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ON THE SECOND EIGENVALUE OF NONLINEAR EIGENVALUE PROBLEMS

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Abstract. This article is devoted to the characterization of the second eigenvalue of nonlinear eigenvalue problems. We propose an abstract approach which allows to treat nonsmooth quasilinear problems and also to recover, in a unified way, previous results concerning the p-Laplacian.

1. Introduction

Consider the nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda V|u|^{p-2}u \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian and V is a possibly sign-changing weight. A real number λ is said to be an *eigenvalue* if (1.1) admits a nontrivial solution u.

The existence of a diverging sequence (λ_k) of eigenvalues has been proved, under quite general assumptions, in [32]. In the case $V = 1$, a different characterization of λ_2 has been provided in [18], in connection with the introduction of a possibly different sequence of eigenvalues. Further characterizations of λ_2, under various sets of assumptions, have been provided in [3] and in particular in [15, 4] by a mountain pass description. More recently, the mountain pass characterization of the second eigenvalue has been proved also for the fractional p-Laplacian in [6]. In all these papers the main techniques involved concern regularity theory for the solutions u of (1.1) and variational methods, as the eigenvalues λ can be characterized as the critical values of the functional

$$f(u) = \int_\Omega |\nabla u|^p \, dx$$

on the manifold

$$M = \big\{u : \int_\Omega V|u|^p \, dx = 1\big\} \cup \big\{u : \int_\Omega V|u|^p \, dx = -1\big\}\,.$$

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More precisely, eigenvalues λ with $\lambda > 0$ are characterized by means of the manifold

$$M = \{u : \int_\Omega V|u|^p \, dx = 1\}$$

and those with $\lambda < 0$ by means of the manifold

$$M = \Big\{u : \int_\Omega V|u|^p \, dx = -1\Big\} \,.$$

In the recent paper [20] the case in which Ω is a p-quasi open set is considered and the mountain pass characterization of the second eigenvalue is proved also in that setting. In this last paper, some typical techniques of critical point theory are replaced by the use of the minimizing movements.

On the other hand, the existence of a diverging sequence (λ_k) of eigenvalues has also been proved in [29] when, more generally, f is a convex functional of the form

$$f(u) = \int_\Omega a(x, \nabla u) \, dx \,.$$

Since $a(x, \cdot)$ is not supposed to be of class C^1, the metric critical point theory developed independently in [13, 16] and in [21, 22] is applied in this case.

The main purpose of this paper is to extend the characterizations of the second eigenvalue to the case treated in [29] by an abstract approach, based on techniques of metric critical point theory, which allows to recover in a unified way also the previous results on the second eigenvalue of the p-Laplacian.

After recalling the main tools of metric critical point theory in Section 2, we will propose in Section 3 our abstract setting and prove in Section 4 the main results. They will be applied in Section 5 to the setting of [29], while Section 6 is devoted to problems on p-quasi open sets as in [20], but without the use of minimizing movements, and Section 7 to the fractional p-Laplacian considered in [6].

2. Metric critical point theory

Let M be a metric space endowed with the distance d and $f : M \to \mathbb{R}$ a continuous function. We will denote by $B_\delta(u)$ the open ball of center u and radius δ.

2.1. **First basic facts.**

The next notion has been independently introduced in [13, 16] and in [22], while a variant has been considered in [21].

Definition 2.1. For every $u \in M$, we denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B_\delta(u) \times [0, \delta] \to M$$

satisfying

$$d(\mathcal{H}(v, t), t) \leq t \,, \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t \,,$$

for every $v \in B_\delta(u)$ and $t \in [0, \delta]$. The extended real number $|df|(u)$ is called the *weak slope* of f at u.

Remark 2.2. Let M be an open subset of a normed space and let $f : M \to \mathbb{R}$ be of class C^1. Then $|df|(u) = \|f'(u)\|$ for any $u \in M$.

Remark 2.3. Let $u \in M$ be a local minimum of f. Then $|df|(u) = 0$.

Remark 2.4. Let $\widehat{M}$ be another metric space and $\Psi : \widehat{M} \to M$ a homeomorphism which is Lipschitz continuous of constant L. Then, for every $u \in \widehat{M}$, we have

$$|d(f \circ \Psi)|(u) \leq L\,|df|(\Psi(u))\,.$$

Example 2.5. Let $M = \mathbb{R}$, $\widehat{M} = [0, +\infty[$ and $f : M \to \mathbb{R}$ defined as $f(u) = -|u|$. Then $|df|(0) = 0$, while $\left|d(f\big|_{[0,+\infty[})\right|(0) = 1$. On the other hand, the inclusion map $[0, +\infty[\subseteq \mathbb{R}$ is Lipschitz continuous, but it is not a homeomorphism.

Definition 2.6. We say that $u \in M$ is a *(lower) critical point* of f if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a *(lower) critical value* of f if there exists $u \in M$ such that $f(u) = c$ and $|df|(u) = 0$.

Definition 2.7. Given $c \in \mathbb{R}$, we say that f satisfies the *Palais-Smale condition at level c* ($(PS)_c$, for short), if every sequence (u_k) in M, with $f(u_k) \to c$ and $|df|(u_k) \to 0$, admits a convergent subsequence in M.

The next concept was first introduced in [7].

Definition 2.8. Let $\hat{u} \in M$. Given $c \in \mathbb{R}$, we say that f satisfies the *Cerami-Palais-Smale condition at level c* ($(CPS)_c$, for short), if every sequence (u_k) in M, with $f(u_k) \to c$ and $(1 + d(u_k, \hat{u}))|df|(u_k) \to 0$, admits a convergent subsequence in M.

Since

$$(1 + d(u_k, \hat{u}))|df|(u_k) \leq (1 + d(u_k, \check{u}))|df|(u_k) + d(\check{u}, \hat{u})|df|(u_k)\,,$$

it is easily seen that $(CPS)_c$ is independent of the choice of $\hat{u}$. It is also clear that $(PS)_c$ implies $(CPS)_c$.

When f is smooth, the next result is contained in [30, Theorem 1], which in turn developed some variants of the celebrated Mountain Pass Theorem (see [1, 31]).

Theorem 2.9. *Let $v \in M$ be a local minimum of f, let $w \in M$ with $w \neq v$ and $f(w) \leq f(v)$ and set*

$$\Phi = \{\varphi \in C([-1, 1]; M) : \varphi(-1) = v\,,\ \varphi(1) = w\}\,.$$

Assume that M is complete, $\Phi \neq \emptyset$ and that f satisfies $(CPS)_c$ at the level

$$c = \inf_{\varphi \in \Phi} \max_{-1 \leq t \leq 1} f(\varphi(t))\,.$$

Then there exists a critical point u of f with $u \neq v$, $u \neq w$ and $f(u) = c$.

Proof. According to [11, Theorem 4.1], the $(CPS)_c$ condition is just the $(PS)_c$ condition with respect to an auxiliary distance which keeps the completeness of M and does not change the critical points of f and the topology of M. Therefore, we may assume without loss of generality that f satisfies $(PS)_c$.

Let $r > 0$ be such that $d(w, v) > r$ and

$$f(z) \geq f(v) \quad \text{whenever } d(z, v) \leq r\,.$$

If we set

$$A = \{z \in M : d(z, v) = r\}\,,$$

we infer from [13, Theorem 3.7] that there exists a critical point u of f with $f(u) = c$ and, moreover, that $u \in A$ if $c = \inf_A f$. In both cases $c > \inf_A f$ and $c = \inf_A f$ we have that $u \neq v$, $u \neq w$ and the assertion follows. $\square$

Theorem 2.10. *Let*
$$b > a := \inf_M f > -\infty\,.$$
Assume that M is complete and that f has no critical value in $]a,b[$ and satisfies $(CPS)_c$ for every $c \in [a,b[$. Suppose also that either the set $f^{-1}(a)$ is finite or each $v \in f^{-1}(a)$ admits a path connected neighborhood in $\{w \in M : f(w) \leq b\}$. Then, for every $u \in M$ with $f(u) \leq b$ and $|df|(u) \neq 0$, there exists a continuous map $\varphi : [-1,1] \to M$ such that $\varphi(-1) = u$, $f(\varphi(1)) = a$ and $f(\varphi(t)) \leq b$ for any $t \in [-1,1]$.

Proof. As before, we may assume without loss of generality that f satisfies $(PS)_c$ for every $c \in [a,b[$. Let $u \in M$ with $f(u) \leq b$ and $|df|(u) \neq 0$. By Definition 2.1 there exists a continuous map $\psi : [-1,0] \to M$ with $\psi(-1) = u$, $f(\psi(0)) < b$ and $f(\psi(t)) \leq b$ for any $t \in [-1,0]$. Let $f(\psi(0)) < \beta < b$.

Suppose first that $f^{-1}(a)$ is finite. Then, by the Second Deformation Lemma (see [12, Theorem 4] and also [10, Theorem 2.10]), there exists a continuous map
$$\eta : \{w \in M : f(w) \leq \beta\} \times [0,1] \to \{w \in M : f(w) \leq \beta\}$$
such that $\eta(w,0) = w$ and $f(\eta(w,1)) = a$. In particular
$$\varphi(t) = \begin{cases} \psi(t) & \text{if } -1 \leq t \leq 0\,,\\ \eta(\psi(0),t) & \text{if } 0 \leq t \leq 1\,,\end{cases}$$
has the required properties.

Assume now that each $v \in f^{-1}(a)$ admits a path connected neighborhood V_v in the sublevel $\{w \in M : f(w) \leq b\}$ and set
$$W = \cup_{v\in f^{-1}(a)} V_v\,.$$
From the Deformation Theorem (see [13, Theorem 2.14]) we infer that there exists $\alpha \in]a,\beta[$ such that
$$\{w \in M : f(w) \leq \alpha\} \subseteq W\,.$$
Then, by the Noncritical Interval Theorem (see [13, Theorem 2.15]), there exists a continuous map
$$\eta : \{w \in M : f(w) \leq \beta\} \times [0,1] \to \{w \in M : f(w) \leq \beta\}$$
such that $\eta(w,0) = w$ and $f(\eta(w,1)) \leq \alpha$. Finally, since $\eta(\psi(0),1) \in W$ there exists a continuous map
$$\xi : [0,1] \to \{w \in M : f(w) \leq b\}$$
such that $\xi(0) = \eta(\psi(0),1)$ and $f(\xi(1)) = a$. Then
$$\varphi(t) = \begin{cases} \psi(t) & \text{if } -1 \leq t \leq 0\,,\\ \eta(\psi(0),2t) & \text{if } 0 \leq t \leq 1/2\,,\\ \xi(2t-1) & \text{if } 1/2 \leq t \leq 1\,,\end{cases}$$
has the required properties. □

Remark 2.11. Let
$$M = \big\{(x,y) \in \mathbb{R}^2 : x \neq 0\,,\ y = \sin\frac{1}{x}\big\} \cup (\{0\} \times [-2,2])$$
and let $f : M \to \mathbb{R}$ be defined as $f(x,y) = x^2$. Then the set of minima is infinite and there are minima without a path connected neighborhood, while the other

assumptions of Theorem 2.10 are satisfied for any $b > 0$. On the other hand, there is no path connecting points (x, y) with $f(x, y) > 0$ and a minimum of f.

2.2. **A case with symmetry.** Let now $\Psi : M \to M$ be an isometry such that $\Psi \circ \Psi = \mathrm{Id}$. We assume that f is also Ψ-*invariant*, namely that $f(\Psi(u)) = f(u)$ for any $u \in M$, and we set

$$\mathrm{Fix}(M) = \{u \in M : \Psi(u) = u\}\,.$$

Definition 2.12. A subset A of M is said to be Ψ-*invariant* if $\Psi(A) \subseteq A$. A map $\varphi : A \to \mathbb{R}^k$, where A is a Ψ-invariant subset of M, is said to be Ψ-*equivariant* if $\varphi(\Psi(u)) = -\varphi(u)$ for any $u \in A$. Finally, a map $\varphi : S \to M$, where $S \subseteq \mathbb{R}^k$ is symmetric with respect to the origin, is said to be Ψ-*equivariant* if $\varphi(-u) = \Psi(\varphi(u))$ for any $u \in S$.

For every nonempty Ψ-invariant subset A of M, we set

$$\begin{aligned}\gamma(A) = \min\big\{&k \geq 1 : \text{there exists a } \Psi\text{-equivariant and continuous map}\\ &\varphi : A \to \mathbb{R}^k \setminus \{0\}\big\}\,.\end{aligned}$$

We agree that $\gamma(A) = \infty$ if there is no such k and we set $\gamma(\emptyset) = 0$.

We also set

$$\begin{aligned}\overline{\gamma}(A) = \sup\big\{&k \geq 1 : \text{there exists a } \Psi\text{-equivariant and continuous map}\\ &\varphi : \mathbb{R}^k \setminus \{0\} \to A\big\}\,.\end{aligned}$$

Again, we set $\overline{\gamma}(\emptyset) = 0$.

It is well known (see e.g. [8, 25]) that

$$\overline{\gamma}(A) \leq \gamma(A) \quad \text{for every } \Psi\text{-invariant subset } A \text{ of } M$$

and it is clear that $\overline{\gamma}(A) = \gamma(A) = \infty$ whenever $A \cap \mathrm{Fix}(M) \neq \emptyset$.

Then, for every $k \geq 1$, we set

$$\begin{aligned}\underline{c}_k = \inf\big\{&\max_A f : A \text{ is a compact and } \Psi\text{-invariant subset of } M\\ &\text{with } \gamma(A) \geq k\big\}\,,\\ \overline{c}_k = \inf\big\{&\max_A f : A \text{ is a compact and } \Psi\text{-invariant subset of } M\\ &\text{with } \overline{\gamma}(A) \geq k\big\}\,,\end{aligned}$$

where we agree that $\underline{c}_k = +\infty$ (resp. $\overline{c}_k = +\infty$) if there is no A with $\gamma(A) \geq k$ (resp. $\overline{\gamma}(A) \geq k$).

It is easily seen that

$$\underline{c}_k \leq \underline{c}_{k+1}\,, \quad \overline{c}_k \leq \overline{c}_{k+1}\,, \quad \underline{c}_k \leq \overline{c}_k\,, \quad \text{for every } k \geq 1\,,$$
$$\underline{c}_1 = \overline{c}_1 = \inf_M f\,.$$

Theorem 2.13. *Assume that M is complete. Then the following facts hold:*

(a) *if*
$$-\infty < \underline{c}_k < \inf\{f(u) : u \in \mathrm{Fix}(M)\}$$
and f satisfies $(CPS)_{\underline{c}_k}$, then $\underline{c}_k$ is a critical value of f (we agree that $\inf \emptyset = +\infty$);

(b) *if*
$$-\infty < \overline{c}_k < \inf\{f(u) : u \in \mathrm{Fix}(M)\}$$
and f satisfies $(CPS)_{\overline{c}_k}$, then $\overline{c}_k$ is a critical value of f;

(c) *if*
$$-\infty < \underline{c}_k = \cdots = \underline{c}_{k+m-1} < \inf\{f(u) : u \in \mathrm{Fix}(M)\}$$
and f satisfies $(CPS)_{\underline{c}_k}$, then
$$\gamma(\{u \in M : f(u) = \underline{c}_k\,,\ |df|(u) = 0\}) \geq m\,;$$

(d) *if f is bounded from below,*
$$b < \inf\{f(u) : u \in \mathrm{Fix}(M)\}$$
and f satisfies $(CPS)_c$ for every $c \leq b$, then we have
$$\gamma(\{u \in M : f(u) \leq b\}) < \infty\,.$$

Proof. Again, the proof of [11, Theorem 4.1] is compatible with the symmetry structure. Therefore one can assume $(PS)_c$ instead of $(CPS)_c$. Then the argument is the same as in the proof of [17, Theorem 2.5]. □

2.3. **Constrained problems.** Let now X be a real Banach space. In the following, $\partial f(u)$ will denote Clarke's subdifferential and $f^0(u;v)$ the associated generalized directional derivative (see [9]).

If f is locally Lipschitz, we have
$$f^0(u;v) := \limsup_{z\to u\,,\ t\to 0^+} \frac{f(z+tv)-f(z)}{t} = \limsup_{z\to u\,,\ w\to v\,,\ t\to 0^+} \frac{f(z+tw)-f(z)}{t}\,,$$
$$\partial f(u) = \{\alpha \in X' : \langle \alpha, v\rangle \leq f^0(u;v) \text{ for any } v \in X\}\,.$$
If f is locally Lipschitz and convex, we also have that
$$f^0(u;v) = \lim_{t\to 0^+} \frac{f(u+tv)-f(u)}{t} = \lim_{w\to v\,,\ t\to 0^+} \frac{f(u+tw)-f(u)}{t}\,,$$
and $\partial f(u)$ agrees with the subdifferential of convex analysis.

Theorem 2.14. *Let U be an open subset of X, $f, g : U \to \mathbb{R}$ two locally Lipschitz functions,*
$$M = \{v \in U : g(v) = 0\}$$
and $u \in M$ with $0 \notin \partial g(u)$. Then we have
$$\big|d(f\big|_M)\big|(u) \geq \min\{\|\alpha - \lambda\beta\| : \alpha \in \partial f(u)\,,\ \beta \in \partial g(u)\,,\ \lambda \in \mathbb{R}\}\,.$$

Proof. Since $0 \notin \partial g(u)$, there exists $v \in X$ such that $g^0(u;v) < 0$, namely
$$g^0(u;u_- - u) < 0\,,\quad g^0(u;u-u_+) < 0\,,$$
if we set $u_- = u + v$ and $u_+ = u - v$. Then the assertion follows from [17, Theorem 3.5]. □

Theorem 2.15. *Let U be a convex and open subset of X, $f : U \to \mathbb{R}$ a lower semicontinuous and convex function, $g : U \to \mathbb{R}$ a function of class C^1,*
$$M = \{v \in U : g(v) = 0\}$$
and $u \in M$ with $g'(u) \neq 0$. Then f is locally Lipschitz and we have
$$\big|d(f\big|_M)\big|(u) = \min\{\|\alpha - \lambda g'(u)\| : \alpha \in \partial f(u)\,,\ \lambda \in \mathbb{R}\}\,.$$

Proof. By [19, Corollaries 2.5 and 2.4], f is locally Lipschitz. Then, from Theorem 2.14 we infer that

$$\big|d(f\big|_M)\big|(u) \geq \min\{\|\alpha - \lambda g'(u)\| : \alpha \in \partial f(u)\,,\ \lambda \in \mathbb{R}\}\,.$$

Let now $\alpha \in \partial f(u)$, $\lambda \in \mathbb{R}$ and let

$$\mathcal{H} : (B_\delta(u) \cap M) \times [0, \delta] \to M$$

be as in Definition 2.1. Let

$$g(v) = g(u) + \langle g'(u), v - u\rangle + \|v - u\|\omega(v)\,,$$

where ω is continuous with $\omega(u) = 0$. Then we have

$$(f - \lambda g)(\mathcal{H}(u, t)) - (f - \lambda g)(u) = f(\mathcal{H}(u, t)) - f(u) \leq -\sigma t$$

and

$$\begin{aligned}
&(f - \lambda g)(\mathcal{H}(u, t)) - (f - \lambda g)(u)\\
&\geq \langle \alpha - \lambda g'(u), \mathcal{H}(u, t) - u\rangle - \lambda\|\mathcal{H}(u, t) - u\|\,\omega(\mathcal{H}(u, t))\\
&\geq -(\|\alpha - \lambda g'(u)\| + |\lambda|\,|\omega(\mathcal{H}(u, t))|)\,\|\mathcal{H}(u, t) - u\|\\
&\geq -(\|\alpha - \lambda g'(u)\| + |\lambda|\,|\omega(\mathcal{H}(u, t))|)\,t\,.
\end{aligned}$$

We infer that

$$\sigma \leq \|\alpha - \lambda g'(u)\| + |\lambda|\,|\omega(\mathcal{H}(u, t))| \quad \text{for every } t \in]0, \delta]\,.$$

Going to the limit as $t \to 0$, we conclude that

$$\sigma \leq \|\alpha - \lambda g'(u)\|$$

and the assertion follows. □

Proposition 2.16. *Let U be an open subset of X, $f, g : U \to \mathbb{R}$ two functions of class C^1,*

$$M = \{v \in U : g(v) = 0\}$$

and $u \in M$ with $g'(u) \neq 0$. Then we have

$$\big|d(f\big|_M)\big|(u) = \min\{\|f'(u) - \lambda g'(u)\| : \lambda \in \mathbb{R}\}\,.$$

For a proof of the above proposition see [16, Corollary 2.12].

Proposition 2.17. *Let $p \in \mathbb{R}$ and let $g : X \setminus \{0\} \to \mathbb{R}$ be a locally Lipschitz function which is positively homogeneous of degree p. Then we have*

$$g^0(u; u) = p\,g(u)\,, \quad g^0(u; -u) = -p\,g(u) \quad \textit{for any } u \neq 0\,,$$
$$\langle \alpha, u\rangle = p\,g(u) \quad \textit{for any } u \neq 0 \textit{ and } \alpha \in \partial g(u)\,.$$

Proof. If L is a Lipschitz constant in a neighborhood of u, for v close to u and t close to 0 we have

$$\begin{aligned}
\frac{g(v + tu) - g(v)}{t} &= \frac{g(v + tv) - g(v)}{t} + \frac{g(v + tu) - g(v + tv)}{t}\\
&= \frac{(1 + t)^p - 1}{t}\,g(v) + \frac{g(v + tu) - g(v + tv)}{t}\,,
\end{aligned}$$

whence

$$\Big\|\frac{g(v + tu) - g(v)}{t} - \frac{(1 + t)^p - 1}{t}\,g(v)\Big\| \leq L\|v - u\|\,.$$

It follows

$$g^0(u;u) = \limsup_{v\to u\,,\ t\to 0^+} \frac{g(v+tu)-g(v)}{t} = p\,g(u)\,.$$

The generalized directional derivative $g^0(u;-u)$ can be treated in a similar way. Then we also have

$$p\,g(u) = -g^0(u;-u) \le \langle \alpha, u\rangle \le g^0(u;u) = p\,g(u)$$

and the proof is complete. □

Corollary 2.18. *Let $f, g : X \setminus \{0\} \to \mathbb{R}$ be two locally Lipschitz functions which are positively homogeneous of the same degree $p \neq 0$. Then the following facts hold:*

(a) *for every $u \in X \setminus \{0\}$ with $g(u) \neq 0$, we have $0 \notin \partial g(u)$;*
(b) *if $u \in X \setminus \{0\}$ and $\alpha \in \partial f(u)$, $\beta \in \partial g(u)$, $\lambda \in \mathbb{R}$ satisfy $\alpha = \lambda\beta$, then*

$$f(u) = \lambda\, g(u)\,.$$

Proof. (a) If $p > 0$ and $g(u) > 0$, we have $g^0(u;-u) = -p\,g(u) < 0$ by Proposition 2.17, whence $0 \notin \partial g(u)$. The other cases can be treated in a similar way. By Proposition 2.17 we have

$$p\,f(u) = \langle \alpha, u\rangle = \lambda\langle \beta, u\rangle = \lambda\, p\, g(u)\,,$$

whence the assertion. □

3. General facts on nonlinear eigenvalue problems

Let X be a real Banach space with $X \neq \{0\}$ and let $f, g : X \to \mathbb{R}$ be two functions such that:

(i) f and g are even, continuous and positively homogeneous of the same degree $p > 0$.

Definition 3.1. We say that $u \in X$ is an *eigenvector* if $g(u) \neq 0$ and u is a critical point of $f\big|_{M_u}$, where

$$M_u = \{v \in X : g(v) = g(u)\}\,.$$

In such a case we say that

$$\lambda = \frac{f(u)}{g(u)}$$

is the *eigenvalue* associated with the eigenvector u.

Proposition 3.2. *If u is an eigenvector with eigenvalue λ then, for every $t \neq 0$, we have that tu is an eigenvector with the same eigenvalue.*

Proof. Since $\Psi(u) = tu$ is a homeomorphism such that Ψ and Ψ^{-1} are both Lipschitz continuous, it follows from Remark 2.4 that u is a critical point of f restricted to M_u if and only if tu is a critical point of f restricted to

$$\{v \in X : g(t^{-1}v) = g(u)\}\,.$$

Then the assertion easily follows. □

Definition 3.3. An eigenvalue λ is said to be *simple*, if it is not associated with two linearly independent eigenvectors.

In the following, we will only consider eigenvectors with $g(u) > 0$. Observe that λ is an eigenvalue associated with an eigenvector u with $g(u) > 0$ if and only if λ is a critical value of f restricted to

$$M := \{v \in X : g(v) = 1\} .$$

We also assume that

(ii) the set M is not empty and the function $f\big|_M$ is bounded from below and satisfies $(CPS)_c$ for any $c \in \mathbb{R}$.

This section is devoted to the consequences of (i) and (ii). We consider the isometry $\Psi : X \to X$ defined as $\Psi(u) = -u$ and define $\gamma(A)$ and $\overline{\gamma}(A)$, for every symmetric subset A of X, according to Section 2.

Then, for every $k \geq 1$, we set

$$\begin{aligned} \underline{\lambda}_k &= \inf \big\{ \max_A f : A \text{ is a compact and symmetric subset of } M \\ &\qquad \text{with } \gamma(A) \geq k \big\} , \\ \overline{\lambda}_k &= \inf \big\{ \max_A f : A \text{ is a compact and symmetric subset of } M \\ &\qquad \text{with } \overline{\gamma}(A) \geq k \big\} , \end{aligned}$$

where we agree that $\underline{\lambda}_k = +\infty$ (resp. $\overline{\lambda}_k = +\infty$) if there is no A with $\gamma(A) \geq k$ (resp. $\overline{\gamma}(A) \geq k$).

According to Section 2, we have that

$$\underline{\lambda}_k \leq \underline{\lambda}_{k+1} , \quad \overline{\lambda}_k \leq \overline{\lambda}_{k+1} , \quad \underline{\lambda}_k \leq \overline{\lambda}_k , \quad \text{for every } k \geq 1 ,$$
$$\underline{\lambda}_1 = \overline{\lambda}_1 = \inf_M f .$$

Since $\underline{\lambda}_1 = \overline{\lambda}_1$, in the following we will simply write λ_1.

Theorem 3.4. *The following facts hold:*

(a) *if $\underline{\lambda}_k < +\infty$, then $\underline{\lambda}_k$ is an eigenvalue;*
(b) *if $\overline{\lambda}_k < +\infty$, then $\overline{\lambda}_k$ is an eigenvalue;*
(c) $\inf_M f$ *is achieved, so that $\lambda_1 = \min_M f$;*
(d) *if λ_1 is simple, then $\lambda_1 < \underline{\lambda}_2$;*
(e) *for every $b \in \mathbb{R}$, we have*

$$\gamma(\{u \in M : f(u) \leq b\}) < \infty ,$$

whence $\lim_k \underline{\lambda}_k = +\infty$.

Proof. Assertions (a), (b) and (e) follow from (a), (b) and (d) of Theorem 2.13, while assertion (c) is a particular case of (a) or (b).

(d) If $\lambda_1 = \underline{\lambda}_2$, we have

$$\gamma(\{u \in M : f(u) = \lambda_1\}) \geq 2$$

by (c) of Theorem 2.13, while $\gamma(\{u, -u\}) = 1$ for every $u \in M$. $\square$

Definition 3.5. Let $u \in M$ be an eigenvector with eigenvalue λ_1 and let Φ be the set of the continuous maps $\varphi : [-1, 1] \to M$ such that $\varphi(-1) = -u$ and $\varphi(1) = u$. If $\Phi \neq \emptyset$, we define the "mountain pass eigenvalue" associated with u

$$\lambda_{mp}(u) = \inf_{\varphi \in \Phi} \max_{-1 \leq t \leq 1} f(\varphi(t)) ,$$

otherwise we set $\lambda_{mp}(u) = +\infty$.

Theorem 3.6. *For every eigenvector $u \in M$ with eigenvalue λ_1, the following facts hold:*

(a) *if $\lambda_{mp}(u) < +\infty$, then $\lambda_{mp}(u)$ is an eigenvalue;*
(b) *we have $\lambda_1 \leq \underline{\lambda}_2 \leq \overline{\lambda}_2 \leq \lambda_{mp}(u)$;*
(c) *if $b \in \mathbb{R}$ and there exists an odd and continuous map $\varphi : \mathbb{R}^2 \setminus \{0\} \to M$ such that $u \in \varphi(\mathbb{R}^2 \setminus \{0\})$ and $f(\varphi(t_1, t_2)) \leq b$ for any $(t_1, t_2) \in \mathbb{R}^2 \setminus \{0\}$, then*
$$\lambda_{mp}(u) \leq b\,.$$

Proof. Assertion (a) follows from Theorem 2.9.

(b) We already know that $\lambda_1 \leq \underline{\lambda}_2 \leq \overline{\lambda}_2$. If $\varphi : [-1, 1] \to M$ is a continuous map such that $\varphi(-1) = -u$ and $\varphi(1) = u$, then $\psi : \mathbb{R}^2 \setminus \{0\} \to M$ defined as
$$\psi(t_1, t_2) = \begin{cases} \varphi\Big(\frac{t_1}{\sqrt{t_1^2+t_2^2}}\Big) & \text{if } t_2 \geq 0\,, \\ -\varphi\Big(-\frac{t_1}{\sqrt{t_1^2+t_2^2}}\Big) & \text{if } t_2 \leq 0\,, \end{cases}$$
is continuous and odd, whence $\overline{\lambda}_2 \leq \lambda_{mp}(u)$.

(c) Let $u = \varphi(\tau_1, \tau_2)$ with $(\tau_1, \tau_2) \in \mathbb{R}^2 \setminus \{0\}$, whence $-u = \varphi(-\tau_1, -\tau_2)$. There exists a continuous map $\psi : [-1, 1] \to \mathbb{R}^2 \setminus \{0\}$ such that $\psi(-1) = (-\tau_1, -\tau_2)$ and $\psi(1) = (\tau_1, \tau_2)$. Then $(\varphi\circ\psi) : [-1, 1] \to M$ is continuous and satisfies $(\varphi\circ\psi)(-1) = -u$, $(\varphi\circ\psi)(1) = u$ and $f((\varphi\circ\psi)(t)) \leq b$ for any $t \in [-1, 1]$, whence $\lambda_{mp}(u) \leq b$. $\square$

Example 3.7. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be defined as
$$f(x, y, z) = 8z^6 - 15(x^2 + y^2 + z^2)z^4 + 6(x^2 + y^2 + z^2)^2 z^2 + 2(x^2 + y^2 + z^2)^3\,,$$
$$g(x, y, z) = (x^2 + y^2 + z^2)^3\,.$$
Then we have
$$\lambda_1 = 1\,, \quad \underline{\lambda}_2 = \overline{\lambda}_2 = 2\,,$$
while $\pm u$ with $u = (0, 0, 1)$ are the eigenvectors in M with eigenvalue λ_1. On the other hand, $\lambda_{mp}(u) = 43/16$ so that
$$\lambda_1 < \underline{\lambda}_2 = \overline{\lambda}_2 < \lambda_{mp}(u)\,.$$

The proof of [6, Proposition 4.2] has suggested us the next concept.

Definition 3.8. Let $u \in X$ with $g(u) > 0$. We say that (u_1, u_2) is a *decomposition of* u, if $u_1, u_2 \in X$, $u = u_1 + u_2$, $g(u_j) > 0$ for $j = 1, 2$ and
$$g(t_1u_1 + t_2u_2) \geq g(t_1u_1) + g(t_2u_2)\,,$$
$$f(t_1u_1 + t_2u_2) \leq \frac{f(u)}{g(u)}\, g(t_1u_1 + t_2u_2)\,,$$
for every $t_1, t_2 \in \mathbb{R}$.

An element $u \in X$ with $g(u) > 0$ is said to be *decomposable*, if it admits a decomposition (u_1, u_2).

Proposition 3.9. *Let $b \in \mathbb{R}$ and let $u_1, u_2 \in X$ with $g(u_j) > 0$ for $j = 1, 2$ and*
$$g(t_1u_1 + t_2u_2) \geq g(t_1u_1) + g(t_2u_2)\,,$$
$$f(t_1u_1 + t_2u_2) \leq b\, g(t_1u_1 + t_2u_2)\,,$$

for every $t_1, t_2 \in \mathbb{R}$*. Then there exists an odd and continuous map* $\varphi : \mathbb{R}^2\setminus\{0\} \to M$*, such that*

$$\frac{u_1}{g(u_1)^{1/p}}\,,\ \frac{u_2}{g(u_2)^{1/p}}\,,\ \frac{u_1+u_2}{g(u_1+u_2)^{1/p}} \in \varphi(\mathbb{R}^2\setminus\{0\})$$

and

$$f(\varphi(t_1,t_2)) \le b \quad \textit{for every } (t_1,t_2) \in \mathbb{R}^2\setminus\{0\}\,.$$

Proof. Since

$$g(t_1u_1+t_2u_2) \ge g(t_1u_1)+g(t_2u_2) = |t_1|^p g(u_1) + |t_2|^p g(u_2)\,,$$

we can define an odd and continuous map $\varphi : \mathbb{R}^2\setminus\{0\} \to M$ as

$$\varphi(t_1,t_2) = \frac{t_1u_1+t_2u_2}{g(t_1u_1+t_2u_2)^{1/p}}\,.$$

Of course,

$$\frac{u_1}{g(u_1)^{1/p}} = \varphi(1,0)\,,\ \frac{u_2}{g(u_2)^{1/p}} = \varphi(0,1)\,,\ \frac{u_1+u_2}{g(u_1+u_2)^{1/p}} = \varphi(1,1)$$

and

$$f(\varphi(t_1,t_2)) = \frac{f(t_1u_1+t_2u_2)}{g(t_1u_1+t_2u_2)} \le b\,,$$

whence the assertion. □

Theorem 3.10. *If* λ *is an eigenvalue which admits a decomposable eigenvector, then* $\lambda \ge \overline{\lambda}_2$*.*

Proof. Let u be a decomposable eigenvector with eigenvalue λ and let (u_1,u_2) be a decomposition of u. By Proposition 3.9, there exists an odd and continuous map $\varphi : \mathbb{R}^2\setminus\{0\} \to M$ such that

$$f(\varphi(t_1,t_2)) \le \frac{f(u)}{g(u)} = \lambda \quad \text{for every } (t_1,t_2) \in \mathbb{R}^2\setminus\{0\}\,,$$

whence $\overline{\lambda}_2 \le \lambda$. □

4. Main results

Let again X be a real Banach space and $f, g : X \to \mathbb{R}$ be two functions satisfying (i) and (ii). As before, we will consider only eigenvectors u with $g(u) > 0$.

Throughout this section, we also assume that:

- (iii) if u is an eigenvector with eigenvalue λ and v is an eigenvector with eigenvalue μ (possibly with $\lambda = \mu$), such that u and v are linearly independent and v is not decomposable, then one at least of the following facts holds:
  - (a) we have
  $$g(t_1u+t_2v) \ge g(t_1u)+g(t_2v)\,,$$
  $$f(t_1u+t_2v) \le \max\{\lambda,\mu\}\, g(t_1u+t_2v)\,,$$
  for every $t_1, t_2 \in \mathbb{R}$.
  - (b) u is decomposable and admits a decomposition (u_1,u_2) such that
  $$g(t_1u_1+t_2v) \ge g(t_1u_1)+g(t_2v)\,,$$
  $$f(t_1u_1+t_2v) \le \max\{\lambda,\mu\}\, g(t_1u_1+t_2v)\,,$$
  for every $t_1, t_2 \in \mathbb{R}$;

(c) u is decomposable and admits a decomposition (u_1, u_2) such that u_1 is not an eigenvector.

This section is devoted to study the consequences of (i), (ii) and (iii).

Theorem 4.1. *The following facts are equivalent:*

(a) λ_1 *is simple;*
(b) *we have* $\lambda_1 < \underline{\lambda}_2$*;*
(c) *each eigenvector with eigenvalue* λ_1 *is not decomposable.*

Proof. By (d) of Theorem 3.4 and Theorem 3.10, it is enough to prove that (c) $\Rightarrow$ (a). Assume, for a contradiction, that u, v are two linearly independent eigenvectors with eigenvalue λ_1. We know that u and v are not decomposable. Then assertion (a) of assumption (iii) holds. It easily follows that $g(u+v) > 0$. Moreover, if $w \in X$ with $g(w) = g(u+v)$, we have

$$f(w) \geq \lambda_1 g(w) = \lambda_1 g(u+v) \geq f(u+v)\,.$$

By Remark 2.3, we infer that $(u+v)$ is an eigenvector. Of course λ_1 is the associated eigenvalue and $u+v$ admits the decomposition (u, v), whence a contradiction. □

Theorem 4.2. *There is no eigenvalue* λ *satisfying* $\lambda_1 < \lambda < \overline{\lambda}_2$*. Moreover, we have* $\underline{\lambda}_2 = \overline{\lambda}_2$*.*

Proof. Assume, for a contradiction, that λ is an eigenvalue such that $\lambda_1 < \lambda < \overline{\lambda}_2$ and let u be an eigenvector with eigenvalue λ and v an eigenvector with eigenvalue λ_1. By Proposition 3.2 we have that u and v are linearly independent. From Theorem 3.10 we infer that u and v are not decomposable, so that assertion (a) of assumption (iii) holds. By Proposition 3.9 there exists an odd and continuous map

$$\varphi : \mathbb{R}^2 \setminus \{0\} \to M$$

such that

$$f(\varphi(t_1, t_2)) \leq \max\{\lambda_1, \lambda\} = \lambda \quad \text{for every } (t_1, t_2) \in \mathbb{R}^2 \setminus \{0\}\,,$$

whence $\overline{\lambda}_2 \leq \lambda$ and a contradiction follows.

If $\lambda_1 = \overline{\lambda}_2$, it is obvious that $\lambda_1 = \underline{\lambda}_2 = \overline{\lambda}_2$. If $\lambda_1 < \overline{\lambda}_2$, it follows from Theorems 3.10 and 4.1 that $\lambda_1 < \underline{\lambda}_2$, whence $\underline{\lambda}_2 = \overline{\lambda}_2$. □

Theorem 4.3. *If* λ_1 *is simple, then* λ_1 *is isolated in the set of the eigenvalues.*

The above theorem follows from Theorems 4.1 and 4.2. Now we can prove the main result of the paper.

Theorem 4.4. *For every eigenvector* $u \in M$ *with eigenvalue* λ_1*, we have*

$$\lambda_{mp}(u) = \underline{\lambda}_2 = \overline{\lambda}_2\,.$$

In particular, $\lambda_{mp}(u)$ *is independent of* u*.*

Proof. Let $u \in M$ be an eigenvector with eigenvalue λ_1. By (b) of Theorem 3.6, it is sufficient to prove that $\lambda_{mp}(u) \leq \underline{\lambda}_2$. We deal with several possible scenarios.

Case 1: λ_1 is not simple.

Subcase 1.1: u is decomposable. Let (u_1, u_2) be a decomposition of u. By Proposition 3.9, there exists an odd and continuous map $\varphi : \mathbb{R}^2 \setminus \{0\} \to M$ such that

$$\frac{u_1}{g(u_1)^{1/p}}\,,\ \frac{u_2}{g(u_2)^{1/p}}\,,\ u \in \varphi(\mathbb{R}^2 \setminus \{0\})$$

and

$$f(\varphi(t_1,t_2)) \le \lambda_1 \quad \text{for every } (t_1,t_2) \in \mathbb{R}^2 \setminus \{0\}\,.$$

Actually, in this case equality holds and u_1, u_2 also are eigenvectors with eigenvalue λ_1. Taking into account assertion (c) of Theorem 3.6, we conclude that

$$\lambda_{mp}(u) \le \lambda_1 \le \underline{\lambda}_2\,.$$

Subcase 1.2: u is not decomposable. Then, by Theorem 4.1, the eigenvalue λ_1 admits another eigenvector v which is decomposable. Clearly u and v are linearly independent and we take into account assumption (iii). As in the previous case, if (v_1, v_2) is a decomposition of v, then v_1 and v_2 also are eigenvectors with eigenvalue λ_1. Therefore assertion (c) of assumption (iii) cannot hold.

If (a) holds, by Proposition 3.9 there exists an odd and continuous map $\varphi : \mathbb{R}^2 \setminus \{0\} \to M$ such that $u \in \varphi(\mathbb{R}^2 \setminus \{0\})$ and

$$f(\varphi(t_1,t_2)) = \lambda_1 \quad \text{for every } (t_1,t_2) \in \mathbb{R}^2 \setminus \{0\}\,.$$

Again, taking into account assertion (c) of Theorem 3.6, we conclude that

$$\lambda_{mp}(u) \le \lambda_1 \le \underline{\lambda}_2\,.$$

If (v_1, v_2) is a decomposition of v as in (b), again by Proposition 3.9 there exists an odd and continuous map

$$\varphi : \mathbb{R}^2 \setminus \{0\} \to M$$

with the same properties as in the previous case, whence

$$\lambda_{mp}(u) \le \lambda_1 \le \underline{\lambda}_2\,.$$

Case 2: λ_1 is simple. Now, by Theorem 4.1, it is $\lambda_1 < \underline{\lambda}_2$ and u is not decomposable. If $\underline{\lambda}_2 = +\infty$ it is obvious that $\lambda_{mp}(u) \le \underline{\lambda}_2$. Otherwise, let $\underline{\lambda}_2 < +\infty$ and let v be an eigenvector associated with $\underline{\lambda}_2$. Since $\lambda_1 \ne \underline{\lambda}_2$, from Proposition 3.2 it follows that u and v are linearly independent.

This time, all the three scenarios (a), (b) and (c) of assumption (iii) are possible. In the cases (a) and (b) we find again, by Proposition 3.9, an odd and continuous map $\varphi : \mathbb{R}^2 \setminus \{0\} \to M$ such that $u \in \varphi(\mathbb{R}^2 \setminus \{0\})$ and

$$f(\varphi(t_1,t_2)) \le \max\{\lambda_1, \underline{\lambda}_2\} = \underline{\lambda}_2 \quad \text{for every } (t_1,t_2) \in \mathbb{R}^2 \setminus \{0\}\,.$$

By assertion (c) of Theorem 3.6 we conclude that

$$\lambda_{mp}(u) \le \underline{\lambda}_2\,.$$

Finally, let (v_1, v_2) be a decomposition of v as in (c) of assumption (iii). Without loss of generality, we may assume that $g(v_1) = 1$.

By Proposition 3.9 there exists an odd and continuous map $\varphi : \mathbb{R}^2 \setminus \{0\} \to M$ such that $v_1 \in \varphi(\mathbb{R}^2 \setminus \{0\})$ and

$$f(\varphi(t_1,t_2)) \le \underline{\lambda}_2 \quad \text{for every } (t_1,t_2) \in \mathbb{R}^2 \setminus \{0\}\,.$$

If $v_1 = \varphi(\tau_1, \tau_2)$, then $-v_1 = \varphi(-\tau_1, -\tau_2)$ and there exists a continuous map $\psi : [-1,1] \to \mathbb{R}^2 \setminus \{0\}$ such that $\psi(-1) = (-\tau_1, -\tau_2)$ and $\psi(1) = (\tau_1, \tau_2)$. Then $(\varphi\circ\psi) : [-1,1] \to M$ satisfies $(\varphi\circ\psi)(-1) = -v_1$, $(\varphi\circ\psi)(1) = v_1$ and $f((\varphi\circ\psi)(t)) \le \underline{\lambda}_2$ for any $t \in [-1,1]$.

On the other hand, it follows from Theorem 4.2 that $f\big|_M$ has no critical value in $]\lambda_1, \underline{\lambda}_2[$. Furthermore, it is $f^{-1}(\lambda_1) = \{u, -u\}$ and $f(v_1) \leq \underline{\lambda}_2$ with $\big|d(f\big|_M)\big|(v_1) \neq 0$, as v_1 is not an eigenvector.

From Theorem 2.10 with $a = \lambda_1$ and $b = \underline{\lambda}_2$, we infer that there exists a continuous map $\eta : [-1, 1] \to M$ such that $\eta(-1) = v_1$, $f(\eta(1)) = \lambda_1$ and $f(\eta(t)) \leq \underline{\lambda}_2$ for any $t \in [-1, 1]$. It follows that $\eta(1)$ is either u or $-u$.

If we define $\zeta : [-1, 1] \to M$ by

$$\zeta(t) = \begin{cases} -\eta(-3 - 4t) & \text{if } -1 \leq t \leq -1/2\,, \\ (\varphi \circ \psi)(2t) & \text{if } -1/2 \leq t \leq 1/2\,, \\ \eta(4t - 3) & \text{if } 1/2 \leq t \leq 1\,, \end{cases}$$

then it is easily seen that ζ is a continuous map connecting $-u$ and u with $f(\zeta(t)) \leq \underline{\lambda}_2$ for any $t \in [-1, 1]$, whence $\lambda_{mp}(u) \leq \underline{\lambda}_2$ and the proof is complete. $\square$

5. Nonsmooth quasilinear elliptic problems

This section is devoted to the setting of [29]. In the following, we set

$$s^+ = \max\{s, 0\}\,, \quad s^- = \max\{-s, 0\}\,.$$

Let Ω be an open subset of $\mathbb{R}^n$ and let $1 < p < \infty$. Let $V \in L^1_{loc}(\Omega)$ and $a : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfy:

- (h1) the function $a(\cdot, \xi)$ is measurable for every $\xi \in \mathbb{R}^n$ and the function $a(x, \cdot)$ is strictly convex for a.e. $x \in \Omega$;
- (h2) there exist $b \geq \nu > 0$ such that
$$\nu|\xi|^p \leq a(x, \xi) \leq b|\xi|^p$$
for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$;
- (h3) we have $a(x, t\xi) = |t|^p a(x, \xi)$ for a.e. $x \in \Omega$ and every $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$;
- (h4) we have $V > 0$ on a set of positive measure and
  - if $p < n$, then $V^+ \in L^{n/p}(\Omega)$;
  - if $p = n$, then Ω is bounded and $V^+ \in L^q(\Omega)$ for some $q > 1$;
  - if $p > n$, then Ω is bounded and $V^+ \in L^1(\Omega)$.

By [19, Corollaries 2.3 and 2.4], the function $a(x, \cdot)$ is locally Lipschitz for a.e. $x \in \Omega$. According to Section 2, we set

$$a^0(x, \xi; \eta) = \lim_{t \to 0^+} \frac{a(x, \xi + t\eta) - a(x, \xi)}{t} = \lim_k k\left[a(x, \xi + (1/k)\eta) - a(x, \xi)\right]\,.$$

It follows that

$$\{x \mapsto a^0(x, U(x); W(x))\}$$

is measurable, whenever $U, W : \Omega \to \mathbb{R}^n$ are measurable.

We denote by $D^{1,p}_0(\Omega)$ the completion of $C^\infty_c(\Omega)$ with respect to the norm

$$\|\nabla u\|_p = \Big(\int_\Omega |\nabla u|^p\, dx\Big)^{1/p}\,.$$

Then we consider

$$X = \big\{u \in D^{1,p}_0(\Omega) : V^-|u|^p \in L^1(\Omega)\big\}$$

endowed with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p\, dx + \int_\Omega V^-|u|^p\, dx\Big)^{1/p}$$

and define $f, g : X \to \mathbb{R}$ by

$$f(u) = \int_\Omega a(x, \nabla u)\, dx\,, \quad g(u) = \frac{1}{p} \int_\Omega V|u|^p\, dx\,.$$

We also denote by $L^\infty_c(\Omega)$ the set of functions $u \in L^\infty(\Omega)$ vanishing a.e. outside some compact subset of Ω.

Theorem 5.1. *The following facts hold:*

(a) *X is a Banach space naturally embedded in $W^{1,p}_{loc}(\Omega)$;*
(b) *f and g satisfy assumptions (i) and (ii); moreover, f is convex and locally Lipschitz, while g is of class C^1;*
(c) *for every $u \in X$, we have that u is an eigenvector in the sense of Definition 3.1 if and only if $u \neq 0$ and there exist $\lambda \in \mathbb{R}$ and $\alpha \in L^{p'}(\Omega; \mathbb{R}^n)$ such that $\alpha \in \partial_\xi a(x, \nabla u)$ a.e. in Ω and*

$$\int_\Omega \alpha \cdot \nabla w\, dx = \lambda \int_\Omega V|u|^{p-2} uw\, dx \quad \text{for any } w \in X\,.$$

Moreover, λ is the associated eigenvalue in the sense of Definition 3.1.

Proof. Assertions (a) and (b) are proved in [29]. Since $f(u) = 0$ only if $u = 0$, assertion (c) follows from Corollary 2.18, Theorem 2.15 and [29, Lemma 3.1]. $\square$

According to Sections 3 and 4, we will consider only eigenvectors u with

$$\int_\Omega V|u|^p\, dx > 0\,.$$

Several basic properties of eigenvalues and eigenvectors, such as the simplicity of the first eigenvalue and a Strong Maximum Principle, are already proved in [29]. Moreover, it is shown that $\underline{\lambda}_k < +\infty$ for any $k \geq 1$, so that $(\underline{\lambda}_k)$ is a diverging sequence of eigenvalues.

We aim first to prove also the extension of a well known property (see [2, 26, 14, 23, 28, 24, 27, 5]), namely that only the first eigenvalue admits an eigenvector with constant sign, if Ω is connected.

Lemma 5.2. *Let $\underline{a} : \mathbb{R}^n \to \mathbb{R}$ be a convex function which is positively homogeneous of degree p. Then $\underline{a}$ is locally Lipschitz and we have*

$$\underline{a}(\xi_1) \geq \underline{a}(\xi_0) + \frac{1}{p}\, \underline{a}^0(\xi_0; ps^{p-1}\xi_1 - (p-1)s^p\xi_0 - \xi_0)$$

for every $\xi_0, \xi_1 \in \mathbb{R}^n$ and $s \in \mathbb{R}$ such that either $s > 0$ or $s = 0$ and $\xi_1 = 0$.

Proof. As before, $\underline{a}$ is locally Lipschitz. Assume first that $s > 0$. As in the proof of [5, Lemma 2.1], for every $t \in [0, 1]$ we have

$$\begin{aligned} \underline{a}\Big(\frac{(1-t)\xi_0 + ts^{p-1}\xi_1}{((1-t) + ts^p)^{\frac{p-1}{p}}}\Big) &= ((1-t) + ts^p)\underline{a}\Big(\frac{1-t}{(1-t) + ts^p}\, \xi_0 + \frac{ts^p}{(1-t) + ts^p}\, \frac{\xi_1}{s}\Big) \\ &\leq (1-t)\underline{a}(\xi_0) + ts^p \underline{a}\Big(\frac{\xi_1}{s}\Big) \\ &= (1-t)\underline{a}(\xi_0) + t\, \underline{a}(\xi_1)\,. \end{aligned}$$

On the other hand, if we set

$$\eta(t) = \frac{(1-t)\xi_0 + ts^{p-1}\xi_1}{((1-t) + ts^p)^{\frac{p-1}{p}}}\,,$$

it is easily seen that

$$\eta'(0) = s^{p-1}\xi_1 - \frac{p-1}{p}\, s^p \xi_0 - \frac{1}{p}\, \xi_0 \,,$$

whence

$$\underline{a}(\xi_1) - \underline{a}(\xi_0) \geq \lim_{t\to 0^+} \frac{\underline{a}(\eta(t)) - \underline{a}(\xi_0)}{t} = \frac{1}{p}\, \underline{a}^0(\xi_0; p s^{p-1}\xi_1 - (p-1)s^p\xi_0 - \xi_0)\,.$$

In the case $s = 0$ and $\xi_1 = 0$, by Proposition 2.17 we have

$$\underline{a}^0(\xi_0; -\xi_0) = -p\, \underline{a}(\xi_0)\,,$$

whence

$$\underline{a}(\xi_1) = 0 = \underline{a}(\xi_0) + \frac{1}{p}\, \underline{a}^0(\xi_0; -\xi_0)$$

and the proof is complete. □

Theorem 5.3. *If u is an eigenvector, then the following facts hold:*

(a) *if $u > 0$ a.e. in Ω, then the associated eigenvalue is λ_1;*
(b) *if $u \geq 0$ a.e. in Ω and Ω is connected, then the associated eigenvalue is λ_1.*

Proof. Let u be an eigenvector with eigenvalue λ such that $u \geq 0$ a.e. in Ω. For every $w \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$ with $\nabla w \in L^p(\Omega;\mathbb{R}^N)$ and $w \geq 0$ a.e. in Ω and for every $\varepsilon > 0$, it is easily seen that

$$\frac{w^p}{(u+\varepsilon)^{p-1}} \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$$

with

$$\nabla \frac{w^p}{(u+\varepsilon)^{p-1}} = p\, \frac{w^{p-1}}{(u+\varepsilon)^{p-1}}\, \nabla w - (p-1)\, \frac{w^p}{(u+\varepsilon)^p}\, \nabla u \in L^p(\Omega;\mathbb{R}^n)\,.$$

From Lemma 5.2 we infer that

$$\int_\Omega a(x,\nabla w)\, dx - \int_\Omega a(x,\nabla u)\, dx \geq \frac{1}{p} \int_\Omega a^0\Big(x,\nabla u; \nabla \frac{w^p}{(u+\varepsilon)^{p-1}} - \nabla u\Big)\, dx\,.$$

Let now $w \in D^{1,p}_0(\Omega) \cap L^\infty_c(\Omega)$ with $w \geq 0$ a.e. in Ω, so that

$$\frac{w^p}{(u+\varepsilon)^{p-1}} \in X\,.$$

Taking into account (c) of Theorem 5.1, it follows that

$$\begin{aligned}
\int_\Omega a(x,\nabla w)\, dx - \int_\Omega a(x,\nabla u)\, dx &\geq \frac{1}{p} \int_\Omega a^0(x,\nabla u; \nabla \frac{w^p}{(u+\varepsilon)^{p-1}} - \nabla u)\, dx \\
&\geq \frac{1}{p} \int_\Omega \alpha \cdot (\nabla \frac{w^p}{(u+\varepsilon)^{p-1}} - \nabla u)\, dx \\
&= \frac{\lambda}{p} \int_\Omega V u^{p-1} (\frac{w^p}{(u+\varepsilon)^{p-1}} - u)\, dx \\
&= \frac{\lambda}{p} \int_\Omega V \frac{u^{p-1}}{(u+\varepsilon)^{p-1}}\, w^p\, dx - \frac{\lambda}{p} \int_\Omega V u^p\, dx \\
&= \frac{\lambda}{p} \int_\Omega V \frac{u^{p-1}}{(u+\varepsilon)^{p-1}}\, w^p\, dx - \int_\Omega a(x,\nabla u)\, dx\,,
\end{aligned}$$

whence
$$\int_\Omega a(x,\nabla w)\,dx \geq \frac{\lambda}{p}\int_\Omega V\,\frac{u^{p-1}}{(u+\varepsilon)^{p-1}}\,w^p\,dx\,.$$
Now let $w \in X$ with $w \geq 0$ a.e. in Ω, let $(\hat{w}_k)$ be a sequence in $C_c^\infty(\Omega)$ converging to w in $D_0^{1,p}(\Omega)$ and let
$$w_k = \min\{\hat{w}_k^+, w\}\,.$$
Then $w_k \in D_0^{1,p}(\Omega) \cap L_c^\infty(\Omega)$ with $0 \leq w_k \leq w$ a.e. in Ω, whence
$$\int_\Omega a(x,\nabla w_k)\,dx \geq \frac{\lambda}{p}\int_\Omega V\,\frac{u^{p-1}}{(u+\varepsilon)^{p-1}}\,w_k^p\,dx\,.$$
Going to the limit as $k \to \infty$ and $\varepsilon \to 0$, we obtain
$$\int_\Omega a(x,\nabla w)\,dx \geq \frac{\lambda}{p}\int_{\{u>0\}} V\,w^p\,dx \quad \text{for every } w \in X \text{ with } w \geq 0 \text{ a.e. in } \Omega\,.$$
Now, if $u > 0$ a.e. in Ω we actually have
$$\int_\Omega a(x,\nabla w)\,dx \geq \frac{\lambda}{p}\int_\Omega V\,w^p\,dx \quad \text{for every } w \in X \text{ with } w \geq 0 \text{ a.e. in } \Omega\,.$$
By [29, Proposition 6.1], the eigenvalue λ_1 admits an eigenvector v with $v \geq 0$ a.e. in Ω. Without loss of generality, we may assume that $g(v) = 1$. Then we have
$$\lambda_1 = \int_\Omega a(x,\nabla v)\,dx \geq \lambda$$
and assertion (a) follows.

If $u \geq 0$ a.e. in Ω and Ω is connected, from the Strong Maximum Principle (see [29, Proposition 5.1]) we infer that $u > 0$ a.e. in Ω and assertion (b) follows from assertion (a). □

Lemma 5.4. *The following facts hold:*

(a) *if $u \in X$ is an eigenvector and u is sign-changing, then u is decomposable and $(u^+, -u^-)$ is a decomposition of u;*

(b) *if $u \in X$ is an eigenvector with eigenvalue λ and there exists a connected component ω of Ω such that u is not a.e. vanishing on ω and on $\Omega \setminus \omega$, then u is decomposable and (u_1, u_2) given by*
$$u_1 = u\chi_\omega\,,\ u_2 = u\chi_{\Omega\setminus\omega}$$
is a decomposition of u satisfying
$$f(u_j) = \lambda\, g(u_j) \quad \text{for } j = 1, 2\,;$$

(c) *if $u, v \in X$ are two linearly independent eigenvectors and there exists a connected component ω of Ω such that*
$$u\chi_{\Omega\setminus\omega} = v\chi_{\Omega\setminus\omega} = 0\,,$$
then one at least, say u, is sign-changing and u^+, $-u^-$ are not eigenvectors.

Proof. Since $u^\pm \in X$ whenever $u \in X$, as in the proof of [29, Proposition 6.1] we infer that
$$f(u^\pm) = \lambda\, g(u^\pm)\,.$$
Then assertion (a) easily follows.

If ω is a connected component of Ω and $u_1 = u\chi_\omega$, $u_2 = u\chi_{\Omega\setminus\omega}$, then $u_1, u_2 \in X$ and in a similar way it turns out that

$$f(u_j) = \lambda\, g(u_j)\,.$$

Then assertion (b) also follows.

Finally, let u, v be two eigenvectors as in assertion (c). Without loss of generality, we may assume that $\omega = \Omega$ with Ω connected. First of all we claim that one at least is sign-changing. Assume, for a contradiction, that u and v are both of constant sign. From Theorem 5.3 we infer they are both with eigenvalue λ_1. But this fact contradicts the simplicity of the first eigenvalue (see [29, Proposition 6.4]). Assume that u is sign-changing. By assertion(a) we have that $(u^+, -u^-)$ is a decomposition of u and from the Strong Maximum Principle (see [29, Proposition 5.1]) we infer that u^+, $-u^-$ are not eigenvectors. □

Theorem 5.5. *The functions f and g satisfy also assumption (iii).*

Proof. Let u be an eigenvector with eigenvalue λ and v an eigenvector with eigenvalue μ such that u and v are linearly independent and v is not decomposable.

By (a) and (b) of Lemma 5.4, we have that v has constant sign and there exists a connected component ω of Ω such that $v\chi_{\Omega\setminus\omega} = 0$.

If $u\chi_\omega = 0$, then assertion (a) of assumption (iii) holds. If $u\chi_\omega$ and $u\chi_{\Omega\setminus\omega}$ are both different from 0, then by (b) of Lemma 5.4

$$u_1 = u\chi_{\Omega\setminus\omega}\,, \quad u_2 = u\chi_\omega$$

provide a decomposition of u satisfying assertion (b) of assumption (iii).

Finally, assume that $u\chi_{\Omega\setminus\omega} = 0$. By (c) of Lemma 5.4 we have that $(u^+, -u^-)$ is a decomposition of u and u^+, $-u^-$ are not eigenvectors. Therefore, assertion (c) of assumption (iii) holds. □

Remark 5.6. If Ω is connected, then assertion (c) of assumption (iii) always holds.

Now all the results of Section 4 can be applied. Let us point out that Ω is not assumed to be connected. In particular, let us summarize the results concerning the second eigenvalue.

Theorem 5.7. *For every eigenvector $u \in M$ with eigenvalue λ_1, we have*

$$\lambda_{mp}(u) = \underline{\lambda}_2 = \overline{\lambda}_2 = \min\{\lambda \in \mathbb{R} : \lambda \textit{ is an eigenvalue with } \lambda \neq \lambda_1\}\,.$$

Proof. Since $(\underline{\lambda}_k)$ is a diverging sequence of eigenvalues, the set

$$\{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue with } \lambda \neq \lambda_1\}$$

is not empty. Then the assertion follows from Theorems 4.2 and 4.4. □

6. A problem on quasi open sets

In this section we show that the abstract setting of Section 4 can be applied also to the p-Laplacian on p-quasi open sets. In this way we provide a different proof, without the use of minimizing movements, of the main result of [20].

Let $1 < p < \infty$ and let Ω be a p-quasi open subset of $\mathbb{R}^n$ with finite measure (we refer the reader to [20] for definitions and main results concerning this class of sets). Define $f, g : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$f(u) = \int_\Omega |\nabla u|^p\, dx\,, \quad g(u) = \int_\Omega |u|^p\, dx\,.$$

Theorem 6.1. *The following facts hold:*

(a) *f and g satisfy assumptions (i) and (ii); moreover, f and g are of class C^1;*
(b) *for every $u \in W_0^{1,p}(\Omega)$, we have that u is an eigenvector in the sense of Definition 3.1 if and only if $u \neq 0$ and there exists $\lambda \in \mathbb{R}$ such that*
$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \, dx = \lambda \int_\Omega |u|^{p-2} uw \, dx \quad \text{for any } w \in W_0^{1,p}(\Omega)\,.$$
Moreover, λ is the associated eigenvalue in the sense of Definition 3.1.

Proof. Assertion (a) is proved in [20]. Then assertion (b) follows from Proposition 2.16 and Corollary 2.18. □

To prove condition (iii), we will follow the same scheme of the previous section. However, this time the task will be simpler, because [20] already provides all the basic information on eigenvectors and eigenvalues.

Lemma 6.2. *The following facts hold:*

(a) *if $u \in W_0^{1,p}(\Omega)$ is an eigenvector and u is sign-changing, then u is decomposable and $(u^+, -u^-)$ is a decomposition of u;*
(b) *if $u \in W_0^{1,p}(\Omega)$ is an eigenvector with eigenvalue λ and there exists a p-quasi connected component ω of Ω such that u is not a.e. vanishing on ω and on $\Omega \setminus \omega$, then u is decomposable and (u_1, u_2) given by*
$$u_1 = u\chi_\omega\,, \quad u_2 = u\chi_{\Omega\setminus\omega}$$
is a decomposition of u satisfying
$$f(u_j) = \lambda\, g(u_j) \quad \text{for } j = 1, 2\,;$$
(c) *if $u, v \in W_0^{1,p}(\Omega)$ are two linearly independent eigenvectors and there exists a p-quasi connected component ω of Ω such that*
$$u\chi_{\Omega\setminus\omega} = v\chi_{\Omega\setminus\omega} = 0\,,$$
then one at least, say u, is sign-changing and u^+, $-u^-$ are not eigenvectors.

Proof. Since $u^\pm \in W_0^{1,p}(\Omega)$ whenever $u \in W_0^{1,p}(\Omega)$, assertion (a) easily follows. On the other hand, it is proved in [20, Lemma 2.9] that $u\chi_\omega \in W_0^{1,p}(\Omega)$ whenever $u \in W_0^{1,p}(\Omega)$ and ω is a p-quasi connected component of Ω. Of course, this implies that also $u\chi_{\Omega\setminus\omega} \in W_0^{1,p}(\Omega)$. Then assertion (b) also easily follows.

Finally, let u, v be two eigenvectors as in assertion (c). Again by [20, Lemma 2.9] we have that $u\big|_\omega, v\big|_\omega \in W_0^{1,p}(\omega)$ are eigenvectors with respect to ω. We claim that one at least is sign-changing. Assuming for a contradiction that they are both of constant sign, it follows from [20, Theorem 3.3 and Lemma 3.9] that $u\big|_\omega$ and $v\big|_\omega$ are associated with the first eigenvalue of ω. By [20, Proposition 3.12] the first eigenvalue of ω is simple and a contradiction follows. If u is sign-changing, then u^+ and $-u^-$ cannot be eigenvectors again by [20, Theorem 3.3]. □

Theorem 6.3. *The functions f and g satisfy also assumption (iii).*

Proof. The argument is the same of Theorem 5.5, with connected components replaced by p-quasi connected components. □

Remark 6.4. If Ω is p-quasi connected, then assertion (c) of assumption (iii) always holds.

Also in this setting all the results of Section 4 can be applied. In particular, we provide a different proof of [20, Theorem 3.14]. Let us point out that, in our case, u is not required to be supported in a p-quasi connected component of Ω.

Theorem 6.5. *For every eigenvector $u \in M$ with eigenvalue λ_1, we have*

$$\lambda_{mp}(u) = \underline{\lambda}_2 = \overline{\lambda}_2 = \min\{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue with } \lambda \neq \lambda_1\}\,.$$

Proof. Since the set

$$\{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue with } \lambda \neq \lambda_1\}$$

is not empty, the assertion follows from Theorems 4.2 and 4.4. □

7. A problem with a fractional operator

Finally, let us show that the setting of Section 4 can be applied to the fractional p-Laplacian treated in [6].

Let Ω be a bounded and open subset of $\mathbb{R}^n$, let $1 < p < \infty$, $0 < s < 1$ and let X be the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\Big(\int_\Omega |u|^p\,dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy\Big)^{1/p}\,.$$

Define $f, g : X \to \mathbb{R}$ by

$$f(u) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy\,, \quad g(u) = \int_\Omega |u|^p\,dx\,.$$

Theorem 7.1. *The following facts hold:*

(a) *f and g satisfy assumptions (i) and (ii); moreover, f and g are of class C^1;*
(b) *for every $u \in X$, we have that u is an eigenvector in the sense of* [6] *and λ is the associated eigenvalue if and only if the same holds in the sense of Definition 3.1.*

Proof. Assertion (a) is proved in [6]. Then assertion (b) follows from Proposition 2.16 and Corollary 2.18. □

With respect to Sections 5 and 6, the proof of condition (iii) requires some modifications, because the fractional operator has different features, as shown in [6]. Because of the nonlocal character, even if Ω is not connected, the behavior is that of the connected case.

Lemma 7.2. *The following facts hold:*

(a) *if $u \in X$ is an eigenvector and u is sign-changing, then u is decomposable and $(u^+, -u^-)$ is a decomposition of u such that u^+ and $-u^-$ are not eigenvectors;*
(b) *if $u, v \in X$ are two linearly independent eigenvectors, then one at least is sign-changing.*

Proof. (a) If $u \in X$ is a sign-changing eigenvector, just the proof of [6, Proposition 4.2] shows that $(u^+, -u^-)$ is a decomposition of u. From [6, Proposition 2.6] we infer that u^+ and $-u^-$ cannot be eigenvectors.

(b) follows from [6, Theorem 2.8]. □

Theorem 7.3. *The functions f and g satisfy also assumption (iii). More precisely, they always satisfy assertion (c) of assumption (iii).*

Proof. Let u be an eigenvector with eigenvalue λ and v an eigenvector with eigenvalue μ such that u and v are linearly independent and v is not decomposable. By (a) of Lemma 7.2, we have that v has constant sign. Then u must be sign-changing by (b) of Lemma 7.2 and assertion (c) of assumption (iii) follows from (a) of Lemma 7.2. □

Finally, also [6, Theorem 5.3] can be proved in the setting of Section 4.

Theorem 7.4. *For every eigenvector $u \in M$ with eigenvalue λ_1, we have*

$$\lambda_{mp}(u) = \underline{\lambda}_2 = \overline{\lambda}_2 = \min\{\lambda \in \mathbb{R} : \lambda \textit{ is an eigenvalue with } \lambda \neq \lambda_1\}\,.$$

Proof. Again, since the set

$$\{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue with } \lambda \neq \lambda_1\}$$

is not empty, the assertion follows from Theorems 4.2 and 4.4. □

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