Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 20, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS TO BURGERS EQUATIONS IN A NON-PARABOLIC DOMAIN

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Communicated by Mokhtar Kirane

Abstract. In this article, we study the semilinear Burgers equation with time variable coefficients, subject to boundary condition in a non-parabolic domain. Some assumptions on the boundary of the domain and on the coefficients of the equation will be imposed. The right-hand side of the equation is taken in $L^2(\Omega)$. The method we used is based on the approximation of the non-parabolic domain by a sequence of subdomains which can be transformed into regular domains. This paper is an extension of the work [2].

1. Introduction

The Burgers equation is a fundamental partial differential equation in modeling many physical phenomena, such as fluid mechanics, acoustics, turbulence [3, 6], traffic flow, growth of interfaces, and financial mathematics [7, 12].

In [11], the author studied a linear parabolic equation in a domain similar to the one considered in this work. Other references on the analysis of linear parabolic problems in non-regular domains are discussed for example in [1, 5, 8, 9].

The work by Clark et al. [4] is devoted to the homogeneous Burgers equation in non-parabolic domains which can be transformed into rectangle. In the same domains, we have established the existence, uniqueness and the optimal regularity of the solution to the non-homogeneous Burgers equation with time variable coefficients in an anisotropic Sobolev space (see [2]). The present paper is an extension of this last work to another type of non-regular domains.

Let $\Omega \subset \mathbb{R}^2$ be the "triangular" domain

$$\Omega = \{(t,x) \in \mathbb{R}^2;\ 0 < t < T,\ x \in I_t\},$$

where T is a positive number and

$$I_t = \{x \in \mathbb{R};\ \varphi_1(t) < x < \varphi_2(t),\ t \in (0,T)\},$$

with

$$\varphi_1(0) = \varphi_2(0). \tag{1.1}$$

The functions φ_1, φ_2 are defined on $[0,T]$, and belong to $\mathcal{C}^1(0,T)$.

2010 *Mathematics Subject Classification.* 35K58, 35Q35.
Key words and phrases. Burgers equation; existence; uniqueness; non-parabolic domains; anisotropic Sobolev space.

Submitted December 14, 2017. Published January 15, 2018.

The most interesting point of the problem studied here is the fact that $\varphi_1(0) = \varphi_2(0)$, because the domain is not rectangular and cannot be transformed into a regular domain without the appearance of some degenerate terms in the equation.

In Ω, we consider the boundary-value problem for the non-homogeneous Burgers equation with variable coefficient

$$\begin{gathered} \partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = f(t,x) \quad (t,x) \in \Omega, \\ u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T), \end{gathered} \tag{1.2}$$

where $f \in L^2(\Omega)$ and $c(t)$ is given.

We look for some conditions on the functions $c(t)$, $\varphi_1(t)$ and $\varphi_2(t)$ such that (1.2) admits a unique solution u belonging to the anisotropic Sobolev space

$$H^{1,2}(\Omega) = \{u \in L^2(\Omega); \partial_t u, \partial_x u, \partial_x^2 u \in L^2(\Omega)\}.$$

In the sequel, we assume that there exist positive constants c_1 and c_2, such that

$$c_1 \le c(t) \le c_2, \quad \text{for all } t \in (0,T), \tag{1.3}$$

and we note that

$$\|u\|_{L^2(I_t)} = \Big(\int_{\varphi_1(t)}^{\varphi_2(t)} |u(t,x)|^2 \,\mathrm{d}x \Big)^{1/2},$$

$$\|u\|^2_{L^\infty(I_t)} = \sup_{x \in I_t} |u(t,x)|.$$

To establish the existence of a solution to (1.2), we also assume that

$$|\varphi'(t)| \le \gamma \quad \text{for all } t \in [0,T], \tag{1.4}$$

where γ is a positive constant and $\varphi(t) = \varphi_2(t) - \varphi_1(t)$ for all $t \in [0,T]$.

Remark 1.1. Once problem (1.2) is solved, we can deduce the solution of the problem

$$\begin{gathered} \partial_t u(t,x) + a(t)u(t,x)\partial_x u(t,x) - b(t)\partial_x^2 u(t,x) = f(t,x) \quad (t,x) \in \Omega, \\ u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T). \end{gathered} \tag{1.5}$$

Indeed, consider the case where $a(t)$ and $b(t)$ are positive and bounded functions for all $t \in [0,T]$. Let h be defined by $h : [0,T] \to [0,T']$

$$h(t) = \int_0^t b(s)\mathrm{d}s,$$

we put $\psi_i = \varphi_i \circ h^{-1}$ where $i = 1,2$. Using the change of variables $t' = h(t)$, $v(t',x) = u(t,x)$, (1.5) becomes equivalent to (1.2), because it may be written as follows

$$\begin{gathered} \partial_{t'} v(t',x) + c(t')v(t',x)\partial_x v(t',x) - \partial_x^2 v(t',x) = g(t',x) \quad (t',x) \in \Omega', \\ v(t',\psi_1(t')) = v(t',\psi_2(t')) = 0, \quad t' \in (0,T'), \end{gathered}$$

where $c(t') = \frac{a(t)}{b(t)}$, $g(t',x) = \frac{f(t,x)}{b(t)}$, $\Omega' = \{(t',x) \in \mathbb{R}^2;\ 0 < t' < T',\ x \in I_{t'}\}$ and $T' = \int_0^T b(s)\mathrm{d}s$.

For the study of problem (1.2) we will follow the method used in [11], which consists in observing that this problem admits a unique solution in domains that can be transformed into rectangles, i.e., when $\varphi_1(0) \neq \varphi_2(0)$.

The paper is organized as follows. In the next section we study problem (1.2) in domain that can be transformed into a rectangle. When φ_1 and φ_2 are monotone on $(0,T)$, we solve in Section 3 the problem in a triangular domain: We approximate this domain by a sequence of subdomains $(\Omega_n)_{n\in\mathbb{N}}$. Then we establish an a priori estimate of the type

$$\|u_n\|^2_{H^{1,2}(\Omega_n)} \leq K\|f_n\|^2_{L^2(\Omega_n)} \leq K\|f\|^2_{L^2(\Omega)},$$

where u_n is the solution of (1.2) in Ω_n and K is a constant independent of n. This inequality allows us to pass to the limit in n. Finally, Section 4 is devoted to problem (1.2) in the case when φ_1 and φ_2 are monotone only near 0.

Our main result is as follows.

Theorem 1.2. *Assume that c and $(\varphi_i(t))_{i=1,2}$ satisfy the conditions* (1.1)*,* (1.3) *and* (1.4)*. Then, the problem*

$$\begin{gathered}\partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = f(t,x) \quad (t,x)\in\Omega,\\ u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t\in(0,T),\end{gathered}$$

admits in the triangular domain Ω a unique solution $u \in H^{1,2}(\Omega)$ in the following cases:
Case 1. *φ_1 (resp φ_2) is a decreasing (resp increasing) function on $(0,T)$.*
Case 2. *φ_1 (resp φ_2) is a decreasing (resp increasing) function only near 0.*

Theses cases will be proved in Section 3 and Section 4, respectively.

2. Solution in a domain that can be transformed into a rectangle

Let $\Omega \subset \mathbb{R}^2$ be the domain

$$\begin{gathered}\Omega = \{(t,x)\in\mathbb{R}^2 : 0<t<T,\ x\in I_t\},\\ I_t = \{x\in\mathbb{R} : \varphi_1(t) < x < \varphi_2(t),\ t\in(0,T)\}.\end{gathered}$$

In this section, we assume that $\varphi_1(0) \neq \varphi_2(0)$. In other words

$$\varphi_1(t) < \varphi_2(t) \quad \text{for all } t\in[0,T]. \tag{2.1}$$

Theorem 2.1. *If $f\in L^2(\Omega)$ and $c(t)$, $(\varphi_i)_{i=1,2}$ satisfy the assumptions* (1.3)*,* (1.4) *and* (2.1)*, then the problem*

$$\begin{gathered}\partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = f(t,x) \quad (t,x)\in\Omega,\\ u(0,x) = 0 \quad x\in J = (\varphi_1(0),\varphi_2(0)),\\ u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t\in(0,T),\end{gathered} \tag{2.2}$$

admits a solution $u\in H^{1,2}(\Omega)$.

Proof. The change of variables: $\Omega \to R$

$$(t,x)\mapsto(t,y) = \Big(t, \frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}\Big)$$

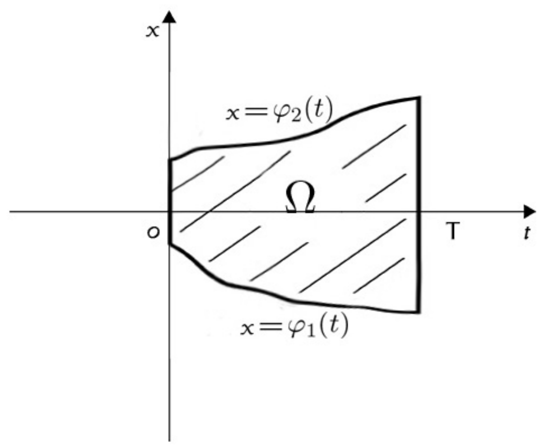

FIGURE 1. Domain that can be transformed into a rectangle.

transforms Ω into the rectangle $R = (0,T) \times (0,1)$. Putting $u(t,x) = v(t,y)$ and $f(t,x) = g(t,y)$, problem (2.2) becomes

$$\begin{gathered} \partial_t v(t,y) + p(t)v(t,y)\partial_y v(t,y) - q(t)\partial_y^2 v(t,y) + r(t,y)\partial_y v(t,y) \\ = g(t,y) \quad (t,y) \in R, \\ v(0,y) = 0 \quad y \in (0,1), \\ v(t,0) = v(t,1) = 0 \quad t \in (0,T), \end{gathered} \tag{2.3}$$

where

$$\varphi(t) = \varphi_2(t) - \varphi_1(t), \quad p(t) = \frac{c(t)}{\varphi(t)},$$
$$q(t) = \frac{1}{\varphi^2(t)}, \quad r(t,y) = -\frac{y\varphi'(t) + \varphi_1'(t)}{\varphi(t)}.$$

This change of variables preserves the spaces $H^{1,2}$ and L^2. In other words

$$f \in L^2(\Omega) \Leftrightarrow g \in L^2(R),$$
$$u \in H^{1,2}(\Omega) \Leftrightarrow v \in H^{1,2}(R).$$

According to (1.3) and (1.4), the functions p, q and r satisfy the following conditions

$$\alpha < p(t) < \beta, \quad \forall t \in [0,T],$$
$$\alpha < q(t) < \beta, \quad \forall t \in [0,T],$$
$$|\partial_y r(t,y)| \le \beta, \quad \forall (t,y) \in R,$$

where α and β are positive constants.

So, problem (2.2) is equivalent to problem (2.3), and by [2] problem (2.3) admits a solution $v \in H^{1,2}(R)$. Then, problem (2.2) in the domain Ω admits a solution $u \in H^{1,2}(\Omega)$. $\square$

3. Proof of Theorem 1.2, Case 1

Let

$$\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ x \in I_t\},$$
$$I_t = \{x \in \mathbb{R} : \varphi_1(t) < x < \varphi_2(t),\ t \in (0,T)\},$$

with $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$.

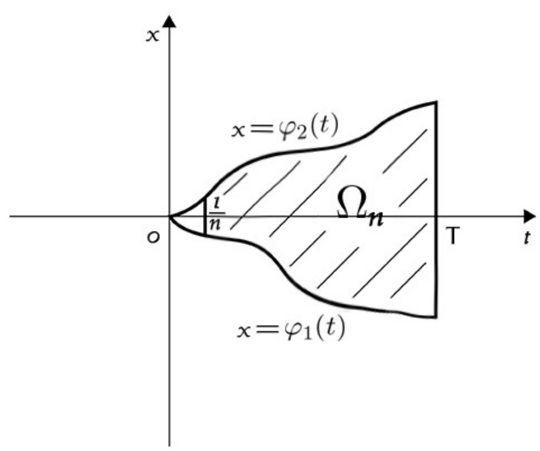

Figure 2. Non-parabolic domain.

For each $n \in \mathbb{N}^\star$, we define

$$\Omega_n = \{(t,x) \in \mathbb{R}^2 : \frac{1}{n} < t < T,\ x \in I_t\},$$

and we set $f_n = f_{|\Omega_n}$, where f is given in $L^2(\Omega)$. By Theorem 2.1 there exists a solution $u_n \in H^{1,2}(\Omega_n)$ of the problem

$$\begin{gathered}
\partial_t u_n(t,x) + c(t)u_n(t,x)\partial_x u_n(t,x) - \partial_x^2 u_n(t,x) \\
= f_n(t,x) \quad (t,x) \in \Omega_n, \\
u_n(\frac{1}{n},x) = 0, \quad \varphi_1(\frac{1}{n}) < x < \varphi_2(\frac{1}{n}), \\
u_n(t,\varphi_1(t)) = u_n(t,\varphi_2(t)) = 0 \quad t \in [\frac{1}{n},T],
\end{gathered} \tag{3.1}$$

in Ω_n.

To prove Case 1 of Theorem 1.2, we have to pass to the limit in (3.1). For this purpose we need the following result.

Proposition 3.1. *There exists a positive constant K independent of n such that*

$$\|u_n\|^2_{H^{1,2}(\Omega_n)} \le K\|f_n\|^2_{L^2(\Omega_n)} \le K\|f\|^2_{L^2(\Omega)}.$$

To prove this proposition we need some preliminary results.

Lemma 3.2. *There exists a positive constant K_1 independent of n such that*

$$\|u_n\|^2_{L^2(\Omega_n)} \le K_1\|\partial_x u_n\|^2_{L^2(\Omega_n)}, \tag{3.2}$$

$$\|\partial_x u_n\|^2_{L^2(\Omega_n)} \le K_1\|f_n\|^2_{L^2(\Omega_n)}. \tag{3.3}$$

Proof. We have

$$|u_n|^2 = \Big|\int_{\varphi_1(t)}^{x} \partial_x u_n \,\mathrm{d}s\Big|^2 \le (x-\varphi_1(t))\int_{\varphi_1(t)}^{x} |\partial_x u_n|^2 \,\mathrm{d}s.$$

integrating from $\varphi_1(t)$ to $\varphi_2(t)$, we obtain

$$\int_{\varphi_1(t)}^{\varphi_2(t)} |u_n|^2 \,\mathrm{d}x \le \int_{\varphi_1(t)}^{\varphi_2(t)} \Big((x-\varphi_1(t))\int_{\varphi_1(t)}^{x} |\partial_x u_n|^2 \,\mathrm{d}s\Big)\,\mathrm{d}x,$$

hence

$$\int_{\varphi_1(t)}^{\varphi_2(t)} |u_n|^2 \,\mathrm{d}x \le (\varphi_2(t)-\varphi_1(t))\int_{\varphi_1(t)}^{\varphi_2(t)}\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \,\mathrm{d}x\,\mathrm{d}x,$$

and

$$\int_{\varphi_1(t)}^{\varphi_2(t)} |u_n|^2 \,\mathrm{d}x \le (\varphi_2(t)-\varphi_1(t))^2\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \,\mathrm{d}x.$$

Then, there exists a positive constant K_1 independent of n such that

$$\|u_n\|^2_{L^2(I_t)} \le K_1\|\partial_x u_n\|^2_{L^2(I_t)},$$

integrating between $\frac{1}{n}$ and T we obtain inequality (3.2).

Now, multiplying both sides of (3.1) by u_n and integrating between $\varphi_1(t)$ and $\varphi_2(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\varphi_1(t)}^{\varphi_2(t)} (u_n)^2 \,\mathrm{d}x + c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n u_n^2 \,\mathrm{d}x - \int_{\varphi_1(t)}^{\varphi_2(t)} u_n\partial_x^2 u_n \,\mathrm{d}x = \int_{\varphi_1(t)}^{\varphi_2(t)} f_n u_n \,\mathrm{d}x.$$

Integration by parts gives

$$c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n u_n^2 \,\mathrm{d}x = \frac{c(t)}{3}\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x (u_n)^3 \,\mathrm{d}x = 0;$$

then

$$\frac{1}{2}\frac{d}{dt}\int_{\varphi_1(t)}^{\varphi_2(t)} (u_n)^2 \,\mathrm{d}x + \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \,\mathrm{d}x = \int_{\varphi_1(t)}^{\varphi_2(t)} f_n u_n \,\mathrm{d}x. \tag{3.4}$$

By integrating (3.4) from $1/n$ to T, we find that

$$\begin{aligned}&\frac{1}{2}\|u_n(T,x)\|^2_{L^2(I_T)} + \int_{1/n}^{T} \|\partial_x u_n(s)\|^2_{L^2(I_t)} \,\mathrm{d}s\\ &\le \int_{1/n}^{T} \|f_n(s)\|_{L^2(I_t)}\|u_n(s)\|_{L^2(I_t)} \,\mathrm{d}s.\end{aligned}$$

Using the elementary inequality

$$|rs| \le \frac{\varepsilon}{2}r^2 + \frac{s^2}{2\varepsilon}, \quad \forall r,s\in R,\ \forall\varepsilon>0, \tag{3.5}$$

with $\varepsilon = K_1$, we obtain

$$\begin{aligned}&\frac{1}{2}\|u_n(T,x)\|^2_{L^2(I_T)} + \int_{1/n}^{T} \|\partial_x u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s\\&\leq \frac{K_1}{2}\int_{1/n}^{T}\|f_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s + \frac{1}{2K_1}\int_{1/n}^{T}\|u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s.\end{aligned}$$

Thanks to (3.2), we have

$$\|u_n(T,x)\|^2_{L^2(I_T)} + \int_{1/n}^{T}\|\partial_x u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s \leq K_1\int_{1/n}^{T}\|f_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s, \tag{3.6}$$

so,

$$\|\partial_x u_n\|^2_{L^2(\Omega_n)} \leq K_1\|f_n\|^2_{L^2(\Omega_n)}.$$

□

Corollary 3.3. *There exists a positive constant K_2 independent of n, such that for all $t \in [1/n, T]$,*

$$\|\partial_x u_n\|^2_{L^2(I_t)} + \int_{1/n}^{T}\|\partial_x^2 u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s \leq K_2.$$

Proof. Multiplying both sides of (3.1) by $\partial_x^2 u_n$ and integrating between $\varphi_1(t)$ and $\varphi_2(t)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\varphi_1(t)}^{\varphi_2(t)}(\partial_x u_n)^2\mathrm{d}x + \int_{\varphi_1(t)}^{\varphi_2(t)}(\partial_x^2 u_n)^2\,\mathrm{d}x\\&= -\int_{\varphi_1(t)}^{\varphi_2(t)} f_n\partial_x^2 u_n\,\mathrm{d}x + c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u_n\partial_x u_n\partial_x^2 u_n\,\mathrm{d}x.\end{aligned} \tag{3.7}$$

Using Cauchy-Schwartz inequality, (3.5) with $\varepsilon = \frac{1}{2}$ leads to

$$\begin{aligned}|\int_{\varphi_1(t)}^{\varphi_2(t)} f_n\partial_x^2 u_n\,\mathrm{d}x| &\leq \Big(\int_{\varphi_1(t)}^{\varphi_2(t)}|\partial_x^2 u_n|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_{\varphi_1(t)}^{\varphi_2(t)}|f_n|^2\,\mathrm{d}x\Big)^{1/2}\\&\leq \frac{1}{4}\int_{\varphi_1(t)}^{\varphi_2(t)}|\partial_x^2 u_n|^2\,\mathrm{d}x + \int_{\varphi_1(t)}^{\varphi_2(t)}|f_n|^2\,\mathrm{d}x.\end{aligned} \tag{3.8}$$

Now, we have to estimate the last term of (3.7). An integration by parts gives

$$\int_{\varphi_1(t)}^{\varphi_2(t)} u_n\partial_x u_n\partial_x^2 u_n\,\mathrm{d}x = \int_{\varphi_1(t)}^{\varphi_2(t)} u_n\partial_x\Big(\frac{1}{2}(\partial_x u_n)^2\Big)\,\mathrm{d}x = -\frac{1}{2}\int_{\varphi_1(t)}^{\varphi_2(t)}(\partial_x u_n)^3\,\mathrm{d}x.$$

Since $\partial_x u_n$ satisfies $\int_{\varphi_1(t)}^{\varphi_2(t)}\partial_x u_n\,\mathrm{d}x = 0$ we deduce that the continuous function $\partial_x u_n$ is zero at some point $\xi(t) \in (\varphi_1(t), \varphi_2(t))$, and by integrating $2\partial_x u_n\partial_x^2 u_n$ between $\xi(t)$ and x, we obtain

$$2\int_{\xi(t)}^{x}\partial_x u_n\partial_x^2 u_n\,\mathrm{d}s\int_{\xi(t)}^{x} = \partial_x(\partial_x u_n)^2\,\mathrm{d}s = (\partial_x u_n)^2,$$

the Cauchy-Schwartz inequality gives

$$\|\partial_x u_n\|^2_{L^\infty(I_t)} \leq 2\|\partial_x u_n\|_{L^2(I_t)}\|\partial_x^2 u_n\|_{L^2(I_t)},$$

but

$$\|\partial_x u_n\|^3_{L^3(I_t)} \leq \|\partial_x u_n\|^2_{L^2(I_t)}\|\partial_x u_n\|_{L^\infty(I_t)},$$

so, (1.3) yields

$$|\int_{\varphi_1(t)}^{\varphi_2(t)} c(t)u_n\partial_x u_n\partial_x^2 u_n \,\mathrm{d}x| \le \Big(\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \,\mathrm{d}x\Big)^{1/4}\Big(c_2^{4/5}\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \,\mathrm{d}x\Big)^{5/4}.$$

Finally, by Young's inequality $|AB| \le \frac{|A|^p}{p} + \frac{|B|^{p'}}{p'}$, with $1 < p < \infty$ and $p' = \frac{p}{p-1}$. Choosing $p = 4$ (then $p' = \frac{4}{3}$)

$$A = \Big(\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \,\mathrm{d}x\Big)^{1/4}, \quad B = \Big(c_2^{4/5}\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \,\mathrm{d}x\Big)^{5/4},$$

the estimate of the last term of (3.7) becomes

$$\begin{aligned} &|\int_{\varphi_1(t)}^{\varphi_2(t)} c(t)u_n\partial_x u_n\partial_x^2 u_n \,\mathrm{d}x| \\ &\le \frac{1}{4}\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \,\mathrm{d}x + \frac{3}{4}c_2^{4/3}\Big(\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \,\mathrm{d}x\Big)^{5/3}. \end{aligned} \tag{3.9}$$

Let us return to (3.7): By integrating between $\frac{1}{n}$ and t, from the estimates (3.8) and (3.9), we obtain

$$\begin{aligned} &\|\partial_x u_n\|_{L^2(I_t)}^2 + \int_{1/n}^t \|\partial_x^2 u_n(s)\|_{L^2(I_t)}^2 \,\mathrm{d}s \\ &\le 2\int_{1/n}^t \|f_n(s)\|_{L^2(I_t)}^2 \,\mathrm{d}s + \frac{3}{2}c_2^{4/3}\int_{1/n}^t \Big(\|\partial_x u_n(s)\|_{L^2(I_t)}^2\Big)^{5/3} \,\mathrm{d}s. \end{aligned}$$

$f_n \in L^2(\Omega_n)$, then there exists a constant c_3 such that

$$\begin{aligned} &\|\partial_x u_n\|_{L^2(I_t)}^2 + \int_{1/n}^t \|\partial_x^2 u_n(s)\|_{L^2(I_t)}^2 \,\mathrm{d}s \\ &\le c_3 + \frac{3}{2}c_2^{4/3}\int_{1/n}^t \Big(\|\partial_x u_n(s)\|_{L^2(I_t)}^2\Big)^{2/3}\|\partial_x u_n(s)\|_{L^2(I_t)}^2 \,\mathrm{d}s. \end{aligned}$$

Consequently, the function

$$\varphi(t) = \|\partial_x u_n\|_{L^2(I_t)}^2 + \int_{1/n}^t \|\partial_x^2 u_n(s)\|_{L^2(I_t)}^2 \,\mathrm{d}s$$

satisfies the inequality

$$\varphi(t) \le c_3 + \int_{1/n}^t \Big(\frac{3}{2}c_2^{4/3}\|\partial_x u_n(s)\|_{L^2(I_t)}^{4/3}\Big)\varphi(s)\mathrm{d}s,$$

Gronwall's inequality shows that

$$\varphi(t) \le c_3 \exp\Big(\int_{1/n}^t (\frac{3}{2}c_2^{4/3}\|\partial_x u_n(s)\|_{L^2(I_t)}^{4/3})\mathrm{d}s\Big).$$

According to Lemma 3.2 the integral $\int_{1/n}^t \|\partial_x u_n\|_{L^2(I_t)}^{4/3}\mathrm{d}s$ is bounded by a constant independent of n. So there exists a positive constant K_2 such that

$$\|\partial_x u_n\|_{L^2(I_t)}^2 + \int_{1/n}^T \|\partial_x^2 u_n(s)\|_{L^2(I_t)}^2 \,\mathrm{d}s \le K_2.$$

□

Lemma 3.4. *There exists a constant K_3 independent of n such that*

$$\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \le K_3\|f_n\|^2_{L^2(\Omega_n)}.$$

Then Theorem 3.1 is a direct consequence of Lemmas 3.2 and 3.4.

Proof. To prove Lemma 3.4, we develop the inner product in $L^2(\Omega_n)$,

$$\begin{aligned}
\|f_n\|^2_{L^2(\Omega_n)} &= (\partial_t u_n + c(t)u_n\partial_x u_n - \partial_x^2 u_n, \partial_t u_n + c(t)u_n\partial_x u_n - \partial_x^2 u_n)_{L^2(\Omega_n)}\\
&= \|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} + \|c(t)u_n\partial_x u_n\|^2_{L^2(\Omega_n)}\\
&\quad - 2(\partial_t u_n, \partial_x^2 u_n)_{L^2(\Omega_n)} + 2(\partial_t u_n, c(t)u_n\partial_x u_n)_{L^2(\Omega_n)}\\
&\quad - 2(c(t)u_n\partial_x u_n, \partial_x^2 u_n)_{L^2(\Omega_n)},
\end{aligned}$$

so,

$$\begin{aligned}
&\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)}\\
&= \|f_n\|^2_{L^2(\Omega_n)} - \|c(t)u_n\partial_x u_n\|^2_{L^2(\Omega_n)} + 2(c(t)u_n\partial_x u_n, \partial_x^2 u_n)_{L^2(\Omega_n)}\\
&\quad - 2(\partial_t u_n, c(t)u_n\partial_x u_n)_{L^2(\Omega_n)} + 2(\partial_t u_n, \partial_x^2 u_n)_{L^2(\Omega_n)}.
\end{aligned} \tag{3.10}$$

Using (1.3) and (3.5) with $\varepsilon = 1/2$, we obtain

$$\left|-2(\partial_t u_n, c(t)u_n\partial_x u_n)_{L^2(\Omega_n)}\right| \le \frac{1}{2}\|\partial_t u_n\|^2_{L^2(\Omega_n)} + 2c_2^2\|u_n\partial_x u_n\|^2_{L^2(\Omega_n)}, \tag{3.11}$$

and

$$\left|2(c(t)u_n\partial_x u_n, \partial_x^2 u_n)_{L^2(\Omega_n)}\right| \le 2c_2^2\|u_n\partial_x u_n\|^2_{L^2(\Omega_n)} + \frac{1}{2}\|\partial_x^2 u_n\|^2_{L^2(\Omega_n)}. \tag{3.12}$$

Now calculating the last term of (3.10),

$$\begin{aligned}
(\partial_t u_n, \partial_x^2 u_n)_{L^2(\Omega_n)} &= -\int_{1/n}^T \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_t(\partial_x u_n)\partial_x u_n \,\mathrm{d}x\mathrm{d}t + \int_{1/n}^T [\partial_t u_n \partial_x u_n]_{\varphi_1(t)}^{\varphi_2(t)}\,\mathrm{d}t\\
&= -\frac{1}{2}\int_{1/n}^T \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_t(\partial_x u_n)^2 \,\mathrm{d}x\mathrm{d}t + \int_{1/n}^T [\partial_t u_n \partial_x u_n]_{\varphi_1(t)}^{\varphi_2(t)}\,\mathrm{d}t\\
&= -\frac{1}{2}\Big[\int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \,\mathrm{d}x\Big]_{1/n}^T + \int_{1/n}^T [\partial_t u_n \partial_x u_n]_{\varphi_1(t)}^{\varphi_2(t)}\,\mathrm{d}t\\
&= -\frac{1}{2}\int_{\varphi_1(T)}^{\varphi_2(T)} (\partial_x u_n)^2(T,x)\,\mathrm{d}x + \frac{1}{2}\int_{\varphi_1(\frac{1}{n})}^{\varphi_2(\frac{1}{n})} (\partial_x u_n)^2(\frac{1}{n},x)\,\mathrm{d}x\\
&\quad + \int_{1/n}^T \partial_t u_n(t,\varphi_2(t))\partial_x u_n(t,\varphi_2(t))\,\mathrm{d}t\\
&\quad - \int_{1/n}^T \partial_t u_n(t,\varphi_1(t))\partial_x u_n(t,\varphi_1(t))\,\mathrm{d}t.
\end{aligned}$$

According to the boundary conditions, we have

$$\partial_t u_n(t,\varphi_i(t)) + \varphi_i'(t)\partial_x u_n(t,\varphi_i(t)) = 0, \quad i = 1,2,$$

so

$$(\partial_t u_n, \partial_x^2 u_n)_{L^2(\Omega_n)} = -\frac{1}{2}\int_{\varphi_1(T)}^{\varphi_2(T)} (\partial_x u_n)^2(T,x)\,\mathrm{d}x - \int_{1/n}^T \varphi_2'(t)(\partial_x u_n(t,\varphi_2(t)))^2\,\mathrm{d}t$$

$$+\int_{1/n}^{T} \varphi_1'(t)(\partial_x u_n(t,\varphi_1(t)))^2\,\mathrm{d}t,$$

it follows that

$$(\partial_t u_n, \partial_x^2 u_n) \le 0. \tag{3.13}$$

From (3.11), (3.12) and (3.13), (3.10) becomes

$$\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \le 2\|f_n\|^2_{L^2(\Omega_n)} + 10c_2^2\|u_n\partial_x u_n\|^2_{L^2(\Omega_n)}. \tag{3.14}$$

On the other hand, using the injection of $H_0^1(I_t)$ in $L^\infty(I_t)$, we obtain

$$\begin{aligned}
\Big|\int_{1/n}^{T}\int_{\varphi_1(t)}^{\varphi_2(t)} (u_n\partial_x u_n)^2\,\mathrm{d}x\,\mathrm{d}t\Big| &\le \int_{1/n}^{T}\Big(\|u_n\|^2_{L^\infty(I_t)}\int_{\varphi_1(t)}^{\varphi_2(t)}|\partial_x u_n|^2\,\mathrm{d}x\Big)\,\mathrm{d}t\\
&\le \int_{1/n}^{T}\|u_n\|^2_{H_0^1(I_t)}\|\partial_x u_n\|^2_{L^2(I_t)}\,\mathrm{d}t\\
&\le \|u_n\|^2_{L^\infty(\frac{1}{n},T;H_0^1(I_t))}\|\partial_x u_n\|^2_{L^2(\Omega_n)},
\end{aligned}$$

According to Corollary 3.3, $\|u_n\|^2_{L^\infty(\frac{1}{n},T;H_0^1(I_t))}$ is bounded, then by (3.3) and (3.14), there exists a constant K_3 independent of n, such that

$$\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \le K_3\|f_n\|^2_{L^2(\Omega_n)}.$$

However,

$$\|f_n\|^2_{L^2(\Omega_n)} \le \|f\|^2_{L^2(\Omega)},$$

then, from lemmas 3.2 and 3.4 , there exists a constant K independent of n, such that

$$\|u_n\|^2_{H^{1,2}(\Omega_n)} \le K\|f_n\|^2_{L^2(\Omega_n)} \le K\|f\|^2_{L^2(\Omega)}.$$

This completes the proof. □

Existence and uniqueness. Choose a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of the domains defined previously, such that $\Omega_n \subseteq \Omega$, as $n \longrightarrow +\infty$ then $\Omega_n \longrightarrow \Omega$.

Consider $u_n \in H^{1,2}(\Omega_n)$ the solution of

$$\begin{gathered}
\partial_t u_n(t,x) + c(t)u_n(t,x)\partial_x u_n(t,x) - \partial_x^2 u_n(t,x) = f_n(t,x) \quad (t,x)\in\Omega_n,\\
u_n(\frac{1}{n},x) = 0 \quad \varphi_1(\frac{1}{n}) < x < \varphi_2(\frac{1}{n}),\\
u_n(t,\varphi_1(t)) = u_n(t,\varphi_2(t)) = 0 \quad t\in]\frac{1}{n},T[.
\end{gathered}$$

We know that a solution u_n exists by the Theorem 2.1. Let $\widetilde{u_n}$ be the extension by zero of u_n outside Ω_n. From the proposition 3.1 results the inequality

$$\|\widetilde{u_n}\|^2_{L^2(\Omega_n)} + \|\partial_t\widetilde{u_n}\|^2_{L^2(\Omega_n)} + \|\partial_x\widetilde{u_n}\|^2_{L^2(\Omega_n)} + \|\partial_x^2\widetilde{u_n}\|^2_{L^2(\Omega_n)} \le C\|f\|^2_{L^2(\Omega)}.$$

This implies that $\widetilde{u_n}$, $\partial_t\widetilde{u_n}$ and $\partial_x^j\widetilde{u_n}$, $j=1,2$ are bounded in $L^2(\Omega_n)$, from Corollary 3.3 $\widetilde{u_n\partial_x u_n}$ is bounded in $L^2(\Omega_n)$. So, it is possible to extract a subsequence from u_n, still denoted u_n such that

$$\begin{gathered}
\partial_t\widetilde{u_n} \to \partial_t u \quad \text{weakly in} \quad L^2(\Omega_n),\\
\widetilde{\partial_x^2 u_n} \to \partial_x^2 u \quad \text{weakly in} \quad L^2(\Omega_n),\\
\widetilde{u_n}\partial_x\widetilde{u_n}u_n \to u\partial_x u \quad \text{weakly in } L^2(\Omega_n).
\end{gathered}$$

Then $u \in H^{1,2}(\Omega)$ is solution to problem (1.2).

For the uniqueness, let us observe that any solution $u \in H^{1,2}(\Omega)$ of problem (1.2) is in $L^\infty(0,T,H_0^1(I_t))$. Indeed, by the same way as in Corollary 3.3, we prove that there exists a positive constant K_2 such that for all $t \in [0,T]$

$$\|\partial_x u\|^2_{L^2(I_t)} + \int_0^T \|\partial_x^2 u(s)\|^2_{L^2(I_t)}\,\mathrm{d}s \le K_2.$$

Let $u_1, u_2 \in H^{1,2}(\Omega)$ be two solutions of (1.2). We put $u = u_1 - u_2$. It is clear that $u \in L^\infty(0,T,H_0^1(I_t))$. The equations satisfied by u_1 and u_2 leads to

$$\int_{\varphi_1(t)}^{\varphi_2(t)} [\partial_t u w + c(t) u w \partial_x u_1 + c(t) u_2 w \partial_x u + \partial_x u \partial_x w]\,\mathrm{d}x = 0.$$

Taking, for $t \in [0,T]$, $w = u$ as a test function, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I_t)} + \|\partial_x u\|^2_{L^2(I_t)}\\ &= -c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u^2 \partial_x u_1\,\mathrm{d}x - c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u_2 u \partial_x u\,\mathrm{d}x.\end{aligned} \tag{3.15}$$

An integration by parts gives

$$c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u^2 \partial_x u_1\,\mathrm{d}x = -2c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u \partial_x u u_1\,\mathrm{d}x,$$

then (3.15) becomes

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I_t)} + \|\partial_x u\|^2_{L^2(I_t)} = \int_{\varphi_1(t)}^{\varphi_2(t)} c(t)(2u_1 - u_2) u \partial_x u\,\mathrm{d}x.$$

By (1.3) and inequality (3.5) with $\varepsilon = 2$, we obtain

$$\begin{aligned}&\Big|\int_{\varphi_1(t)}^{\varphi_2(t)} c(t)(2u_1 - u_2) u \partial_x u\,\mathrm{d}x\Big|\\ &\le \frac{1}{4}c_2^2\big(2\|u_1\|_{L^\infty(0,T,H_0^1(I_t))} + \|u_2\|_{L^\infty(0,T,H_0^1(I_t))}\big)^2\|u\|^2_{L^2(I_t)} + \|\partial_x u\|^2_{L^2(I_t)}.\end{aligned}$$

So, we deduce that there exists a non-negative constant D, such as

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I_t)} \le D\|u\|^2_{L^2(I_t)},$$

and Gronwall's lemma leads to $u = 0$. This completes the proof of Theorem 1.2, Case 1.

4. Proof of Theorem 1.2, Case 2

In this case we set $\Omega = Q_1 \cup Q_2 \cup \Gamma_{T_1}$ where

$$Q_1 = \{(t,x) \in \mathbb{R}^2 : 0 < t < T_1,\ x \in I_t\},$$
$$Q_2 = \{(t,x) \in \mathbb{R}^2 : T_1 < t < T,\ x \in I_t\},$$
$$\Gamma_{T_1} = \{(T_1,x) \in \mathbb{R}^2 : x \in I_{T_1}\},$$

with T_1 small enough. $f \in L^2(\Omega)$ and $f_i = f_{|Q_i}$, $i = 1,2$.

Theorem 1.2, Case 1, applied to the domain Q_1, shows that there exists a unique solution $u_1 \in H^{1,2}(Q_1)$ of the problem

$$\partial_t u_1(t,x) + c(t) u_1(t,x) \partial_x u_1(t,x) - \partial_x^2 u_1(t,x)$$

$$= f_1(t,x) \quad (t,x) \in Q_1,$$
$$u_1(t,\varphi_1(t)) = u_1(t,\varphi_2(t)) = 0 \quad t \in (0,T_1).$$

Lemma 4.1. *If $u \in H^{1,2}((T_1,T)\times(0,1))$, then $u_{|t=T_1} \in H^1(\{T_1\}\times(0,1))$.*

The above lemma is a special case of [10, Theorem 2.1, Vol. 2]. Using the transformation $[T_1,T]\times[0,1] \to Q_2$,

$$(t,x) \mapsto (t,y) = (t,(\varphi_2(t)-\varphi_1(t))x + \varphi_1(t))$$

we deduce from Lemma 4.1 the following result.

Lemma 4.2. *If $u \in H^{1,2}(Q_2)$, then $u_{|\Gamma_{T_1}} \in H^1(\Gamma_{T_1})$.*

We denote the trace $u_{1|\Gamma_{T_1}}$ by u_0 which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $u_1 \in H^{1,2}(Q_1)$.

Theorem 2.1 applied to the domain Q_2, shows that there exists a unique solution $u_2 \in H^{1,2}(Q_2)$ of the problem

$$\partial_t u_2(t,x) + c(t)u_2(t,x)\partial_x u_2(t,x) - \partial_x^2 u_2(t,x) = f_2(t,x) \quad (t,x) \in Q_2,$$
$$u_2(0,x) = u_0(x) \quad \varphi_1(T_1) < x < \varphi_2(T_1),$$
$$u_2(t,\varphi_1(t)) = u_2(t,\varphi_2(t)) = 0 \quad t \in [T_1,T],$$

putting

$$u = \begin{cases} u_1 & \text{in} \quad Q_1, \\ u_2 & \text{in} \quad Q_2, \end{cases}$$

we observe that $u \in H^{1,2}(\Omega)$ because $u_{1|\Gamma_{T_1}} = u_{2|\Gamma_{T_1}}$ and is a solution of the problem

$$\partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = f(t,x) \quad (t,x) \in \Omega,$$
$$u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T).$$

We prove the uniqueness of the solution by the same way as in Case 1.

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