COMPOSITION AND CONVOLUTION THEOREMS FOR
\(\mu\)-STEPANOV PSEUDO ALMOST PERIODIC FUNCTIONS AND
APPLICATIONS TO FRACTIONAL INTEGRO-DIFFERENTIAL
EQUATIONS

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Abstract. In this article we establish new convolution and composition theorems for \(\mu\)-Stepanov pseudo almost periodic functions. We prove that the space of vector-valued \(\mu\)-Stepanov pseudo almost periodic functions is a Banach space. As an application, we prove the existence and uniqueness of \(\mu\)-pseudo almost periodic mild solutions for the fractional integro-differential equation

\[
D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t,u(t)),
\]

where \(A\) generates an \(\alpha\)-resolvent family \(\{S_\alpha(t)\}_{t \geq 0}\) on a Banach space \(X\), \(a \in L^1_{\text{loc}}(\mathbb{R}_+), \alpha > 0\), the fractional derivative is understood in the sense of Weyl and the nonlinearity \(f\) is a \(\mu\)-Stepanov pseudo almost periodic function.

1. Introduction

Ezzinbi et al. [1] defined the space of \(\mu\)-SP\(^p\)-pseudo almost periodic functions. This space contains the space of Stepanov-like weighted pseudo almost periodic functions (see [8, 11]) and the space of \(\mu\)-pseudo almost periodic functions (see [5]). Several composition theorems and their applications in the context of Stepanov-like almost periodic, Stepanov-like pseudo almost periodic and Stepanov-like weighted pseudo almost periodic functions appear for example in [2, 9, 10, 12, 14]. Here we generalize the composition theorem given by Zhao et al. for the space of Stepanov-like weighted pseudo almost periodic functions (see [14, Th. 2.15]). Also, we recover the composition result given by Ezzinbi et al. for \(\mu\)-SP\(^p\)-pseudo almost periodic functions (see [1, Th. 2.29]). Moreover, we establish another composition theorem that does not require Lipschitzian nonlinearities (Theorem 3.5 and Theorem 3.8).

In Theorem 3.10 we prove that the convolution of a strongly continuous family \(\{S(t)\}_{t \geq 0}\) with a \(\mu\)-SP\(^p\)-pseudo almost periodic function \(F\), \((S * f)(t) = \int_{-\infty}^{t} S(t-s)F(s)ds\), is a \(\mu\)-pseudo almost periodic function. We prove that the collection of \(\mu\)-SP\(^p\)-pseudo almost periodic functions is a Banach space with a natural norm (Theorem 3.3), and combine our results to prove the existence and uniqueness

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of \( \mu \)-pseudo almost periodic solutions to a class of abstract fractional differential equations
\[
D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, u(t)),
\]
where \( A \) generates an \( \alpha \)-resolvent family \( \{S_\alpha(t)\}_{t \geq 0} \) on a Banach space \( X \), \( a \in L^1_{\text{loc}}(\mathbb{R}_+) \), \( \alpha > 0 \), the fractional derivative is understood in the sense of Weyl and provided that the nonlinear term \( f \) is \( \mu \)-Stepanov pseudo almost periodic.

2. Preliminaries

Throughout this article \( (X, \| \cdot \|_X) \) and \( (Y, \| \cdot \|_Y) \) denote complex Banach spaces and \( B(X,Y) \) the Banach space of bounded linear operators from \( X \) to \( Y \); when \( X = Y \) we write \( B(X) \).

We denote by \( BC(\mathbb{R}, X) \) the Banach space of \( X \)-valued bounded and continuous functions on \( \mathbb{R} \), with norm
\[
\|f\| = \sup\{\|f(t)\|_X : t \in \mathbb{R}\}.
\]

**Definition 2.1** \( (\text{H}) \). A function \( f \in C(\mathbb{R}, X) \) is called (Bohr) almost periodic if for each \( \epsilon > 0 \) there exists \( l = l(\epsilon) > 0 \) such that every interval of length \( l \) contains a number \( \tau \) with the property that
\[
\|f(t+\tau) - f(t)\| < \epsilon \quad (t \in \mathbb{R}).
\]
The collection of all such functions will be denoted by \( AP(\mathbb{R}, X) \).

This definition is equivalent to the so-called Bochner’s criterion, namely, \( f \in AP(\mathbb{R}, X) \) if and only if for every sequence of reals \( (s_n^\prime) \) there exists a subsequence \( (s_n) \) such that \( (f(n + s_n)) \) is uniformly convergent on \( \mathbb{R} \).

**Definition 2.2** \( (\text{H}) \). A function \( f \in C(\mathbb{R} \times Y, X) \) is called (Bohr) almost periodic in \( t \in \mathbb{R} \) uniformly in \( y \in K \) where \( K \subset Y \) is any compact subset if for each \( \epsilon > 0 \) there exists \( l = l(\epsilon) > 0 \) such that every interval of length \( l \) contains a number \( \tau \) with the property that
\[
\|f(t+\tau,y) - f(t,y)\| < \epsilon \quad (t \in \mathbb{R}, \ y \in K).
\]
The collection of such functions will be denoted by \( AP(\mathbb{R} \times Y, X) \).

Let \( \mathcal{B} \) denote the Lebesgue \( \sigma \)-field of \( \mathbb{R} \), see \( \text{[4]} \). Let \( \mathcal{M} \) stand for the set of all positive measures \( \nu \) on \( \mathcal{B} \) satisfying \( \mu(\mathbb{R}) = \infty \) and \( \mu([a,b]) < \infty \) for all \( a, b \in \mathbb{R} \).

Throughout this paper will consider the following hypotheses:

(H1) For all \( a, b \) and \( c \) in \( \mathbb{R} \), such that \( 0 \leq a < b \leq c \), there exist \( \tau_0 \geq 0 \) and \( \alpha_0 > 0 \) such that
\[
|\tau| \leq \tau_0 \Rightarrow \mu((a+\tau, b+\tau)) \geq \alpha_0 \mu([\tau, c+\tau]).
\]

(H2) For all \( \tau \in \mathbb{R} \), there exist \( \beta > 0 \) and a bounded interval \( I \) such that
\[
\mu\{a + \tau, a \in A\} \leq \beta \mu(A) \text{ if } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.
\]

Note that Hypothesis (H2) implies (H1), see \( \text{[5]} \) Lemma 2.1.

**Definition 2.3** \( (\text{H}) \). Let \( \mu \in \mathcal{M} \). A function \( f \in BC(\mathbb{R}, X) \) is said to be \( \mu \)-ergodic if
\[
\lim_{T \to +\infty} \frac{1}{\mu([-T,T])} \int_{[-T,T]} \|f(t)\| d\mu(t) = 0.
\]
We denote by $\mathcal{E}(\mathbb{R}, X, \mu)$ the set of such functions. A function $f \in BC(\mathbb{R} \times X, X)$ is said to be $\mu$-ergodic if

$$\lim_{T \to +\infty} \frac{1}{\mu([-T, T])} \int_{[-T, T]} \|f(t, x)\|d\mu(t) = 0,$$

uniformly in $x \in X$. Denote by $\mathcal{E}(\mathbb{R} \times X, X, \mu)$ the set of such functions.

**Definition 2.4** ([5]). Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, X)$ is said to be $\mu$-pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$. Denote by $PAP(\mathbb{R}, X, \mu)$ the collection of such functions.

**Definition 2.5** ([11]). The Bochner transform $f^b(t, s)$ with $t \in \mathbb{R}, s \in [0, 1]$ of a function $f : \mathbb{R} \to X$ is defined by

$$f^b(t, s) := f(t + s).$$

**Definition 2.6** ([11]). The Bochner transform $f^b(t, s, u)$ with $t \in \mathbb{R}, s \in [0, 1], u \in X$ of a function $f : \mathbb{R} \times X \to X$ is defined by

$$f^b(t, s, u) := f(t + s, u) \quad \text{for all } u \in X.$$

**Definition 2.7** ([11]). Let $p \in [1, \infty)$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with exponent $p$, consist of all measurable functions $f : \mathbb{R} \to X$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0, 1; X))$. This is a Banach space with the norm

$$\|f\|_{BS^p(\mathbb{R}, X)} := \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left( \int_{t}^{t+1} \|f(\tau)\|^p d\tau \right)^{1/p}.$$

**Definition 2.8** ([8]). A function $f \in BS^p(\mathbb{R}, X)$ is called Stepanov almost periodic if $f^b \in AP(\mathbb{R}, L^p(0, 1; X))$. We denote the set of all functions by $APS^p(\mathbb{R}, X)$.

**Definition 2.9** ([8]). A function $f : \mathbb{R} \times X \to Y$ with $f(\cdot, u) \in BS^p(\mathbb{R}, Y)$, for each $u \in X$, is called Stepanov almost periodic function in $t \in \mathbb{R}$ uniformly for $u \in X$ if, for each $\epsilon > 0$ and each compact set $K \subset X$ there exists a relatively dense set $P = P(\epsilon, f, K) \subset \mathbb{R}$ such that

$$\sup_{t \in \mathbb{R}} \left( \int_{0}^{1} \|f(t + s + \tau, u) - f(t + s, u)\| ds \right)^{1/p} < \epsilon,$$

for each $\tau \in P$ and each $u \in K$. We denote by $APS^p(\mathbb{R} \times X, Y)$ the set of such functions.

**Definition 2.10** ([1]). Let $\mu \in \mathcal{M}$. A function $f \in BS^p(\mathbb{R}, X)$ is said $\mu$-Stepanov-like pseudo almost periodic (or $\mu$-$S^p$-pseudo almost periodic) if it can be expressed as $f = g + \phi$, where $g \in APS^p(\mathbb{R}, X)$ and $\phi^b \in \mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)$. In other words, a function $f \in L^p_{loc}(\mathbb{R}, X)$ is said $\mu$-$S^p$-pseudo almost periodic relatively to measure $\mu$, if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; X)$ is $\mu$-pseudo almost periodic in the sense that there exist two functions $g, \phi : \mathbb{R} \to X$ such that $f = g + \phi$, where $g \in APS^p(\mathbb{R}, X)$ and $\phi^b \in \mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)$, that is $\phi^b \in BC(\mathbb{R}, L^p(0, 1; X))$ and

$$\lim_{T \to +\infty} \frac{1}{\mu([-T, T])} \int_{[-T, T]} \left( \int_{t}^{t+1} \|\phi(s)\|^p ds \right)^{1/p} d\mu(t) = 0.$$

We denote by $PAPS^p(\mathbb{R}, X, \mu)$ the set of all such functions.
Definition 2.11 ([1]). Let $\mu \in \mathcal{M}$. A function $f: \mathbb{R} \times Y \to X$ with $f(\cdot, u) \in L^p_{\text{loc}}(\mathbb{R}, X)$ for each $u \in Y$, is said to be $\mu$-Stepanov-like pseudo almost periodic (or $\mu$-$S^p$-pseudo almost periodic) if it can be expressed as $f = g + \phi$, where $g \in AP^p(\mathbb{R} \times Y, X)$ and $\phi \in \mathcal{E}(\mathbb{R} \times Y, L^p(0,1; X), \mu)$. We denote by $PAPS^p(\mathbb{R} \times Y, X, \mu)$ the set of all such functions.

3. Main results

For $1 \leq p < \infty$, we define $\mathcal{B}: BS^p(\mathbb{R}, X) \to L^\infty(\mathbb{R}, L^p(0,1; X))$ by

$$f \mapsto (\mathcal{B}f)(t) = f^p(t, s) = f(t + s) \quad (t \in \mathbb{R}, \ s \in [0,1]),$$

see [2].

Remark 3.1. It follows from its definition that the operator $\mathcal{B}$ is a linear isometry between $BS^p(\mathbb{R}, X)$ and $L^\infty(\mathbb{R}, L^p(0,1; X))$. More precisely,

$$\|\mathcal{B}f\|_{L^\infty(\mathbb{R}, L^p)} = \|f\|_{BS^p(\mathbb{R}, X)}.$$

Remark 3.2. The definition of $\mu$-Stepanov-like pseudo almost periodic functions can be written using the preceding notation. Thus, for $\mu \in \mathcal{M}$, we say that a function $f$ is said to be $\mu$-Stepanov-like pseudo almost periodic (or $\mu$-$S^p$-pseudo almost periodic) if and only if $f \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1; X))) + \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1; X), \mu)).$

Thus,

$$PAPS^p(\mathbb{R}, X, \mu) = \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1; X))) + \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1; X), \mu)). \tag{3.1}$$

Also, assume that $\mu$ satisfies (H1). Since $\mathcal{B}$ is an isometry and $AP(\mathbb{R}, L^p(0,1; X)) \cap \mathcal{E}(\mathbb{R}, L^p(0,1; X), \mu) = \{0\}$ by [5] Cor. 2.29 we have that the sum is direct, that is,

$$PAPS^p(\mathbb{R}, X, \mu) = \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1; X))) \oplus \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1; X), \mu)).$$

Based on the definition of the operator $\mathcal{B}$, next we prove that $PAPS^p(\mathbb{R}, X, \mu)$ is a Banach space.

Theorem 3.3. If $\mu \in \mathcal{M}$ satisfies (H1), then $PAPS^p(\mathbb{R}, X, \mu)$ is a Banach space with the norm

$$\|f\|_{PAPS^p(\mathbb{R}, X, \mu)} = \|g\|_{BS^p(\mathbb{R}, X)} + \|h\|_{BS^p(\mathbb{R}, X)},$$

where $f = g + h$ with $g \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1; X)))$, $h \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1; X), \mu))$.

Proof. Let $(f_n)$ be a Cauchy sequence in $PAPS^p(\mathbb{R}, X, \mu)$. Then

$$\|f_n - f_m\|_{PAPS^p(\mathbb{R}, X, \mu)} \to 0 \quad \text{as} \ n, m \to \infty.$$

Let $f_n = g_n + h_n$ and $f_m = g_m + h_m$ with $g_n, g_m \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1; X)))$ and $h_n, h_m \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1; X), \mu))$. If $n, m \to \infty$, then

$$\|\mathcal{B}g_n - \mathcal{B}g_m\|_{L^\infty(\mathbb{R}, L^p)} = \|g_n - g_m\|_{BS^p(\mathbb{R}, X)} \leq \|f_n - f_m\|_{PAPS^p(\mathbb{R}, X, \mu)} \to 0,$$

$$\|\mathcal{B}h_n - \mathcal{B}h_m\|_{L^\infty(\mathbb{R}, L^p)} = \|h_n - h_m\|_{BS^p(\mathbb{R}, X)} \leq \|f_n - f_m\|_{PAPS^p(\mathbb{R}, X, \mu)} \to 0.$$}

This implies that $(\mathcal{B}g_n)$ and $(\mathcal{B}h_n)$ are Cauchy sequences in $AP(\mathbb{R}, L^p(0,1; X))$ and $\mathcal{E}(\mathbb{R}, L^p(0,1; X), \mu)$ respectively. Since $AP(\mathbb{R}, L^p(0,1; X))$ is a closed subspace of $BC(\mathbb{R}, L^p(0,1; X))$ then it is a Banach space. Also, it follows from [5] Cor. 2.31] that $\mathcal{E}(\mathbb{R}, L^p(0,1; X), \mu)$ is a Banach space. Then there exist $g \in AP(\mathbb{R}, L^p(0,1; X))$ and $h \in \mathcal{E}(\mathbb{R}, L^p(0,1; X), \mu)$ such that

$$\|\mathcal{B}g_n - g\|_{L^\infty(\mathbb{R}, L^p)} \to 0, \quad \|\mathcal{B}h_n - h\|_{L^\infty(\mathbb{R}, L^p)} \to 0 \quad (n \to \infty).$$


Let
\[ f_1 := B^{-1}(\{g\}) \in B^{-1}(AP(\mathbb{R}, L^p(0, 1; X))) \]
\[ f_2 := B^{-1}(\{h\}) \in B^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)). \]
Note that \( f_1 \) and \( f_2 \) are well defined because \( B \) is injective. Let \( f := f_1 + f_2 \in PAPS^p(\mathbb{R}, X, \mu). \) Thus
\[
\left\| f_n - f \right\|_{PAPS^p(\mathbb{R}, X, \mu)} = \left\| (g_n + h_n) - (f_1 + f_2) \right\|_{PAPS^p(\mathbb{R}, X, \mu)} \\
= \left\| g_n - f_1 \right\|_{BS^p(\mathbb{R}, X)} + \left\| h_n - f_2 \right\|_{BS^p(\mathbb{R}, X)} \\
= \left\| Bg_n - Bf_1 \right\|_{L^\infty(\mathbb{R}, L^p)} + \left\| Bh_n - Bf_2 \right\|_{L^\infty(\mathbb{R}, L^p)} \\
= \left\| Bg_n - g \right\|_{L^\infty(\mathbb{R}, L^p)} + \left\| Bh_n - h \right\|_{L^\infty(\mathbb{R}, L^p)} \rightarrow 0 \quad (n \rightarrow \infty).
\]
Hence \( PAPS^p(\mathbb{R}, X, \mu) \) is a Banach space.

The following theorem is taken from [7, Theorem 2.1].

**Theorem 3.4.** Let \( \mu \in \mathcal{M} \) and \( I \) be a bounded interval (eventually \( \emptyset \)). Assume that \( f(\cdot) \in BS^p(\mathbb{R}, X) \). Then the following assertions are equivalent.

(a) \( f^p(\cdot) \in \mathcal{E}(\mathbb{R}, L^p(0, 1; X)), \mu). \)

(b) \[
\lim_{T \to \infty} \frac{1}{\mu([-T, T] \setminus I)} \int_{\mu([-T, T] \setminus I)} \left( \int_t^{t+1} \| f(s) \|^p ds \right)^{1/p} dt = 0.
\]

(c) For any \( \epsilon > 0, \)
\[
\lim_{T \to \infty} \frac{\mu\left( \left\{ t \in [-T, T] \setminus I : \left( \int_t^{t+1} \| f(s) \|^p ds \right)^{1/p} > \epsilon \right\} \right)}{\mu([-T, T] \setminus I)} = 0.
\]

The following theorem about composition of Stepanov-like type pseudo almost periodic functions generalizes [13, Theorem 2.15].

**Theorem 3.5.** Let \( \mu \in \mathcal{M} \) and let \( f = g + \phi \in PAPS^p(\mathbb{R} \times X, X, \mu) \) with \( g \in B^{-1}(AP(\mathbb{R} \times X, L^p(0, 1; X))) \) and \( \phi \in B^{-1}(\mathcal{E}(\mathbb{R} \times X, L^p(0, 1; X)), \mu) \). Assume the following conditions.

(a) \( f(t, x) \) is uniformly continuous in any bounded set \( K' \subset X \) uniformly for \( t \in \mathbb{R}, \)

(b) \( g(t,x) \) is uniformly continuous in any bounded set \( K' \subset X \) uniformly for \( t \in \mathbb{R}, \)

(c) for every bounded subset \( K' \subset X, \) the set \( \{ f(\cdot, x) : x \in K' \} \) is bounded in \( PAPS^p(\mathbb{R} \times X, X, \mu). \)

If \( x = \alpha + \beta \in PAPS^p(\mathbb{R}, X, \mu) \cap B(\mathbb{R}, X), \) with \( \alpha \in B^{-1}(AP(\mathbb{R}, L^p(0, 1; X))), \)
\( \beta \in B^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)) \) and \( Q = \{ x(t) : t \in \mathbb{R} \}, \) \( Q_1 = \{ \alpha(t) : t \in \mathbb{R} \} \) are compact, then \( f(\cdot, x(\cdot)) \in PAPS^p(\mathbb{R}, X, \mu). \)

**Proof.** Let \( f(t, x(t)) = G(t) + H(t) + W(t), \) where
\[
G(t) = g(t, \alpha(t)), \quad H(t) = f(t, x(t)) - f(t, \alpha(t)), \quad W(t) = \phi(t, \alpha(t)).
\]
Since \( g \) satisfies condition (b) and \( Q_1 = \{ \alpha(t) : t \in \mathbb{R} \} \) is compact, by [3] Prop. 1 we have \( G \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X))) \). To show that \( f(\cdot, x(\cdot)) \in \mathcal{PAP}^p(\mathbb{R}, X, \mu) \) it is sufficient to show that \( H, W \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X))) \).

First, we see that \( H \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X))) \). Since \( x(\cdot) \) and \( \alpha(\cdot) \) are bounded, we can choose a bounded subset \( K' \subset X \) such that \( x(\mathbb{R}), \alpha(\mathbb{R}) \subset K' \). By assumption \((c)\) we have that \( H(\cdot) \in BS^p(\mathbb{R}, X) \) and by assumption \((a)\) we obtain that \( f \) is uniformly continuous on the bounded set \( K' \subset X \) uniformly \( t \in \mathbb{R} \). Then, given \( \epsilon > 0 \), there exists \( \delta > 0 \), such that \( u, v \in K' \) and \( \|u - v\| < \delta \) imply that \( \|f(t, u) - f(t, v)\| \leq \epsilon \) for all \( t \in \mathbb{R} \). Then, we have

\[
\left( \int_t^{t+1} \|f(s, u) - f(s, v)\|^p ds \right)^{1/p} \leq \epsilon.
\]

Hence, for each \( t \in \mathbb{R} \), \( \|\beta(s)\|_{BS^p(\mathbb{R}, X)} < \delta \), \( s \in [t, t+1] \) implies that for all \( t \in \mathbb{R} \),

\[
\left( \int_t^{t+1} \|H(s)\|^p ds \right)^{1/p} = \left( \int_t^{t+1} \|f(s, x(s)) - f(s, \alpha(s))\|^p ds \right)^{1/p} \leq \epsilon.
\]

Therefore,

\[
\mu \left( t \in [-T, T] : \left( \int_t^{t+1} \|f(s, x(s)) - f(s, \alpha(s))\|^p ds \right)^{1/p} > \epsilon \right) \leq \frac{\mu \left( t \in [-T, T] : \left( \int_t^{t+1} \|\beta(s)\|^p ds \right)^{1/p} > \delta \right)}{\mu([-T, T])}.
\]

Since \( \beta \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu) \), then Theorem 3.4 implies that for the above mentioned \( \delta \) we have

\[
\lim_{T \to \infty} \frac{\mu \left( t \in [-T, T] : \left( \int_t^{t+1} \|f(s, x(s)) - f(s, \alpha(s))\|^p ds \right)^{1/p} > \epsilon \right)}{\mu([-T, T])} = 0.
\]

By Theorem 3.4 we have that \( H \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X))) \).

Now, we prove that \( W \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X))) \). Since \( f \) and \( g \) satisfy \((a)\) and \((b)\) respectively, then, given \( \epsilon > 0 \), exists \( \delta > 0 \), such that \( u, v \in Q_1 \), \( \|u - v\| < \delta \) imply that

\[
\left( \int_t^{t+1} \|f(s, u) - f(s, v)\|^p ds \right)^{1/p} \leq \frac{\epsilon}{16}, \quad t \in \mathbb{R},
\]

\[
\left( \int_t^{t+1} \|g(s, u) - g(s, v)\|^p ds \right)^{1/p} \leq \frac{\epsilon}{16}, \quad t \in \mathbb{R}.
\]

Let \( \delta_0 := \min\{\epsilon, \delta\} \). Then

\[
\left( \int_t^{t+1} \|\phi(s, u) - \phi(s, v)\|^p ds \right)^{1/p} \\
\leq \left( \int_t^{t+1} \|f(s, u) - f(s, v)\|^p ds \right)^{1/p} + \left( \int_t^{t+1} \|g(s, u) - g(s, v)\|^p ds \right)^{1/p} \\
\leq \frac{\epsilon}{8}
\]

for all \( t \in \mathbb{R}, \) and \( u, v \in Q_1 \), \( \|u - v\| < \delta_0 \).

Since \( Q_1 = \{ \alpha(t) : t \in \mathbb{R} \} \) is compact, there exist open balls \( O_k \) \((k = 1, 2, \ldots, m)\) with center in \( u_k \in Q_1 \) and radius \( \delta_0 \) given above, such that \( \{ \alpha(t) : t \in \mathbb{R} \} \subset \)
Define and choose \( B_k \) such that \( B_k := \{ t \in \mathbb{R} : \| \alpha(t) - u_k \| < \delta_0 \}, \) \( k = 1, 2, \ldots, m, \mathbb{R} = \bigcup_{k=1}^m B_k \) and set \( C_1 = B_1, C_k = B_k \backslash (\bigcup_{j=1}^{k-1} B_j) \) \( (k = 2, 3, \ldots, m). \) Then \( \mathbb{R} = \bigcup_{k=1}^m C_k \) where \( C_i \cap C_j = \emptyset, i \neq j, 1 \leq i, j \leq m. \) Let us define the function \( \pi : \mathbb{R} \to X \) by \( \pi(t) = u_k \) for \( t \in C_k, k = 1, \ldots, m. \) Then \( \| \alpha(t) - \pi \| < \delta_0 \) for all \( t \in \mathbb{R} \) and
\[
(\sum_{k=1}^m \int_{C_k \cap [t,t+1]} \| \phi(s, \alpha(s)) - \phi(s, u_k) \|^p ds)^{1/p} = \left( \int_t^{t+1} \| \phi(s, \alpha(s)) - \phi(s, \pi(s)) \|^p ds \right)^{1/p} < \frac{\epsilon}{8}.
\]

Since \( \phi \in B^{-1}(\mathcal{E}(\mathbb{R} \times X, L^p(0, 1; X)), \mu), \) there exists \( T_0 > 0 \) such that
\[
\frac{1}{\mu([-T,T])} \int_{[-T,T]} \left( \int_t^{t+1} \| \phi(s, u_k) \|^p ds \right)^{1/p} d\mu(t) < \frac{\epsilon}{8m^2},
\]
for all \( T > T_0 \) and \( 1 \leq k \leq m. \) Therefore,
\[
\frac{1}{\mu([-T,T])} \int_{[-T,T]} \left( \sum_{k=1}^m \int_{C_k \cap [t,t+1]} \| \phi(s, \alpha(s)) - \phi(s, u_k) \|^p ds \right)^{1/p} d\mu(t)
\]
\[
= \frac{1}{\mu([-T,T])} \int_{[-T,T]} \left( \sum_{k=1}^m \int_{C_k \cap [t,t+1]} \| \phi(s, u_k) \|^p ds \right)^{1/p} d\mu(t)
\]
\[
\leq \frac{2^{1+\frac{1}{p}}}{\mu([-T,T])} \int_{[-T,T]} \left( \sum_{k=1}^m \int_{C_k \cap [t,t+1]} \| \phi(s, \alpha(s)) - \phi(s, u_k) \|^p ds \right)^{1/p} d\mu(t)
\]
\[
+ \frac{2^{1+\frac{1}{p}}}{\mu([-T,T])} \int_{[-T,T]} \left( \sum_{k=1}^m \int_{C_k \cap [t,t+1]} \| \phi(s, u_k) \|^p ds \right)^{1/p} d\mu(t)
\]
\[
< \frac{\epsilon}{2} + m^{1/p} \frac{\epsilon}{2m} < \epsilon.
\]

Hence \( W \in B^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X))). \) The conclusion follows.

From Theorem 3.5 we obtain the following result of [1].

**Corollary 3.6.** Let \( \mu \in \mathcal{M} \) and let \( f = g + \phi \in \text{PAP}^p(\mathbb{R} \times X, \mu) \) that satisfies a Lipschitz condition in \( x \in X \) uniformly in \( t \in \mathbb{R}, \) that is, there is a constant \( L \geq 0 \) such that \( \| f(t,x) - f(t,y) \| \leq L \| x - y \|, \) for all \( x, y \in X \) and \( t \in \mathbb{R}. \) If \( x \in \text{PAP}(\mathbb{R}, X, \mu), \) then \( f(\cdot, x(\cdot)) \in \text{PAP}^p(\mathbb{R}, X, \mu). \)

To prove the next composition theorem, we need the following lemma.

**Lemma 3.7 ([1]).** Suppose that

(a) \( f \in \text{APS}^p(\mathbb{R} \times X, X) \) with \( p > 1 \) and there exists a function \( L_f \in B^s(\mathbb{R}, \mathbb{R}) \) \( (r \geq \max\{p, p/\lambda - 1\}) \) such that
\[
\| f(t,u) - f(t,v) \| \leq L_f(t) \| u - v \| \quad t \in \mathbb{R}, u, v \in X.
\]

(b) \( x \in \text{APS}^p(\mathbb{R}, X), \) and there exist a set \( E \subset \mathbb{R} \) with \( \text{meas}(E) = 0 \) such that
\[
K = \{ x(t) : t \in \mathbb{R} \setminus E \}
\]
is compact in \( X. \)
Then there exist \( q \in [1, p) \) such that \( f(\cdot, x(\cdot)) \in \APS^q(\mathbb{R}, X) \).

The next result of composition is new.

**Theorem 3.8.** Let \( \mu \in \mathcal{M}, p > 1, f = \varphi \in \PAPS^p(\mathbb{R} \times X, X, \mu) \) with \( g \in \mathcal{B}^{-1}(\mathcal{A}(\mathbb{R} \times X, L^p(0, 1; X))) \) and \( \phi \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R} \times X, L^p(0, 1; X), \mu)) \). Assume that

(i) there exist nonnegative functions \( L_f, L_g \) in the space \( \APS^r(\mathbb{R}, \mathbb{R}) \), with \( r \geq \max\{p, p/p - 1\} \), such that

\[
\|f(t, u) - f(t, v)\| \leq L_f(t)\|u - v\|, \quad \|g(t, u) - g(t, v)\| \leq L_g(t)\|u - v\|
\]

for \( t \in \mathbb{R} \) and \( u, v \in X \).

(ii) \( h = \alpha + \beta \in \PAPS^p(\mathbb{R}, X, \mu) \) with

\[
\alpha \in \mathcal{B}^{-1}(\mathcal{A}(\mathbb{R}, L^p(0, 1; X))), \quad \beta \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu))
\]

and there exist a set \( E \subset \mathbb{R} \) with \( \text{meas}(E) = 0 \) such that the set \( K = \{\alpha(t) : t \in \mathbb{R} \setminus E\} \) is compact in \( X \).

Then there exist \( q \in [1, p) \) such that \( f(\cdot, h(\cdot)) \in \APS^q(\mathbb{R}, X, \mu) \).

**Proof.** We can decompose

\[
f(t, h(t)) = g(t, \alpha(t)) + f(t, h(t)) - f(t, \alpha(t)) + \phi(t, \alpha(t)).
\]

Set

\[
F(t) := g(t, \alpha(t)), \quad G(t) := f(t, h(t)) - f(t, \alpha(t)), \quad H(t) := \phi(t, \alpha(t)).
\]

Since \( r \geq \frac{p}{p - 1} \) then there exists \( q \in [1, p) \) such that \( r = \frac{p}{p - q} \). Let \( p' = p/p - q \) and \( q' = p/q \). Therefore \( \frac{1}{p'} + \frac{1}{q'} = 1 \). Since \( \alpha \in \APS^p(\mathbb{R}, X) \) and \( g \in \APS^p(\mathbb{R} \times X, X) \) then by assumptions and Lemma 3.7 we obtain that \( F \in \mathcal{B}^{-1}(\mathcal{A}(\mathbb{R}, L^p(0, 1; X))) \).

Next we show that \( G \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)) \). By Holder inequality we have

\[
\int_t^{t+1} \|G(\sigma)\|_q^q \, d\sigma = \int_t^{t+1} \|f(\sigma, h(\sigma)) - f(\sigma, \alpha(\sigma))\|_q^q \, d\sigma
\]

\[
\leq \int_t^{t+1} L^q_f(\sigma)\|h(\sigma) - \alpha(\sigma)\|_q^q \, d\sigma
\]

\[
= \int_t^{t+1} L^q_f(\sigma)\|\beta(\sigma)\|_q^q \, d\sigma
\]

\[
\leq \left( \int_t^{t+1} L^{q'}_f(\sigma) \, d\sigma \right)^{1/p'} \left( \int_t^{t+1} \|\beta(\sigma)\|^{q'q} \, d\sigma \right)^{1/q'}
\]

\[
= \left[ \left( \int_t^{t+1} L^q_f(\sigma) \, d\sigma \right)^{1/r} \right]^{r/p'} \left[ \left( \int_t^{t+1} \|\beta(\sigma)\|_p \, d\sigma \right)^{1/p} \right]^{p/q'}
\]

\[
\leq \|L_f\|_{BS^r} \left[ \left( \int_t^{t+1} \|\beta(\sigma)\|_p \, d\sigma \right)^{1/p} \right]^{q}.\]

Then

\[
\frac{1}{\mu([-T, T])} \int_{[-T, T]} \left( \int_t^{t+1} \|G(\sigma)\|_q^q \, d\sigma \right)^{1/q} \, d\mu(t)
\]

\[
\leq \frac{\|L_f\|_{BS^r}}{\mu([-T, T])} \int_{[-T, T]} \left( \int_t^{t+1} \|\beta(\sigma)\|_p^p \, d\sigma \right)^{1/p} \, d\mu(t).
\]
Since $\beta \in B^{-1}(E(\mathbb{R}, L^p(0,1;X),\mu))$ we obtain that $G \in B^{-1}(E(\mathbb{R}, L^q(0,1;X),\mu))$.

Next, we prove that $H \in B^{-1}(E(\mathbb{R}, L^q(0,1;X),\mu))$.

Since $\phi \in B^{-1}(E(\mathbb{R}, L^p(0,1;X),\mu))$, for each $\epsilon > 0$ there exist $T_0 > 0$ such that $T > T_0$ implies that

$$
\frac{1}{\mu([-T,T])} \int_{[-T,T]} \left( \int_t^{t+1} \|\phi(t,u)\|^p \, d\sigma \right)^{1/p} \, d\mu(t) < \epsilon \quad (u \in X).
$$

Since $K$ is compact, we can find finite open balls $O_k (k = 1,2,3,\ldots,m)$ with center $x_k$ such that $K \subset \bigcup_{k=1}^m O_k$. Thus, for all $u \in K$ there exist $x_k$ such that

$$
\|\phi(t+\sigma,u)\| \\
\leq \|\phi(t+\sigma,u) - \phi(t+\sigma,x_k)\| + \|\phi(t+\sigma,x_k)\| \\
\leq \|f(t+\sigma,u) - f(t+\sigma,x_k)\| + \|g(t+\sigma,u) - g(t+\sigma,x_k)\| + \|\phi(t+\sigma,x_k)\| \\
\leq L_f(t+\sigma)e + L_g(t+\sigma)e + \|\phi(t+\sigma,x_k)\| \quad (t \in \mathbb{R}, \sigma \in [0,1]).
$$

Hence

$$
\sup_{u \in K} \|\phi(t+\sigma,u)\| \leq L_f(t+\sigma)e + L_g(t+\sigma)e + \sum_{k=1}^m \|\phi(t+\sigma,x_k)\|.
$$

Since $r \geq p$ then $L_f, L_g \in APS^r(\mathbb{R},\mathbb{R}) \subset APS^p(\mathbb{R},\mathbb{R}) \subset BS^p(\mathbb{R},\mathbb{R})$.

By Minkowski’s inequality, we obtain

$$
\left( \int_0^1 \left( \sup_{u \in K} \|\phi(t+\sigma,u)\| \right)^p \, d\sigma \right)^{1/p} \\
\leq (\|L_f\|_{BS^p} + \|L_g\|_{BS^p}) \epsilon + \sum_{k=1}^m \left( \int_0^1 \left( \sup_{u \in K} \|\phi(t+\sigma,u)\| \right)^p \, d\sigma \right)^{1/p}.
$$

For $T > T_0$ we have

$$
\frac{1}{\mu([-T,T])} \int_{[-T,T]} \left( \int_0^1 \left( \sup_{u \in K} \|\phi(t+\sigma,u)\| \right)^p \, d\sigma \right)^{1/p} \, d\mu(t) \\
\leq (\|L_f\|_{BS^p} + \|L_g\|_{BS^p} + m) \epsilon.
$$

Hence

$$
\lim_{T \to \infty} \frac{1}{\mu([-T,T])} \int_{[-T,T]} \left( \int_0^1 \left( \sup_{u \in K} \|\phi(t+\sigma,u)\| \right)^p \, d\sigma \right)^{1/p} \, d\mu(t) = 0.
$$

On the other hand

$$
\frac{1}{\mu([-T,T])} \int_{[-T,T]} \|H^b(t)\|_{q} \, d\mu(t) \\
\leq \frac{1}{\mu([-T,T])} \int_{[-T,T]} \|H^b(t)\|_{p} \, d\mu(t) \\
= \frac{1}{\mu([-T,T])} \int_{[-T,T]} \left( \int_0^1 \|\phi(t+\sigma,\alpha(t+\sigma))\|_{p} \, d\sigma \right)^{1/p} \, d\mu(t) \\
\leq \frac{1}{\mu([-T,T])} \int_{[-T,T]} \left( \int_0^1 \left( \sup_{u \in K} \|\phi(t+\sigma,u)\| \right)^{p} \, d\sigma \right)^{1/p} \, d\mu(t) \to 0.
$$

as $T \to \infty$. Hence $H \in B^{-1}(E(\mathbb{R}, L^q(0,1;X),\mu))$. It proves that $f(\cdot, h(\cdot)) = F(\cdot) + [G(\cdot) + H(\cdot)] \in APS^q(\mathbb{R},X,\mu)$. □
We recall the following convolution theorem.

**Theorem 3.9** ([2 Theorem 3.1]). Let \( S : \mathbb{R} \to B(X) \) be strongly continuous. Suppose that there exists a function \( \phi \in L^1(\mathbb{R}) \) such that

(a) \( \|S(t)\| \leq \phi(t), \quad t \in \mathbb{R} \);
(b) \( \phi(t) \) is nonincreasing;
(c) \( \sum_{n=1}^{\infty} \phi(n) < \infty \).

If \( g \in APS_p(\mathbb{R}, X) \), then

\[
(S*g)(t) := \int_{-\infty}^{t} S(t-s)g(s) \, ds \in AP(\mathbb{R}, X). 
\]

The next result is one of the original contributions of this work.

**Theorem 3.10.** Let \( \mu \in \mathcal{M} \) be given and let \( S : \mathbb{R} \to B(X) \) be strongly continuous. Suppose that there exists a function \( \phi \in L^1(\mathbb{R}) \) such that

(a) \( \|S(t)\| \leq \phi(t), \quad t \in \mathbb{R} \);
(b) \( \phi(t) \) is nonincreasing;
(c) \( \sum_{n=1}^{\infty} \phi(n) < \infty \).

If \( f = g + h \in PAPS_p(\mathbb{R}, X, \mu) \) with \( g \in B^{-1}(AP(\mathbb{R}, L^p(0,1;X))) \) and \( h \in B^{-1}(E(\mathbb{R}, L^p(0,1;X))) \), then

\[
(S*f)(t) := \int_{-\infty}^{t} S(t-s)f(s) \, ds \in PAP(\mathbb{R}, X, \mu). 
\]

**Proof.** Since

\[
(S*g)(t) := \int_{-\infty}^{t} S(t-s)g(s) \, ds = \int_{-\infty}^{t} S(t-s)h(s) \, ds, 
\]

and, from Theorem 3.9 \((S*g) \in AP(\mathbb{R}, X)\) it remains to show that \((S*h) \in E(\mathbb{R}, X, \mu)\). Set

\[
H(t) := \int_{-\infty}^{t} S(t-s)h(s) \, ds = \int_{-\infty}^{t} S(s)h(t-s) \, ds, 
\]

and

\[
H_n(t) := \int_{t-n}^{t-n+1} S(t-\sigma)h(\sigma) \, d\sigma, \quad n = 1, 2, \ldots. 
\]

Note that \( H_n(t) \) is continuous and

\[
\|H_n(t)\| \leq \int_{t-n}^{t-n+1} \|S(t-\sigma)\| \|h(\sigma)\| \, d\sigma \\
= \int_{n-1}^{n} \|S(\sigma)\| \|h(t-\sigma)\| \, d\sigma \\
\leq \int_{n-1}^{n} \phi(\sigma) \|h(t-\sigma)\| \, d\sigma \\
\leq \phi(n-1) \left( \int_{n-1}^{n} \|h(t-\sigma)\|^p \, d\sigma \right)^{1/p}. 
\]

Hence, for \( T > 0 \),

\[
\frac{1}{\mu([-T,T])} \int_{[-T,T]} \|H_n(t)\| \, d\mu(t) 
\]
\[ \leq \phi(n - 1) \frac{1}{\mu([-T, T])} \int_{[-T,T]} \left( \int_{n-1}^{n} \|h(t - \sigma)\|^p \, d\sigma \right)^{1/p} \, d\mu(t). \]

Using the fact that the space \( \mathcal{E}(\mathbb{R}, X, \mu) \) is translation invariant, it follows that \( t \to h(t - \sigma) \) belongs to \( \mathcal{E}(\mathbb{R}, X, \mu) \). The above inequality leads to \( H_n \in \mathcal{E}(\mathbb{R}, X, \mu) \) for each \( n = 1, 2, \ldots \). The above estimate implies

\[ \|H_n(t)\| \leq \phi(n - 1)\|h\|_{BS^p(\mathbb{R}, X)}. \]

By hypothesis we have

\[ \sum_{n=1}^{\infty} \|H_n(t)\| \leq \sum_{n=1}^{\infty} \phi(n - 1)\|h\|_{BS^p(\mathbb{R}, X)} < C\|h\|_{BS^p(\mathbb{R}, X)} < \infty. \]

It follows from Weierstrass test that the series \( \sum_{n=1}^{\infty} H_n(t) \) is uniformly convergent on \( \mathbb{R} \). Moreover

\[ H(t) = \int_{-\infty}^{t} S(t - s)h(s) \, ds = \sum_{n=1}^{\infty} H_n(t). \]

Since \( H \in C(\mathbb{R}, X) \) and

\[ \|H(t)\| \leq \sum_{n=1}^{\infty} \|H_n(t)\| \leq C\|h\|_{BS^p(\mathbb{R}, X)}, \]

we have

\[ \frac{1}{\mu([-T, T])} \int_{[-T,T]} \|H(t)\| \, d\mu(t) \leq \frac{1}{\mu([-T, T])} \int_{[-T,T]} \|H(t) - \sum_{k=1}^{n} H_k(t)\| \, d\mu(t) + \sum_{k=1}^{n} \frac{1}{\mu([-T, T])} \int_{[-T,T]} \|H_k(t)\| \, d\mu(t). \]

Since \( H_k(t) \in \mathcal{E}(\mathbb{R}, X, \mu) \) and \( \sum_{k=1}^{n} H_n(t) \) converges uniformly to \( H(t) \), it follows that

\[ \lim_{T \to \infty} \frac{1}{\mu([-T, T])} \int_{[-T,T]} \|H(t)\| \, d\mu(t) = 0. \]

Hence \( H(\cdot) = \sum_{n=1}^{\infty} H_n(t) \in \mathcal{E}(\mathbb{R}, X, \mu) \). Therefore, \( (S*f)(t) = \int_{-\infty}^{t} S(t-s)f(s) \, ds \) is \( \mu \)-pseudo almost periodic.

\[ \square \]

4. An application to fractional integro-differential equations

Given a function \( g : \mathbb{R} \to X \), the \textit{Weyl fractional integral} of order \( \alpha > 0 \) is defined by

\[ D^{-\alpha}g(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1}g(s) \, ds, \quad t \in \mathbb{R}, \]

when this integral is convergent. The \textit{Weyl fractional derivative} \( D^\alpha g \) of order \( \alpha > 0 \) is defined by

\[ D^\alpha g(t) := \frac{d^n}{dt^n} D^{-(n-\alpha)}g(t), \quad t \in \mathbb{R}, \]

where \( n = [\alpha] + 1 \). It is known that \( D^\alpha D^{-\alpha}g = g \) for any \( \alpha > 0 \), and \( D^n = \frac{d^n}{dt^n} \) holds with \( n \in \mathbb{N} \).
**Definition 4.1 (H3).** Let $A$ be a closed and linear operator with domain $D(A)$ defined on a Banach space $X$, and $\alpha > 0$. Given $a \in L^1_{\text{loc}}(\mathbb{R}^+)$, we say that $A$ is the generator of an $\alpha$-resolvent family if there exist $\omega \geq 0$ and a strongly continuous family $S_\alpha : [0, \infty) \to \mathcal{B}(X)$ such that \[ \{ \frac{\lambda^\alpha}{1 + \hat{a}(\lambda)} : \text{Re } \lambda > \omega \} \subset \rho(A) \] and for all $x \in X$,

\[ (\lambda^\alpha - (1 + \hat{a}(\lambda))A)^{-1} x = \frac{1}{1 + \hat{a}(\lambda)} \left( \frac{\lambda^\alpha}{1 + \hat{a}(\lambda)} - A \right)^{-1} x = \int_0^\infty e^{-\lambda t}S_\alpha(t)x\,dt, \]

for $\text{Re } \lambda > \omega$. In this case, $\{S_\alpha(t)\}_{t \geq 0}$ is called the $\alpha$-resolvent family generated by $A$.

Next, we consider the existence and uniqueness of $\mu$-pseudo almost periodic mild solutions for the fractional integro-differential equations

\[ D^\alpha u(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)\,ds + f(t, u(t)), \quad (4.1) \]

where $A$ generates an $\alpha$-resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ on a Banach space $X$, $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ and $f \in PAPS^p(\mathbb{R} \times X, \mu)$ satisfies the Lipschitz condition.

**Definition 4.2.** A function $u : \mathbb{R} \to X$ is said to be a mild solution of (4.1) if

\[ u(t) = \int_{-\infty}^t S_\alpha(t-s)f(s,u(s))\,ds \quad (t \in \mathbb{R}) \]

where $\{S_\alpha(t)\}_{t \geq 0}$ is the $\alpha$-resolvent family generated by $A$.

**Theorem 4.3.** Let $\mu \in \mathcal{M}$, and assume (H2) holds. Let $p > 1$ and $f \in PAPS^p(\mathbb{R} \times X, X, \mu)$ be given. Suppose that

(H3) There exists $L_f \geq 0$ such that

\[ \|f(t,u) - f(t,v)\| \leq L_f\|u - v\|, \quad t \in \mathbb{R}, \; u, v \in X. \]

(H4) Operator $A$ generates an $\alpha$-resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ such that $\|S_\alpha(t)\| \leq \varphi_\alpha(t)$, for all $t \geq 0$, where $\varphi_\alpha(\cdot) \in L^1(\mathbb{R}^+)$ is nonincreasing such that $\varphi_0 := \sum_{n=0}^{\infty} \varphi_\alpha(n) < \infty$.

If $L_f < \|\varphi_\alpha\|^{-1}$, then (4.1) has a unique mild solution in $PAP(\mathbb{R}, X, \mu)$.

**Proof.** Consider the operator $Q : PAP(\mathbb{R}, X, \mu) \to PAP(\mathbb{R}, X, \mu)$ defined by

\[ (Qu)(t) := \int_{-\infty}^t S(t-s)f(s,u(s))\,ds, \quad t \in \mathbb{R}. \]

First, we show that $Q(PAP(\mathbb{R}, X, \mu)) \subset PAP(\mathbb{R}, X, \mu)$. Let $u \in PAP(\mathbb{R}, X, \mu)$. Since $f \in PAPS^p(\mathbb{R} \times X, X, \mu)$ and satisfy (H3) we have from Corollary 3.3 that $f(\cdot, u(\cdot)) \in PAPS^p(\mathbb{R}, X, \mu)$. Then, by assumption (h4) we obtain from Theorem 3.10 that $Qu \in PAP(\mathbb{R}, X, \mu)$.

Let $u, v \in PAP(\mathbb{R}, X, \mu)$. By conditions (H3) and (H4) we have

\[ \|Qu - Qv\|_{\infty} = \sup_{t \in \mathbb{R}} \| (Qu)(t) - (Qv)(t) \| \]

\[ = \sup_{t \in \mathbb{R}} \| \int_{-\infty}^t S(t-s)[f(s,u(s)) - f(s,v(s))]\,ds \|

\[ \leq L_f \sup_{t \in \mathbb{R}} \int_0^\infty \|S(s)\|\|u(t-s) - v(t-s)\|\,ds \]
\[
\begin{align*}
\leq L_f \|u - v\|_{\infty} & \int_{0}^{\infty} \varphi(t) \, ds \\
& = L_f \|\varphi\|_1 \|u - v\|_{\infty}.
\end{align*}
\]

This proves that \( Q \) is a contraction, so by the Banach Fixed Point Theorem we conclude that \( Q \) has unique fixed point. It follows that \( Qu = u \in PAP(\mathbb{R}, X, \mu) \) and it is unique. Hence \( u \) is the unique mild solution of \((4.1)\) which belongs to \( PAP(\mathbb{R}, X, \mu) \).

\textbf{Theorem 4.4.} Let \( \mu \in M \). Assume that (H2) holds. Let \( p > 1 \) and \( f = g + h \in PAPS^p(\mathbb{R} \times X, X, \mu) \) be given. Suppose that:

(H5) There exist nonnegative functions \( L_f(\cdot), L_g(\cdot) \in APS^r(\mathbb{R}, \mathbb{R}) \) with \( r \geq \max\{p, \frac{p}{p-1}\} \) such that
\[
\|f(t, u) - f(t, v)\| \leq L_f(t) \|u - v\|, \quad \|g(t, u) - g(t, v)\| \leq L_g(t) \|u - v\|,
\]
for \( t \in \mathbb{R} \) and \( u, v \in X \).

(H6) Operator \( A \) generates an \( \alpha \)-resolvent family \( \{S_\alpha(t)\}_{t \geq 0} \) such that \( \|S_\alpha(t)\| \leq Me^{-\omega t} \), for all \( t \geq 0 \) and
\[
\|L_f\|_{B^{\alpha \prime}} < \frac{1 - e^{-\omega \rho_0}}{M} \left( \frac{\omega \rho_0}{1 - e^{-\omega \rho_0}} \right)^{1/\rho_0}
\]
where \( \frac{1}{\rho} + \frac{1}{\rho_0} = 1 \).

Then \((4.1)\) has a unique mild solution in \( PAP(\mathbb{R}, X, \mu) \).

\textbf{Proof.} Let \( u = u_1 + u_2 \in PAP(\mathbb{R}, X, \mu) \) where \( u_1 \in AP(\mathbb{R}, X) \) and \( u_2 \in \mathcal{E}(\mathbb{R}, X, \mu) \). Then \( u \in PAPS^p(\mathbb{R}, X, \mu) \). Since the range of almost periodic functions is relatively compact set, then \( K = \{u_1(t) : t \in \mathbb{R}\} \) is compact in \( X \). Thus, by conditions (H5) and (H6) we have that all the hypotheses of Theorem 3.8 fulfilled, then there exists \( q \in [1, p) \) such that \( f(\cdot, u(\cdot)) \in PAPS^q(\mathbb{R}, X, \mu) \).

Consider the operator \( Q : PAP(\mathbb{R}, X, \mu) \to PAP(\mathbb{R}, X, \mu) \) such that
\[
(Qu)(t) := \int_{-\infty}^{t} S(t-s)f(s, u(s)) \, ds, \quad (t \in \mathbb{R}).
\]

Since \( f(\cdot, u(\cdot)) \in PAPS^q(\mathbb{R}, X, \mu) \) it follows from Theorem 3.10 that \( Q \) maps \( PAP(\mathbb{R}, X, \mu) \) into \( PAP(\mathbb{R}, X, \mu) \).

For any \( u, v \in PAP(\mathbb{R}, X, \mu) \) we have
\[
\|(Qu)(t) - (Qv)(t)\| \leq \int_{-\infty}^{t} \|S(t-s)\| \|f(s, u(s)) - f(s, v(s))\| \, ds
\]
\[
\leq \int_{-\infty}^{t} Me^{-\omega(t-s)} L_f(s) \|u(s) - v(s)\| \, ds
\]
\[
\leq \|u - v\| \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} Me^{-\omega(t-s)} L_f(s) \, ds
\]
\[
\leq \|u - v\| \sum_{k=1}^{\infty} \left( \int_{t-k}^{t-k+1} Me^{-\omega(t-s)} \right)^{1/\rho_0} \|L_f(s)\|_{B^{\alpha \prime}}
\]
\[
= \frac{M M^{r_0} e^{-\omega(t-s)}}{1 - e^{-\omega \rho_0}} \|u - v\| \|L_f(s)\|_{B^{\alpha \prime}}.
\]
From Banach contraction mapping principle we have that $Q$ has a unique fixed point in $PAP(\mathbb{R}, X, \mu)$ which is the unique mild solution of Equation (4.1).

**Example 4.5.** We put $A = -\varrho$ in $X = \mathbb{R}$, $a(t) = \frac{\varrho t^{\alpha-1}}{4 \Gamma(\alpha)}$, $\varrho > 0$, $0 < \alpha < 1$, and $f(t,u) = g(t,u) + h(t,u)$ where

$$g(t,u(t,x)) = [\sin t + \sin(\sqrt{2} t)] \sin(u(t,x)), \quad h(t,u(t,x)) = \phi(t) \sin(u(t,x)),$$

and $\phi(t)$ is such that $|\phi(t)e^t| \leq K$ with $K > 0$.

Consider the measure $\mu$ whose Radon-Nikodým derivative is $\rho(t) = e^t$. Then $\mu \in \mathcal{M}$ and satisfies the (H2) (see [4, Ex. 3.6]). Note that $g \in B^{-1}(AP(\mathbb{R}, L^p(0;1;X)))$ and $h \in B^{-1}(\mathcal{E}(\mathbb{R}, L^p(0;1;X), \mu))$. Hence $f \in PAPS([\mathbb{R} \times X, X, \mu]$). Furthermore,

$$|f(t,u) - f(t,v)| \leq L|u - v|,$$

where $L := \max\{2, K\}$. Therefore $f$ satisfies (C1).

Now, note that Equation (4.1) takes the form

$$D^\alpha u(t) = -gu(t) - \frac{\varrho^2}{4} \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds + f(t,u(t)), \quad t \in \mathbb{R}. \quad (4.2)$$

It follows from [13] Example 4.17 that $A$ generates an $\alpha$-resolvent family $\{S_{\alpha}(t)\}_{t \geq 0}$ such that

$$\hat{S}_{\alpha}(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + 2/\varrho)^2 (\lambda^\alpha + 2/\varrho + \lambda^{\alpha/2})} \frac{\lambda^{\alpha/2}}{(\lambda^\alpha + 2/\varrho + \lambda^{\alpha/2})^2}.$$}

Thus, we obtain explicitly

$$S_{\alpha}(t) = (r * r)(t) \quad t > 0,$$

with $r(t) = t^{\frac{\alpha}{2}-1}E_{\frac{\alpha}{2}}(-\frac{\alpha}{2}t^\alpha)$, and where $E_{\frac{\alpha}{2}}(\cdot)$ is the Mittag-Leffler function.

By properties of the Mittag-Leffler function we obtain that (H4) holds. Then, by Theorem 4.3 (4.2) has a unique mild solution $u \in PAP(\mathbb{R}, X, \mu)$ provided $\|S_{\alpha}\| < \frac{1}{2}$. Finally we note that, for $0 < \alpha < 1$, $\varrho > 0$ may be chosen so that $\|S_{\alpha}\| < \frac{1}{2}$ as in the proof of [13] Lemma 3.9.

**References**


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