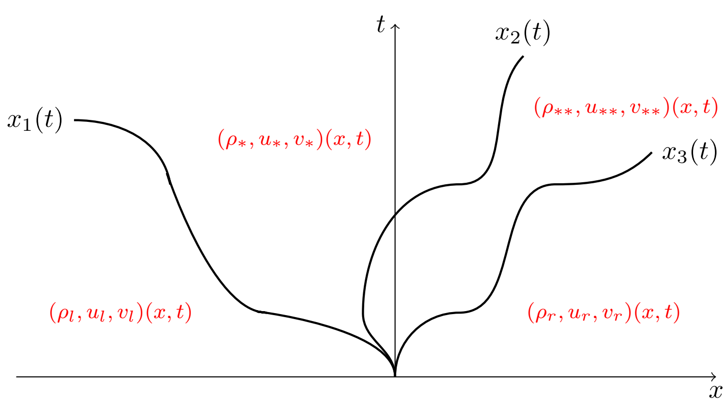

FIGURE 2. Solution for the generalized Riemann problem.

As in Section 2, we have the following result.

Theorem 3.1. *Given left and right states* $(\rho_l^0(x), u_l^0(x), v_l^0(x))$ *and* $(\rho_r^0(x), u_r^0(x), v_r^0(x))$, *respectively, such that the initial data* (3.1) *satisfy the generalized Lax shock condition* (1.3) *and the total variations of* $u_0(x) \pm sv_0(x)$ *are bounded. Then, the generalized Riemann problem for the Suliciu relaxation system* (1.1)–(3.1) *has an unique entropy solution.*

Figure 2 corresponds to the solution for the generalized Riemann problem in Eulerian coordinates.

3.1. **Example of a interaction of waves.** Now, we are interested in the interaction of elementary waves for the generalized Riemann problem associated with the Suliciu relaxation system (1.1). In this sense, we consider (1.1) with initial data

$$(\rho_0,u_0,v_0)(x)=\begin{cases}(\rho_l,u_l,v_l), & \text{if } x<0,\\ (\rho_m(x),u_m,v_m), & \text{if } 0<x<b,\\ (\rho_r,u_r,v_r), & \text{if } x>b,\end{cases}$$

where ρ_l,ρ_r,u_i,v_i, $i=l,m$ or r, are constants and $\rho_m(x)=e^x$.

Thereby, the solution on the left, right and middle states is given by

$$\begin{gathered}\rho_l(x,t)=\rho_l,\quad \rho_m(x,t)=e^{x-u_mt},\quad \rho_r(x,t)=\rho_r,\\ u_l(x,t)=u_l,\quad u_m(x,t)=u_m,\quad u_r(x,t)=u_r,\\ v_l(x,t)=v_l,\quad v_m(x,t)=v_m,\quad v_r(x,t)=v_r,\end{gathered}$$

and the intermediate states by

$$\begin{gathered}\rho_*(x,t)=\frac{1}{\frac{1}{\rho_l}+\frac{u_m-u_l}{2s}-\frac{v_m-v_l}{2}}=\rho_*,\\ \rho_{**}(x,t)=\frac{1}{\frac{1}{e^{x-u_mt}}+\frac{u_m-u_l}{2s}+\frac{v_m-v_l}{2}},\\ u_*(x,t)=\frac{u_m+u_l}{2}-s\frac{v_m-v_l}{2}=u_{**}(x,t),\\ v_*(x,t)=\frac{v_m+v_l}{2}-\frac{u_m-u_l}{2s}=v_{**}(x,t).\end{gathered}$$

and

$$\begin{gathered}\widetilde{\rho}_*(x,t)=\frac{1}{\frac{1}{e^{x-u_mt}}+\frac{u_r-u_m}{2s}-\frac{v_r-v_m}{2}},\\ \widetilde{\rho}_{**}(x,t)=\frac{1}{\frac{1}{\rho_r}+\frac{u_r-u_m}{2s}+\frac{v_r-v_m}{2}}=\widetilde{\rho}_{**},\\ \widetilde{u}_*(x,t)=\frac{u_r+u_m}{2}-s\frac{v_r-v_m}{2}=\widetilde{u}_{**}(x,t)\text{ and}\\ \widetilde{v}_*(x,t)=\frac{v_r+v_m}{2}-\frac{u_r-u_m}{2s}=\widetilde{u}_{**}(x,t).\end{gathered}$$

Also, the curves $x_i=x_i(t)$, $\widetilde{x}_i=\widetilde{x}_i(t)$ for $i=1,2,3$, are

$$\begin{gathered}x_1(t)=\Big(u_l-\frac{s}{\rho_l}\Big)t,\\ x_2(t)=\Big(\frac{u_m+u_l}{2}-s\frac{v_m-v_l}{2}\Big)t,\\ x_3(t)=u_mt+\ln(st+1),\end{gathered}$$

and

$$\begin{gathered}\widetilde{x}_1(t)=u_mt+\ln(e^b-st),\\ \widetilde{x}_2(t)=\Big(\frac{u_r+u_m}{2}-s\frac{v_r-v_m}{2}\Big)t+b,\\ \widetilde{x}_3(t)=\Big(u_r+\frac{s}{\rho_r}\Big)t+b.\end{gathered}$$

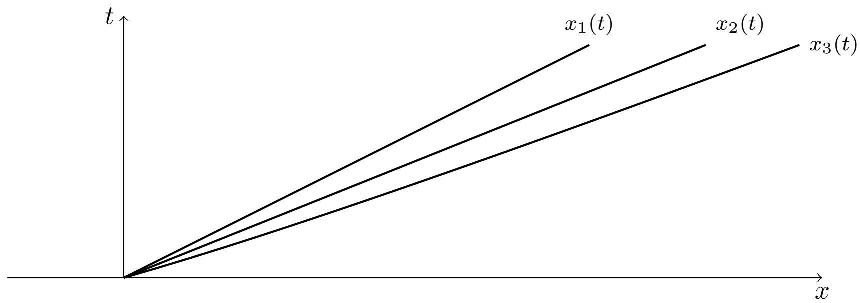

FIGURE 3. The curves $x_i = x_i(t)$ in the plane $t - x$.

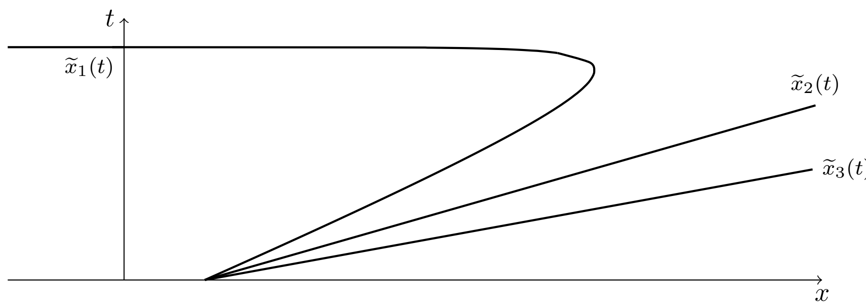

FIGURE 4. The curves $\widetilde{x}_i = \widetilde{x}_i(t)$ in the plane $t - x$.

Now, we observe that the curves $x_3 = x_3(t)$ and $\widetilde{x}_1 = \widetilde{x}_1(t)$ intersect at point (t_1, x_1) defined by

$$\begin{gathered} t_1 = \frac{e^b - 1}{2s}, \\ x_1 = u_m \frac{e^b - 1}{2s} + \ln\Big(\frac{e^b + 1}{2}\Big) = u_m t_1 + \ln\Big(\frac{e^b + 1}{2}\Big). \end{gathered} \tag{3.3}$$

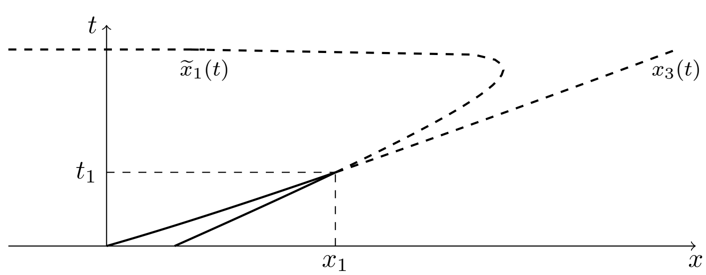

FIGURE 5. Point of intersection (t_1, x_1) of the curves $x_3(t)$ and $\widetilde{x}_1(t)$.

Following the way of getting the interaction of elementary waves, we propose to solve the following Riemann problem associated with the Suliciu relaxation system

(1.1) with initial data

$$
\begin{aligned}
&(\rho_{**}, u_{**}, v_{**})(x, t_1), \quad \text{if } x < x_1,\\
&(\widetilde{\rho}_*, \widetilde{u}_*, \widetilde{v}_*)(x, t_1), \quad \text{if } x > x_1,
\end{aligned}
$$

where x_1 and t_1 is given by (3.3). For time $t > t_1$, we must find the solution in four new regions until some time t_2 as shown in Figure 6.

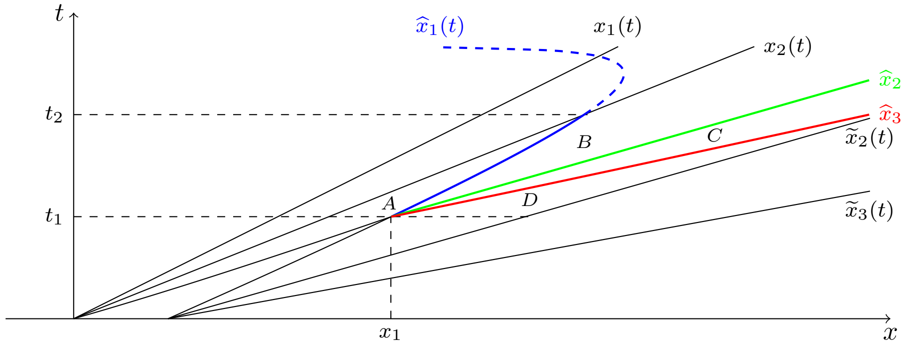

FIGURE 6. Regions A, B, C and D.

In the region $A = \{(x,t) : t_1 < t < t_2, x_2(t) < x < \widehat{x}_1(t) \text{ and } x_2(t_2) = \widehat{x}_1(t_2)\}$, we have

$$
\begin{gathered}
\frac{1}{\rho_{**}(x,t)} = \frac{1}{e^{x-u_m t_1 - u_*(t-t_1)}} + C_1,\\
u_{**}(x,t) = \frac{u_m + u_l}{2} - s\frac{v_m - v_l}{2} = u_* = u_{**},\\
v_{**}(x,t) = \frac{v_m + v_l}{2} - \frac{u_m - u_l}{2s} = v_* = v_{**}.
\end{gathered}
$$

Moreover, the curve $\widehat{x}_1 = \widehat{x}_1(t)$ is

$$
\widehat{x}_1(t) = \begin{cases} u_m t_1 + u_*(t-t_1) + \ln\left(\frac{1+e^{C_1 s(t-t_1)}(C_1 e^{x_1 - u_m t_1} - 1)}{C_1}\right), & \text{if } C_1 \neq 0,\\ u_m t_1 + u_*(t-t_1) + \ln\left(e^{x_1 - u_m t_1} - s(t-t_1)\right), & \text{if } C_1 = 0, \end{cases}
$$

where

$$
C_1 = \frac{u_m - u_l}{2s} + \frac{v_m - v_l}{2}.
$$

In the region $D = \{(x,t) : t_1 < t < t_3, \widehat{x}_3(t) < x < \widetilde{x}_2(t) \text{ and } \widetilde{x}_2(t_3) = \widehat{x}_3(t_3)\}$, we have

$$
\begin{gathered}
\frac{1}{\widetilde{\rho}_*(x,t)} = \frac{1}{e^{x-u_m t_1 - \widetilde{u}_*(t-t_1)}} + C_2,\\
\widetilde{u}_*(x,t) = \frac{u_r + u_m}{2} - s\frac{v_r - v_m}{2} = \widetilde{u}_* = \widetilde{u}_{**},\\
\widetilde{v}_*(x,t) = \frac{v_r + v_m}{2} - \frac{u_r - u_m}{2s} = \widetilde{v}_* = \widetilde{v}_{**},
\end{gathered}
$$

and the curve $\widehat{x}_3 = \widehat{x}_3(t)$ is given by

$$
\widehat{x}_3(t) = \begin{cases} u_m t_1 + \widetilde{u}_*(t-t_1) + \ln\left(\frac{e^{C_2 s(t-t_1)}(C_2 e^{x_1 - u_m t_1} + 1) - 1}{C_2}\right), & \text{if } C_2 \neq 0,\\ u_m t_1 + \widetilde{u}_*(t-t_1) + \ln\left(s(t-t_1) + e^{x_1 - u_m t_1}\right), & \text{if } C_2 = 0, \end{cases}
$$

with
$$C_2 = \frac{u_r - u_m}{2s} - \frac{v_r - v_m}{2}.$$
In the regions B and C, we obtain that
$$\begin{gathered}
\frac{1}{\widehat{\rho}_*(x,t)} = \frac{1}{e^{x-u_m t_1 - u_*(t-t_1)}} + C_1 + \frac{\widetilde{u}_* - u_*}{2s} - \frac{\widetilde{v}_* - v_*}{2}, \\
\frac{1}{\widehat{\rho}_{**}(x,t)} = \frac{1}{e^{x-u_m t_1 - \widetilde{u}_*(t-t_1)}} + C_1 + \frac{\widetilde{u}_* - u_*}{2s} + \frac{\widetilde{v}_* - v_*}{2}, \\
\widehat{u}_*(x,t) = \frac{\widetilde{u}_* + u_*}{2} - s\frac{\widetilde{v}_* - v_*}{2} = \widehat{u}_{**}(x,t), \\
\widehat{v}_*(x,t) = \frac{\widetilde{v}_* + v_*}{2} - \frac{\widetilde{u}_* - u_*}{2s} = \widehat{v}_{**}(x,t),
\end{gathered}$$
and where the curve $\widehat{x}_2 = \widehat{x}_2(t)$ is
$$\widehat{x}_2(t) = x_1 + \Big(\frac{\widetilde{u}_* + u_*}{2} - s\frac{\widetilde{v}_* - v_*}{2}\Big)(t - t_1).$$

Observe that, at time t_2 the curves $x_2(t)$ and $\widehat{x}_1(t)$ intersect and a new Riemann problem should be considered for the Suliciu relaxation system with initial data
$$\begin{aligned}
&(\rho_*, u_*, v_*)(x, t_2), \quad \text{if } x < x_2, \\
&(\widehat{\rho}_*, \widehat{u}_*, \widehat{v}_*)(x, t_2), \quad \text{if } x > x_2.
\end{aligned}$$
In each new intersection, we obtain a Riemann problem which can be solve of natural form.

3.2. **Numerical solutions.** Now, we show some numerical solutions for the generalized Riemann problem for the system (1.1) in Eulerian coordinates. Our numerical evidences were studied for the interaction of elementary waves. Similar numerical results can be obtained for Lagrangian coordinates, we shall omit them. For the system (1.1), we denote $m = \rho u$ $w = \rho v$, $U = (\rho, m, w)$ and the initial condition by $U_0 = (\rho_0, m_0, w_0)$. The Lax-Friedrichs scheme is obtained in the following way: let h, k be positive numbers satisfying the CFL condition
$$\frac{k}{h} \max_{U \in \Sigma} \{\lambda_1(U), \lambda_2(U), \lambda_3(U)\} < 1$$
where Σ is some region containing the initial data. Then we define the grid points $t_n = nk, x_j = jh$ and $x_{j+1/2} = (j + 1/2)h$ where $n \in \mathbb{N}$, $j \in \mathbb{Z}$. The initial data $U_0 = (\rho_0, m_0, w_0)$ is approximated by
$$U_0^h(x) := \sum_j U_j^0 \chi_{(x_{j-1/2}, x_{j+1/2}]}(x),$$
where U_j^0 are constant states in Σ such that U_0^h converges weakly to U_0 as h approaches zero, e.g.
$$U_j^0 = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} U_0(x) dx.$$
Now suppose that the approximate solution U^h has been defined in some strip $\mathbb{R} \times [0, t_n)$, $n \geq 1$. Then, in each rectangle $R_{j,n} = (x_{j-1/2}, x_{j+1/2}] \times [t_n, t_{n+1})$ we define $U^h(x,t)$ as the constant U_j^n where, for the system (1.1), it is given by
$$\rho_j^n = \frac{1}{2}\left(\rho_{j-1}^{n-1} + \rho_{j+1}^{n-1}\right) - \frac{k}{2h}\left(m_{j+1}^{n-1} - m_{j-1}^{n-1}\right),$$

$$
\begin{aligned}
m_j^n &= \frac{1}{2}\left(m_{j-1}^{n-1}+m_{j+1}^{n-1}\right)-\frac{k}{2h}\Big(\frac{(m_{j+1}^{n-1})^2+s^2w_{j+1}^{n-1}}{\rho_{j+1}^{n-1}}-\frac{(m_{j-1}^{n-1})^2+s^2w_{j-1}^{n-1}}{\rho_{j-1}^{n-1}}\Big),\\
w_j^n &= \frac{1}{2}\left(w_{j-1}^{n-1}+w_{j+1}^{n-1}\right)-\frac{k}{2h}\Big(\frac{m_{j+1}^{n-1}(w_{j+1}^{n-1}+1)}{\rho_{j+1}^{n-1}}-\frac{m_{j-1}^{n-1}(w_{j-1}^{n-1}+1)}{\rho_{j-1}^{n-1}}\Big).
\end{aligned}
$$

The CFL condition guarantees that $\rho_j^n > 0$. We also shall set $u_j^n = m_j^n/\rho_j^n$ and $v_j^n = w_j^n/\rho_j^n$. Now, we consider the Riemann problem for the system (1.1) with $s = 1$ and initial data

$$
(\rho_0, u_0, v_0)(x) = \begin{cases} (1, 3, \frac{7}{2}), & \text{if } x < 0,\\ (e^x, \frac{5}{2}, 4), & \text{if } 0 < x < \ln(4),\\ (\frac{1}{2}, \frac{7}{2}, 3), & \text{if } x > \ln(4). \end{cases} \tag{3.4}
$$

For CFL= 0.994 and final time $t = 1.0$, the numerical results are show in the Figure 7.

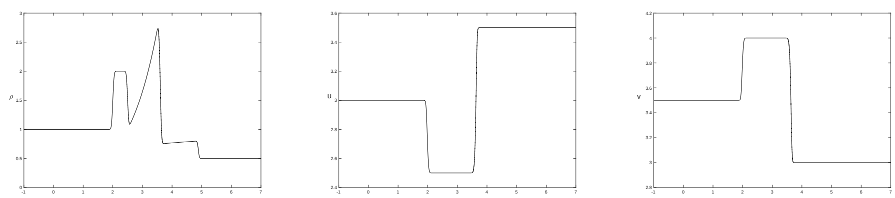

FIGURE 7. Numerical solution for the generalized Riemann problem (1.1)–(3.4). On the left, ρ at the time $t = 1.0$; On the middle, u at the time $t = 1.0$; On the right, v at the time $t = 1.0$.

From subsection 3.1, for $s = 1$ and $0 < t < 3/2$, the solution for problem (1.1)–(3.4) is given by

$$
\begin{aligned}
&(\rho(x,t), u(x,t), v(x,t))\\
&= \begin{cases} (1, 3, \frac{7}{2}), & \text{if } x < 2t,\\ (2, \frac{5}{2}, 4), & \text{if } 2t < x < \frac{5}{2}t,\\ (e^{x-\frac{5}{2}t}, \frac{5}{2}, 4), & \text{if } \frac{5}{2}t < x < \frac{5}{2}t + \ln(t+1),\\ (e^x, \frac{5}{2}, 4), & \text{if } \frac{5}{2}t + \ln(t+1) < x < \frac{5}{2}t + \ln(4-t),\\ (e^{x-\frac{5}{2}t}/(1+e^{x-\frac{5}{2}t}), \frac{7}{2}, 3), & \text{if } \frac{5}{2}t + \ln(4-t) < x < \frac{7}{2}t + \ln(4),\\ (\frac{1}{2}, \frac{7}{2}, 3), & \text{if } \frac{7}{2}t + \ln(4) < x < \frac{11}{2}t + \ln(4),\\ (\frac{1}{2}, \frac{7}{2}, 3), & \text{if } x > \frac{11}{2}t + \ln(4), \end{cases}
\end{aligned}
$$

which are in correspondence with the results presented in the Figure 7.

Conclusions. In this work, we studied the generalized Riemann problem for the Suliciu relaxation system. In [5] the uniqueness of solutions for the Cauchy problem is proved. However, generally is difficult the construction of explicit solutions for a particular initial data. For additional information about the behavior of the solution, we solve the generalized Riemann problem and we show an example of the interaction of the elementary waves.

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