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SPECTRAL PROPERTIES OF FRACTIONAL DIFFERENTIATION OPERATORS

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Abstract. We consider the fractional differentiation operators in a variety of senses. We show that the strong accretive property is the common property of fractional differentiation operators. Also we prove that the sectorial property holds for operators second order with fractional derivative in lower terms, we explore the location of spectrum and resolvent sets and show that the generalized spectrum is discrete. We prove that there is two-sided estimate for eigenvalues of real component of operators second order with fractional derivative in lower terms.

1. Introduction

The term accretive applicable to a linear operator T acting in a Hilbert space H was introduced by Friedrichs in the work [5], and means that the operator has the following property: The numeric domain of values $\Theta(T)$ (see [8, p.335]) is a subset of the right half-plane i.e.

$$\mathrm{Re}\langle Tu, u\rangle_H \geq 0, \quad u \in \mathfrak{D}(T).$$

Accepting a notation [9] we assume that Ω is convex domain of the n dimensional Euclidean space $\mathbb{E}^n$, P is a fixed point of the boundary $\partial\Omega$, $Q(r,\vec{e})$ is an arbitrary point of Ω; we denote by $\vec{e}$ as a unit vector having the direction from P to Q, denote by $r = |P - Q|$ as a Euclidean distance between points P and Q. We will consider classes of Lebesgue $L_p(\Omega)$, $1 \leq p < \infty$ complex valued functions. In polar coordinates, the summability f on Ω of degree p means that

$$\int_\Omega |f(Q)|^p dQ = \int_\omega d\chi \int_0^{d(\vec{e})} |f(Q)|^p r^{n-1} dr < \infty, \tag{1.1}$$

where $d\chi$ is the element of the solid angle the surface of a unit sphere in $\mathbb{E}^n$ and ω is a surface of this sphere, $d := d(\vec{e})$ is the length of segment of ray going from point P in the direction $\vec{e}$ within the domain Ω. Without lose of generality, we consider only those directions of $\vec{e}$ for which the inner integral on the right side of equality (1.1)

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exists and is finite, is well known that this is almost all directions. Notation $\operatorname{Lip}\mu$, $0<\mu\leq 1$ means the set of functions satisfying the Holder-Lipschitz condition

$$\operatorname{Lip}\mu := \big\{\rho(Q) : |\rho(Q)-\rho(P)|\leq Mr^{\mu},\ P,Q\in\bar{\Omega}\big\}.$$

The operator of fractional differentiation in the sense of Kipriyanov defined in [10] by formal expression

$$\mathfrak{D}^{\alpha}(Q)=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^r\frac{[f(Q)-f(P+\vec{e}t)]}{(r-t)^{\alpha+1}}\big(\frac{t}{r}big)^{n-1}dt+C_n^{(\alpha)}f(Q)r^{-\alpha},\quad P\in\partial\Omega,$$

where $C_n^{(\alpha)}=(n-1)!/\Gamma(n-\alpha)$, according to [10, Theorem 2] acting as follows

$$\mathfrak{D}^{\alpha}:\overset{\circ}{W}{}_p^l(\Omega)\to L_q(\Omega),\quad lp\leq n,\quad 0<\alpha<l-\frac{n}{p}+\frac{n}{q},\quad p\leq q<\frac{np}{n-lp}. \tag{1.2}$$

If in the condition (1.2) we have the strict inequality $q>p$, then for sufficiently small $\delta>0$ the next inequality holds

$$\|\mathfrak{D}^{\alpha}f\|_{L_q(\Omega)}\leq\frac{K}{\delta^{\nu}}\|f\|_{L_p(\Omega)}+\delta^{1-\nu}\|f\|_{L_p^l(\Omega)}, \tag{1.3}$$

where

$$\nu=\frac{n}{l}\Big(\frac{1}{p}-\frac{1}{q}\Big)+\frac{\alpha+\beta}{l}. \tag{1.4}$$

The constant K is independent of δ,f, and point $P\in\partial\Omega\beta$ is an arbitrarily small fixed positive number. Further we assume that $(0<\alpha<1)$. Using the terminology of [20], the left-side, right-side classes of functions representable by the fractional integral on the segment we will denote respectively by $I_{a+}^{\alpha}(L_p(a,b))$, $I_{b-}^{\alpha}(L_p(a,b))$, $1\leq p\leq\infty$. Denote $\operatorname{diam}\Omega=\mathfrak{d}$; C,C_i are constants for $i\in\mathbb{N}_0$. We use for inner product of points $P=(P_1,P_2,\dots,P_n)$ and $Q=(Q_1,Q_2,\dots,Q_n)$ which belong to $\mathbb{E}^n$ a contracted notations $P\cdot Q=P^iQ_i=\sum_{i=1}^n P_iQ_i$. As usually D_iu denotes the generalized derivative of function u with respect to coordinate variable with index $1\leq i\leq n$. We will assume that all functions has a zero extension outside of $\bar{\Omega}$. Symbols: $\mathrm{D}(L),\mathrm{R}(L)$ denote respectively the domain of definition, and range of values of operator L. Everywhere, if not stated otherwise we will use the notations of [9], [10], [20]. Let us define the operators by the following integral constructions

$$(\mathfrak{I}_{0+}^{\alpha}g)(Q):=\frac{1}{\Gamma(\alpha)}\int_0^r\frac{g(P+t\vec{e})}{(r-t)^{1-\alpha}}\big(\frac{t}{r}\big)^{n-1}dt,(\mathfrak{I}_{d-}^{\alpha}g)(Q):=\frac{1}{\Gamma(\alpha)}\int_r^d\frac{g(P+t\vec{e})}{(t-r)^{1-\alpha}}dt,$$
$$g\in L_p(\Omega),\ 1\leq p\leq\infty.$$

In this way was defined operators we will call respectively the left-sided, right-sided operator of fractional integration in the direction. We introduce the classes of functions representable by the fractional integral in the direction of $\vec{e}$

$$\mathfrak{I}_{0+}^{\alpha}(L_p):=\{u:\ u(Q)=(\mathfrak{I}_{0+}^{\alpha}g)(Q),\ g\in L_p(\Omega),\ 1\leq p\leq\infty\}, \tag{1.5}$$
$$\mathfrak{I}_{d-}^{\alpha}(L_p)=\{u:\ u(Q)=(\mathfrak{I}_{d-}^{\alpha}g)(Q),\ g\in L_p(\Omega),\ 1\leq p\leq\infty\}. \tag{1.6}$$

We define the families of operators $\psi_{\varepsilon}^{+},\ \psi_{\varepsilon}^{-},\ \varepsilon>0$ as follows: $\mathrm{D}(\psi_{\varepsilon}^{+}),\mathrm{D}(\psi_{\varepsilon}^{-})\subset L_p(\Omega)$. In the left-side case

$$(\psi_{\varepsilon}^{+}f)(Q)=\begin{cases}\int_0^{r-\varepsilon}\frac{f(P+\vec{e}r)r^{n-1}-f(P+\vec{e}t)t^{n-1}}{(r-t)^{\alpha+1}r^{n-1}}dt, & \varepsilon\leq r\leq d,\\ \frac{f(Q)}{\alpha}\big(\frac{1}{\varepsilon^{\alpha}}-\frac{1}{r^{\alpha}}\big), & 0\leq r<\varepsilon.\end{cases} \tag{1.7}$$

In the right-side case

$$(\psi_\varepsilon^- f)(Q) = \begin{cases} \int_{r+\varepsilon}^{d} \frac{f(P+\vec{e}r)-f(P+\vec{e}t)}{(t-r)^{\alpha+1}} dt, & 0 \le r \le d-\varepsilon, \\ \frac{f(Q)}{\alpha}\big(\frac{1}{\varepsilon^\alpha} - \frac{1}{(d-r)^\alpha}\big), & d-\varepsilon < r \le d. \end{cases}$$

Following [20, p.181] we define a truncated fractional derivative similarly the derivative in the sense of Marchaud, in the left-side case

$$(\mathfrak{D}_{0+,\varepsilon}^{\alpha} f)(Q) = \frac{1}{\Gamma(1-\alpha)} f(Q) r^{-\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} (\psi_\varepsilon^+ f)(Q), \tag{1.8}$$

in the right-side case

$$(\mathfrak{D}_{d-,\varepsilon}^{\alpha} f)(Q) = \frac{1}{\Gamma(1-\alpha)} f(Q)(d-r)^{-\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} (\psi_\varepsilon^- f)(Q).$$

Left-side and right-side fractional derivatives accordingly will be understood as a limits in the sense of norm $L_p(\Omega)$, $1 \le p < \infty$ of truncated fractional derivatives

$$\mathfrak{D}_{0+}^{\alpha} f = \lim_{\substack{\varepsilon\to 0 \\ (L_p)}} \mathfrak{D}_{0+,\varepsilon}^{\alpha} f, \; \mathfrak{D}_{d-}^{\alpha} f = \lim_{\substack{\varepsilon\to 0 \\ (L_p)}} \mathfrak{D}_{d-,\varepsilon}^{\alpha} f.$$

We need several auxiliary propositions, which we will present in the next section.

2. Results

We have the following theorem on the boundedness of operators fractional integration in a direction.

Theorem 2.1. *Operators of fractional integration in the direction are bounded in $L_p(\Omega)$, $1 \le p < \infty$, the following estimates holds*

$$\|\mathfrak{I}_{0+}^{\alpha} u\|_{L_p(\Omega)} \le C\|u\|_{L_p(\Omega)}, \; \|\mathfrak{I}_{d-}^{\alpha} u\|_{L_p(\Omega)} \le C\|u\|_{L_p(\Omega)}, \; C = \mathfrak{d}^{\alpha}/\Gamma(\alpha+1). \tag{2.1}$$

Proof. Let us prove the first estimate of (2.1), the proof of the second estimate is analogous. Using the generalized Minkowski inequality, we have

$$\begin{aligned} \|\mathfrak{I}_{0+}^{\alpha} u\|_{L_p(\Omega)} &= \frac{1}{\Gamma(\alpha)} \Big(\int_\Omega \Big| \int_0^r \frac{g(P+t\vec{e})}{(r-t)^{1-\alpha}} \big(\frac{t}{r}\big)^{n-1} dt \Big|^p dQ \Big)^{1/p} \\ &= \frac{1}{\Gamma(\alpha)} \Big(\int_\Omega \Big| \int_0^r \frac{g(Q-\tau\vec{e})}{\tau^{1-\alpha}} \big(\frac{r-\tau}{r}\big)^{n-1} d\tau \Big|^p dQ \Big)^{1/p} \\ &\le \frac{1}{\Gamma(\alpha)} \Big(\int_\Omega \Big(\int_0^{\mathfrak{d}} \frac{|g(Q-\tau\vec{e})|}{\tau^{1-\alpha}} d\tau \Big)^p dQ \Big)^{1/p} \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{d}} \tau^{\alpha-1} d\tau \Big(\int_\Omega |g(Q-\tau\vec{e})|^p dQ \Big)^{1/p} \\ &\le \frac{\mathfrak{d}^{\alpha}}{\Gamma(\alpha+1)} \|u\|_{L_p(\Omega)}. \end{aligned}$$

□

Theorem 2.2. *Assume $f \in L_p(\Omega)$ and exists $\lim_{\varepsilon\to 0} \psi_\varepsilon^+ f$ or $\lim_{\varepsilon\to 0} \psi_\varepsilon^- f$ in the sense of norm $L_p(\Omega)$, $1 \le p < \infty$. Then respectively $f \in \mathfrak{I}_{0+}^{\alpha}(L_p)$ or $f \in \mathfrak{I}_{d-}^{\alpha}(L_p)$.*

Proof. Let $f \in L_p(\Omega)$ and $\lim_{\substack{\varepsilon\to 0\\(L_p)}} \psi^+_\varepsilon f = \psi$. Consider the function

$$(\varphi^+_\varepsilon f)(Q) = \frac{1}{\Gamma(1-\alpha)}\big\{\frac{f(Q)}{r^\alpha} + \alpha(\psi^+_\varepsilon f)(Q)\big\}.$$

Note (1.7), we can easily see that $\varphi^+_\varepsilon f \in L_p(\Omega)$. From fundamental property family of functions $\{\varphi^+_\varepsilon f\}$ follows, that there is exists a limit $\varphi^+_\varepsilon f \to \varphi \in L_p(\Omega)$. In consequence of proved in theorem 2.1 continuous property of operator $\mathfrak{I}^\alpha_{0+}$ in the space $L_p(\Omega)$, for completing this theorem sufficient to show that $\lim_{\varepsilon\to 0,\,(L_p)} \mathfrak{I}^\alpha_{0+}\varphi^+_\varepsilon f = f$. For $\varepsilon \le r \le d$, we have

$$\begin{aligned}
&(\mathfrak{I}^\alpha_{0+}\varphi^+_\varepsilon f)(Q)\frac{\pi r^{n-1}}{\sin\alpha\pi}\\
&= \int_\varepsilon^r \frac{f(P+y\vec{e})y^{n-1-\alpha}}{(r-y)^{1-\alpha}}dy\\
&\quad + \alpha\int_\varepsilon^r (r-y)^{\alpha-1}dy\int_0^{y-\varepsilon}\frac{f(P+y\vec{e})y^{n-1} - f(P+t\vec{e})t^{n-1}}{(y-t)^{\alpha+1}}dt\\
&\quad + \frac{1}{\varepsilon^\alpha}\int_0^\varepsilon f(P+y\vec{e})(r-y)^{\alpha-1}y^{n-1}dy = I.
\end{aligned}$$

After conversion in the second summand, we have

$$\begin{aligned}
I = \frac{1}{\varepsilon^\alpha}&\int_0^r f(P+y\vec{e})(r-y)^{\alpha-1}y^{n-1}dy\\
&- \alpha\int_\varepsilon^r (r-y)^{\alpha-1}dy\int_0^{y-\varepsilon}\frac{f(P+t\vec{e})}{(y-t)^{\alpha+1}}t^{n-1}dt.
\end{aligned}\tag{2.2}$$

Making a change variable in the second integral, changing the order of integration and going back to the previous variable, we obtain

$$\begin{aligned}
&\alpha\int_\varepsilon^r (r-y)^{\alpha-1}dy\int_0^{y-\varepsilon}\frac{f(P+t\vec{e})}{(y-t)^{\alpha+1}}t^{n-1}dt\\
&= \alpha\int_0^{r-\varepsilon}(r-y-\varepsilon)^{\alpha-1}dy\int_0^y\frac{f(P+t\vec{e})}{(y+\varepsilon-t)^{\alpha+1}}t^{n-1}dt\\
&= \alpha\int_0^{r-\varepsilon} f(P+t\vec{e})t^{n-1}dt\int_t^{r-\varepsilon}\frac{(r-y-\varepsilon)^{\alpha-1}}{(y+\varepsilon-t)^{\alpha+1}}dy\\
&= \alpha\int_0^{r-\varepsilon} f(P+t\vec{e})t^{n-1}dt\int_{t+\varepsilon}^r (r-y)^{\alpha-1}(y-t)^{-\alpha-1}dy.
\end{aligned}\tag{2.3}$$

Applying [20, (13.18) p.184] we have

$$\int_{t+\varepsilon}^r (r-y)^{\alpha-1}(y-t)^{-\alpha-1}dy = \frac{1}{\alpha\varepsilon^\alpha}\frac{(r-t-\varepsilon)^\alpha}{r-t}.\tag{2.4}$$

Rewrite (2.2) taking into account the relations (2.3), (2.4), after that make change a variable $t = r - \varepsilon\tau$, we obtain

$$\begin{aligned}
&(\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon} f)(Q)\frac{\pi r^{n-1}}{\sin\alpha\pi} \\
&= \frac{1}{\varepsilon^{\alpha}}\Big\{\int_0^r f(P+y\vec{e})(r-y)^{\alpha-1}y^{n-1}dy \\
&\quad - \int_0^{r-\varepsilon}\frac{f(P+t\vec{e})(r-t-\varepsilon)^{\alpha}}{r-t}t^{n-1}dt\Big\} \\
&= \frac{1}{\varepsilon^{\alpha}}\int_0^r \frac{f(P+t\vec{e})\left[(r-t)^{\alpha}-(r-t-\varepsilon)^{\alpha}_{+}\right]}{r-t}t^{n-1}dt \\
&= \int_0^{r/\varepsilon}\frac{\tau^{\alpha}-(\tau-1)^{\alpha}_{+}}{\tau}f(P+[r-\varepsilon\tau]\vec{e})(r-\varepsilon\tau)^{n-1}d\tau,
\end{aligned} \tag{2.5}$$

where $\tau_+ = \begin{cases}\tau, & \tau \geq 0;\\ 0, & \tau < 0.\end{cases}$

Consider the auxiliary function $\mathcal{K}$ defined in [20, p.105] and having the next properties

$$\mathcal{K}(t) = \frac{\sin\alpha\pi}{\pi}\frac{t^{\alpha}_{+}-(t-1)^{\alpha}_{+}}{t} \in L_p(\mathbb{R}^1), \quad \int_0^{\infty}\mathcal{K}(t)dt = 1, \quad \mathcal{K}(t) > 0. \tag{2.6}$$

From (2.5), (2.6), since f has a zero extension outside of $\bar{\Omega}$, we have

$$(\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon} f)(Q) - f(Q) = \int_0^{\infty}\mathcal{K}(t)\{f(P+[r-\varepsilon t]\vec{e})(1-\varepsilon t/r)^{n-1}_{+} - f(P+r\vec{e})\}dt. \tag{2.7}$$

If $0 \leq r < \varepsilon$, then in accordance with (1.7) after the changing a variable, we obtain

$$\begin{aligned}
&(\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon} f)(Q) - f(Q) \\
&= \frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}}\int_0^r \frac{f(P+t\vec{e})}{(r-t)^{1-\alpha}}\big(\frac{t}{r}big)^{n-1}dt - f(Q) \\
&= \frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}}\int_0^r \frac{f(P+[r-t]\vec{e})}{t^{1-\alpha}}\big(\frac{r-t}{r}\big)^{n-1}dt - f(Q).
\end{aligned} \tag{2.8}$$

Consider the domains

$$\Omega_{\varepsilon} := \{Q \in \Omega : d(\vec{e}) \geq \varepsilon\}, \quad \Omega_{-\varepsilon} = \Omega \setminus \Omega_{\varepsilon}. \tag{2.9}$$

Accordingly with this definition we can divide the surface ω into two parts ω' and ω'', where ω' is the subset of ω for which $d(\vec{e}) \geq \varepsilon$, ω'' is the subset of ω for which $d(\vec{e}) < \varepsilon$. Taking into account (2.7), (2.8), we obtain

$$\begin{aligned}
&\|(\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon} f) - f\|^p_{L_p(\Omega)} \\
&= \int_{\omega'}d\chi\int_{\varepsilon}^{d}\Big|\int_0^{\infty}\mathcal{K}(t)[f(Q-\varepsilon t\vec{e})(1-\varepsilon t/r)^{n-1}_{+} - f(Q)]dt\Big|^p r^{n-1}dr \\
&\quad + \int_{\omega'}d\chi\int_0^{\varepsilon}\Big|\frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}}\int_0^r \frac{f(P+[r-t]\vec{e})}{t^{1-\alpha}}\big(\frac{r-t}{r}\big)^{n-1}dt - f(Q)\Big|^p r^{n-1}dr \\
&\quad + \int_{\omega''}d\chi\int_0^{d}\Big|\frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}}\int_0^r \frac{f(P+[r-t]\vec{e})}{t^{1-\alpha}}\big(\frac{r-t}{r}\big)^{n-1}dt - f(Q)\Big|^p r^{n-1}dr \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{2.10}$$

Consider I_1; using the generalized Minkovski's inequality we obtain

$$I_1^{1/p} \le \int_0^\infty \mathcal{K}(t)\Big(\int_{\omega'} d\chi \int_\varepsilon^d |f(Q-\varepsilon t\vec{e})(1-\varepsilon t/r)_+^{n-1} - f(Q)|^p r^{n-1} dr\Big)^{1/p} dt.$$

Let us introduce the notation

$$\mathcal{K}(t)\Big(\int_{\omega'} d\chi \int_\varepsilon^d |f(Q-\varepsilon t\vec{e})(1-\varepsilon t/r)_+^{n-1} - f(Q)|^p r^{n-1} dr\Big)^{1/p} dt = h(\varepsilon, t),$$

we have the inequality

$$|h(\varepsilon,t)| \le 2\mathcal{K}(t)\|f\|_{L_p(\Omega)}, \quad \forall \varepsilon > 0. \tag{2.11}$$

Note that

$$\begin{aligned}
|h(\varepsilon,t)| &\le \Big(\int_{\omega'} d\chi \int_\varepsilon^d \big|(1-\varepsilon t/r)_+^{n-1}[f(Q-\varepsilon t\vec{e}) - f(Q)]\big|^p r^{n-1} dr\Big)^{1/p} dt \\
&\quad + \Big(\int_{\omega'} d\chi \int_0^d |f(Q)[1-(1-\varepsilon t/r)_+^{n-1}]|^p r^{n-1} dr\Big)^{1/p} dt \\
&= I_{11} + I_{12}.
\end{aligned}$$

In consequence of property continuity on average in space $L_p(\Omega)$, for all fixed $0 < t < \infty$, we have $I_{11} \to 0$, $\varepsilon \to 0$. Consider I_{12}, it is obvious that for all fixed $0 < t < \infty$, almost everywhere in Ω the following relations holds

$$h_1(\varepsilon,t,r) = \big|f(Q)[1-(1-\varepsilon t/r)_+^{n-1}]\big| \le |f(Q)|, \; h_1(\varepsilon,t,r) \to 0, \quad \varepsilon \to 0.$$

Applying the majorant theorem of Lebesgue we obtain $I_{12} \to 0$, as $\varepsilon \to 0$. It implies that for all fixed $0 < t < \infty$, we have

$$\lim_{\varepsilon \to 0} h(\varepsilon, t) = 0. \tag{2.12}$$

Note (2.11), (2.12), again by applying majorant theorem of Lebesgue, we obtain

$$I_1 \to 0, \quad \text{as } \varepsilon \to 0.$$

Using Mincovski's inequality we can estimate I_2,

$$\begin{aligned}
I_2^{1/p} &\le \frac{\sin\alpha\pi}{\pi\varepsilon^\alpha}\Big(\int_{\omega'} d\chi \int_0^\varepsilon \Big|\int_0^r \frac{f(Q-t\vec{e})}{t^{1-\alpha}}\big(\frac{r-t}{r}\big)^{n-1} dt\Big|^p r^{n-1} dr\Big)^{1/p} \\
&\quad + \Big(\int_{\omega'} d\chi \int_0^\varepsilon |f(Q)|^p r^{n-1} dr\Big)^{1/p} = I_{21} + I_{22}.
\end{aligned}$$

Applying the generalized Mincovski's inequality, we obtain

$$\begin{aligned}
&I_{21}\frac{\pi}{\sin\alpha\pi} \\
&= \frac{1}{\varepsilon^\alpha}\Big(\int_{\omega'} d\chi \int_0^\varepsilon \Big|\int_0^r \frac{f(Q-t\vec{e})}{t^{1-\alpha}}\big(\frac{r-t}{r}\big)^{n-1} dt\Big|^p r^{n-1} dr\Big)^{1/p} \\
&\le \frac{1}{\varepsilon^\alpha}\Big\{\int_{\omega'}\Big[\int_0^\varepsilon t^{\alpha-1}\Big(\int_t^\varepsilon |f(Q-t\vec{e})|^p \big(\frac{r-t}{r}\big)^{(p-1)(n-1)}(r-t)^{n-1} dr\Big)^{1/p} dt\Big]^p d\chi\Big\}^{1/p} \\
&\le \frac{1}{\varepsilon^\alpha}\Big\{\int_{\omega'}\Big[\int_0^\varepsilon t^{\alpha-1}\Big(\int_t^\varepsilon |f(P+[r-t]\vec{e})|^p (r-t)^{n-1} dr\Big)^{1/p} dt\Big]^p d\chi\Big\}^{1/p} \\
&\le \frac{1}{\varepsilon^\alpha}\Big\{\int_{\omega'}\Big[\int_0^\varepsilon t^{\alpha-1}\Big(\int_0^\varepsilon |f(P+r\vec{e})|^p r^{n-1} dr\Big)^{1/p} dt\Big]^p d\chi\Big\}^{1/p} \\
&= \frac{1}{\alpha}\|f\|_{L_p(\Delta_\varepsilon)},
\end{aligned}$$

where $\Delta_\varepsilon := \{Q \in \Omega_\varepsilon,\ r < \varepsilon\}$. Note that $\operatorname{meas}\Delta_\varepsilon \to 0$, $\varepsilon \to 0$, hence $I_{21}, I_{22} \to 0$. It implies that $I_2 \to 0$. Applying analogous reasoning we can get that $I_3 \to 0$, $\varepsilon \to 0$. According to the note given above we came to conclusion that $\mathfrak{I}^\alpha_{0+}\varphi^+_\varepsilon f \to f$ in L_p. From the remark at the beginning of this proof, we complete the proof corresponding to the left-side case. The proof for the right-side case is analogous. We have to show that $\lim_{\varepsilon\to 0}\mathfrak{I}^\alpha_{d-}\varphi^-_\varepsilon f = f$ in the sense of L_p-norm, for this purpose we must repeating the previous reasoning with minor technical differences. □

Theorem 2.3. *Let $f = \mathfrak{I}^\alpha_{0+}\psi$ or $f = \mathfrak{I}^\alpha_{d-}\psi$, $\psi \in L_p(\Omega)$, $1 \le p < \infty$. Then, respectively, $\mathfrak{D}^\alpha_{0+}f = \psi$ or $\mathfrak{D}^\alpha_{d-}f = \psi$, in the sense of norm L_p.*

Proof. Consider the difference

$$\begin{aligned}
&r^{n-1}f(Q) - (r-\tau)^{n-1}f(Q-\tau\vec{e})\\
&= \frac{1}{\Gamma(\alpha)}\int_0^r \frac{\psi(Q-t\vec{e})}{t^{1-\alpha}}(r-t)^{n-1}dt - \frac{1}{\Gamma(\alpha)}\int_\tau^r \frac{\psi(Q-t\vec{e})}{(t-\tau)^{1-\alpha}}(r-t)^{n-1}dt\\
&= \tau^{\alpha-1}\int_0^r \psi(Q-t\vec{e})k(\frac{t}{\tau})(r-t)^{n-1}dt,\ k(t)\\
&= \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}, & 0<t<1;\\ t^{\alpha-1}-(t-1)^{\alpha-1}, & t>1.\end{cases}
\end{aligned}$$

Hence with the assumptions $\varepsilon \le r \le d$, we have

$$\begin{aligned}
(\psi^+_\varepsilon f)(Q) &= \int_\varepsilon^r \frac{r^{n-1}f(Q)-(r-\tau)^{n-1}f(Q-\tau\vec{e})}{r^{n-1}\tau^{\alpha+1}}d\tau\\
&= \int_\varepsilon^r \tau^{-2}d\tau\int_0^r \psi(Q-t\vec{e})k(\frac{t}{\tau})(1-t/r)^{n-1}dt\\
&= \int_0^r \psi(Q-t\vec{e})(1-t/r)^{n-1}dt\int_\varepsilon^r k(\frac{t}{\tau})\tau^{-2}d\tau\\
&= \int_0^r \psi(Q-t\vec{e})(1-t/r)^{n-1}t^{-1}dt\int_{t/r}^{t/\varepsilon} k(s)ds.
\end{aligned}$$

Applying [20, (6.12) p.106], we obtain

$$(\psi^+_\varepsilon f)(Q)\cdot\frac{\alpha}{\Gamma(1-\alpha)} = \int_0^r \psi(Q-t\vec{e})(1-t/r)^{n-1}\big[\frac{1}{\varepsilon}\mathcal{K}(\frac{t}{\varepsilon}) - \frac{1}{r}\mathcal{K}(\frac{t}{r})\big]dt,$$

since in accordance with (2.6) we have

$$\mathcal{K}(\frac{t}{r}) = [\Gamma(1-\alpha)\Gamma(\alpha)]^{-1}(\frac{t}{r})^{\alpha-1},$$

it follows that

$$(\psi^+_\varepsilon f)(Q)\cdot\frac{\alpha}{\Gamma(1-\alpha)} = \int_0^{r/\varepsilon}\mathcal{K}(t)\psi(Q-\varepsilon t\vec{e})(1-\varepsilon t/r)^{n-1}dt - \frac{f(Q)}{\Gamma(1-\alpha)r^\alpha}.$$

Since the function $\psi(Q)$ extended by zero outside of $\bar{\Omega}$, then if we note (1.8),(2.6), we obtain

$$(\mathfrak{D}^\alpha_{0+,\varepsilon}f)(Q) - \psi(Q) = \int_0^\infty \mathcal{K}(t)[\psi(Q-\varepsilon t\vec{e})(1-\varepsilon t/r)^{n-1}_+ - \psi(Q)]dt,$$

for $\varepsilon \le r \le d$. For values r such that $0 \le r < \varepsilon$ by (1.7), we have

$$(\mathfrak{D}^{\alpha}_{0+,\varepsilon} f)(Q) - \psi(Q) = \frac{f(Q)}{\varepsilon^{\alpha}\Gamma(1-\alpha)} - \psi(Q).$$

Using generalized Mincovski's inequality, we can get the estimate

$$\begin{aligned}\|(\mathfrak{D}^{\alpha}_{0+,\varepsilon} f)(Q) - \psi(Q)\|_{L_p(\Omega)} &\le \int_0^{\infty} \mathcal{K}(t)\|\psi(Q-\varepsilon t\vec{e})(1-\varepsilon t/r)_+^{n-1} - \psi(Q)\|_{L_p(\Omega)}dt \\ &\quad + \frac{1}{\Gamma(1-\alpha)\varepsilon^{\alpha}}\|f\|_{L_p(\Delta'_\varepsilon)} + \|\psi\|_{L_p(\Delta'_\varepsilon)},\end{aligned}$$

where $\Delta'_\varepsilon = \Delta_\varepsilon \cup \Omega_{-\varepsilon}$. Note that as it shown for the right side of (2.10), we can conclude that all tree summands of the right side of last inequality tends to zero as $\varepsilon \to 0$. □

Theorem 2.4. *Let $\rho \in \operatorname{Lip}\lambda$, $\alpha < \lambda \le 1$, $f \in H^1_0(\Omega)$, then $\rho f \in \mathfrak{I}^{\alpha}_{0+}(L_2) \cap \mathfrak{I}^{\alpha}_{d-}(L_2)$.*

Proof. We provide a proof only for the left-side case, because the proof corresponding to the right-side case is analogous. Suppose that all functions have a zero extension outside of $\bar{\Omega}$. At first assume $f \in C_0^{\infty}(\Omega)$ and in terms of notations (2.9) let us consider domains $\Omega' = \Omega_{\varepsilon_1}$, $\Omega'' = \Omega_{-\varepsilon_1}$, we have

$$\|\psi^+_{\varepsilon_1} f - \psi^+_{\varepsilon_2} f\|_{L_2(\Omega)} \le \|\psi^+_{\varepsilon_1} f - \psi^+_{\varepsilon_2} f\|_{L_2(\Omega')} + \|\psi^+_{\varepsilon_1} f - \psi^+_{\varepsilon_2} f\|_{L_2(\Omega'')}. \tag{2.13}$$

Denote: $\rho(P+\vec{e}t)t^{n-1} = \sigma(P+\vec{e}t)$ and consider

$$\begin{aligned}&\|\psi^+_{\varepsilon_1} f - \psi^+_{\varepsilon_2} f\|_{L_2(\Omega')} \\ &\le \Big(\int_{\omega'} d\chi \int_{\varepsilon_1}^{d} \Big|\int_{r-\varepsilon_1}^{r-\varepsilon_2} \frac{(\sigma f)(Q) - (\sigma f)(P+\vec{e}t)}{r^{n-1}(r-t)^{\alpha+1}} dt\Big|^2 r^{n-1}dr\Big)^{1/2} \\ &\quad + \Big(\int_{\omega'} d\chi \int_{\varepsilon_2}^{\varepsilon_1} \Big|\int_0^{r-\varepsilon_1} \frac{(\sigma f)(Q)}{r^{n-1}(r-t)^{\alpha+1}} dt \\ &\quad - \int_0^{r-\varepsilon_2} \frac{(\sigma f)(Q) - (\sigma f)(P+\vec{e}t)}{r^{n-1}(r-t)^{\alpha+1}} dt\Big|^2 r^{n-1}dr\Big)^{1/2} \\ &\quad + \Big(\int_{\omega'} d\chi \int_0^{\varepsilon_2} \Big|\int_0^{r-\varepsilon_1} \frac{(\sigma f)(Q)}{r^{n-1}(r-t)^{\alpha+1}} dt \\ &\quad - \int_0^{r-\varepsilon_2} \frac{(\sigma f)(Q)}{r^{n-1}(r-t)^{\alpha+1}} dt\Big|^2 r^{n-1}drBig)^{1/2} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Note since $f \in C_0^{\infty}(\Omega)$, then for sufficient small $\varepsilon_1 : f(Q) = 0$, $r < \varepsilon_1$. This implies that $I_2 + I_3 = 0$ and also implies that second summand of (2.13) equals zero. Making the change of variable in I_1, we have

$$I_1 = \Big(\int_{\omega'} d\chi \int_{\varepsilon_1}^{d} \Big|\int_{\varepsilon_1}^{\varepsilon_2} \frac{(\sigma f)(Q) - (\sigma f)(Q-\vec{e}t)}{r^{n-1}t^{\alpha+1}} dt\Big|^2 r^{n-1}dr\Big)^{1/2}.$$

Using the generalized Minkowski's inequality we obtain

$$\begin{aligned}I_1 &\le \int_{\varepsilon_2}^{\varepsilon_1} t^{-\alpha-1}\Big(\int_{\omega'} d\chi \int_{\varepsilon_1}^{d} \Big|(\rho f)(Q) - (1-t/r)^{n-1}(\rho f)(Q-\vec{e}t)\Big|^2 r^{n-1}dr\Big)^{1/2} dt \\ &\le \int_{\varepsilon_2}^{\varepsilon_1} t^{-\alpha-1}\Big(\int_{\omega'} d\chi \int_{\varepsilon_1}^{d} \Big|(\rho f)(Q) - (\rho f)(Q-\vec{e}t)\Big|^2 r^{n-1}dr\Big)^{1/2} dt\end{aligned}$$

$$
\begin{aligned}
&+\int_{\varepsilon_2}^{\varepsilon_1} t^{-\alpha-1}\Big(\int_{\omega'} d\chi \int_{\varepsilon_1}^{d}\big[1-(1-t/r)^{n-1}\big]\,|(\rho f)(Q-\vec{e}t)|^2 r^{n-1}dr\Big)^{1/2}dt\\
&\le C_1\int_{\varepsilon_2}^{\varepsilon_1} t^{\lambda-\alpha-1}dt\\
&\quad+\int_{\varepsilon_2}^{\varepsilon_1} t^{-\alpha}\Big(\int_{\omega'} d\chi \int_{\varepsilon_1}^{d}\Big|\frac{1}{r}\sum_{i=0}^{n-2}\big(\frac{t}{r}\big)^{i}(\rho f)(Q-\vec{e}t)\Big|^2 r^{n-1}dr\Big)^{1/2}dt.
\end{aligned}
$$

In consequence of the boundendedness property of function f, exists constant δ such that $f(Q-\vec{e}t)=0,\ r<\delta$. Finally we have the estimate

$$
I_1 \le \frac{C_1}{\lambda-\alpha}(\varepsilon_1^{\lambda-\alpha}-\varepsilon_2^{\lambda-\alpha}) + \|f\|_{L_2(\Omega)}\frac{(n-1)}{\delta(1-\alpha)}(\varepsilon_1^{1-\alpha}-\varepsilon_2^{1-\alpha}).
$$

By theorem 2.1 we have the inclusion $\rho f\in \mathfrak{I}_{0+}^{\alpha}(L_2),\ f\in C_0^{\infty}(\Omega)$.

Let $f\in H_0^1(\Omega)$, then there exists a sequence $\{f_n\}\subset C_0^{\infty}(\Omega),\ \rho f_n \xrightarrow{L_2} \rho f$. According to the part proved above, we have $\rho f_n = \mathfrak{I}_{0+}^{\alpha}\varphi_n,\ \{\varphi_n\}\in L_2(\Omega)$, therefore,

$$
\mathfrak{I}_{0+}^{\alpha}\varphi_n \xrightarrow{L_2} \rho f. \tag{2.14}
$$

We will show that exists $\varphi\in L_2(\Omega)$ such that $\varphi_n \xrightarrow{L_2} \varphi$. Note, that by theorem 2.2 we have $\mathfrak{D}_{0+}^{\alpha}\rho f_n=\varphi_n$. Thus introducing the notation $f_{n+m}-f_n=c_{n,m}$, we obtain

$$
\begin{aligned}
\|\varphi_{n+m}-\varphi_n\|_{L_2(\Omega)} &\le \frac{\alpha}{\Gamma(1-\alpha)}\Big(\int_{\Omega}\Big|\int_0^r \frac{(\sigma c_{n,m})(Q)-(\sigma c_{n,m})(P+\vec{e}t)}{r^{n-1}(t-r)^{\alpha+1}}dt\Big|^2 dQ\Big)^{1/2}\\
&\quad+\frac{1}{\Gamma(1-\alpha)}\Big(\int_{\Omega}\Big|\frac{(\rho c_{n,m})(Q)}{r^{\alpha}}\Big|^2 dQ\Big)^{1/2} = I_1+I_2.
\end{aligned}
$$

Let us estimate I_1,

$$
\begin{aligned}
\frac{\Gamma(1-\alpha)}{\alpha}I_1 &\le \Big\{\int_{\Omega}\Big|\int_0^r \frac{(\rho c_{n,m})(Q)-(\rho c_{n,m})(Q-\vec{e}t)}{t^{\alpha+1}}dt\Big|^2 dQ\Big\}^{1/2}\\
&\quad+\Big\{\int_{\Omega}\Big|\int_0^r \frac{(\rho c_{n,m})(Q-\vec{e}t)[1-(1-t/r)^{n-1}]}{t^{1+\alpha}}dt\Big|^2 dQ\Big\}^{1/2}\\
&= I_{01}+I_{02}.
\end{aligned}
$$

Now we consider I_{01}. It is obvious that

$$
\begin{aligned}
I_{01} &\le \sup_{Q\in\Omega}|\rho(Q)|\Big\{\int_{\Omega}\Big(\int_0^r \frac{|c_{n,m}(Q)-c_{n,m}(Q-\vec{e}t)|}{t^{\alpha+1}}dt\Big)^2 dQ\Big\}^{1/2}\\
&\quad+\Big\{\int_{\Omega}\Big|\int_0^r \frac{c_{n,m}(Q-\vec{e}t)[\rho(Q)-\rho(Q-\vec{e}t)]}{t^{\alpha+1}}dt\Big|^2 dQ\Big\}^{1/2}\\
&= I_{11}+I_{21}.
\end{aligned}
$$

Applying the generalized Minkowski's inequality, and representing the function under the integral by the derivative in the direction of $\vec{e}$, we obtain

$$
\begin{aligned}
I_{11} &\le C_1\int_0^{\mathfrak{d}} t^{-\alpha-1}\Big(\int_{\Omega}\big|c_{n,m}(Q)-c_{n,m}(Q-\vec{e}t)\big|^2 dQ\Big)^{1/2}dt\\
&= C_1\int_0^{\mathfrak{d}} t^{-\alpha-1}\Big(\int_{\Omega}\Big|\int_0^t c'_{n,m}(Q-\vec{e}\tau)d\tau\Big|^2 dQ\Big)^{1/2}dt.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, and Fubini's theorem, we have

$$\begin{aligned} I_{11} &\leq C_1 \int_0^{\mathfrak{d}} t^{-\alpha-1}\Big(\int_\Omega dQ \int_0^t \big|c'_{n,m}(Q-\vec{e}\tau)\big|^2 d\tau \int_0^t d\tau\Big)^{1/2} dt \\ &= C_1 \int_0^{\mathfrak{d}} t^{-\alpha-1/2}\Big(\int_0^t d\tau \int_\Omega \big|c'_{n,m}(Q-\vec{e}\tau)\big|^2 dQ\Big)^{1/2} dt \\ &\leq C_1 \frac{\mathfrak{d}^{1-\alpha}}{1-\alpha}\|c'_{n,m}\|_{L_2(\Omega)}. \end{aligned}$$

Using Holder's property of function ρ analogously to the previous reasoning, we have the following estimate

$$I_{21} \leq M \int_0^{\mathfrak{d}} t^{\lambda-\alpha-1}\Big(\int_\Omega |c_{n,m}(Q-\vec{e}t)|^2 \, dQ Big)^{1/2} dt \leq M \frac{\mathfrak{d}^{\lambda-\alpha}}{\lambda-\alpha}\|c_{n,m}\|_{L_2(\Omega)}.$$

Applying trivial estimates, we have

$$\begin{aligned} I_{02} &\leq C_1\Big\{\int_\Omega \Big|\int_0^r |c_{n,m}(Q-\vec{e}t)| \sum_{i=0}^{n-2} (\frac{t}{r})^i r^{-1} t^{-\alpha} dt\Big|^2 dQ\Big\}^{1/2} \\ &\leq C_2\Big\{\int_\Omega \Big|\int_0^r |c_{n,m}(Q-\vec{e}t)| r^{-1} t^{-\alpha} dt\Big|^2 dQ\Big\}^{1/2} \\ &\leq C_2\Big\{\int_\Omega \Big(\int_0^r t^{-\alpha} dt \int_t^r \big|c'_{n,m}(Q-\vec{e}\tau)\big| \, d\tau\Big)^2 r^{-2} dQ\Big\}^{1/2} \\ &= C_2\Big\{\int_\Omega (\int_0^r \big|c'_{n,m}(Q-\vec{e}\tau)\big| \, d\tau \int_0^\tau t^{-\alpha} dt)^2 r^{-2} dQ\Big\}^{1/2} \\ &= \frac{C_2}{1-\alpha}\Big\{\int_\Omega \Big(\int_0^r \big|c'_{n,m}(Q-\vec{e}\tau)\big| (\frac{\tau}{r})\tau^{-\alpha} d\tau Big)^2 dQ\Big\}^{1/2} \\ &\leq \frac{C_2}{1-\alpha}\Big\{\int_\Omega \Big(\int_0^r \big|c'_{n,m}(Q-\vec{e}\tau)\big| \tau^{-\alpha} d\tau\Big)^2 dQ\Big\}^{1/2}. \end{aligned}$$

Using the generalized Minkowski's inequality, we have

$$I_{02} \leq C_3 \int_0^{\mathfrak{d}} \tau^{-\alpha} d\tau \Big(\int_\Omega \big|c'_{n,m}(Q-\vec{e}\tau)\big|^2 dQ\Big)^{1/2} \leq C_3 \frac{\mathfrak{d}^{1-\alpha}}{1-\alpha}\|c'_{n,m}\|_{L_2(\Omega)}.$$

Consider I_2, representing the function under the integral by the derivative in the direction of $\vec{e}$,

$$\begin{aligned} I_2 &\leq \frac{C_1}{\Gamma(1-\alpha)}\Big(\int_\Omega |c_{n,m}(Q)|^2 r^{-2\alpha} dQ\Big)^{\frac{1}{2}} \\ &= \frac{C_1}{\Gamma(1-\alpha)}(\int_\Omega r^{-2\alpha}\Big|\int_0^r c'_{n,m}(Q-\vec{e}t) dt\Big|^2 dQ\Big)^{1/2} \\ &\leq \frac{C_1}{\Gamma(1-\alpha)}\Big(\int_\Omega \Big|\int_0^r c'_{n,m}(Q-\vec{e}t) t^{-\alpha} dt\Big|^2 dQ\Big)^{1/2}. \end{aligned}$$

Using the generalized Minkowski's inequality, then applying obvious estimates we have

$$\begin{aligned} I_2 &\leq C_4\Big\{\int_\omega \Big[\int_0^d t^{-\alpha} dt \Big(\int_t^d |c'_{n,m}(Q-\vec{e}t)|^2 r^{n-1} dr\Big)^{1/2}\Big]^2 d\chi\Big\}^{1/2} \\ &\leq C_4\Big\{\int_\omega \Big[\int_0^{\mathfrak{d}} t^{-\alpha} dt \Big(\int_0^d |c'_{n,m}(Q-\vec{e}t)|^2 r^{n-1} dr\Big)^{1/2}\Big]^2 d\chi\Big\}^{1/2} \end{aligned}$$

$$
\begin{aligned}
&= C_4 \int_0^{\mathfrak{d}} t^{-\alpha} dt \Big(\int_\omega d\chi \int_0^d |c'_{n,m}(Q - \vec{e}t)|^2 r^{n-1} dr \Big)^{1/2} \\
&\le C_4 \frac{\mathfrak{d}^{1-\alpha}}{1-\alpha} \|c'_{n,m}\|_{L_2(\Omega)}.
\end{aligned}
$$

From the fundamental property of the sequences $\{c_{n,m}\}$, $\{c'_{n,m}\}$, it follows that $I_1, I_2 \to 0$. Hence the sequence $\{\varphi_n\}$ is fundamental and in consequence of completeness property of the space $L_2(\Omega)$ there is a limit of sequence $\{\varphi_n\}$, a some function $\varphi \in L_2(\Omega)$. In consequence of theorem 2.1 the operator of fractional integration in the direction is boundary acting in the space $L_2(\Omega)$, hence

$$
\mathfrak{I}^\alpha_{0+} \varphi_n \xrightarrow{L_2} \mathfrak{I}^\alpha_{0+} \varphi.
$$

Since (2.14) holds, we have $\rho f = \mathfrak{I}^\alpha_{0+} \varphi$. $\square$

Lemma 2.5. *The operator $\mathfrak{D}^\alpha$ is a contraction of operator $\mathfrak{D}^\alpha_{0+}$, exactly $\mathfrak{D}^\alpha \subset \mathfrak{D}^\alpha_{0+}$.*

Proof. We show that the equality holds

$$
(\mathfrak{D}^\alpha f)(Q) = (\mathfrak{D}^\alpha_{0+} f)(Q), \quad f \in \overset{0}{W}{}^l_p(\Omega). \tag{2.15}
$$

It follows from the next obvious conversions:

$$
\begin{aligned}
&r^{n-1}\mathfrak{D}^\alpha v \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^r \frac{v(Q) - v(P + \vec{e}t)}{(r-t)^{\alpha+1}} t^{n-1} dt + \frac{C_n^{(\alpha)}}{\Gamma(1-\alpha)} v(Q) r^{n-1-\alpha} \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^r \frac{r^{n-1} v(Q) - t^{n-1} v(P + \vec{e}t)}{(r-t)^{\alpha+1}} dt \\
&\quad - v(Q) \frac{\alpha}{\Gamma(1-\alpha)} \int_0^r \frac{r^{n-1} - t^{n-1}}{(r-t)^{\alpha+1}} dt + \frac{(n-1)!}{\Gamma(n-\alpha)} v(Q) r^{n-1-\alpha} \\
&= (\mathfrak{D}^\alpha_{0+} t^{n-1} v)(Q) - \frac{\alpha v(Q)}{\Gamma(1-\alpha)} \sum_{i=0}^{n-2} r^{n-2-i} \int_0^r \frac{t^i}{(r-t)^\alpha} dt \\
&\quad + \frac{(n-1)!}{\Gamma(n-\alpha)} v(Q) r^{n-1-\alpha} - \frac{1}{\Gamma(1-\alpha)} v(Q) r^{n-1-\alpha} \\
&= (\mathfrak{D}^\alpha_{0+} t^{n-1} v)(Q) - I_1 + I_2 - I_3.
\end{aligned} \tag{2.16}
$$

Let us conduct the following conversions using the formula of fractional integration of the exponential function [20, (2.44) p.47]

$$
\begin{aligned}
I_1 &= \frac{\alpha v(Q)}{\Gamma(1-\alpha)} r^{n-2} \int_0^r \frac{1}{(r-t)^\alpha} dt + \frac{\alpha v(Q)}{\Gamma(1-\alpha)} \sum_{i=1}^{n-2} r^{n-2-i} \int_0^r \frac{t^i}{(r-t)^\alpha} dt \\
&= v(Q) \frac{\alpha}{\Gamma(2-\alpha)} r^{n-1-\alpha} + v(Q) \alpha \sum_{i=1}^{n-2} r^{n-2-i} (I^{1-\alpha}_{0+} t^i)(r) \\
&= v(Q) \frac{\alpha}{\Gamma(2-\alpha)} r^{n-1-\alpha} + v(Q) \alpha \sum_{i=1}^{n-2} r^{n-1-\alpha} \frac{i!}{\Gamma(2-\alpha+i)}.
\end{aligned}
$$

Consequently

$$\begin{aligned} r^{-n+1+\alpha}(I_1+I_3)/v(Q) &= \frac{1}{\Gamma(2-\alpha)} + \alpha\sum_{i=1}^{n-2}\frac{i!}{\Gamma(2-\alpha+i)} \\ &= \frac{2}{\Gamma(3-\alpha)} + \alpha\sum_{i=2}^{n-2}\frac{i!}{\Gamma(2-\alpha+i)} \\ &= \frac{3!}{\Gamma(4-\alpha)} + \alpha\sum_{i=3}^{n-2}\frac{i!}{\Gamma(2-\alpha+i)} \\ &= \frac{(n-2)!}{\Gamma(n-1-\alpha)} + \alpha\frac{(n-2)!}{\Gamma(n-\alpha)} = \frac{(n-1)!}{\Gamma(n-\alpha)}. \end{aligned} \tag{2.17}$$

Hence $I_2 - I_1 - I_3 = 0$, and equality (2.15) follows from (2.16),(2.17). The proof of the fact of difference the operators $\mathfrak{D}^\alpha$ and $\mathfrak{D}^\alpha_{0+}$ implies from the following reasoning. Let $f \in \mathfrak{I}^\alpha_{0+}\varphi,\ \varphi \in L_p(\Omega)$, then in consequence of theorem 2.2 we have $\mathfrak{D}^\alpha_{0+}\mathfrak{I}^\alpha_{0+}\varphi = \varphi$. Hence $\mathfrak{I}^\alpha_{0+}(L_p) \subset \mathrm{D}(\mathfrak{D}^\alpha_{0+})$. Given the above remains to note that there exists $f \in \mathfrak{I}^\alpha_{0+}(L_p)$, such that

$$f(\Lambda) \neq 0,\ \Lambda \subset \partial\Omega, \quad \operatorname{meas}\Lambda \neq 0,$$

at the same time

$$f(\partial\Omega) = 0, \quad \forall f \in \mathrm{D}(\mathfrak{D}^\alpha).$$

□

Lemma 2.6. *The following equality holds*

$$\mathfrak{D}^{\alpha^*}_{0+} = \mathfrak{D}^\alpha_{d-},$$

where

$$\mathrm{D}(\mathfrak{D}^\alpha_{0+}) = \mathfrak{I}^\alpha_{0+}(L_2), \quad \mathrm{D}(\mathfrak{D}^\alpha_{d-}) = \mathfrak{I}^\alpha_{d-}(L_2).$$

Proof. We show the next relation

$$\begin{gathered} (\mathfrak{D}^\alpha_{0+}f, g)_{L_2(\Omega)} = (f, \mathfrak{D}^\alpha_{d-}g)_{L_2(\Omega)}, \\ f \in \mathfrak{I}^\alpha_{0+}(L_2),\ g \in \mathfrak{I}^\alpha_{d-}(L_2). \end{gathered} \tag{2.18}$$

Note that as a consequence of theorem 2.3 the next equalities holds: $\mathfrak{D}^\alpha_{0+}\mathfrak{I}^\alpha_{0+}\varphi = \varphi \in L_2(\Omega)$, $\mathfrak{D}^\alpha_{d-}\mathfrak{I}^\alpha_{d-}\psi = \psi \in L_2(\Omega)$. Given the above and in consequence of theorem 2.1, we obtain that the left and right side of (2.18) exist and are finite. Using the Fubini's theorem we can perform the following conversions

$$\begin{aligned} (\mathfrak{D}^\alpha_{0+}f, g)_{L_2(\Omega)} &= \int_\omega d\chi \int_0^d \varphi(P+\vec{e}r)\overline{(\mathfrak{I}^\alpha_{d-}\psi)(Q)}r^{n-1}dr \\ &= \frac{1}{\Gamma(\alpha)}\int_\omega d\chi \int_0^d \varphi(P+\vec{e}r)r^{n-1}dr \int_r^d \frac{\overline{\psi(P+\vec{e}t)}}{(t-r)^{1-\alpha}}\,dt \\ &= \frac{1}{\Gamma(\alpha)}\int_\omega d\chi \int_0^d \overline{\psi(P+\vec{e}t)}t^{n-1}dt \int_0^t \frac{\varphi(P+\vec{e}r)}{(t-r)^{1-\alpha}}(\frac{r}{t})^{n-1}dr \\ &= \int_\Omega (\mathfrak{I}^\alpha_{0+}\varphi)(Q)\,\overline{\psi(Q)}\,dQ = (f, \mathfrak{D}^\alpha_{d-}g)_{L_2(\Omega)}. \end{aligned} \tag{2.19}$$

Inequality (2.18) is proved. From equality (2.18) follows that $\mathrm{D}(\mathfrak{D}^\alpha_{d-}) \subset \mathrm{D}(\mathfrak{D}^{\alpha^*}_{0+})$. Since $\mathrm{R}(\mathfrak{D}^\alpha_{d-}) = L_2$, then $\mathrm{R}(\mathfrak{D}^{\alpha^*}_{0+}) = L_2$. We will show that $\mathrm{D}(\mathfrak{D}^{\alpha^*}_{0+}) \subset \mathrm{D}(\mathfrak{D}^\alpha_{d-})$

thus completing the proof. In accordance with the definition of conjugate operator, for all element $f \in \mathrm{D}(\mathfrak{D}^{\alpha}_{0+})$ and pars of elements $g \in \mathrm{D}(\mathfrak{D}^{\alpha^*}_{0+})$, $g^* \in \mathrm{R}(\mathfrak{D}^{\alpha^*}_{0+})$, the integral equality holds

$$\langle \mathfrak{D}^{\alpha}_{0+} f, g\rangle_{L_2(\Omega)} = \langle f, g^*\rangle_{L_2(\Omega)}.$$

Suppose $f = \mathfrak{I}^{\alpha}_{0+}\varphi$, $\varphi \in L_2(\Omega)$. Using the Fubini's theorem and performing the conversion similar to (2.19), we have

$$\langle \mathfrak{D}^{\alpha}_{0+} f, g - \mathfrak{I}^{\alpha}_{d-} g^*\rangle_{L_2(\Omega)} = 0.$$

By theorem 2.3 the image of operator $\mathfrak{D}^{\alpha}_{0+}$ coincides with the space $L_2(\Omega)$. Hence the element $(g - \mathfrak{I}^{\alpha}_{d-} g^*) \in L_2$ equals zero. It implies that $\mathrm{D}(\mathfrak{D}^{\alpha^*}_{0+}) \subset \mathrm{D}(\mathfrak{D}^{\alpha}_{d-})$. □

3. Strong accretiveness property

The following theorem establishes the strong accretive property (see [8, p. 352]) for the operator of fractional differentiation in the sense of Kipriyanov acting in the complex weight space of Lebesgue summable with squared functions.

Theorem 3.1. *Let $n \geq 2$, $\rho(Q)$ is non-negative real function in class $\mathrm{Lip}\,\mu$, $\mu > \alpha$. Then for the operator of fractional differentiation in the sense of Kipriyanov the inequality of a strong accretiveness holds*

$$\mathrm{Re}\langle f, \mathfrak{D}^{\alpha} f\rangle_{L_2(\Omega,\rho)} \geq \frac{1}{\lambda^2}\|f\|^2_{L_2(\Omega,\rho)}, \quad f \in H^1_0(\Omega). \tag{3.1}$$

Proof. First we assume that f is real. For $f \in C^{\infty}_0(\Omega)$ consider the following difference in which the second summand exists due to theorem 2.4,

$$\begin{aligned}
&\rho(Q) f(Q)(\mathfrak{D}^{\alpha} f)(Q) - \frac{1}{2}(\mathfrak{D}^{\alpha}\rho f^2)(Q)\\
&= \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^r \frac{\rho(Q)[f(P+\vec{e}r) - f(P+\vec{e}t)]^2}{(r-t)^{\alpha+1}}\big(\frac{t}{r}\big)^{n-1} dt\\
&\quad + \frac{C^{(\alpha)}_n}{2}\rho(Q)|f(Q)|^2 r^{-\alpha} dr \geq 0.
\end{aligned}$$

Therefore,

$$\rho(Q) f(Q)(\mathfrak{D}^{\alpha} f)(Q) \geq \frac{1}{2}(\mathfrak{D}^{\alpha}\rho f^2)(Q). \tag{3.2}$$

Integrating the left and right sides of inequality (3.2), then using a Fubini's theorem we obtain

$$\begin{aligned}
&\int_0^d f(Q)(\mathfrak{D}^{\alpha} f)(Q)\rho(Q) r^{n-1} dr\\
&\geq \frac{1}{2}\int_0^d (\mathfrak{D}^{\alpha}\rho f^2)(Q) r^{n-1} dr\\
&= \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^d t^{n-1} dt \int_t^d \frac{(\rho f^2)(Q) - (\rho f^2)(P+\vec{e}t)}{(r-t)^{\alpha+1}} dr\\
&\quad + \frac{C^{(\alpha)}_n}{2}\int_0^d (\rho f^2)(Q) r^{n-1-\alpha} dr\\
&= -\frac{1}{2}\int_0^d (\mathfrak{D}^{\alpha}_{d-}\rho f^2)(Q) r^{n-1} dr + \frac{C^{(\alpha)}_n}{2}\int_0^d (\rho f^2)(Q) r^{n-1-\alpha} dr
\end{aligned}$$

$$+\frac{1}{2\Gamma(1-\alpha)}\int_0^d(\rho f^2)(Q)r^{n-1}(d-r)^{-\alpha}dr=I.$$

Let us rewrite the first summand of the last sum using the formula of fractional integration of exponential function [20, (2.44) p.47], we obtain

$$\int_0^d(\mathfrak{D}^\alpha_{d-}\rho f^2)(Q)r^{n-1}dr=\frac{(n-1)!}{\Gamma(n-\alpha)\Gamma(\alpha)}\int_0^d(\mathfrak{D}^\alpha_{d-}\rho f^2)(Q)dr\int_0^r\frac{t^{n-1-\alpha}}{(r-t)^{1-\alpha}}dt.$$

Note that by theorems 2.3 and 2.4 we have $\rho f^2=\mathfrak{I}^\alpha_{d-}(\mathfrak{D}^\alpha_{d-}\rho f^2)$. Using Fubini's theorem, we obtain

$$\begin{aligned}
&\frac{(n-1)!}{\Gamma(n-\alpha)\Gamma(\alpha)}\int_0^d(\mathfrak{D}^\alpha_{d-}\rho f^2)(Q)dr\int_0^r\frac{t^{n-1-\alpha}}{(r-t)^{1-\alpha}}dt\\
&=\frac{(n-1)!}{\Gamma(n-\alpha)}\int_0^d\big[\mathfrak{I}^\alpha_{d-}(\mathfrak{D}^\alpha_{d-}\rho f^2)\big](P+\vec{e}t)t^{n-1-\alpha}dt\\
&=C_n^{(\alpha)}\int_0^d(\rho f^2)(Q)r^{n-1-\alpha}dr.
\end{aligned}$$

Therefore,

$$I=\frac{1}{2\Gamma(1-\alpha)}\int_0^d(\rho f^2)(Q)r^{n-1}(d-r)^{-\alpha}dr\geq\frac{\mathfrak{d}^{-\alpha}}{2\Gamma(1-\alpha)}\int_0^d(\rho f^2)(Q)r^{n-1}dr.$$

Finally for any direction $\vec{e}$, we obtain the inequality

$$\int_0^{d(\vec{e})}f(Q)(\mathfrak{D}^\alpha f)(Q)\rho(Q)r^{n-1}dr\geq\frac{\mathfrak{d}^{-\alpha}}{2\Gamma(1-\alpha)}\int_0^{d(\vec{e})}(\rho f^2)(Q)r^{n-1}dr.$$

Integrating the left and right sides of the last inequality we obtain

$$\langle f,\mathfrak{D}^\alpha f\rangle_{L_2(\Omega,\rho)}\geq\frac{1}{\lambda^2}\|f\|^2_{L_2(\Omega,\rho)},\quad f\in C_0^\infty(\Omega),\quad \lambda^2=2\Gamma(1-\alpha)\mathfrak{d}^\alpha. \tag{3.3}$$

Suppose that $f\in H_0^1(\Omega)$. There is a sequence $\{f_k\}\in C_0^\infty(\Omega)$ such that $f_k\xrightarrow{H^1}f$. The conditions imposed on the weight function ρ implies the equivalence of norms $L_2(\Omega)$ and $L_2(\Omega,\rho)$, hence $f_k\xrightarrow{L_2(\Omega,\rho)}f$. Using the smoothness of weight function ρ, the embedding of spaces $L_p(\Omega)$, $p\geq 1$, and the inequality (1.3), we obtain the estimate

$$\|\mathfrak{D}^\alpha f\|_{L_2(\Omega,\rho)}\leq C_1\|\mathfrak{D}^\alpha f\|_{L_q(\Omega)}\leq C_2\|f\|^2_{H_0^1(\Omega)},$$

where $2<q<2n/(2\alpha-2+n)$, $C_i>0$, $(i=1,2)$. Therefore $\mathfrak{D}^\alpha f_k\xrightarrow{L_2(\Omega,\rho)}\mathfrak{D}^\alpha f$. Hence from the continuity properties of the inner product in the Hilbert space, we obtain

$$\langle f_k,\mathfrak{D}^\alpha f_k\rangle_{L_2(\Omega,\rho)}\to\langle f,\mathfrak{D}^\alpha f\rangle_{L_2(\Omega,\rho)}.$$

Passing to the limit in the left and right side of inequality (3.3), we obtain the inequality (3.1) in the real case.

Now consider the case when f is complex-valued. Note that

$$\operatorname{Re}\langle f,\mathfrak{D}^\alpha f\rangle_{L_2(\Omega,\rho)}=\langle u,\mathfrak{D}^\alpha u\rangle_{L_2(\Omega,\rho)}+\langle v,\mathfrak{D}^\alpha v\rangle_{L_2(\Omega,\rho)}, \tag{3.4}$$

$$u=\operatorname{Re}f,\ v=\operatorname{Im}f. \tag{3.5}$$

Obviously inequality (3.1) follows from relations (3.3) and (3.4). □

4. Sectorial property

Consider a uniformly elliptic operator with real-valued coefficients and fractional derivative in the sense of Kipriyanov in lower terms, defined by the expression

$$\begin{gathered}Lu := -D_j(a^{ij}D_iu) + p\,\mathfrak{D}^{\alpha}u, \quad (i,j=\overline{1,n}),\\ \mathrm{D}(L) = H^2(\Omega)\cap H_0^1(\Omega),\\ a^{ij}(Q)\in C^1(\bar\Omega), \quad a^{ij}\xi_i\xi_j \ge a_0|\xi|^2, \quad a_0>0, \qquad (4.1)\\ p(Q)>0, \quad p(Q)\in \operatorname{Lip}\mu, \quad (0<\alpha<\mu). \qquad (4.2)\end{gathered}$$

We also will consider the formal conjugate operator

$$L^+u := -D_i(a^{ij}D_ju) + \mathfrak{D}^{\alpha}_{d-}pu, \quad \mathrm{D}(L^+) = \mathrm{D}(L),$$

and the operator

$$H = \frac{1}{2}(L+L^+).$$

We will use the special case of the Green's formula

$$-\int_\Omega D_j(a^{ij}D_iu)\,\bar v\,dQ = \int_\Omega a^{ij}D_iu\,\overline{D_jv}\,dQ\,, \quad u\in H^2(\Omega),\ v\in H_0^1(\Omega).$$

The following Lemma establishes a property of the closure of operator L.

Lemma 4.1. *The operators L, L^+, H have closure $\tilde L, \tilde L^+, \tilde H$, the domains of definition of this operators is included in $H_0^1(\Omega)$.*

Proof. At first consider operator L. For a function $f\in \mathrm{D}(L)$ using the Green's formula, we have with the notations (3.5)

$$\begin{aligned}-\int_\Omega D_j(a^{ij}D_if)\bar f dQ &= \int_\Omega a^{ij}D_if\overline{D_jf}dQ\\ &= \int_\Omega a^{ij}(D_iuD_ju + D_ivD_jv)dQ \qquad (4.3)\\ &\quad + \imath\int_\Omega a^{ij}(D_ivD_ju - D_iuD_jv)dQ.\end{aligned}$$

Applying condition (4.1), we obtain

$$\begin{aligned}\mathrm{Re}(a^{ij}D_if, D_jf)_{L_2(\Omega)} &= (a^{ij}D_iu, D_ju)_{L_2(\Omega)} + (a^{ij}D_iv, D_jv)_{L_2(\Omega)}\\ &\ge a_0(\|u\|^2_{L_2^1(\Omega)} + \|v\|^2_{L_2^1(\Omega)}) \qquad (4.4)\\ &= a_0\|f\|^2_{L_2^1(\Omega)}, \quad f\in H_0^1(\Omega).\end{aligned}$$

From (4.3), (4.4) it follows that

$$-\mathrm{Re}(D_j[a^{ij}D_if], f)_{L_2(\Omega)} \ge a_0\ |f\|^2_{L_2^1(\Omega)}, \quad f\in \mathrm{D}(L). \qquad (4.5)$$

Choose an arbitrary $\varepsilon>0$. From (4.5), theorem 3.1, (4.2), using a Jung's inequality it is easy to show that for sequence $\{f_n\}\subset \mathrm{D}(L)$, the next two-sided estimate holds

$$\begin{aligned}C_1\|f_n\|^2_{L_2^1(\Omega)} + C_2\|f_n\|^2_{L_2(\Omega)} &\le 2\ \mathrm{Re}(f_n, Lf_n)_{L_2(\Omega)}\\ &\le \frac{1}{\varepsilon}\|Lf_n\|^2_{L_2(\Omega)} + \varepsilon\|f_n\|^2_{L_2(\Omega)},\end{aligned} \qquad (4.6)$$

where $C_i > 0$, $i = 1, 2$. In consequence of [8, theorem 3.4 p. 337], from lower estimate (4.6), follows that operator L has a closure. Let $u \in \mathrm{D}(\tilde{L})$, then by definition, exists a sequence $\{f_n\} \subset \mathrm{D}(L)$ such as that $f_n \xrightarrow{L_2} f$, $\{Lf_n\}$ is fundamental sequence in the sense of the norm $L_2(\Omega)$. Hence the inequality (4.6) implies that a sequence $\{f_n\}$ is fundamental in the sense of the norm $H_0^1(\Omega)$. In consequence of the completeness property of the space $H_0^1(\Omega)$, we have $u \in H_0^1(\Omega)$. The proof corresponding to the case of operator L completed. For proving this result for the case of operators L^+, H we must note that

$$
\begin{gathered}
(Lf, g)_{L_2(\Omega)} = (f, L^+ g)_{L_2(\Omega)}, \quad \mathrm{Re}(Lf, f)_{L_2(\Omega)} = (Hf, f)_{L_2(\Omega)}, \\
f, g \in \mathrm{D}(L),
\end{gathered} \tag{4.7}
$$

and then repeat the previous proof using this remark. □

We have the following theorem describing the spectral properties of the closed operator $\tilde{L}$.

Theorem 4.2. *Operators $\tilde{L}$, $\tilde{L}^+$ is strongly accretive, the numerical range of values belongs to the sector*

$$\mathfrak{S} := \{\zeta \in \mathbb{C} : \arg(\zeta - \gamma)| \le \theta\},$$

where θ and γ defined by the coefficients of operator L.

Proof. Consider operator L. Applying estimate (4.5) we can get the inequality

$$\mathrm{Re}\langle f_n, Lf_n\rangle_{L_2(\Omega)} \ge a_0 \|f_n\|^2_{L_2^1(\Omega)} + \mathrm{Re}\langle f_n, \mathfrak{D}^\alpha f_n\rangle_{L_2(\Omega,p)}, \quad \{f_n\} \subset \mathrm{D}(L). \tag{4.8}$$

Assume $f \in \mathrm{D}(\tilde{L})$. Note that exists sequence $f_n \underset{L}{\longrightarrow} f$, $\{f_n\} \subset \mathrm{D}(L)$ and in consequences of lemma 2.6: $f \in H_0^1(\Omega)$. Using the continuity property of the inner product and passing to the limit in the left and right side of inequality (4.8), we obtain

$$\mathrm{Re}\langle f, \tilde{L}f\rangle_{L_2(\Omega)} \ge a_0 \|f\|^2_{L_2^1(\Omega)} + \mathrm{Re}\langle f, \mathfrak{D}^\alpha f\rangle_{L_2(\Omega,p)}, \quad f \in \mathrm{D}(\tilde{L}). \tag{4.9}$$

By theorem 3.1 we can rewrite the previous inequality in the form

$$\mathrm{Re}\langle f, \tilde{L}f\rangle_{L_2(\Omega)} \ge a_0 \|f\|^2_{L_2^1(\Omega)} + \frac{1}{\lambda^2}\|f\|^2_{L_2(\Omega,p)}, \quad f \in \mathrm{D}(\tilde{L}). \tag{4.10}$$

Applying the inequality of Friedrichs - Poincare to the first summand of the right side (4.10), we will get the inequality of a strong accretiveness for operator $\tilde{L}$,

$$\mathrm{Re}\langle f, \tilde{L}f\rangle_{L_2(\Omega)} \ge \frac{1}{\mu^2}\|f\|^2_{L_2(\Omega)}, \quad f \in \mathrm{D}(\tilde{L}), \quad \mu^{-2} = a_0 + \lambda^{-2} \inf_{Q\in\Omega} |p(Q)|. \tag{4.11}$$

Consider imaginary component of the form generated by the operator L. For $f \in \mathrm{D}(L)$ we obtain

$$
\begin{aligned}
|\operatorname{Im}\langle f, Lf\rangle_{L_2(\Omega)}| &\le \Big| \int_\Omega (a^{ij} D_i u D_j v - a^{ij} D_i v D_j u) dQ \Big| \\
&\quad + \big| \langle u, \mathfrak{D}^\alpha v\rangle_{L_2(\Omega,p)} - \langle v, \mathfrak{D}^\alpha u\rangle_{L_2(\Omega,p)} \big| \\
&= I_1 + I_2.
\end{aligned} \tag{4.12}
$$

Using the Cauchy-Schwarz inequality for a sum, then the Jung's inequality, we have

$$\begin{gathered} a^{ij} D_i u D_j v \le a(Q)|Du||Dv| \le \frac{1}{2} a(Q)(|Du|^2 + |Dv|^2), \\ a(Q) = \Big(\sum_{i,j=1}^{n} |a_{ij}(Q)|^2 \Big)^{1/2}. \end{gathered} \tag{4.13}$$

Hence

$$I_1 \le a_1 \|f\|^2_{L^1_2(\Omega)}, \ a_1 = \sup_{Q \in \Omega} |a(Q)|.$$

Applying inequality (1.3), and Jung's inequality, we obtain

$$\begin{aligned} &|\langle u, \mathfrak{D}^\alpha v \rangle_{L_2(\Omega,p)}| \\ &\le C\|u\|_{L_2(\Omega)} \|\mathfrak{D}^\alpha v\|_{L_q(\Omega)} \le C\|u\|_{L_2(\Omega)} \Big\{ \frac{K}{\delta^\nu} \|v\|_{L_2(\Omega)} + \delta^{1-\nu} \|v\|_{L^1_2(\Omega)} \Big\} \\ &\le \frac{1}{\varepsilon} \|u\|^2_{L_2(\Omega)} + \varepsilon \big(\frac{KC}{\sqrt{2}\delta^\nu} \big)^2 \|v\|^2_{L_2(\Omega)} + \frac{\varepsilon}{2} (C\delta^{1-\nu})^2 \|v\|^2_{L^1_2(\Omega)}, \end{aligned} \tag{4.14}$$

$$2 < q < \frac{2n}{2\alpha - 2 + n}, \quad C = (\operatorname{meas} \Omega)^{\frac{q-2}{q}} \sup_{Q \in \Omega} p(Q). \tag{4.15}$$

Hence

$$\begin{aligned} I_2 &\le \big| \langle u, \mathfrak{D}^\alpha v \rangle_{L_2(\Omega,p)} \big| + \big| \langle v, \mathfrak{D}^\alpha u \rangle_{L_2(\Omega,p)} \big| \\ &\le \frac{1}{\varepsilon} \big(\|u\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(\Omega)} \big) + \varepsilon \big(\frac{KC}{\sqrt{2}\delta^\nu} \big)^2 (\|u\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(\Omega)}) \\ &\quad + \frac{\varepsilon}{2} (C\delta^{1-\nu})^2 (\|u\|^2_{L^1_2(\Omega)} + \|v\|^2_{L^1_2(\Omega)}) \\ &= \big(\varepsilon \delta^{-2\nu} C_1 + \frac{1}{\varepsilon} \big) \|f\|^2_{L_2(\Omega)} + \varepsilon \delta^{2-2\nu} C_2 \|f\|^2_{L^1_2(\Omega)}. \end{aligned} \tag{4.16}$$

Using (4.16), (4.12) and applying a reasoning, analogous to the one in the proof of (4.9), we have the estimate

$$\begin{aligned} &|\operatorname{Im} \langle f, \tilde{L} f \rangle_{L_2(\Omega)}| \\ &\le (\varepsilon \delta^{-2\nu} C_1 + \frac{1}{\varepsilon}) \|f\|^2_{L_2(\Omega)} + (\varepsilon \delta^{2-2\nu} C_2 + a_1) \|f\|^2_{L^1_2(\Omega)}, f \in \mathrm{D}(\tilde{L}). \end{aligned}$$

Thus in consequence of (4.11) for arbitrary $k > 0$, the next inequality holds

$$\begin{aligned} &\operatorname{Re} \langle f, \tilde{L} f \rangle_{L_2(\Omega)} - k |\operatorname{Im} \langle f, \tilde{L} f \rangle_{L_2(\Omega)}| \\ &\ge \big(a_0 - k[\varepsilon \delta^{2-2\nu} C_2 + a_1] \big) \|f\|^2_{L^1_2(\Omega)} + \big(\frac{1}{\mu^2} - k[\varepsilon \delta^{-2\nu} C_1 + \frac{1}{\varepsilon}] \big) \|f\|^2_{L_2(\Omega)}. \end{aligned}$$

Choose $k = a_0(\varepsilon \delta^{2-2\nu} C_2 + a_1)^{-1}$, we obtain

$$\begin{gathered} |\operatorname{Im} \langle f, (\tilde{L} - \gamma) f \rangle_{L_2(\Omega)}| \le \frac{1}{k} \operatorname{Re} \langle f, (\tilde{L} - \gamma) f \rangle_{L_2(\Omega)}, \\ \gamma = \frac{1}{\mu^2} - k[\varepsilon \delta^{-2\nu} C_1 + \frac{1}{\varepsilon}]. \end{gathered} \tag{4.17}$$

The last inequality implies that the numerical range of values $\Theta(\tilde{L})$ belongs to the sector with top in γ and half-angle $\theta = \arctan(1/k)$. The prove for the case corresponding to the operator $\tilde{L}^+$ is obvious if we note the first relation (4.7). $\square$

We will not research detailed conditions for the coefficients of operator L under which $\gamma > 0$ holds, just we note that it follows from the second relation of (4.17), that can be easy formulated. In further reasoning , we assume that the coefficients δ, ε of the operator L are chosen according to the second relation of (4.17) so that $\gamma > 0$.

Theorem 4.3. *The operators $\tilde{L}, \tilde{L}^{+}, \tilde{H}$ are m-sectorial, and operator $\tilde{H}$ is self-adjoint.*

Proof. Let us prove the theorem for the case corresponding to the operator $\tilde{L}$. By theorem 4.2 we know that the operator $\tilde{L}$ is sectorial i.e. the numerical range of values of $\tilde{L}$ belongs to the sector $\mathfrak{S}$. By [8, Theorem 3.2 p. 336] we arrive to the conclusion that $\mathrm{R}(\tilde{L} - \zeta)$ is a closed space for any $\zeta \in \mathbb{C} \setminus \mathfrak{S}$ and the next relation holds

$$\operatorname{def}(\tilde{L} - \zeta) = \mu, \ \mu = \text{const}. \tag{4.18}$$

Since (4.11) holds, then on the subspace $\mathrm{R}(\tilde{L} + \zeta)$, $\operatorname{Re}\zeta > 0$ defined the inverse operator. Accordingly condition [8, (3.38) p.350], we need to show that

$$\operatorname{def}(\tilde{L} + \zeta) = 0, \quad \|(\tilde{L} + \zeta)^{-1}\| \le (\operatorname{Re}\zeta)^{-1}, \quad \operatorname{Re}\zeta > 0. \tag{4.19}$$

Let us prove the first relation of (4.19). Suppose that the parameters: δ, ε are chosen from the second relation (4.17) so that $\gamma > 0$. Hence, the left half-plane has included in to the complement of the sector $\mathfrak{S}$ in the complex plain.

Let $\zeta_0 \in \mathbb{C} \setminus \mathfrak{S}$, $\operatorname{Re}\zeta_0 < 0$. In consequence of inequality (4.11), we have

$$\operatorname{Re}\langle f, (\tilde{L} - \zeta) f\rangle_{L_2(\Omega)} \ge (\mu^{-2} - \operatorname{Re}\zeta)\|f\|^2_{L_2(\Omega)}. \tag{4.20}$$

Since the operator $\tilde{L} - \zeta_0$ has a closed range of values $\mathrm{R}(\tilde{L} - \zeta_0)$, it follows that

$$L_2 = \mathrm{R}(\tilde{L} - \zeta_0) \oplus \mathrm{R}(\tilde{L} - \zeta_0)^{\perp}.$$

Note that intersection of sets $C_0^\infty(\Omega)$ and $\mathrm{R}(\tilde{L} - \zeta_0)^{\perp}$ is empty, because if we assume otherwise, then applying inequality (4.20) for any element $u \in C_0^\infty(\Omega) \cap \mathrm{R}(\tilde{L} - \zeta_0)^{\perp}$, we obtain

$$(\mu^{-2} - \operatorname{Re}\zeta_0)\|u\|^2_{L_2(\Omega)} \le \operatorname{Re}\langle u, (\tilde{L} - \zeta_0) u\rangle_{L_2(\Omega)} = 0,$$

hence $u = 0$. Thus intersection of sets $C_0^\infty(\Omega)$ and $\mathrm{R}(\tilde{L} - \zeta_0)^{\perp}$ is empty, it implies that

$$(g, v)_{L_2(\Omega)} = 0, \quad \forall g \in \mathrm{R}(\tilde{L} - \zeta_0)^{\perp}, \ \forall v \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense set in $L_2(\Omega)$, it follows that $\mathrm{R}(\tilde{L} - \zeta_0)^{\perp} = 0$. It implies that $\operatorname{def}(\tilde{L} - \zeta_0) = 0$ and if we note (4.18) we arrive to the conclusion that $\operatorname{def}(\tilde{L} - \zeta) = 0$, $\zeta \in \mathbb{C} \setminus \mathfrak{S}$. Hence $\operatorname{def}(\tilde{L} + \zeta) = 0$, $\operatorname{Re}\zeta > 0$ and proof the first relation of (4.19) is complete.

For proving the second relation (4.19) we note that

$$\begin{gathered}(\mu^{-2} + \operatorname{Re}\zeta)\|f\|^2_{L_2(\Omega)} \le \operatorname{Re}\langle f, (\tilde{L} + \zeta) f\rangle_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)} \|(\tilde{L} + \zeta)\|_{L_2(\Omega)}, \\ f \in \mathrm{D}(\tilde{L}), \quad \operatorname{Re}\zeta > 0.\end{gathered}$$

Using the first relation of (4.19) we have

$$\|(\tilde{L} + \zeta)^{-1} g\|_{L_2(\Omega)} \le (\mu^{-2} + \operatorname{Re}\zeta)^{-1}\|g\|_{L_2(\Omega)} \le (\operatorname{Re}\zeta)^{-1}\|g\|_{L_2(\Omega)}, \quad g \in L_2(\Omega).$$

This implies

$$\|(\tilde{L} + \zeta)^{-1}\| \le (\operatorname{Re}\,\zeta)^{-1}, \quad \operatorname{Re}\zeta > 0.$$

The proof of the case corresponding to $\tilde{L}$ is complete. The proof for the case corresponding to $\tilde{L}^{+}$ is analogous if we note the first relation of (4.7).

Consider operator $\tilde{H}$. Obviously that $\tilde{H}$ is symmetric operator. Hence the numerical range of values of operator $\tilde{H}$ belongs to the real axis. Note (4.7),(4.8), using H-convergence and passing to the limit, analogously way of obtaining (4.11) we can get the inequality

$$\langle f, \tilde{H} f\rangle_{L_2(\Omega)} \geq \frac{1}{\mu^2}\|f\|^2_{L_2(\Omega)}.$$

Reasoning as in the proof corresponding to the case of operator $\tilde{L}$ and applying [8, Theorem 3.2 p.336], we conclude that $\operatorname{def}(\tilde{H}-\zeta)=0$, $\operatorname{Im}\zeta \neq 0$ and

$$\operatorname{def}(\tilde{H}+\zeta)=0, \quad \|(\tilde{H}+\zeta)^{-1}\| \leq (\operatorname{Re}\zeta)^{-1}, \quad \operatorname{Re}\zeta>0.$$

The last relations implies that operator $\tilde{H}$ is m-accretive. as well as sectorial, then m-sectorial. In consequence of [8, Theorem 3.16 p.340] operator $\tilde{H}$ is self-adjoint. $\square$

5. Main theorems

Further reasoning would require the use of theory sesquilinear sectorial forms. If not stated, otherwise, we use the definitions and notation from [8]. Consider the forms

$$\begin{gathered}
\mathbf{t}[u,v]=\int_{\Omega} a^{ij} D_i u \overline{D_j v}\, dQ+\int_{\Omega} p\, \mathfrak{D}^{\alpha} u\, \bar{v}\, dQ,\ u,v \in H_0^1(\Omega),\\
\mathbf{t}^*[u,v]=\overline{\mathbf{t}[v,u]}=\int_{\Omega} a^{ij} D_j u \overline{D_i v}\, dQ+\int_{\Omega} u p\, \overline{\mathfrak{D}^{\alpha} v}\, dQ,\\
h=\frac{1}{2}(\mathbf{t}+\mathbf{t}^*)=\mathfrak{Re}\,\mathbf{t}.
\end{gathered}$$

Lemma 5.1. *The form* $\mathbf{t}$ *is a closed sectorial form, moreover* $\mathbf{t}=\tilde{\mathfrak{f}}$*, where*

$$\mathfrak{f}[u,v]=(\tilde{L}u,v)_{L_2}, \quad u,v \in \mathrm{D}(\tilde{L}).$$

Proof. At first we will prove that the following two-sided inequality holds

$$C_0\|f\|^2_{H_0^1} \leq |\mathbf{t}[f]| \leq C_1\|f\|^2_{H_0^1}, \quad f \in H_0^1(\Omega). \tag{5.1}$$

Note that from (4.4), theorem 3.1, (4.2), we obtain a lower estimate (5.1)

$$C_0\|f\|^2_{H_0^1} \leq \operatorname{Re}\mathbf{t}[f] \leq |\mathbf{t}[f]|, \quad f \in H_0^1(\Omega). \tag{5.2}$$

Applying (4.13),(4.14), we obtain the upper estimate (5.1),

$$|\mathbf{t}[f]| \leq \left|(a^{ij} D_i f, D_j f)_{L_2(\Omega)}\right|+\left|(p\, \mathfrak{D}^{\alpha} f, f)_{L_2(\Omega)}\right| \leq C_1\|f\|^2_{H_0^1(\Omega)}, \quad f \in H_0^1(\Omega). \tag{5.3}$$

The proof of the estimates (5.1) is complete. By definition of form $\mathbf{t}$ (5), we have $H_0^1(\Omega)=\mathrm{D}(\mathbf{t}) \subset \mathrm{D}(\tilde{\mathbf{t}})$. If $f \in \mathrm{D}(\tilde{\mathbf{t}})$ then there is a sequence $f_n \underset{\mathbf{t}}{\rightarrow} f$, applying lower estimate (5.1) we can conclude that $f_n \xrightarrow{H_0^1} f$. Hence $f \in H_0^1(\Omega)$, and $\mathrm{D}(\tilde{\mathbf{t}}) \subset H_0^1(\Omega)$. It implies that $\mathrm{D}(\tilde{\mathbf{t}})=\mathrm{D}(\mathbf{t})$ and $\mathbf{t}$ is closed form. Proof of the sectorial property is contained in the proof of theorem 4.2.

Let us prove that $\mathbf{t}=\tilde{\mathfrak{f}}$. At first we need to show that

$$\mathfrak{f}[u,v]=\mathbf{t}[u,v], \quad u,v \in \mathrm{D}(\mathfrak{f}). \tag{5.4}$$

Using the Green's formula, we have

$$(Lu,v)_{L_2} = \mathbf{t}[u,v], \quad u,v \in \mathrm{D}(L). \tag{5.5}$$

Hence we can rewrite the relation (5.1) as

$$C_0\|f\|^2_{H^1_0} \le |(Lf,f)_{L_2}| \le C_1\|f\|^2_{H^1_0}, \quad f \in \mathrm{D}(L). \tag{5.6}$$

Suppose $f \in \mathrm{D}(\tilde{L})$. Since conditions of L-convergence holds then exists sequence $\{f_n\} \subset \mathrm{D}(L)$ such that we have $f_n \underset{L}{\to} f$. From (5.6), (5.1) follows that $f_n \underset{\mathbf{t}}{\to} f$. Now consider the elements $u,v \in \mathrm{D}(\tilde{L})$, as shown above exists a sequences $\{u_n\},\{v_n\}$ including in $\mathrm{D}(L)$ and L-converging, $\mathbf{t}$-converging to u,v respectively. Passing to the limit in the left and right side of (5.5), we obtain (5.4). Hence from (5.4),(5.1) follows

$$C_0\|f\|^2_{H^1_0} \le |\mathfrak{f}[f]| \le C_1\|f\|^2_{H^1_0}, \quad f \in \mathrm{D}(\mathfrak{f}). \tag{5.7}$$

Now we are ready to prove that $\mathbf{t} = \tilde{\mathfrak{f}}$. Note that in consequence of theorem 4.2 operator $\tilde{L}$ is sectorial, hence by theorem 1.27 [8, p. 399], the form $\mathfrak{f}$ has a closure. If $f_0 \in \mathrm{D}(\tilde{\mathfrak{f}})$ then we have $f_n \underset{\mathfrak{f}}{\to} f_0$, $\{f_n\} \subset \mathrm{D}(\tilde{L})$. Applying the estimates (5.7),(5.1) we can easy to see that $f_n \underset{\mathbf{t}}{\to} f_0$ and since $\mathbf{t}$ is closed form, it follows that $f_0 \in \mathrm{D}(\mathbf{t})$. On the other hand let $f_0 \in \mathrm{D}(\mathbf{t})$. Note that the set $C_0^\infty(\Omega) = \mathrm{D}(\mathbf{t})$ is a kernel of form $\mathbf{t}$ in the sense of definition [8, p. 397], it follows from (5.1). Hence exists sequence $\{f_n\} \subset \mathrm{D}(\mathbf{t})$, $f_n \underset{\mathbf{t}}{\to} f_0$. Applying estimates (5.1), (5.7), we have $f_n \underset{\mathfrak{f}}{\to} f_0$, it implies that $f_0 \in \mathrm{D}(\tilde{\mathfrak{f}})$. Now let $u,v \in \mathrm{D}(\tilde{\mathfrak{f}})$, using given above, [8, Theorem 1.17 p. 395] and passing to the limits in the left and right side of inequality (5.4), we obtain

$$\tilde{\mathfrak{f}}[u,v] = \mathbf{t}[u,v], \quad u,v \in \mathrm{D}(\tilde{\mathfrak{f}}).$$

On the other hand if $u,v \in \mathrm{D}(\mathbf{t})$, by analogous way, we obtain

$$\mathbf{t}[u,v] = \tilde{\mathfrak{f}}[u,v], \quad u,v \in \mathrm{D}(\mathbf{t}).$$

Hence $\mathbf{t} = \tilde{\mathfrak{f}}$. □

Lemma 5.2. *The form h is a closed symmetric sectorial form, moreover $h = \tilde{\mathfrak{k}}$, where*

$$\mathfrak{k}[u,v] = (\tilde{H}u,v)_{L_2}, \quad u,v \in \mathrm{D}(\tilde{H}).$$

Proof. The implementation of the symmetric property in the sense of [8, (1.5) p.387] follows from the definition of form h. It is sufficient to note that

$$h[u,v] = \frac{1}{2}(t[u,v] + \overline{t[v,u]}) = \frac{1}{2}\overline{(t[v,u] + \overline{t[u,v]})} = \overline{h[v,u]}, \quad u,v \in \mathrm{D}(h).$$

It is obvious that $h[f] = \mathrm{Re}\,\mathbf{t}[f]$. Hence applying the lower estimate of (5.2), estimate (5.3), we have

$$C_0\|f\|_{H^1_0} \le h[f] \le C_1\|f\|_{H^1_0}, \quad f \in H^1_0(\Omega). \tag{5.8}$$

Using h-convergence, it is easy to see that from lower estimate of (5.8) consequences that form h is closed. Proof of the sectorial property contains in the proof of theorem 4.2.

Let us prove that $h = \tilde{\mathfrak{k}}$. At first we need to show that

$$\mathfrak{k}[u,v] = h[u,v], \quad u,v \in \mathrm{D}(\mathfrak{k}). \tag{5.9}$$

Note the definition of operator H, applying the Green's formula, lemmas 2.5 and 2.6, we have

$$(Hu,v)_{L_2} = h[u,v], \quad u,v \in \mathrm{D}(H). \tag{5.10}$$

Using the previous equality we can rewrite estimate (5.8) as follows

$$C_0\|f\|_{H_0^1} \le (Hf,f)_{L_2} \le C_1\|f\|_{H_0^1}, \quad f \in \mathrm{D}(H). \tag{5.11}$$

Note that in consequence of lemma 4.1 operator H has a closure and $\mathrm{D}(\mathfrak{k}) \subset H_0^1(\Omega)$. Suppose $f \in \mathrm{D}(\tilde{H})$, since conditions of H-convergence holds, then exists $\{f_n\} \subset \mathrm{D}(H)$, such that we have $f_n \underset{H}{\longrightarrow} f$. From (5.11) and(5.8) it follows that $f_n \underset{h}{\to} f$. Now consider the elements $u,v \in \mathrm{D}(\tilde{H})$, as shown above exists sequences $\{u_n\},\{v_n\}$ including in $\mathrm{D}(H)$ and H-converging, h-converging to u,v respectively. Passing to the limit on the left and right side of (5.10) we obtain (5.9). From inequalities (5.9) and (5.8) it follows

$$C_0\|f\|_{H_0^1} \le \mathfrak{k}[f] \le C_1\|f\|_{H_0^1}, \quad f \in \mathrm{D}(\tilde{H}). \tag{5.12}$$

Now we are ready to prove that $h = \tilde{\mathfrak{k}}$. Note that by theorem 4.2 operator $\tilde{H}$ is sectorial, hence by [8, Theorem 1.27 p. 399] the form $\mathfrak{k}$ has a closure. If $f_0 \in \mathrm{D}(\tilde{\mathfrak{k}})$ then we have $f_n \underset{\mathfrak{k}}{\to} f_0$, $\{f_n\} \subset \mathrm{D}(\tilde{H})$. Applying the estimates (5.12) and (5.8) we can easy to see that $f_n \underset{h}{\to} f_0$ and since h is a closed form, then $f_0 \in \mathrm{D}(h)$. On the other hand let $f_0 \in \mathrm{D}(h)$. Note that the set $C_0^\infty(\Omega) = \mathrm{D}(h)$ is the kernel of form h in the sense of definition [8, p. 397], it follows from (5.8). Hence exists sequence $\{f_n\} \subset \mathrm{D}(h)$, $f_n \underset{h}{\to} f_0$. Applying estimates (5.8),(5.12), we have $f_n \underset{\mathfrak{k}}{\to} f_0$, it implies that $f_0 \in \mathrm{D}(\tilde{\mathfrak{k}})$. Now let $u,v \in \mathrm{D}(\tilde{\mathfrak{k}})$, using given above, [8, Theorem 1.17 p. 395] and passing to the limits in the left and right side of inequality (5.9), we obtain

$$\tilde{\mathfrak{k}}[u,v] = h[u,v],\ u,v \in \mathrm{D}(\tilde{\mathfrak{k}}).$$

On the other hand if $u,v \in \mathrm{D}(h)$, by an analogous way, we obtain

$$h[u,v] = \tilde{\mathfrak{k}}[u,v], \quad u,v \in \mathrm{D}(h).$$

This implies $h = \tilde{\mathfrak{k}}$. □

Theorem 5.3. *Operator $\tilde{H}$ has a compact resolvent, discrete spectrum, and the following estimate for eigenvalues of operator $\tilde{H}$ holds*

$$\lambda_n(L_0) \le \lambda_n(\tilde{H}) \le \lambda_n(L_1),\ n \in \mathbb{N}, \tag{5.13}$$

where $\lambda_n(L_k)$ are respectively eigenvalues of operators with constant coefficients defined by operator L

$$L_k f = -a_k^{ij} D_j D_i f + p_k f, \quad f \in \mathrm{D}(L), \quad k = 0,1. \tag{5.14}$$

Proof. First we need to prove three propositions.

(i) Operator $\tilde{H}$ is positive defined. For proving this fact note that operator $\tilde{H}$ has domain of definition is dense in $L_2(\Omega)$; the symmetric property of operator $\tilde{H}$ follows from the definition; in consequently of theorem 3.1, $\tilde{H}$ is positive and bounded below. Hence one is positive defined.

(ii) The space $H_0^1(\Omega)$ coincides as a set of elements, with energetic spaces $\mathfrak{H}_{\tilde{H}}, \mathfrak{H}_{L_k}$, $k=0,1$. We must note that
$$\|f\|_{\mathfrak{H}_{\tilde{H}}} = h[f], \quad f \in H_0^1(\Omega), \tag{5.15}$$
it follows from reasoning of lemma 5.2.

(iii) We have a following estimates for energetic norms
$$\|f\|_{\mathfrak{H}_{L_0}} \leq \|f\|_{\mathfrak{H}_{\tilde{H}}} \leq \|f\|_{\mathfrak{H}_{L_1}},\, f \in H_0^1(\Omega). \tag{5.16}$$

Applying reasoning that was used to obtain the inequality (5.1), we have equivalence of norms H_0^1 and $\mathfrak{H}_{L_k}$, $k=0,1$. In particular we can easy to see that exists operators (5.14) L_0, L_1, such that the next inequalities holds
$$\|f\|_{\mathfrak{H}_{L_0}} \leq C_0\|f\|_{H_0^1}, \quad C_1\|f\|_{H_0^1} \leq \|f\|_{\mathfrak{H}_{L_1}}, \quad f \in H_0^1(\Omega). \tag{5.17}$$
Hence from (5.8), (5.15) and (5.17), it follows (5.16).

Now we can prove the main statements of this theorem. In (i) we proved that the operators $L_0, L_1, \tilde{H}$ is positive defined. Note given above, it is easy to see that the norms H_0^1, $\mathfrak{H}_{\tilde{H}}$, $\mathfrak{H}_{L_k}$, $k=0,1$ are equivalence. Applying the Rellich-Kondrashov theorem we have that the energetic spaces: $\mathfrak{H}_{\tilde{H}}$, $\mathfrak{H}_{L_k}$, $k=0,1$ are compactly embedded in $L_2(\Omega)$. Using [17, Theorem 3 p.216] we have that the operators $L_0, L_1, \tilde{H}$ has a discrete spectrum. In consequence of theorem 3.1 operator $\tilde{H}$ is semibounded from below with positive constant, in consequence of theorem 4.3 one is self-adjoint. Hence a zero belongs to a resolvent set of operator $\tilde{H}$. In consequence of [17, Theorem 5 p.222] operator $\tilde{H}$ has a compact resolvent at point zero. Hence in accordance with [8, Theorem 6.29 p.237] operator $\tilde{H}$ has a compact resolvent on the resolvent set.

Using (i), (ii), (iii), we have
$$L_0 \leq \tilde{H} \leq L_1,$$
where order relation is understood in terms of [17, p.225]. Since as mentioned above operators $L_0, L_1, \tilde{H}$ has a discrete spectrum, then using theorem 1 [17, p.225] we obtain (5.13). □

Theorem 5.4. *Operator $\tilde{L}$ has a compact resolvent, and discrete spectrum.*

Proof. Note that by theorem 4.3 operators $\tilde{L}, \tilde{H}$ is m-sectorial, operator $\tilde{H}$ is self-adjoint. Applying lemma 5.1, lemma 5.2, [8, Theorem 2.9, p.409] we have that $T_{\mathbf{t}} = \tilde{L}$, $T_h = \tilde{H}$, where $T_{\mathbf{t}}$, T_h are respectively extensions by Fridrichs of operators $\tilde{L}$, $\tilde{H}$ (see [8, p.409]). Since in accordance with definition [8, p.424] operator $\tilde{H}$ is a real part of operator $\tilde{L}$, then in consequences of Theorem 5.3 and [8, Theorem 3.3 p.424] operator $\tilde{L}$ has a compact resolvent, hence by [8, Theorem 6.29, p.237] spectrum of operator $\tilde{L}$ is discrete. □

Remark 5.5. It is easy to see that the Kypriaynov operator in the one-dimensional case reduces to the Marchaud operator, along with that the results of this work are true only for dimensions $2 \leq n < \infty$, it follows from from the conditions of theorem 3.1. However we can apply obtained technique for the cases corresponding to the Marchaud operator and Riemann-Liouville operator by using [14, Corollary 1], which establishes the strong accretive property for these operators.

Conclusions. This paper presents results cocerning the field of spectral theory of operators of fractional differentiation. It proves a number of propositions which represent independent interest in the theory of fractional calculus. It also introduces a new construction of multidimensional fractional integral in a direction. Having formulated the sufficient conditions of representability functions by the fractional integral in a direction, in particular it proves the embedding of a Sobolev space in classes of functions representable by the fractional integral in direction.

Note that the technique of proof borrowed from the one-dimensional case is of particular interest. It should be noted that the constructed extension of Kipriyanov operator, was found as a conjugate operator. These all create a complete picture reflecting the qualitative properties of fractional differential operators. It should be noted that in as main new results, there were proven the following theorems: the theorem establishing a strong accretive property for the operator of fractional differentiation in the Kyprianov sense, the theorem establishing a sectorial property for differential operator second order with operator of fractional differentiation in lower terms. For this operator it was proven the theorem establishing the maximum accretive property. Proven a theorem on the discreteness of the spectrum of real part of operator, obtained two-sided estimate of its eigenvalues. As the main result a theorem on the discreteness of the spectrum of differential operator second order with fractional derivative in the lower terms was proven. With the help of the theory of bilinear forms we obtained general theoretical results for differential operators second order with fractional derivative in lower terms. In this paper we consider the proof corresponding to the multidimensional case, however, is possible to have reduction to the one-dimensional case. For example the one-dimensional case was described in [13]. It should also be noted that the results in this direction can be treated on the real axis. It is noteworthy that the use of bilinear forms as a tool to study differential operator second order with fractional derivative in lower terms, this gives the opportunity to see a dominant of senior term in the manifestation of the functional properties of operator. This technique is new and can be used for study the spectrum of the perturbed operator of fractional differentiation. Therefore, regardless of the results the idea of the proof may be of interest.

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