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BOUNDED SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS WITH TIME DELAY

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ABSTRACT. We consider the initial value problem

$$\frac{d^2u}{dt^2} + Au(t) = f(u(t), u(t-w)), \quad t > 0,$$
$$u(t) = \varphi(t), \quad -w \le t \le 0$$

for a nonlinear hyperbolic equation with time delay in a Hilbert space with the self adjoint positive definite operator A. We establish the existence and uniqueness of a bounded solution, and show application of the main theorem for four nonlinear partial differential equations with time delay. We present first and second order accuracy difference schemes for the solution of one dimensional nonlinear hyperbolic equation with time delay. Numerical results are also given.

1. INTRODUCTION

Delay differential equations are used to model biological, physical, and sociological processes, as well as naturally occurring oscillatory systems (see, for example [2, 3, 14, 16, 21, 29, 31]). It is known that, in delay differential equations, the presence of the delay term causes the difficulties in analysis of differential equations. Lu [23], studies monotone iterative schemes for finite-difference solutions of reaction-diffusion systems with time delays and gives modified iterative schemes by combing the method of upper-lower solutions and the Jacobi method or the Gauss-Seidel method.

Ashyralyev and Sobolevskii [13], consider the initial-value problem for linear delay partial differential equations of the parabolic type and give a sufficient condition for the stability of the solution of this initial-value problem. They obtain the stability estimates in Hölder norms for the solutions of the problem.

Ashyralyev and Agirseven [1, 5, 6, 7, 8, 9, 10] investigated several types of initial and boundary value problems for linear delay parabolic equations. They give theorems on stability and convergence of difference schemes for the numerical solution of initial and boundary value problems for linear parabolic equations with time delay.

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Moreover, Ashyralyev, Agirseven and Ceylan [11], are interested in finding sufficient conditions for the existence of a unique bounded solution of the initial value problem

$$\begin{gathered} \frac{du}{dt} + Au(t) = f(u(t), u(t-w)), \quad t > 0, \\ u(t) = \varphi(t), \quad -w \le t \le 0 \end{gathered} \tag{1.1}$$

for the differential equation in a Banach space E with the positive operator A with dense domain $D(A)$. The main theorem on the existence and uniqueness of a bounded solution of problem (1.1) was established for a nonlinear evolutionary equation with time delay. The application of the main theorem for four different nonlinear partial differential equations with time delay was shown. Numerical results were given.

Henriquez, Cuevas and Caicedo [19] study the existence of almost periodic solutions for linear retarded functional differential equations with finite delay. They consider the existence of almost periodic solutions with the stabilization of distributed control systems.

Hao, Fan, Cao and Sun [18] proposed a linearized quasi-compact finite difference scheme for semilinear space-fractional diffusion equations with a fixed time delay. Under the local Lipschitz conditions, they proved the solvability and convergence of the scheme in the discrete maximum norm by the energy method.

Liang [22] is concerned with the convergence and asymptotic stability of semidiscrete and full discrete schemes for linear parabolic equations with delay. She proved that the semidiscrete scheme, backward Euler and Crank-Nicolson full discrete schemes can unconditionally preserve the delay-independent asymptotic stability with some additional restrictions on time and spatial stepsizes of the forward Euler full discrete scheme.

Bhrawy, Abdelkawy and Mallawi [15] investigated the Chebyshev Gauss-Lobatto pseudospectral scheme in spatial directions for solving one-dimensional, coupled, and two-dimensional parabolic partial differential equations with time delays. They also develop an efficient numerical algorithm based on the Chebyshev pseudospectral algorithm to obtain the two spatial variables in solving the two-dimensional time delay parabolic equations.

Firstly based on the Vishik's results and using methods of operator theory, Ismailov, Guler and Ipek [20] described all solvable extensions of a minimal operator generated by linear delay differential-operator expression of first order in the Hilbert space of vector-functions in finite interval. They found sharp formulas for the spectrums of these solvable extensions.

Piriadarshani and Sengadir [24] obtain an existence theorem for a semi-linear partial differential equation with infinite delay employing a phase space in which discretizations can naturally be performed. For linear partial differential equations with infinite delay they show that the solutions of the ordinary differential equation with infinite delay obtained by the semi-discretization converge to the original solution.

Castro, Rodriguez, Cabrera and Martin [17] developed an explicit finite difference scheme for a model with coefficients variable in time and studied their properties of convergence and stability.

It is known that various initial-boundary value problems for evolutionary nonlinear delay partial differential equations can be reduced to the initial value problem for the differential equation

$$\begin{aligned} &\frac{d^2u}{dt^2} + Au(t) = f(u(t), u(t-w)), \quad t > 0, \\ &u(t) = \varphi(t), -w \le t \le 0 \end{aligned} \tag{1.2}$$

in a Hilbert space H with the self adjoint positive definite operator A with dense domain $D(A)$. Let $\{c(t), t \ge 0\}$ be a strongly continuous cosine operator-function defined by the formula

$$c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}.$$

Then, from the definition of the sine operator-function $s(t)$,

$$s(t)u = \int_0^t c(s)u\,ds$$

it follows that

$$s(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}.$$

The following estimates hold:

$$\|c(t)\|_{H\to H} \le 1, \|A^{1/2}s(t)\|_{H\to H} \le 1, \quad t > 0. \tag{1.3}$$

In this article, we are interested in finding sufficient conditions for the existence of a unique bounded solution of problem (1.2). The main theorem on the existence and uniqueness of a bounded solution of problem (1.2) is established for a nonlinear evolutionary equation with time delay. The application of the main theorem for four different nonlinear partial differential equations with time delay is shown. In general, it is not possible to get exact solution of nonlinear problems. Therefore, we can not be able to obtain a sharp estimate for the constants figuring in theorems on existence and uniqueness of a bounded solution. Finally, the first and second order of accuracy difference schemes for the solution of one dimensional nonlinear hyperbolic equation with time delay are presented. Numerical results are given. Note that bounded solutions of nonlinear one dimensional parabolic and hyperbolic partial differential equations with time delay have been investigated in earlier papers [25, 26, 27, 28, 32]. The generality of the approach considered in this paper, however, allows for treating a wider class of multidimensional delay nonlinear differential equations.

2. Main existence and uniqueness theorem

The method of proof is based on reducing problem (1.2) to the integral equation

$$\begin{aligned} u(t) &= c(t-(n-1)w)u((n-1)w) + s(t-(n-1)w)\frac{du((n-1)w)}{dt} \\ &\quad + \int_{(n-1)w}^t s(t-y)f(u(y), u(y-w))dy, \\ &(n-1)w \le t \le nw, \quad n = 1, 2, \dots, \quad u(t) = \varphi(t), \quad -w \le t \le 0 \end{aligned}$$

in $[0,\infty)\times H\times H$ and the use of successive approximations. The recursive formula for the solution of problem (1.2) is

$$\begin{gathered}
u_i(t)=c(t-(n-1)w)u_i((n-1)w)+s(t-(n-1)w)\frac{du_i((n-1)w)}{dt}\\
+\int_{(n-1)w}^{t}s(t-y)f(u_{i-1}(y),u_i(y-w))dy,\\
u_0(t)=c(t-(n-1)w)u_i((n-1)w)+s(t-(n-1)w)\frac{du_i((n-1)w)}{dt},\\
(n-1)w\le t\le nw,\quad n=1,2,\dots,\quad i=1,2,\dots,\\
u_i(t)=\varphi(t),\quad -w\le t\le 0.
\end{gathered}\tag{2.1}$$

Theorem 2.1. *Assume the following hypotheses: For each t, $-w\le t\le 0$, we have $\varphi(t)\in D(A)$ and*

$$\|\varphi(t)\|_H\le M,\quad \|A^{-1/2}\varphi'(t)\|_H\le \widetilde{M}.\tag{2.2}$$

The function $f:H\times H\longrightarrow H$ is continuous and bounded, that is

$$\|A^{-1/2}f(u,v)\|_H\le \bar{M}\tag{2.3}$$

in $H\times H$, and the Lipschitz condition holds uniformly with respect to z,

$$\|A^{-1/2}(f(u,z)-f(v,z))\|_H\le L\|u-v\|_H.\tag{2.4}$$

Here, $L,M,\widetilde{M},\bar{M}$ are positive constants. Then there exists a unique solution to problem (1.2) which is bounded in $[0,\infty)\times H\times H$.

Proof. We consider the interval $0\le t\le w$. Problem (1.2) becomes

$$\frac{d^2u}{dt^2}+Au(t)=f(u(t),\varphi(t-w)),\quad u(0)=\varphi(0),u'(0)=\varphi'(0)$$

and it can be written in equivalent integral form

$$u(t)=c(t)\varphi(0)+s(t)\varphi'(0)+\int_0^t s(t-y)f(u(y),\varphi(y-w))dy.\tag{2.5}$$

According to the method of recursive approximation (2.1), we get

$$u_i(t)=c(t)\varphi(0)+s(t)\varphi'(0)+\int_0^t s(t-y)f(u_{i-1}(y),\varphi(y-w))dy,\tag{2.6}$$

for $i=1,2,\dots$. Therefore,

$$u(t)=u_0(t)+\sum_{i=0}^{\infty}(u_{i+1}(t)-u_i(t)),\tag{2.7}$$

where

$$u_0(t)=c(t)\varphi(0)+s(t)\varphi'(0).$$

Applying estimates (1.3) and (2.2), we get

$$\|u_0(t)\|_H\le\|c(t)\|_{H\to H}\|\varphi(0)\|_H+\|A^{1/2}s(t)\|_{H\to H}\|A^{-1/2}\varphi'(0)\|_H\le M+\widetilde{M}.$$

Applying formula (2.6) and estimates (1.3) and (2.3), we get

$$\begin{aligned}
\|u_1(t)-u_0(t)\|_H&\le\int_0^t\|A^{1/2}s(t-y)\|\|A^{-1/2}f(u_0(y),\varphi(y-w))\|_H dy\\
&\le\bar{M}t.
\end{aligned}$$

Using the triangle inequality, we get

$$\|u_1(t)\|_H \leq M + \widetilde{M} + \bar{M}t.$$

Applying formula (2.6) and estimates (2.4), (1.3) and (2.3), we get

$$\begin{aligned}
&\|u_2(t) - u_1(t)\|_H \\
&\leq \int_0^t \|A^{1/2}s(t-y)\| \|A^{-1/2}[f(u_1(y), \varphi(y-w)) - f(u_0(y), \varphi(y-w))]\|_H dy \\
&\leq L\int_0^t \|u_1(y) - u_0(y)\|_H dy \\
&\leq L\bar{M}\int_0^t y dy = \frac{\bar{M}}{L}\frac{(Lt)^2}{2!}.
\end{aligned}$$

Then

$$\|u_2(t)\|_H \leq M + \widetilde{M} + \frac{\bar{M}}{L}\frac{Lt}{1!} + \frac{\bar{M}}{L}\frac{(Lt)^2}{2!}.$$

Let

$$\|u_n(t) - u_{n-1}(t)\|_H \leq \frac{\bar{M}}{L}\frac{(Lt)^n}{n!}.$$

Then, we obtain

$$\begin{aligned}
&\|u_{n+1}(t) - u_n(t)\|_H \\
&\leq \int_0^t \|A^{1/2}s(t-y)\| \|A^{-1/2}[f(u_n(y), \varphi(y-w)) - f(u_{n-1}(y), \varphi(y-w))]\|_H dy \\
&\leq \int_0^t L\|u_n(y) - u_{n-1}(y)\|_H ds \\
&\leq \int_0^t L\frac{\bar{M}}{L}\frac{(Ly)^n}{n!} dy = \frac{\bar{M}}{L}\frac{(Lt)^{n+1}}{(n+1)!}.
\end{aligned}$$

Therefore, for any $n, n \geq 1$, we have

$$\begin{gathered}
\|u_{n+1}(t) - u_n(t)\|_H \leq \frac{\bar{M}}{L}\frac{(Lt)^{n+1}}{(n+1)!}, \\
\|u_{n+1}(t)\|_H \leq M + \widetilde{M} + \frac{\bar{M}}{L}\frac{Lt}{1!} + \dots + \frac{\bar{M}}{L}\frac{(Lt)^{n+1}}{(n+1)!}
\end{gathered}$$

by mathematical induction. From this and formula (2.7) it follows that

$$\begin{aligned}
\|u(t)\|_H &\leq \|u_0(t)\|_H + \sum_{i=0}^{\infty} \|u_{i+1}(t) - u_i(t)\|_H \\
&\leq M + \widetilde{M} + \sum_{i=0}^{\infty} \frac{\bar{M}}{L}\frac{(Lt)^{i+1}}{(i+1)!} \\
&\leq M + \widetilde{M} + \frac{\bar{M}}{L}e^{Lt}, 0 \leq t \leq w
\end{aligned}$$

which proves the existence of a bounded solution of problem (1.2) in $[0, w] \times H \times H$.

Now, we consider solution of problem (1.2) in $w \leq t \leq 2w$. We note that $0 \leq t - w \leq w$. We denote that

$$\varphi_1(t) = u(t-w), w \leq t \leq 2w.$$

Replacing t and $t-w$ and assuming that

$$\|A^{-1/2}f(u_0(t),\varphi_1(t))\|_H \le \bar{M}_1,$$
$$\|\varphi_1(t)\|_H \le M_1, \quad \|A^{-1/2}\varphi_1'(t)\|_H \le \widetilde{M}_1.$$

Therefore,

$$u_0(t) = c(t-w)\varphi_1(w) + s(t-w)\frac{d\varphi_1(w)}{dt},$$
$$\begin{aligned} u_i(t) &= c(t-w)\varphi_1(w) + s(t-w)\frac{d\varphi_1(w)}{dt} \\ &\quad + \int_w^t s(t-y)f(u_{i-1}(y),u_i(y-w))dy, \quad i=1,2,\dots. \end{aligned}$$

In a similar manner, for any $n, n \ge 1$, we obtain

$$\|u_{n+1}(t)-u_n(t)\|_H \le \frac{\bar{M}_1}{L}\frac{(L(t-w))^{n+1}}{(n+1)!},$$
$$\|u_{n+1}(t)\|_H \le M_1 + \widetilde{M}_1 + \frac{\bar{M}_1}{L}\frac{Lt}{1!} + \dots + \frac{\bar{M}_1}{L}\frac{(L(t-w))^{n+1}}{(n+1)!}.$$

From this it follows that

$$\|u(t)\|_H \le M_1 + \widetilde{M}_1 + \frac{\bar{M}_1}{L}e^{L(t-w)}, \quad w \le t \le 2w$$

which proves the existence of a bounded solution of problem (1.2) in $[w,2w]\times H\times H$.

In a similar manner, we can obtain

$$\|u(t)\|_H \le M_n + \widetilde{M_n} + \frac{\bar{M}_n}{L}e^{L(t-nw)}, \quad nw \le t \le (n+1)w,$$

where M_n, $\widetilde{M_n}$ and $\bar{M}_n$ are bounded. This proves the existence of a bounded solution of problem (1.2) in $[nw,(n+1)w]\times H\times H$. In general, the function $u(t)$ constructed is a solution of problem (1.2) which is bounded in $[0,\infty)\times H\times H$.

Now we will prove uniqueness of this solution of problem (1.2). Assume that there is a bounded solution $v(t)$ of problem (1.2) and $v(t) \ne u(t)$. We denote that $z(t) = v(t) - u(t)$. Therefore for $z(t)$, we have

$$\begin{gathered} \frac{d^2z(t)}{dt^2} + Az(t) = f(v(t),v(t-w)) - f(u(t),u(t-w)), \quad t>0, \\ z(t) = 0, \quad -w \le t \le 0. \end{gathered}$$

We consider the interval $0 \le t \le w$. Since $v(t-w) = u(t-w) = \varphi(t-w)$, we have

$$\begin{gathered} \frac{d^2z(t)}{dt^2} + Az(t) = f(v(t),\varphi(t-w)) - f(u(t),\varphi(t-w)), \quad t>0, \\ z(t) = 0, \quad -w \le t \le 0. \end{gathered}$$

Therefore,

$$z(t) = \int_0^t s(t-y)[f(v(y),\varphi(y-w)) - f(u(y),\varphi(y-w))]ds.$$

Applying estimates (1.3) and (2.3), we get

$$\|z(t)\|_H \le \int_0^t \|A^{1/2}s(t-y)\|\|A^{-1/2}[f(v(y),\varphi(y-w)) - f(u(y),\varphi(y-w))]\|_H dy$$

$$\leq L\int_0^t \|v(y)-u(y)\|_H ds \leq L\int_0^t \|z(y)\|_H dy.$$

Using the integral inequality, we get

$$\|z(t)\|_H \leq 0.$$

From that it follows that $z(t) = 0$ which proves the uniqueness of a bounded solution of problem (1.2) in $[0, w] \times H \times H$. Applying same way and mathematical induction, we can prove the uniqueness of a bounded solution of problem (1.2) in $[0, \infty) \times H \times H$. □

Remark 2.2. Method of present paper also enables to prove, under certain assumptions, the existence of a unique bounded solution of the initial value problem for evolutionary nonlinear partial differential equations

$$\begin{gathered}\frac{d^2u}{dt^2} + Au(t) = f(t, u(t), u([t])), \quad t > 0,\\ u(0) = \varphi(0), \quad u'(0) = \varphi'(0)\end{gathered} \tag{2.8}$$

in a Hilbert space H with the self adjoint positive definite operator A with dense domain $D(A)$. Here $[t]$ denotes the greatest-integer function.

3. Applications

First, we consider the initial-boundary value problem for one dimensional nonlinear delay differential equations of hyperbolic type

$$\begin{gathered}\frac{\partial^2 u(t,x)}{\partial t^2} - (a(x)u_x(t,x))_x + \delta u(t,x) = f(x, u(t,x), u(t-w,x)),\\ 0 < t < \infty, x \in (0,l)\\ u(t,x) = \varphi(t,x), \quad \varphi(t,0) = \varphi(t,l), \quad \varphi_x(t,0) = \varphi_x(t,l),\\ -\omega \leq t \leq 0, \quad x \in [0,l],\\ u(t,0) = u(t,l), \quad u_x(t,0) = u_x(t,l), \quad -\omega \leq t < \infty,\end{gathered} \tag{3.1}$$

where $a(x), \varphi(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number. We will assume that $a(x) \geq a > 0$ and $a(l) = a(0)$.

Theorem 3.1. *Assume the following hypotheses:*

(1) *For each $t, -w \leq t \leq 0$, we have*

$$\|\varphi(t,\cdot)\|_{L_2[0,l]} \leq M, \|\varphi'(t,\cdot)\|_{L_2[0,l]} \leq \widetilde{M}. \tag{3.2}$$

(2) *The function $f : (0,l) \times L_2[0,l] \times L_2[0,l] \to L_2[0,l]$ is continuous and bounded, that is*

$$\|f(u,v)\|_{L_2[0,l]} \leq \overline{M} \tag{3.3}$$

and the Lipschitz condition holds uniformly with respect to z

$$\|f(u,z) - f(v,z)\|_{L_2[0,l]} \leq L\|u-v\|_{L_2[0,l]}. \tag{3.4}$$

Here and below, $L, M, \widetilde{M}, \overline{M}$ are positive constants.

Then there exists a unique solution to problem (3.1) *which is bounded in* $[0,\infty) \times L_2[0,l] \times L_2[0,l]$.

The proof of Theorem 3.1 is based on the abstract Theorem 2.1, on the self-adjointness and positivity in $L_2[0,l]$ of a differential operator A^x defined by the formula

$$A^x u = -\frac{d}{dx}\Big(a(x)\frac{du}{dx}\Big) + \delta u \tag{3.5}$$

with domain $D(A^x) = \{u \in W_2^2[0,l] : u(0) = u(l), u'(0) = u'(l)\}$ [4] and on the estimate

$$\|c\{t\}\|_{L_2[0,l]\to L_2[0,l]} \le 1,\ \|(A^x)^{1/2} s\{t\}\|_{L_2[0,l]\to L_2[0,l]} \le 1, \quad t \ge 0. \tag{3.6}$$

Second, we consider the initial nonlocal boundary value problem for one dimensional nonlinear delay differential equations of hyperbolic type with involution

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t^2} - (a(x)u_x(t,x))_x - \beta(a(-x)u_x(t,-x))_x + \delta u(t,x) \\
= f(x, u(t,x), u(t-w,x)), \quad 0 < t < \infty, x \in (-l,l), \\
u(t,x) = \varphi(t,x), \quad \varphi(t,-l) = \varphi(t,l) = 0, \\
-\omega \le t \le 0, \quad x \in [-l,l], \\
u(t,-l) = u(t,l) = 0, \quad -\omega \le t < \infty,
\end{gathered} \tag{3.7}$$

where $a(x)$ and $\varphi(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number. We will assume that $a \ge a(x) = a(-x) \ge \delta > 0$, $\delta - a|\beta| \ge 0$.

Theorem 3.2. *Assume the following hypotheses:*

(1) *For each t, $-w \le t \le 0$, we have*

$$\|\varphi(t,\cdot)\|_{L_2[-l,l]} \le M, \|\varphi'(t,\cdot)\|_{L_2[-l,l]} \le \widetilde{M}.$$

(2) *The function $f : (-l,l) \times L_2[-l,l] \times L_2[-l,l] \to L_2[-l,l]$ is continuous and bounded, that is*

$$\|f(u,v)\|_{L_2[-l,l]} \le \overline{M}$$

and the Lipschitz condition holds uniformly with respect to z,

$$\|f(u,z) - f(v,z)\|_{L_2[-l,l]} \le L\|u - v\|_{L_2[-l,l]}.$$

Then there exists a unique solution to problem (3.7) *which is bounded in* $[0,\infty) \times L_2[-l,l] \times L_2[-l,l]$.

The proof of Theorem 3.2 is based on the abstract Theorem 2.1, on the self-adjointness and positivity in $L_2[-l,l]$ of a differential operator A^x defined by the formula

$$A^x v(x) = -(a(x)v_x(x)_x - \beta(a(-x)v_x(-x))_x + \delta v(x)$$

with the domain $D(A^x) = \{u \in W_2^2[-l,l] : u(-l) = u(l) = 0\}$ [12] and on the estimate

$$\|c\{t\}\|_{L_2[-l,l]\to L_2[-l,l]} \le 1, \quad \|(A^x)^{1/2} s\{t\}\|_{L_2[-l,l]\to L_2[-l,l]} \le 1, \quad t \ge 0.$$

Third, let $\Omega \subset R^n$ be a bounded open domain with smooth boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,\infty)\times\Omega$ we consider the initial boundary value problem for multidimensional nonlinear delay differential equations of hyperbolic type

$$
\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{n} (a_r(x) u_{x_r}) x_r + \delta u(t,x) \\
= f(x, u(t,x), u(t-w,x)), \quad 0 < t < \infty,\ x = (x_1, \dots, x_n) \in \Omega, \\
u(t,x) = \varphi(t,x), \quad -\omega \le t \le 0, \quad x \in \overline{\Omega}, \\
u(t,x) = 0, \quad x \in S, \quad 0 \le t < \infty,
\end{gathered} \tag{3.8}
$$

where $a_r(x)$ and $\varphi(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number and $a_r(x) > 0$.

Theorem 3.3. *Assume the following hypotheses:*

(1) *For each t, $-w \le t \le 0$ we have*

$$
\|\varphi(t,\cdot)\|_{L_2(\overline{\Omega})} \le M, \quad \|\varphi'(t,\cdot)\|_{L_2(\overline{\Omega})} \le \widetilde{M}.
$$

(2) *The function $f : Q \times L_2(\overline{\Omega}) \times L_2(\overline{\Omega}) \to L_2(\overline{\Omega})$ is continuous and bounded, that is*

$$
\|f(u,v)\|_{L_2(\overline{\Omega})} \le \overline{M}
$$

and the Lipschitz condition holds uniformly with respect to z,

$$
\|f(u,z) - f(v,z)\|_{L_2(\overline{\Omega})} \le L \|u - v\|_{L_2(\overline{\Omega})}.
$$

Then there exists a unique solution to problem (3.8) *which is bounded in* $[0,\infty) \times L_2(\overline{\Omega}) \times L_2(\overline{\Omega})$.

The proof of Theorem 3.3 is based on the abstract Theorem 2.1, on the self-adjointness and positivity in $L_2(\overline{\Omega})$ of a differential operator A^x defined by the formula

$$
A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} + \delta u(x) \tag{3.9}
$$

with domain [30]

$$
D(A^x) = \{u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \le r \le n, u(x) = 0, x \in S\}
$$

and on the estimate

$$
\|c\{t\}\|_{L_2(\overline{\Omega}) \to L_2(\overline{\Omega})} \le 1,\ \|(A^x)^{1/2} s\{t\}\|_{L_2(\overline{\Omega}) \to L_2(\overline{\Omega})} \le 1, \quad t \ge 0. \tag{3.10}
$$

Fourth, in $[0,\infty) \times \Omega$ we consider the initial boundary value problem for multi-dimensional nonlinear delay differential equations of hyperbolic type

$$
\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{n} (a_r(x) u_{x_r}) x_r + \delta u(t,x) = f(x, u(t,x), u(t-w,x)), \\
0 < t < \infty, \quad x = (x_1, \dots, x_n) \in \Omega, \\
u(t,x) = \varphi(t,x), \quad -\omega \le t \le 0, \quad x \in \overline{\Omega}, \\
\frac{\partial u}{\partial \vec{n}}(t,x) = 0, \quad x \in S, \quad 0 \le t < \infty,
\end{gathered} \tag{3.11}
$$

where $a_r(x)$ and $\varphi(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number and $a_r(x) > 0$. Here, $\overrightarrow{n}$ is the normal vector to Ω.

Theorem 3.4. *Suppose that assumptions of Theorem 3.3 hold. Then there exists a unique solution to problem* (3.11) *which is bounded in* $[0,\infty) \times L_2(\overline{\Omega}) \times L_2(\overline{\Omega})$.

The proof of Theorem 3.4 is based on the abstract Theorem 2.1, on the self-adjointness and positivity in $L_2(\overline{\Omega})$ of a differential operator A^x defined by the formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \delta u(x)$$

with domain [30]

$$D(A^x) = \{u(x) : u(x), u_{x_r}(x), (a_r(x)u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \le r \le n, \frac{\partial u}{\partial \vec{n}}(x) = 0, x \in S\}$$

and on estimate (3.10).

4. Numerical results

In general, it is not possible to get exact solution of nonlinear problems. Therefore, the first and second order of accuracy difference schemes for the solution of one dimensional nonlinear hyperbolic equation with time delay are presented. Numerical results are provided. We consider the initial-boundary value problem

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = 2e^{-t}\sin x + \cos(u(t,x)u(t-1,x)) \\
- \cos(e^{-t}\sin x u(t-1,x)), \\
0 < t < \infty,\ 0 < x < \pi, \\
u(t,x) = e^{-t}\sin x, \quad 0 \le x \le \pi, \quad -1 \le t \le 0, \\
u(t,0) = u(t,\pi) = 0, \quad t \ge 0
\end{gathered} \tag{4.1}$$

for the nonlinear delay hyperbolic differential equation. The exact solution of this test example is $u(t,x) = e^{-t}\sin x$.

We get the following iterative difference scheme of first order of accuracy in t for the approximate solution of the initial-boundary value problem (4.1),

$$\begin{gathered}
\frac{{}_m u_n^{k+1} - 2({}_m u_n^k) +_m u_n^{k-1}}{\tau^2} - \frac{{}_m u_{n+1}^{k+1} - 2({}_m u_n^{k+1}) +_m u_{n-1}^{k+1}}{h^2} \\
= 2e^{-t_k}\sin x_n + \cos(({}_{m-1}u_n^k)({}_m u_n^{k-N})) - \cos(e^{-t_k}\sin x_n({}_m u_n^{k-N})), \\
t_k = k\tau, \quad x_n = nh, \quad 1 \le k < \infty, \quad 1 \le n \le M-1, \quad N\tau = 1, \quad Mh = \pi, \\
{}_m u_n^k = e^{-t_k}\sin x_n, \frac{{}_m u_n^{k+1} -_m u_n^k}{\tau} = -e^{-t_k}\sin x_n, \\
t_k = k\tau, \quad x_n = nh, 0 \le n \le M, \quad -N \le k \le 0, \\
{}_m u_0^k =_m u_M^k = 0, \quad 0 \le k < \infty,\ m = 1, 2, \dots
\end{gathered} \tag{4.2}$$

for the nonlinear delay hyperbolic equation. Here and in future m denotes the iteration index and an initial guess ${}_0u_n^k, k \ge 1, 0 \le n \le M$ is to be made. For solving difference scheme (4.2), the numerical steps are given below. For $0 \le k < N, 0 \le n \le M$ the algorithm is as follows : the algorithm is as follows :

1. $m = 1$.
2. ${}_{m-1}u_n^k$ is known.
3. ${}_m u_n^k$ is calculated.
4. If the max absolute error between ${}_{m-1}u_n^k$ and ${}_m u_n^k$ is greater than the given tolerance value, take $m = m + 1$ and go to step 2. Otherwise, terminate the iteration process and take ${}_m u_n^k$ as the result of the given problem.

We write (4.2) in the matrix form

$$\begin{gathered}A_m u^{k+1} + B_m u^k + C_m u^{k-1} = R\varphi({}_{m-1}u^k, {}_m u^{k-N}),\\ Nl+1 \le k \le (l+1)N-1, \quad l = 0,1,\dots,\\ {}_m u^k = e^{-t_k}\{\sin x_n\}_{n=0}^M, {}_m u^{k+1} = {}_m u^k - \tau e^{-t_k}\{\sin x_n\}_{n=0}^M, \quad -N \le k \le 0.\end{gathered} \tag{4.3}$$

Here

$$a = -\frac{1}{h^2}, \quad b = \frac{1}{\tau^2} + \frac{2}{h^2}, \quad c = -\frac{2}{\tau^2}, \quad d = \frac{1}{\tau^2}$$

and A, B,and C are $(M+1)\times(M+1)$ matrices given below:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\ a & b & a & 0 & 0 & & 0 & 0 & 0 & 0\\ 0 & a & b & a & 0 & & 0 & 0 & 0 & 0\\ 0 & 0 & a & b & a & & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & a & b & & 0 & 0 & 0 & 0\\ \vdots & & & & & & & & & \vdots\\ 0 & 0 & 0 & 0 & 0 & & & a & b & a\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\ 0 & c & 0 & 0 & 0 & & 0 & 0 & 0 & 0\\ 0 & 0 & c & 0 & 0 & & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & c & 0 & & 0 & 0 & 0 & 0\\ \vdots & & & & & & & & & \vdots\\ 0 & 0 & 0 & 0 & 0 & & 0 & c & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & c & 0\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\ 0 & d & 0 & 0 & 0 & & 0 & 0 & 0 & 0\\ 0 & 0 & d & 0 & 0 & & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & d & 0 & & 0 & 0 & 0 & 0\\ \vdots & & & & & & & & & \vdots\\ 0 & 0 & 0 & 0 & 0 & & 0 & d & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & d & 0\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

and here and below R is the $(M+1)\times(M+1)$ identity matrix, ${}_m u_n^k = e^{-t_k}\sin x_n$ for $-N \le k \le 0$, $\varphi({}_{m-1}u^k, {}_m u^{k-N})$ and ${}_m u^s$ are $(M+1)\times 1$ column vectors as

$$\varphi({}_{m-1}u^k, {}_m u^{k-N}) = \begin{bmatrix} 0\\ {}_m\varphi_1^k\\ \dots\\ {}_m\varphi_{M-1}^k\\ 0 \end{bmatrix}, \quad {}_m u^s = \begin{bmatrix} {}_m u_0^s\\ {}_m u_1^s\\ \dots\\ {}_m u_{M-1}^s\\ {}_m u_M^s \end{bmatrix}, \quad s = k,\ k\pm 1,$$

$${}_m\varphi_n^k = 2e^{-t_k}\sin x_n + \cos(({}_{m-1}u_n^k)({}_m u_n^{k-N})) - \cos(e^{-t_k}\sin x_n({}_m u_n^{k-N}))$$

for $Nl+1 \le k \le (l+1)N-1$, $l = 0,1,\dots$, $1 \le n \le M-1$.

So, we have the first order difference equation with respect to k with matrix coefficients. From (4.3) it follows that

$$
\begin{gathered}
{}_mu^{k+1} = -A^{-1}(B_m u^k - C_m u^{k-1} + A^{-1}R\varphi^k({}_{m-1}u^k, {}_mu^{k-N})), \\
Nl+1 \le k \le (l+1)N-1, \quad l = 0,1,\dots, \\
{}_mu^k = e^{-t_k}\{\sin x_n\}_{n=0}^M, {}_mu^{k+1} \\
= {}_mu^k - \tau e^{-t_k}\{\sin x_n\}_{n=0}^M, \quad -N \le k \le 0.
\end{gathered} \tag{4.4}
$$

Now, we get the following iterative difference scheme of second order of accuracy in t for the approximate solution of the initial-boundary value problem (4.1),

$$
\begin{gathered}
\frac{{}_mu_n^{k+1} - 2({}_mu_n^k) +_m u_n^{k-1}}{\tau^2} - \frac{{}_mu_{n+1}^{k+1} - 2({}_mu_n^{k+1}) +_m u_{n-1}^{k+1}}{2h^2} \\
- \frac{{}_mu_{n+1}^{k-1} - 2({}_mu_n^{k-1}) +_m u_{n-1}^{k-1}}{2h^2} \\
= 2e^{-t_k}\sin x_n + \cos(({}_{m-1}u_n^k)({}_mu_n^{k-N})) - \cos(e^{-t_k}\sin x_n({}_mu_n^{k-N})), \\
t_k = k\tau, \quad x_n = nh, \quad 1 \le k < \infty, \quad 1 \le n \le M-1, \quad N\tau = 1, \quad Mh = \pi, \\
{}_mu_n^k = e^{-t_k}\sin x_n, \quad \frac{{}_mu_n^{k+1} -_m u_n^k}{\tau} = e^{-t_k}(-1 + \frac{\tau}{2})\sin x_n, \\
t_k = k\tau, \quad x_n = nh, \quad 0 \le n \le M, \quad -N \le k \le 0 \\
{}_mu_0^k =_m u_M^k = 0, \quad 0 \le k < \infty, \quad m = 1,2,\dots.
\end{gathered} \tag{4.5}
$$

We have again $(M+1)\times(M+1)$ system of linear equations and we rewrite (4.5) in the matrix form

$$
\begin{gathered}
A_m u^{k+1} + B_m u^k + C_m u^{k-1} = R\varphi({}_{m-1}u^k, {}_mu^{k-N}), \\
Nl+1 \le k \le (l+1)N-1, \quad l = 0,1,\dots, \\
{}_mu^k = e^{-t_k}\{\sin x_n\}_{n=0}^M, {}_mu^{k+1} \\
= {}_mu^k + \big(\frac{\tau^2}{2} - \tau\big)e^{-t_k}\{\sin x_n\}_{n=0}^M, \quad -N \le k \le 0.
\end{gathered} \tag{4.6}
$$

Here

$$
e = -\frac{1}{2h^2}, \quad f = \frac{1}{\tau^2} + \frac{1}{h^2}, \quad g = -\frac{2}{\tau^2}
$$

and A, B, and C are the $(M+1)\times(M+1)$ matrices given below:

$$
A = C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
e & f & e & 0 & 0 & & 0 & 0 & 0 & 0 \\
0 & e & f & e & 0 & & 0 & 0 & 0 & 0 \\
0 & 0 & e & f & e & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e & f & & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & & & e & f & e \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1
\end{bmatrix},
$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g & 0 & & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

and ${}_m u_n^k = e^{-t_k} \sin x_n$ for $-N \le k \le 0$, $\varphi({}_{m-1}u^k, {}_m u^{k-N})$ and ${}_m u^s$ are $(M+1)\times 1$ column vectors as in (4.3). Hence, we have the second order difference equation with respect to k with matrix coefficients. From (4.6) it follows that

$$\begin{gathered} {}_m u^{k+1} = -A^{-1}(B_m u^k - C_m u^{k-1} + A^{-1} R \varphi^k({}_{m-1}u^k, {}_m u^{k-N})), \\ Nl + 1 \le k \le (l+1)N - 1, \quad l = 0, 1, \dots, \\ {}_m u^k = \{\sin x_n\}_{n=0}^M, \quad {}_m u^{k+1} = {}_m u^k + e^{-t_k}(\frac{\tau^2}{2} - \tau)\{\sin x_n\}_{n=0}^M, \\ -N \le k \le 0. \end{gathered} \tag{4.7}$$

In computations for both first and second order of accuracy difference schemes, the initial guess is chosen as ${}_0 u_n^k = e^{-t_k} \sin x_n$ and when the maximum errors between two consecutive results of iterative difference schemes (4.2) and (4.5) become less than 10^{-8}, the iterative process is terminated. We give numerical results for different values of N and M and u_n^k represent the numerical solutions of these difference schemes at (t_k, x_n). Tables are constructed for $N = M = 30, 60, 120$ in $t \in [0,1]$, $t \in [1,2]$, $t \in [2,3]$, respectively and the errors are computed by the formula

$$E_M^N = \max_{lN \le k \le (l+1)N, l=0,1,\dots,1 \le n \le M-1} |u(t_k, x_n) - u_n^k|.$$

As can be seen from tables, these numerical experiments support the theoretical statements. The number of iterations and maximum errors are decreasing with the increase of grid points.

TABLE 1. Comparison of the errors of different difference schemes in $t \in [0,1]$ (m is the iteration number)

Method	$N = M = 30$	$N = M = 60$	$N = M = 120$
(4.2) for (4.1)	$4.1195 \times 10^{-3}, m = 6$	$2.0322 \times 10^{-3}, m = 6$	$1.0098 \times 10^{-3}, m = 6$
(4.5) for (4.1)	$1.7750 \times 10^{-5}, m = 5$	$4.5557 \times 10^{-6}, m = 4$	$1.1532 \times 10^{-6}, m = 4$

TABLE 2. Comparison of the errors of different difference schemes in $t \in [1,2]$ (m is the iteration number)

Method	$N = M = 30$	$N = M = 60$	$N = M = 120$
(4.2) for (4.1)	$2.3014 \times 10^{-3}, m = 6$	$1.1297 \times 10^{-3}, m = 6$	$5.6051 \times 10^{-4}, m = 2$
(4.5) for (4.1)	$1.7751 \times 10^{-5}, m = 5$	$4.5556 \times 10^{-6}, m = 4$	$1.1531 \times 10^{-6}, m = 4$

Table 3. Comparison of the errors of different difference schemes in $t \in [2, 3]$ (m is the iteration number)

Method	$N = M = 30$	$N = M = 60$	$N = M = 120$
(4.2) for (4.1)	$1.0245 \times 10^{-3}, m = 6$	$5.0161 \times 10^{-4}, m = 6$	$2.4864 \times 10^{-4}, m = 6$
(4.5) for (4.1)	$3.6898 \times 10^{-6}, m = 5$	$9.4326 \times 10^{-7}, m = 4$	$2.3890 \times 10^{-7}, m = 4$

In Tables 1–3, as we increase values of M and N each time starting from $M = N = 30$ by a factor of 2 the errors in the first order of accuracy difference scheme decrease approximately by a factor of $1/2$, the errors in the second order of accuracy difference scheme decrease approximately by a factor of $1/4$. The errors presented in the tables indicate the stability of the difference schemes and the accuracy of the results. Thus, the second order of accuracy difference scheme increases faster than the first order of accuracy difference scheme.

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