

## ALMOST PERIODIC FUNCTIONS IN QUANTUM CALCULUS

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ABSTRACT. In this article, we introduce the concepts of Bochner and Bohr almost periodic functions in quantum calculus and show that both concepts are equivalent. Also, we present a correspondence between almost periodic functions defined in quantum calculus and  $\mathbb{N}_0$ , proving several important properties for this class of functions. We investigate the existence of almost periodic solutions of linear and nonlinear  $q$ -difference equations. Finally, we provide some examples of almost periodic functions in quantum calculus.

### 1. INTRODUCTION

The theory of almost periodic functions was introduced by Bohr [8, 9, 10]. Later, Bochner introduced the concept of almost periodic functions taking values in Banach spaces. In 1945, Sobolev established almost periodicity of solutions of the wave equation.

This class of functions is more general than the class of periodic functions and can describe more precisely several interesting models and phenomena in the environment. For instance, these functions play an important role in the field of celestial mechanics, since there are planets in orbits moving with periods that are not commensurable and thus, almost periodic functions are the best choice to describe their motion. See, for instance, [12, 13] and the references therein.

On the other hand, the theory of quantum calculus has attracted the attention of several researchers (see [1, 2, 3, 4, 5, 6, 7, 14, 18, 19] and the references therein), because of its potential for applications, since this theory can be used to investigate black holes, conformal quantum mechanics, nuclear and high energy physics, fractional quantum Hall effect, high- $T_c$  superconductors, thermostatics of  $q$ -bosons and  $q$ -fermions. See [14, 15, 16, 20] and the references therein.

In this article, our goal is to introduce the concept of almost periodic functions in quantum calculus. Li [17] also gives such a concept, but in our work, we offer a different approach and are able to prove new results following from our definitions. We start by introducing this concept in the Bochner sense, and using this, we prove several properties for this class of functions. After that, we introduce the concept of almost periodicity in the Bohr sense and we establish a correspondence between the Bohr almost periodic functions defined in quantum calculus and  $\mathbb{N}_0$ . As an immediate consequence of [11, Theorem 1.27], we obtain a correspondence between

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2010 *Mathematics Subject Classification.* 39A20, 34N05, 34C25, 39A23.

*Key words and phrases.* Quantum calculus; Jackson derivative; periodicity; Bochner almost periodicity; Bohr almost periodicity.

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Submitted February 1, 2018. Published December 12, 2018.

almost periodic functions defined in quantum calculus and  $[0, \infty)$ . Using the first correspondence, we are able to obtain an equivalence between these two concepts of almost periodic functions in quantum calculus.

Also, we investigate the existence of almost periodic solutions of linear and non-linear  $q$ -difference equations. Finally, in the last section, we provide some examples of almost periodic functions in quantum calculus.

## 2. QUANTUM CALCULUS

In this section, our goal is to present some basic concepts concerning the theory of quantum calculus. All the definitions and results of this section can be found in [1, 2, 3, 4, 5, 6, 7, 14]. Throughout this article, we let  $q > 1$  and we use the notation  $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$ . We start by presenting the quantum derivative of a function  $f : \mathbb{T} \rightarrow \mathbb{R}$ .

**Definition 2.1** (See [14]). The expression

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{(q-1)t}, \quad \text{where } \sigma(t) = qt, \quad t \in \mathbb{T},$$

is called the  $q$ -derivative (or Jackson derivative) of the function  $f : \mathbb{T} \rightarrow \mathbb{R}$ .

In what follows, we present some properties of the quantum derivative.

**Theorem 2.2.** If  $\alpha, \beta \in \mathbb{R}$  and  $f, g : \mathbb{T} \rightarrow \mathbb{R}$  are  $q$ -differentiable, then

$$\begin{aligned} (\alpha f + \beta g)^\Delta(t) &= \alpha f^\Delta(t) + \beta g^\Delta(t), \\ (fg)^\Delta(t) &= f(qt)g^\Delta(t) + g(t)f^\Delta(t) = f(t)g^\Delta(t) + g(qt)f^\Delta(t), \\ \left(\frac{f}{g}\right)^\Delta(t) &= \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(qt)} \end{aligned}$$

for all  $t \in \mathbb{T}$ .

For simplicity, let us denote the quantum intervals by  $[a, b]_{\mathbb{T}}$ ,  $[a, b)_{\mathbb{T}}$ , and  $(a, b]_{\mathbb{T}}$  to represent  $[a, b] \cap \mathbb{T}$ ,  $[a, b) \cap \mathbb{T}$ , and  $(a, b] \cap \mathbb{T}$ , respectively. The definite integral of a function on  $\mathbb{T}$  is defined as follows.

**Definition 2.3.** Let  $f : \mathbb{T} \rightarrow \mathbb{R}$  and  $a, b \in \mathbb{T}$  be such that  $a < b$ . The definite integral of the function  $f$  is given by

$$\int_a^b f(t)\Delta t = (q-1) \sum_{t \in [a, b)_{\mathbb{T}}} tf(t).$$

**Remark 2.4.** As a consequence of Definition 2.3, we have that if  $m, n \in \mathbb{N}_0$  with  $m < n$  and  $f : \mathbb{T} \rightarrow \mathbb{R}$ , then

$$\int_{q^m}^{q^n} f(t)\Delta t = (q-1) \sum_{k=m}^{n-1} q^k f(q^k).$$

**Definition 2.5.** We say that a function  $p : \mathbb{T} \rightarrow \mathbb{R}$  is *regressive* provided

$$1 + (q-1)tp(t) \neq 0 \quad \text{for all } t \in \mathbb{T}.$$

The set of all regressive functions will be denoted by  $\mathcal{R}$ .

**Definition 2.6.** If  $p \in \mathcal{R}$ , then the exponential function is defined by

$$e_p(t, s) = \prod_{k=\log_q s}^{\log_q t-1} (1 + (q-1)q^k p(q^k)) \quad \text{for } t, s \in \mathbb{T} \text{ with } t > s.$$

If  $t = s$ , then we define  $e_p(t, s) = 1$ , and if  $t < s$ , then we define  $e_p(t, s) = 1/e_p(s, t)$ .

**Theorem 2.7** (Variation of Constants [4, Theorem 2.77]). *Let  $p \in \mathcal{R}$ ,  $f : \mathbb{T} \rightarrow \mathbb{R}$ ,  $t_0 \in \mathbb{T}$ , and  $y_0 \in \mathbb{R}$ . The unique solution of the initial value problem*

$$y^\Delta(t) = p(t)y + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^t e_p(t, \sigma(s))f(s)\Delta s.$$

**Lemma 2.8.** *Let  $a, b \in \mathbb{T}$  with  $a < b$  and  $t \in \mathbb{T}$ . Then*

$$\int_{at}^{bt} f(s)\Delta s = t \int_a^b f(st)\Delta s.$$

*Proof.* We have

$$\begin{aligned} \int_{at}^{bt} f(s)\Delta s &= \sum_{k=\log_q a+\log_q t}^{\log_q b+\log_q t-1} (q-1)q^k f(q^k) \\ &= \sum_{k=\log_q a}^{\log_q b-1} (q-1)q^{k+\log_q t} f(q^{k+\log_q t}) \\ &= t \sum_{k=\log_q a}^{\log_q b-1} (q-1)q^k f(tq^k) \\ &= t \int_a^b f(st)\Delta s, \end{aligned}$$

obtaining the desired result.  $\square$

Next, we give the definition of an  $\omega$ -periodic function on  $\mathbb{T}$ .

**Definition 2.9** (See [1, Definition 3.1]). Let  $\omega \in \mathbb{N}$ . A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called  $\omega$ -periodic if

$$q^\omega f(q^\omega t) = f(t) \quad \text{for all } t \in \mathbb{T}.$$

### 3. BOCHNER ALMOST PERIODIC FUNCTIONS

In this section, our goal is to introduce Bochner almost periodic functions for quantum calculus and to prove their main properties. We start by presenting the  $q$ -analogue of the concept of almost periodicity introduced by Bochner.

**Definition 3.1.** The function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called *Bochner almost periodic* on  $\mathbb{T}$  if for every sequence  $\{t'_n\} \subset \mathbb{T}$ , there exists a subsequence  $\{t_n\} \subset \mathbb{T}$  such that  $\lim_{n \rightarrow \infty} t_n f(tt_n)$  exists uniformly on  $\mathbb{T}$ . The set of all almost periodic functions  $f : \mathbb{T} \rightarrow \mathbb{R}$  is denoted by  $\text{AP}(\mathbb{T}, \mathbb{R})$ ,  $\text{AP}(\mathbb{T})$ , or simply  $\text{AP}_q$ .

Based on this definition, we are able to prove some important properties of Bochner almost periodic functions defined on  $\mathbb{T}$  as follows.

**Theorem 3.2.** *If  $f, g : \mathbb{T} \rightarrow \mathbb{R}$  are Bochner almost periodic, then*

- (i)  $f + g$  is Bochner almost periodic on  $\mathbb{T}$ ,
- (ii)  $cf$  is Bochner almost periodic on  $\mathbb{T}$ , for every  $c \in \mathbb{R}$ ,
- (iii)  $f_k : \mathbb{T} \rightarrow \mathbb{R}$  defined by  $f_k(t) := f(tq^k)$  is Bochner almost periodic on  $\mathbb{T}$ , for each  $k \in \mathbb{N}_0$ .

*Proof.* If  $f$  and  $g$  are Bochner almost periodic on  $\mathbb{T}$ , then, for every sequence  $\{t'_n\} \subset \mathbb{T}$ , there exists a subsequence  $\{t_n\}$  such that

$$\lim_{n \rightarrow \infty} t_n f(tt_n) \quad \text{and} \quad \lim_{n \rightarrow \infty} t_n g(tt_n)$$

exist uniformly on  $\mathbb{T}$ . Therefore, by the properties of limits, we obtain

$$\lim_{n \rightarrow \infty} t_n (f + g)(tt_n) = \lim_{n \rightarrow \infty} [t_n f(tt_n) + t_n g(tt_n)] = \lim_{n \rightarrow \infty} t_n f(tt_n) + \lim_{n \rightarrow \infty} t_n g(tt_n)$$

exists uniformly on  $\mathbb{T}$ . Thus,  $f + g$  is Bochner almost periodic on  $\mathbb{T}$ . This proves (i). Similarly, (ii) follows directly from the definition and by the properties of limits. Let us prove (iii). Since  $f$  is Bochner almost periodic on  $\mathbb{T}$ , for every sequence  $\{t'_n\} \subset \mathbb{T}$ , there exists a subsequence  $\{t_n\}$  such that

$$\lim_{n \rightarrow \infty} t_n f(tt_n)$$

exists uniformly on  $\mathbb{T}$ . Therefore, for each  $k \in \mathbb{N}_0$ , we have

$$\lim_{n \rightarrow \infty} t_n f_k(tt_n) = \lim_{n \rightarrow \infty} t_n f(tt_n q^k) = \lim_{n \rightarrow \infty} t_n f((tq^k)t_n)$$

exists uniformly on  $\mathbb{T}$ . Thus,  $f_k$  is also Bochner almost periodic on  $\mathbb{T}$ .  $\square$

Before presenting the next result, let us recall the definition of  $q$ -bounded functions.

**Definition 3.3** (See [3]). A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called  $q$ -bounded if there exists  $K > 0$  such that  $t|f(t)| \leq K$  for all  $t \in \mathbb{T}$ .

**Theorem 3.4.** *Bochner almost periodic functions on  $\mathbb{T}$  are  $q$ -bounded.*

*Proof.* In fact, suppose  $f : \mathbb{T} \rightarrow \mathbb{R}$  is a Bochner almost periodic function which is not  $q$ -bounded. Then, there exists a sequence  $\{t'_n\} \subset \mathbb{T}$  such that

$$t'_n |f(t'_n)| \rightarrow \infty,$$

which implies that there is no subsequence  $\{t_n\} \subset \mathbb{T}$  such that

$$t_n |f(tt_n)|$$

converges at  $t = q^0 \in \mathbb{T}$ , contradicting the fact that  $f$  is Bochner almost periodic on  $\mathbb{T}$ .  $\square$

**Remark 3.5.** Throughout the paper, similarly as in [13, Page 3], we also use the notation  $T_{t_n} f = \bar{f}$  to represent that

$$\lim_{n \rightarrow \infty} t_n f(tt_n) = \bar{f}(t) \quad \text{for every } t \in \mathbb{T}.$$

This notation is used only when the limit exists. When we use it, we specify the mode of convergence (e.g., pointwise, uniform).

**Definition 3.6.** The set

$$H(f) = \{g : \mathbb{T} \rightarrow \mathbb{R} \mid \text{there exists } \{t_n\} \subset \mathbb{T} \text{ with } T_{t_n} f = g \text{ uniformly}\}$$

is called the *hull* of  $f : \mathbb{T} \rightarrow \mathbb{R}$ .

**Theorem 3.7.** *If  $f : \mathbb{T} \rightarrow \mathbb{R}$  is regressive and Bochner almost periodic, then, for every sequence  $\{t'_n\} \subset \mathbb{T}$ , there exists a subsequence  $\{t_n\}$  such that for all  $t, s \in \mathbb{T}$ , we have*

$$\lim_{n \rightarrow \infty} e_f(tt_n, st_n) = \begin{cases} e_{\bar{f}}(t, s), & \text{if } \bar{f} \text{ is regressive,} \\ 0, & \text{otherwise,} \end{cases} \quad (3.1)$$

where  $T_{t_n}f = \bar{f}$ .

*Proof.* If  $f : \mathbb{T} \rightarrow \mathbb{R}$  is Bochner almost periodic, then, for every sequence  $\{t'_n\} \subset \mathbb{T}$ , there exists a subsequence  $\{t_n\}$  such that

$$\lim_{n \rightarrow \infty} t_n f(tt_n) = \bar{f}(t) \quad \text{for every } t \in \mathbb{T}$$

uniformly, i.e.,  $T_{t_n}f = \bar{f}$ . Therefore, for  $s < t$ ,

$$\begin{aligned} e_f(tt_n, st_n) &= \prod_{k=\log_q s + \log_q t_n}^{\log_q t + \log_q t_n - 1} (1 + (q-1)q^k f(q^k)) \\ &= \prod_{k=\log_q s}^{\log_q t - 1} (1 + (q-1)q^{k+\log_q t_n} f(q^{k+\log_q t_n})) \\ &= \prod_{k=\log_q s}^{\log_q t - 1} (1 + (q-1)q^k t_n f(q^k t_n)), \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} e_f(tt_n, st_n) = \lim_{n \rightarrow \infty} \prod_{k=\log_q s}^{\log_q t - 1} (1 + (q-1)q^k t_n f(q^k t_n)) = e_{\bar{f}}(t, s)$$

if  $\bar{f}$  is regressive, and otherwise, we obtain

$$\lim_{n \rightarrow \infty} e_f(tt_n, st_n) = 0,$$

proving (3.1). If  $t = s$ , then (3.1) clearly holds. Finally, if  $t < s$  and  $\bar{f}$  is regressive, then

$$e_f(tt_n, st_n) = \frac{1}{e_f(st_n, tt_n)} \rightarrow \frac{1}{e_{\bar{f}}(s, t)} = e_{\bar{f}}(t, s)$$

as  $n \rightarrow \infty$ , so (3.1) holds as well. Otherwise,  $\lim_{n \rightarrow \infty} e_f(tt_n, st_n) = 0$ .  $\square$

**Remark 3.8.** Notice that if we assume that  $f : \mathbb{T} \rightarrow \mathbb{R}$  is a positive function in Theorem 3.7, that is,  $f(t) > 0$  for every  $t \in \mathbb{T}$ , then the regressivity of  $f$  implies that  $\bar{f}$  is also a regressive function.

**Corollary 3.9.** *If  $f : \mathbb{T} \rightarrow \mathbb{R}$  is Bochner almost periodic, then, for every sequence  $\{t'_n\} \subset \mathbb{T}$ , there exists a subsequence  $\{t_n\}$  such that*

$$\lim_{n \rightarrow \infty} \cosh_f(tt_n, st_n) \quad \text{and} \quad \lim_{n \rightarrow \infty} \sinh_f(tt_n, st_n) \quad (3.2)$$

exist uniformly on  $\mathbb{T}$ .

*Proof.* The proof follows directly from Theorem 3.7, and combining Theorem 3.14 and the following definition of  $\cosh_f$  and  $\sinh_f$  (see [4])

$$\cosh_f = \frac{e_f + e_{-f}}{2} \quad \text{and} \quad \sinh_f = \frac{e_f - e_{-f}}{2},$$

proving the result.  $\square$

**Theorem 3.10.** *If  $a, b : \mathbb{T} \rightarrow \mathbb{R}$  are Bochner almost periodic functions,  $x : \mathbb{T} \rightarrow \mathbb{R}$  solves*

$$x^\Delta(t) = a(t)x(t) + \frac{b(t)}{t},$$

*and the condition*

(A1) *for every  $\{t'_n\} \subset \mathbb{T}$ , there exists  $\{t_n\} \subset \{t'_n\}$  such that*

$$\lim_{n \rightarrow \infty} t_n x(t_0 t_n) = x(t_0)$$

*is satisfied, then  $x$  is Bochner almost periodic.*

*Proof.* Since  $a, b : \mathbb{T} \rightarrow \mathbb{R}$  are Bochner almost periodic, for every sequence  $\{t'_n\} \in \mathbb{T}$ , there exists a subsequence  $\{t_n\}$  such that both

$$\lim_{n \rightarrow \infty} t_n a(tt_n) = \bar{a}(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} t_n b(tt_n) = \bar{b}(t)$$

exist uniformly, that is,  $T_{t_n} a = \bar{a}$  and  $T_{t_n} b = \bar{b}$ . Therefore, by Theorems 2.7 and 3.7, and Lemma 2.8, we obtain

$$\begin{aligned} t_n x(tt_n) &= t_n \left[ e_a(tt_n, t_0 t_n) x(t_0 t_n) + \int_{t_0 t_n}^{tt_n} e_a(tt_n, \sigma(s)) \frac{b(s)}{s} \Delta s \right] \\ &= e_a(tt_n, t_0 t_n) t_n x(t_0 t_n) + t_n^2 \int_{t_0}^t e_a(tt_n, \sigma(st_n)) \frac{b(st_n)}{st_n} \Delta s \\ &= e_a(tt_n, t_0 t_n) t_n x(t_0 t_n) + \int_{t_0}^t e_a(tt_n, t_n \sigma(s)) \frac{t_n b(st_n)}{s} \Delta s \\ &\rightarrow e_{\bar{a}}(t, t_0) x(t_0) + \int_{t_0}^t e_{\bar{a}}(t, \sigma(s)) \frac{\bar{b}(s)}{s} \Delta s = y(t), \end{aligned}$$

obtaining the desired result.  $\square$

**Remark 3.11.** We point out that in the proof of Theorem 3.10, it is possible to determine explicitly the function  $y$ , and its relation with  $x$ . Indeed, a careful examination shows us that  $y$  is the solution of

$$y^\Delta(t) = \bar{a}(t)y(t) + \frac{\bar{b}(t)}{t}, \quad y(t_0) = x(t_0),$$

where  $T_{t_n} a = \bar{a}$  and  $T_{t_n} b = \bar{b}$ .

Now, we present the definition of a Bochner almost periodic function on  $\mathbb{T}$  depending on one parameter. This definition is useful for applications to nonlinear  $q$ -difference equations.

**Definition 3.12.** A function  $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$  is called *Bochner almost periodic on  $t$*  for each  $x \in \mathbb{R}$ , if for every sequence  $\{t'_n\} \in \mathbb{T}$ , there exists a subsequence  $\{t_n\} \subset \{t'_n\}$  such that

$$\lim_{n \rightarrow \infty} t_n f(tt_n, x)$$

exists uniformly on  $\mathbb{T}$  for each  $x \in \mathbb{R}$ .

**Remark 3.13.** As before, we use the notation  $T_{t_n} f = \bar{f}$  to represent that

$$\lim_{n \rightarrow \infty} t_n f(tt_n, x) = \bar{f}(t, x) \quad \text{for each } x \in \mathbb{R}.$$

Next, we present a result concerning the properties of Bochner almost periodic functions on  $\mathbb{T}$  with respect the first variable. We omit its proof, since it follows analogously to the proof of Theorem 3.2.

**Theorem 3.14.** *If  $f, g : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$  are Bochner almost periodic with respect to the first variable for each  $x$  in  $\mathbb{R}$ , then*

- (i)  $f + g$  is Bochner almost periodic with respect to the first variable, for each  $x$  in  $\mathbb{R}$ .
- (ii)  $cf$  is Bochner almost periodic for each  $x \in \mathbb{R}$ , where  $c \in \mathbb{R}$ .

Now, we present a result which shows an important property of Bochner almost periodic functions.

**Theorem 3.15.** *Let  $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$  be Bochner almost periodic for each  $x \in \mathbb{R}$  and suppose that  $f$  satisfies Lipschitz condition*

$$|f(t, x) - f(t, y)| \leq L(t)|x - y| \quad \text{for all } t \in \mathbb{T} \quad \text{and } x, y \in \mathbb{R}, \quad (3.3)$$

where  $L : \mathbb{T} \rightarrow (0, \infty)$  is Bochner almost periodic, i.e., for every sequence  $\{t'_n\} \subset \mathbb{T}$ , there exists a subsequence  $\{t_n\}$  such that

$$\lim_{n \rightarrow \infty} t_n L(tt_n) = \tilde{L}(t)$$

exists uniformly for every  $t \in \mathbb{T}$ . Then,  $\bar{f}$  given by  $T_{t_n} f = \bar{f}$  satisfies the Lipschitz condition with the function  $\tilde{L}$ .

*Proof.* Let  $t \in \mathbb{T}$  and  $x, y \in \mathbb{R}$ . Let  $\varepsilon > 0$ . Then, by the Bochner almost periodicity of  $f$  and  $L$ , for every sequence  $\{t'_n\} \in \mathbb{T}$ , there exists a subsequence  $\{t_n\} \subset \{t'_n\}$  such that

$$|\bar{f}(t, x) - t_n f(tt_n, x)| \leq \frac{\varepsilon}{3}, \quad |\bar{f}(t, y) - t_n f(tt_n, y)| \leq \frac{\varepsilon}{3}, \quad (3.4)$$

$$|\tilde{L}(t) - t_n L(tt_n)| \leq \frac{\varepsilon}{3|x - y|} \quad (3.5)$$

for  $n$  sufficiently large. Therefore, we obtain

$$\begin{aligned} & |\bar{f}(t, x) - \bar{f}(t, y)| \\ & \leq |\bar{f}(t, x) - t_n f(tt_n, x)| + |\bar{f}(t, y) - t_n f(tt_n, y)| + |t_n f(tt_n, x) - t_n f(tt_n, y)| \\ & \leq |\bar{f}(t, x) - t_n f(tt_n, x)| + |\bar{f}(t, y) - t_n f(tt_n, y)| + t_n L(tt_n)|x - y| \\ & \stackrel{(3.4)}{\leq} \frac{2\varepsilon}{3} + |\tilde{L}(t) - t_n L(tt_n)||x - y| + \tilde{L}(t)|x - y| \\ & \stackrel{(3.5)}{\leq} \varepsilon + \tilde{L}(t)|x - y|. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0^+$ , we arrive at

$$|\bar{f}(t, x) - \bar{f}(t, y)| \leq \tilde{L}(t)|x - y|.$$

So, (3.3) is satisfied for  $\bar{f}$  and  $\tilde{L}$ . □

## 4. BOHR ALMOST PERIODIC FUNCTIONS

We start this section by introducing the  $q$ -analogue of the concept of almost periodicity introduced by Bohr for quantum calculus.

**Definition 4.1.** We say that  $f : \mathbb{T} \rightarrow \mathbb{R}$  is *Bohr almost periodic* if for every  $\varepsilon > 0$ , there exists  $N_\varepsilon \in \mathbb{N}$  such that any  $N_\varepsilon$  consecutive elements of  $\mathbb{T}$  contain at least one  $s$  with

$$|stf(ts) - tf(t)| < \varepsilon, \quad \text{for all } t \in \mathbb{T}. \quad (4.1)$$

**Remark 4.2.** From this definition, it is clear that if  $f$  is a periodic function on  $\mathbb{T}$ , then  $f$  is Bohr almost periodic function on  $\mathbb{T}$ . Indeed, suppose  $f$  is an  $\omega$ -periodic function on  $\mathbb{T}$ , where  $\omega \in \mathbb{N}_0$ , then for every  $\varepsilon > 0$ , there exists  $N_\varepsilon := \lceil \varepsilon \rceil \omega + 1 \in \mathbb{N}$  such that any  $N_\varepsilon$  consecutive elements of  $\mathbb{T}$  contain at least one  $s$  with

$$|stf(ts) - tf(t)| = 0 < \varepsilon \quad \text{for all } t \in \mathbb{T},$$

obtaining that  $f$  is Bohr almost periodic on  $\mathbb{T}$ .

Next, we establish a correspondence between Bohr almost periodic functions defined on  $\mathbb{T}$  and  $\mathbb{N}_0$ .

**Theorem 4.3.** *A necessary and sufficient condition for a function  $g : \mathbb{T} \rightarrow \mathbb{R}$  to be Bohr almost periodic on  $\mathbb{T}$  is the existence of a Bohr almost periodic sequence  $f : \mathbb{N}_0 \rightarrow \mathbb{R}$  such that  $g(t) = f(\log_q t)/t$  for every  $t \in \mathbb{T}$ .*

*Proof.* First, assume  $f : \mathbb{N}_0 \rightarrow \mathbb{R}$  is Bohr almost periodic sequence in the sense of [11, Page 45]. Let  $\varepsilon > 0$ . Then, there exists  $N_\varepsilon > 0$  such that among any  $N_\varepsilon$  consecutive integers, there exists  $\omega \in \mathbb{N}$  such that

$$|f(n + \omega) - f(n)| < \varepsilon \quad \text{for all } n \in \mathbb{N}_0.$$

Define  $g : \mathbb{T} \rightarrow \mathbb{R}$  by  $g(t) = f(\log_q t)/t$  for  $t \in \mathbb{T}$ . Consider a set of  $N_\varepsilon$  consecutive elements  $t \in \mathbb{T}$ . Then,  $\log_q t \in \mathbb{N}$  are  $N'_\varepsilon$  consecutive integers. Thus, among them, there exists  $\log_q s \in \mathbb{N}$  with

$$|f(n + \log_q s) - f(n)| < \varepsilon \quad \text{for all } n \in \mathbb{N}_0. \quad (4.2)$$

Then, we have

$$|tsg(ts) - tg(t)| = |f(\log_q t + \log_q s) - f(\log_q t)| \stackrel{(4.2)}{<} \varepsilon \quad \text{for all } t \in \mathbb{T}.$$

By Definition 4.1,  $g$  is Bohr almost periodic on  $\mathbb{T}$ . Next, suppose  $g : \mathbb{T} \rightarrow \mathbb{R}$  is Bohr almost periodic on  $\mathbb{T}$ . Let  $\varepsilon > 0$ . Then, there exists  $N_\varepsilon > 0$  such that among any  $N_\varepsilon$  consecutive elements of  $\mathbb{T}$ , there exists  $s \in \mathbb{T}$  with

$$|tsg(st) - tg(t)| < \varepsilon \quad \text{for all } t \in \mathbb{T}.$$

Define  $f : \mathbb{N}_0 \rightarrow \mathbb{R}$  by  $f(n) = q^n g(q^n)$  for  $n \in \mathbb{N}_0$ . Consider a set of  $N_\varepsilon$  consecutive integers  $n \in \mathbb{N}_0$ . Then,  $q^n \in \mathbb{T}$  are  $N'_\varepsilon$  consecutive elements of  $\mathbb{T}$ . Thus, among them, there exists  $s \in \mathbb{T}$  with  $|tsg(st) - tg(t)| < \varepsilon$  for all  $t \in \mathbb{T}$ . Defining  $\omega := \log_q s$ , we obtain

$$|f(n + \omega) - f(n)| = |q^n sg(q^n s) - q^n g(q^n)| < \varepsilon \quad \text{for all } n \in \mathbb{N}_0.$$

This implies that  $f : \mathbb{N}_0 \rightarrow \mathbb{R}$  is a Bohr almost periodic sequence.  $\square$

The next result can be found in [11, Theorem 1.27]. It describes a correspondence between Bohr almost periodic defined in  $\mathbb{Z}$  and  $\mathbb{R}$ .

**Theorem 4.4.** *A necessary and sufficient condition for a sequence to be Bohr almost periodic is the existence of a Bohr almost periodic  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g(n) = f(n)$  for all  $n \in \mathbb{Z}$ .*

As an immediate consequence of Theorems 4.3 and 4.4, we obtain the following correspondence between Bohr almost periodic for functions defined on  $\mathbb{T}$  and  $[0, \infty)$ .

**Theorem 4.5.** *A necessary and sufficient condition for  $g : \mathbb{T} \rightarrow \mathbb{R}$  to be Bohr almost periodic on  $\mathbb{T}$  is the existence of a Bohr almost periodic function  $f : [0, \infty) \rightarrow \mathbb{R}$  such that  $g(t) = f(\log_q t)/t$  for every  $t \in \mathbb{T}$ .*

The next result shows that the class of Bochner almost periodic functions is equivalent to the class of Bohr almost periodic functions in quantum calculus.

**Theorem 4.6.**  *$f : \mathbb{T} \rightarrow \mathbb{R}$  is Bochner almost periodic if and only if  $f$  is Bohr almost periodic.*

*Proof.* Suppose  $f$  is Bochner almost periodic, but  $f$  is not Bohr almost periodic. Therefore, there exists at least one  $\varepsilon > 0$  such that for any  $N_\varepsilon \in \mathbb{N}$ , the set of  $N_\varepsilon$  consecutive numbers in  $\mathbb{T}$  does not contain any element satisfying (4.1).

Let  $\tau \in \mathbb{T}$  and consider an arbitrary number  $\alpha_1 \in \mathbb{N}$ , then there are no elements satisfying (4.1) on  $[\tau, \tau q^{\alpha_1}]_{\mathbb{T}}$ . Take  $\alpha_2 = \log_q(\tau)\alpha_1$ , then there are no elements satisfying (4.1) on  $[\tau q^{\alpha_1}, \tau q^{\alpha_1 + \alpha_2}]_{\mathbb{T}}$ . Proceeding this way, we can construct a sequence  $\{t_k\}_{k=1}^\infty$ , where  $t_k := q^{\alpha_k}$ , such that  $t_k \rightarrow \infty$  when  $k \rightarrow \infty$ . Then, for any  $i, j > 1$ ,  $i > j$ , we obtain

$$\begin{aligned} \sup_{t \in \mathbb{T}} |t_i t f(t_i t) - t_j t f(t_j t)| &= \sup_{t \in \mathbb{T}} |t_j t (t_i(t_j)^{-1} f(t_i t) - f(t_j t))| \\ &= t_j \sup_{t \in \mathbb{T}} |t (t_i(t_j)^{-1} f(t_i t) - f(t_j t))| \\ &= t_j \sup_{t \in \mathbb{T}} |t (t_i(t_j)^{-1} f(t_i(t_j)^{-1} t) - f(t_j(t_j)^{-1} t))| \\ &= t_j \sup_{t \in \mathbb{T}} |t t_i(t_j)^{-1} f(t_i(t_j)^{-1} t) - t f(t)| \\ &\geq \sup_{t \in \mathbb{T}} |t t_i(t_j)^{-1} f(t_i(t_j)^{-1} t) - t f(t)| \geq \varepsilon, \end{aligned}$$

which proves that the sequence  $\{t_n t f(t_n t)\}$  cannot contain any uniformly convergent subsequence. This contradicts the fact that  $f(t)$  is Bochner almost periodic.

Reciprocally, assume  $f : \mathbb{T} \rightarrow \mathbb{R}$  satisfies Definition 4.1. Then, defining  $g : \mathbb{N}_0 \rightarrow \mathbb{R}$  by

$$g(n) = q^n f(q^n), \quad n \in \mathbb{N}_0,$$

we obtain from Theorem 4.3 that  $g : \mathbb{N}_0 \rightarrow \mathbb{R}$  is Bohr almost periodic, and hence, by [11, Theorem 1.26],  $g : \mathbb{N}_0 \rightarrow \mathbb{R}$  is Bochner almost periodic, i.e., for every sequence  $\{n'_k\} \subset \mathbb{N}_0$ , there exists a subsequence  $\{n_k\}$  such that  $\lim_{k \rightarrow \infty} g(n + n_k)$  exists uniformly for every  $n \in \mathbb{N}_0$ . Hence, there exists the uniform limit as  $k \rightarrow \infty$  of

$$g(n + n_k) = q^{n+n_k} f(q^{n+n_k}) = q^n q^{n_k} f(q^n q^{n_k}) \quad \text{for all } n \in \mathbb{N}_0.$$

Now, let  $\{t'_n\} \subset \mathbb{T}$  be a sequence. Then,  $t'_k = q^{n'_k}$  for some  $\{n'_k\} \subset \mathbb{N}_0$ . There exists a subsequence  $\{n_k\}$  such that

$$\lim_{k \rightarrow \infty} q^{n_k} f(q^n q^{n_k}) \quad \text{exists uniformly for all } n \in \mathbb{N}_0.$$

Define  $t_k = q^{n_k}$ ,  $t = q^n$ . Then

$$\lim_{k \rightarrow \infty} t_k f(tt_k) \quad \text{exists uniformly for all } t \in \mathbb{T},$$

obtaining the desired result.  $\square$

**Remark 4.7.** From Theorem 4.6, we obtain that the class of Bohr almost periodic functions and the class of Bochner almost periodic functions in quantum calculus are equal. Therefore, if  $f : \mathbb{T} \rightarrow \mathbb{R}$  satisfies Definition 3.1 or Definition 4.1, we simply call  $f$  *almost periodic* on  $\mathbb{T}$ . Also, all properties which we have proven for Bochner almost periodic functions remain true for Bohr almost periodic functions.

## 5. EXAMPLES

In this section, we present some examples of almost periodic functions in quantum calculus.

**Example 5.1.** The function  $F(t) = (\cos(\log_q t) + \cos(\sqrt{2} \log_q t))/t$  is almost periodic on  $\mathbb{T}$ . Indeed, since the function  $f(t) = \cos t + \cos(\sqrt{2}t)$  is almost periodic on  $\mathbb{R}$  (see [13, Page 3]), it follows by Theorem 4.5 that the function

$$F(t) = \frac{f(\log_q t)}{t} = \frac{\cos(\log_q t) + \cos(\sqrt{2} \log_q t)}{t}$$

is also almost periodic on  $\mathbb{T}$ .

**Example 5.2.** The function  $F(t) = (\sin(\log_q t) + \sin(\pi \log_q t))/t$  is almost periodic on  $\mathbb{T}$ . In fact, since the function  $f(t) = \sin t + \sin(\pi t)$  is almost periodic on  $\mathbb{R}$  (see [11, Page 107]), it follows by Theorem 4.5 that the function

$$F(t) = \frac{f(\log_q t)}{t} = \frac{\sin(\log_q t) + \sin(\pi \log_q t)}{t}$$

is also almost periodic on  $\mathbb{T}$ .

**Example 5.3.** The function

$$F(t) = (\sin(\log_q t) + \sin(\pi \log_q t) + \cos(\log_q t) + \cos(\sqrt{2} \log_q t))/t$$

is almost periodic on  $\mathbb{T}$ . This follows from Examples 5.1 and 5.2, and Theorem 3.2.

**Example 5.4.** The function  $F(t) = (\sin(\pi \log_q t) + 2(-1)^{\log_q t})/t$  is almost periodic on  $\mathbb{T}$ . This follows from Example 5.1, Theorem 3.2 and Theorem 4.3, since  $(-1)^{\log_q t}$  is a periodic function on  $\mathbb{N}_0$ .

**Acknowledgements.** J. G. Mesquita was supported by CNPq 407952/2016-0, by FAPDF 0193-001.300/2016, and by FEMAT-Fundação de Estudos em Ciências Matemáticas Proc. 039/2017.

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