NONEXISTENCE OF NONNEGATIVE SOLUTIONS FOR PARABOLIC INEQUALITIES IN THE HALF-SPACE

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Abstract. Based on the method of nonlinear capacity, we study the nonexistence of nonnegative monotonic solutions for the quasilinear parabolic inequality $u_t - \Delta_p u \geq u^q$. Also we study generalizations in the half-space in terms of parameters $p$ and $q$.

1. Introduction

The question about nonexistence of nontrivial nonnegative global solutions to nonlinear equation $u_t - Au = g(x)u^q$ and the inequality $u_t - Au \geq g(x)u^q$, where $A$ is an elliptic operator, in different domains is of substantial interest. Such inequalities can be understood as nonlinear heat equations with a supplementary external source term $f(x,t) = u_t - Au - g(x)u^q \geq 0$. The aim of the study is to find the range of values of $q$ such that the equation or inequality in question has no-nontrivial nonnegative global solutions, i.e. the extra heat source leads to blow-up of a local solution.

The results in the whole space $\mathbb{R}^n$ go back to Fujita [11] who established that solutions to the equation $u_t - \Delta u = u^q$ do not exist for $1 < q < 1 + \frac{2}{n}$. Similar nonexistence ranges for much more general operators were obtained later in [16]. As for the half-space, up to our knowledge, so far only stationary solutions have been considered. The first results in this direction were obtained by Berestycki, Capuzzo Dolcetta and Nirenberg [2] who proved nonexistence of solutions to the inequality $-\Delta u \geq u^q$ for $1 < q < \frac{n+1}{n-1}$. The optimality of these results was shown by Birindelli and Mitidieri [3]. Inequalities of the form $Au \geq u^q$ with $A = -\Delta_p$, where $p > 1$ and $\Delta_p$ is the $p$-Laplace operator defined by $\Delta_p u := \text{div}(|Du|^{p-2}Du)$, in the half-space with a punched point or a removed neighborhood of a point on the boundary were studied by Bidaut-Véron and Pohozaev [4], and later by Véron and A. Porretta [18]. They obtained results on nonexistence of solutions in the domains under study and consequently in the whole half-space for $p - 1 < q < q_{cr}(p,n)$, where $q_{cr}(p,n) = p - 1 + \frac{p}{\beta_{p,n}}$, and $\beta_{p,n}$ is the growth rate of singular solutions near zero, obtained explicitly only for $n = 2 (\beta_{p,2} = \frac{3-p+\sqrt{(p-1)^2+2p}}{3(p-1)})$. One should also note the papers of Filippucci [10] on critical exponents for semilinear inequalities of

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the form $-\text{div}(u^\alpha |x|^\beta Du) \geq |x|^{\gamma} u^q$ in the half-space, of Dancer, Du and Efendiev [5] and of Zou [20] on nonexistence of solutions to the Dirichlet problem

$$
-\Delta_p u = u^q, \quad x \in \mathbb{R}_n^+,
$$

$$
u(x) = 0, \quad x \in \partial \mathbb{R}_n^+,
$$

for a nonlinear equation with a $p$-Laplace operator in a half-space, as well as those of Farina, Montoro and Sciunzi [6]–[9] on monotonicity of essentially bounded solutions of the same problem, which implies their nonexistence for a certain range of $q$. Elliptic problems with singular coefficients near unbounded sets were considered, in particular, in [12, 13].

In this article we consider the nonexistence of nonnegative solutions for the parabolic inequality $u_t - \Delta_p u \geq ax^\gamma_n u^q$ in the half-space. Based on the method of nonlinear capacity [10, 11], we obtain sufficient conditions for nonexistence of solutions. Similar results for elliptic inequalities and systems can be found in [14].

The rest of this article consists of three sections. §2 has our main results, §3 contains a proof in the semilinear case, and §4 the quasilinear case.

2. Formulation of main results

Denote $\mathbb{R}_n^+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$. Let $p > 1$, $q > p - 1$, $a > 0$, $\gamma \in \mathbb{R}$, and let $u_0 \in C(\mathbb{R}_n^+)$ be a nonnegative function. Consider the problem

$$
u_t - \Delta_p u \geq ax^\gamma_n u^q, \quad (x, t) \in \mathbb{R}_n^+ \times \mathbb{R}_+,$$

$$
u(x, 0) = u_0(x), \quad x \in \mathbb{R}_n^+,$$

$$
u(x, t) \geq 0, \quad (x, t) \in \mathbb{R}_n^+ \times \mathbb{R}_+.
$$

We understand its weak solutions in the following sense.

Definition 2.1. A weak solution of problem (2.1) is a nonnegative function $u \in C^{2,1}(\mathbb{R}_n^+ \times \mathbb{R}_+)$, which satisfies the integral inequality

$$
\int_{\mathbb{R}_n^+} \int_{\mathbb{R}_n^+} (|Du|^{p-2}(Du, D\varphi) - u\varphi) \, dx \, dt \geq \int_{\mathbb{R}_n^+} \int_{\mathbb{R}_n^+} ax^\gamma_n u^q \varphi \, dx \, dt + \int_{\mathbb{R}_n^+} u_0 \varphi \, dx
$$

for any nonnegative $\varphi \in C^\infty(\mathbb{R}_n^+ \times \mathbb{R}_+)$ such that $\varphi(x, t) \equiv 0$ for $(x, t) \in \partial \mathbb{R}_n^+ \times \mathbb{R}_+$ (that is, for $x_n = 0$).

Weak solutions of the problems considered below are defined in a similar way. In the case $p = 2$, we obtain the following result.

Theorem 2.2. Let $a > 0, \gamma > -2$, and $1 < q \leq 1 + \frac{\gamma+2}{n+1}$. Then (2.1) with $p = 2$:

$$
u_t - \Delta u \geq ax^\gamma_n u^q, \quad (x, t) \in \mathbb{R}_n^+ \times \mathbb{R}_+,$$

$$
u(x, 0) = u_0(x), \quad x \in \mathbb{R}_n^+,$$

$$
u(x, t) \geq 0, \quad (x, t) \in \mathbb{R}_n^+ \times \mathbb{R}_+$$

has no nonnegative nontrivial weak solutions $u$.

For other values of $p \neq 2$, we obtain a nonexistence result in a smaller functional class of solutions (with an additional property of monotonicity).

Theorem 2.3. Let $a > 0$, $\gamma > -p$, $q \geq \max(1, p - 1)$, $\gamma(p - 2) > p(1 - q)$, and

$$
[(n + 1)(q - 1) - \gamma](q - p + 1) - p(q - 1) - \gamma(p - 2) < 0.
$$
Then \[(2.1)\] has no nonnegative nontrivial weak solutions \(u\) such that \(u(x', \cdot, t)\) is monotonic in \(x_n\) for each \(x' \in \mathbb{R}^{n-1}\) and \(t > 0\).

**Corollary 2.4.** Let \(a > 0\) and \(\max(1, p - 1) \leq q \leq p - 1 + \frac{p}{n+1}\). Then the problem
\[
\begin{aligned}
&u_t - \Delta_p u \geq au^q, \quad (x, t) \in \mathbb{R}^n_+ \times \mathbb{R}_+, \\
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n_+, \\
u(x, t) \geq 0, \quad (x, t) \in \mathbb{R}^n_+ \times \mathbb{R}_+ 
\end{aligned}
\]  
(that is, \[(2.1)\] with \(\gamma = 0\)) has no nonnegative nontrivial weak solutions \(u\) such that \(u(x', \cdot, t)\) is monotonic in \(x_n\) for each \(x' \in \mathbb{R}^{n-1}\) and \(t > 0\).

Evidently, the above corollary follows from Theorem \(2.3\) in the case \(\gamma = 0\).

**Remark 2.5.** Nonexistence results can be obtained in the same class of monotonic solutions for the problem
\[
\begin{aligned}
&u_t - \Delta_p u \geq ax_n^\gamma u^q, \quad (x, t) \in \mathbb{R}^n_+ \times \mathbb{R}_+, \\
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n_+, \\
u(x, t) \geq 0, \quad (x, t) \in \mathbb{R}^n_+ \times \mathbb{R}_+, 
\end{aligned}
\]
where the operator \(\Delta_p\) has the opposite sign (see \([14]\)). Although the result in \([14]\) is formulated for monotonically nondecreasing solutions, its proof is valid for non-increasing ones as well.

### 3. Proof of Theorem 2.2

We use the method of nonlinear capacity \([16, 17]\). We choose a family of nonnegative test functions \(\xi_{R,T}(x) \in C^1_0(\mathbb{R}^n)\) such that \(\lambda > 0\) (to be specified below), \(R\) and \(T\) are some positive parameters, and \(\xi_{R,T}(x) = \prod_{k=1}^{N-1} \chi_R(x_k) \cdot \chi_R(x_n - 3R) \cdot \chi_T(t)\) with
\[
\chi_R(s) = \begin{cases} 
1 & \text{if } s \leq R, \\
0 & \text{if } s \geq 2R,
\end{cases}
\]  
(3.1)
where
\[
|D\chi_R(s)| \leq cR^{-1}, \quad s \in \mathbb{R}_+. 
\]  
(3.2)
Multiply both sides of \((2.2)\) by \(\xi_{R,T}^\lambda x_n\) and integrate by parts. After elementary transformations we obtain
\[
\begin{aligned}
a &\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} u^q \xi_{R,T}^\lambda x_n^\gamma + 1 \ dx \ dt \\
&\leq \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} u \cdot |\Delta(\xi_{R,T}^\lambda x_n)| \ dx \ dt + \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} u \cdot \frac{\partial \xi_{R,T}^\lambda}{\partial t} \ |x_n| \ dx \ dt.
\end{aligned}
\]  
(3.3)
Application of the parametric Young inequality to both integrals on the right-hand side of \((3.3)\) yields
\[
\begin{aligned}
\frac{a}{2} &\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} u^q \xi_{R,T}^\lambda x_n^\gamma + 1 \ dx \ dt \leq c \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} |D\xi_{R,T}^\lambda| \frac{2n}{\gamma + 1} \xi_{R,T}^\lambda \frac{2n}{\gamma + 1} x_n^\gamma \ dx \ dt \\
&+ c \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \chi_T(t) \frac{2n}{\gamma + 1} \chi_{R'}^{\lambda - \frac{2n}{\gamma + 1}} x_n^{\gamma + 1} \ dx \ dt 
\end{aligned}
\]  
(3.4)
\[
:= I_1(R, T) + I_2(R, T).
\]
For $\lambda > \frac{2n}{q-1}$, the integral $I_1(R, T)$ can be estimated as

$$I_1(R, T) \leq R^{n-\frac{2n+1}{q-1}} T$$

and $I_2(R, T)$ as

$$I_2(R, T) \leq R^{n-\frac{q-1}{q}T^{1-\frac{n}{q}}}.$$ (3.6)

From (3.4)–(3.6) we obtain

$$\frac{a}{2} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^q \xi^\lambda_{R, T} x_n^{\gamma+1} \, dx \, dt \leq c(R^{n-\frac{2n+1}{q-1}} T + R^{n-\frac{q-1}{q}T^{1-\frac{n}{q}}}).$$ (3.7)

Choosing $T = R^\theta$ with $\theta > 0$ such that both terms are of the same order and taking $R \to \infty$, we obtain

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^q x_n^{\gamma+1} \, dx \, dt = 0,$$

which contradicts the assumption of non-triviality of the solution. This completes the proof of Theorem 2.2.

4. PROOF OF THEOREM 2.3

Now, using the same family of test functions $\xi_{R, T}$ as in the previous proof, we multiply both parts of (2.3) by $u^\alpha \xi^\lambda_{R, T} x_n$, where $\alpha < 0$ will be specified below, and integrate by parts. After elementary transformations we obtain

$$a \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{q+\alpha} \xi^\lambda_{R, T} x_n^{\gamma+1} \, dx \, dt + |\alpha| \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{\alpha-1} |Du|^p \xi^\lambda_{R, T} x_n \, dx \, dt$$

$$\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^\alpha |Du|^{p-1} |D\xi^\lambda_{R, T} x_n| \, dx \, dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u_t u^\alpha \xi^\lambda_{R, T} x_n \, dx \, dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{\alpha} |D^2u|^{p-2} \frac{\partial u}{\partial x_n} \xi^\lambda_{R, T} \, dx \, dt.$$ (4.1)

Application of the parametric Young inequality to the first integral on the right-hand side of (4.1) yields

$$a \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{q+\alpha} \xi^\lambda_{R, T} x_n^{\gamma+1} \, dx \, dt + |\alpha| \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{\alpha-1} |Du|^p \xi^\lambda_{R, T} x_n \, dx \, dt$$

$$\leq c \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{\alpha+p-1} |D\xi^\lambda_{R, T}|^p \xi^\lambda_{R, T} x_n \, dx \, dt$$

$$+ \frac{1}{\alpha+1} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{\alpha+1} (\xi^\lambda_{R, T}) t x_n \, dx \, dt$$

$$+ \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^\alpha |D^2u|^{p-2} \frac{\partial u}{\partial x_n} \xi^\lambda_{R, T} \, dx \, dt.$$ (4.2)
Applying the parametric Young inequality to the first two integrals on the right-hand side of (4.2) once more, we obtain

\[
\frac{a}{2} \int_{R^+} \int_{R^n_+} u^{q+\alpha} \xi_{R,T}^{\gamma+1} dx \, dt + \frac{\alpha}{2} \int_{R^+} \int_{R^n_+} u^{\alpha-1} |Du|^{p} \xi_{R,T}^{q} x_n \, dx \, dt \\
\leq c \int_{R^+} \int_{R^n_+} |D\xi_{R,T}| \frac{p(q+\alpha)}{q-p+1} \xi_{R,T}^{p(q+\alpha)-\frac{q+\alpha-(\alpha+p-1)(\gamma+1)}{q-p+1}} x_n \, dx \, dt \\
+ c \int_{R^+} \int_{R^n_+} |\chi_T(t)|^{\frac{q+\alpha}{q-p+1}} \xi_{R,T}^{\frac{q+\alpha-(\alpha+p-1)(\gamma+1)}{q-p+1}} x_n \, dx \, dt \\
+ \int_{R^+} \int_{R^n_+} u^\alpha |Du|^{p-2} \frac{\partial u}{\partial x_n} \xi_{R,T} \, dx \, dt \\
:= I_1(R, T) + I_2(R, T) + I_3(R, T).
\]

For \( \lambda > \frac{q}{q-p+1} \) and

\[
\alpha > \frac{n(q-p+1) - (q + \gamma - 1)(p-1)}{p + \gamma}
\]

the integral \( I_1(R, T) \) and \( I_2(R, T) \) can be estimated as

\[
I_1(R, T) \leq R^{n-\frac{(q-1)(q+\alpha)+2p-1}{q-p+1}} T, \\
I_2(R, T) \leq R^{n-\frac{q+\alpha-(\alpha+p-1)(\gamma+1)}{q-p+1}} T^{1-\frac{q+\alpha}{q-p+1}}.
\]

If \( \frac{\partial u}{\partial x_n} \geq 0 \), then \( I_3(R, T) < 0 \). Estimate the integral \( I_3(R, T) \) in the case \( \frac{\partial u}{\partial x_n} \leq 0 \). In case \( p < 2 \), using the H"older inequality and integrating by parts, we have

\[
I_3(R, T) = - \int_{R^+} \int_{R^n_+} u^\alpha |Du|^{p-2} \frac{\partial u}{\partial x_n} \xi_{R,T} \, dx \, dt \\
\leq \int_{R^+} \int_{R^n_+} u^\alpha \left( - \frac{\partial u}{\partial x_n} \right)^{p-1} \xi_{R,T} \, dx \, dt \\
\leq c \int_{R^+} \int_{R^n_+} \left( - \frac{\partial u}{\partial x_n} \right)^{p-1} \xi_{R,T} \, dx \, dt \\
\leq c \left( \int_{R^+} \int_{R^n_+} \frac{\partial u}{\partial x_n} \xi_{R,T} \, dx \, dt \right)^{p-1} R^{n(2-p)} \\
= c \left( \int_{R^+} \int_{R^n_+} u^{1+\frac{\alpha}{q-p+1}} \xi_{R,T} \, dx \, dt \right)^{p-1} R^{n(2-p)} \\
\leq c \left( \int_{R^+} \int_{R^n_+} u^{1+\frac{\alpha}{q-p+1}} \frac{\partial \xi_{R,T}}{\partial x_n} \, dx \, dt \right)^{p-1} R^{n(2-p)} \\
\leq c \left( \int_{R^+} \int_{R^n_+} u^{\alpha+\alpha} \xi_{R,T} x_n^{\gamma+1} \, dx \, dt \right)^{\frac{\alpha+p-1}{q-p+1}} R^{n(2-p)} \\
\times \left( \int_{R^+} \int_{R^n_+} \left| \frac{\partial \xi_{R,T}}{\partial x_n} \right|^{\frac{(q+\alpha)(q-1)}{q-p+1}} \, dx \, dt \right)^{\frac{q-p+1}{q-p+1}} \\
\times \left( \xi_{R,T}^{q-p+1} - (q+\alpha) x_n^{-(\gamma+1)(\alpha+p-1)} \right) \, dx \, dt \right)^{\frac{q+p-1}{q-p+1}}.
\]
and by the Young inequality, similarly to the previous argument, 
\[
R^n \leq cR\int_{\mathbb{R}_+^n} u^{q+\alpha} \xi_{R,T}^\gamma x_n^{\gamma+1} \, dx \, dt + cR_{n-\frac{q+\alpha}{q-p+1}}^{\frac{q+\alpha}{q-p+1}} T.
\]
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From (4.3)–(4.7) we obtain
\[
I_3(R, T) \leq \frac{a}{4} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} u^{q+\alpha} \xi_{R,T}^\gamma x_n^{\gamma+1} \, dx \, dt + cR_{n-\frac{q+\alpha}{q-p+1}}^{\frac{q+\alpha}{q-p+1}} T.
\]
Choosing \( T = R^\theta \) with \( \theta > 0 \) such that both terms are of the same order and taking \( R \to \infty \), for \( \alpha \) satisfying (4.4) we obtain
\[
\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} u^{q+\alpha} x_n^{\gamma+1} \, dx \, dt = 0,
\]
which contradicts the assumption of non-triviality of the solution. This proves the theorem in the case \( p < 2 \).

In the case \( p > 2 \), estimates (4.3) and (4.5) are still valid, and for the integral \( I_3(R, T) \) in the case \( \frac{\partial u}{\partial x_n} \leq 0 \) we have
\[
I_3(R, T) = \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} u^{\alpha} |Du|^{p-2} \frac{\partial u}{\partial x_n} \xi_{R,T} \, dx \, dt
\]
\[
= - \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} u^{\alpha} |Du|^{p-2} \left( - \frac{\partial u}{\partial x_n} \right) \xi_{R,T} \, dx \, dt
\]
\[
\leq \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} u^{\alpha} |Du|^{p-2} \xi_{R,T} \, dx \, dt
\]
and by the Young inequality, similarly to the previous argument,
\[
I_3(R, T) \leq \frac{|\alpha|}{4} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} u^{\alpha-1} |Du|^{p-2} \xi_{R,T} \, dx \, dt
\]
\[ I_3(R, T) \leq |\alpha| \int_{\mathbb{R}^n_+} u^{\alpha-1} |Du|^p \xi_{R, T}^\lambda x_n^\gamma \, dx \, dt + \frac{a}{4} \int_{\mathbb{R}^n_+} u^{q+\alpha} R_{R, T}^\lambda x_n^{\gamma+1} \, dx \, dt + c R^{n-(p-1)(\gamma+n)+2n-2n-1(\gamma+1)} T, \]

which together with (4.3) and (4.5) yields (4.8) again. The proof can be completed similarly to the previous case.

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