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EXISTENCE OF SOLUTIONS FOR A BVP OF A SECOND ORDER FDE AT RESONANCE BY USING KRASNOSELSKII'S FIXED POINT THEOREM ON CONES IN THE $\mathcal{L}^1$ SPACE

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Abstract. We provide sufficient conditions for the existence of positive solutions of a nonlocal boundary value problem at resonance concerning a second order functional differential equation. The method is developed by inserting an exponential factor which depends on a suitable positive parameter λ. By this way a Green's kernel can be established and the problem is transformed into an operator equation $u = T_\lambda u$. As it can be shown the well known Krasnoselskii's fixed point theorem on cones in the Banach space $C[0,1]$ cannot be applied. More exactly, there is no (positive) value of the parameter λ for which the condensing property $\|T_\lambda u\| \leq \|u\|$, with $\|u\| = \rho(> 0)$ is satisfied. To overcome this fact we enlarge the space $C[0,1]$ and work in $\mathcal{L}^1[0,1]$ where, now, Krasnoselskii's fixed point theorem is applicable. Compactness criteria in this space are, certainly, needed.

1. Introduction

Let p be a function defined on the set $[0,1] \times \mathcal{L}^1[0,1]$ into the positive real numbers. In this paper we are dealing with the existence of positive solutions of a boundary-value problem concerning a second order functional differential equation of the form

$$(Lu)(t) := u''(t) = -p(t,u)u(t), \quad \text{a.a. } t \in [0,1] =: I, \quad u(0) = 0 \tag{1.1}$$

under the boundary value condition

$$(Tu) := u'(1) - \int_0^1 u'(r)dg(r) = 0, \tag{1.2}$$

where g is a nondecreasing function such that

$$\int_0^1 dg(s) = 1. \tag{1.3}$$

Under this condition we observe that

$$\ker(L) \subseteq \ker(T).$$

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Hence no Green's function can directly be computed. (The latter means that any (nontrivial) solution $u(t) = kt$ of the problem $u'' = 0$, $u(0) = 0$ satisfies the boundary conditions.) Thus the problem is at resonance. In the literature such problems are approached in several ways. The more classical one is to decompose the space we work in the form of a direct sum of subspaces, one of which is $ker(L)$ and then to work with the corresponding projections on these spaces. For this method, and for another search on boundary value problems at resonance, one can consult, for instance, [26, 32, 38, 40] and the references therein. In several works one can see the application of the so called coincidence degree theory of Mawhin [32, 33, 36], where the key tool is the following fact:

Let A be the one-dimensional space of linear functions $y_c(t) = ct$, $t \in I$, with $c \in \mathbb{R}$ and let $J : A \to \mathbb{R}$ be the natural isomorphism $J(y_c) = c$. Then for each open and bounded subset U of A with $0 \in U$, it holds

$$\deg(J, U, 0) = sign_{J(y_c)=0} J'(y_c) = sign J'(0) \neq 0.$$

Another interesting way of investigation of such problems, is by a regularization process based on variational methods, see, e.g, [3, 4, 16, 20, 24, 25, 37, 39].

Nonlocal boundary value problems for ordinary differential equations arise in several branches of applied mathematics and physics. The study of such problems with linear and nonlinear ordinary differential equations was, mainly, initiated in [7, 12, 13, 15, 18]. The methods used therein is based on Leray - Schauder continuation theorem, nonlinear alternatives of Leray - Schauder, coincidence degree theory and some fixed point theorems, see, e.g., [1, 2, 6, 17] and references therein.

In this paper we suggest a new method relying on a transformation of the original equation by using an additive factor of the form $e^{\lambda t}$, for a suitable positive parameter λ. The idea was established in a boundary value problem discussed in [19], (where in Assumption (H1) one can get $p = \lambda - \lambda$ and $q = 0$.) The resulting equation for some specific values of the parameter may permit us to obtain a Green's function and then apply the well known Krasnoselskii's fixed point theorem on cones, [31], which states as follows.

Theorem 1.1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$$

be a completely continuous operator such that either

$$\|Au\| \leq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_1 \quad \textit{and} \quad \|Au\| \geq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_2,$$

or

$$\|Au\| \geq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_1 \quad \textit{and} \quad \|Au\| \leq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_2.$$

Then A has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Here we shall apply Theorem 1.1 on the Banach space $\mathcal{L}^1(I)$ and to the best of our knowledge, it is the first paper where a space different than $C(I)$ is used. The actual reason to apply such an idea is written in the footnote of Theorem 3.7. Therefore here we have to spend more space of the paper on notions concerning some topological means (such as closedness and compactness) of $\mathcal{L}^1(I)$.

Recall that an operator $A : X \to Y$ is *completely continuous* if it is continuous and maps bounded sets into precompact sets. A set is precompact if its closure is compact.

Let us agree on the motation we shall use here.

Let $\mathbb{R}$ be the real line with norm $|\cdot|$ and $\mathbb{R}^+$ the set of nonnegative reals. In this work we shall work in two Banach spaces:

The space $C(I)$ of all continuous functions $x : I \to \mathbb{R}$ with norm $\|\cdot\|_{\sup}$, and its superspace $\mathcal{L}^1(I)$ with norm $\|\cdot\|_{\mathcal{L}^1}$. It is known that it holds

$$\|u\|_{\mathcal{L}^1} \le \|u\|_{\sup},$$

for any $u \in C(I)$. We shall denote by $C_+(I)$ the set of nonnegative functions $u \in C(I)$ and by $\mathcal{L}^1{}_+(I)$ the set of all $u \in \mathcal{L}^1(I)$ with $u(t) \ge 0$, a.e. in I.

Our first essential step is to reformulate the boundary value problem into a fixed point problem of an operator acting on the space $\mathcal{L}^1(I)$, thus the fixed point u of the operator will be a function in this space. However, as it is shown, any fixed point u of this operator is a solution of the boundary value problem, thus u must solve the original differential equation (1.1), namely it is a differentiable function. At least u is an element of the space $C(I)$.

Since we use compactness in the $\mathcal{L}^1$-sense, we need to recall the classical well known Kolmogorov - Riesz - Fréchet Compactness Theorem, (see, e.g. the book by Brezis [8, p. 111], or a recent article [14, Theorem 5 and Cor. 8]) which reads as follows: (Notice that a set in a semimetric space is compact if it closed and totally bounded.)

Theorem 1.2 (Kolmogorov - Riesz - Fréchet). *Let $1 \le p < +\infty$. A subset $\mathcal{F}$ of $\mathcal{L}^p(\mathbb{R}^n)$ is totally bounded if and only if*

(i) *$\mathcal{F}$ is bounded,*

(ii) *for every $\epsilon > 0$ there is some R such that for every $f \in \mathcal{F}$ it holds*

$$\int_{|x|>R} |f(x)|^p dx < \epsilon^p,$$

(iii) *the family $\mathcal{F}$ is equicontinuous in mean, in the sense that for every $\epsilon > 0$ there is some $\rho > 0$ so that, for every $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$, with $|y| < \rho$ it holds*

$$\|f(\cdot + y) - f(\cdot)\|_p = \Big[\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p dx\Big]^{1/p} < \epsilon.$$

We shall apply this theorem adjusted to $\mathcal{L}^1(I)$, where, clearly, condition (ii) is satisfied. (Extend any $u \in \mathcal{L}^1(I)$ to $\mathcal{L}^1(\mathbb{R})$, by setting $u(s) = 0$ for $s \in \mathbb{R} \setminus I$).

Problem (1.1)-(1.2) is a nonlocal boundary value problem which includes as special cases multipoint boundary value problems considered by several authors during the previous decade. More details about this problem can be found e.g. in the papers [26, 27, 28, 29, 30] and into their references. Moreover due to the functional dependence of p on u, the equation can have an integro-differential form, or generally, a functional-differential form. In the literature one can see a lot of works concerning boundary value problems with delay or functional dependence, of first, second or third order, see, e.g., [9, 10, 23, 21, 22, 41] and the references therein.

The paper is organized as follows: In Section 2 we present the conditions satisfied by the functions g and p. Then we reformulate the boundary value problem and built an integral operator equation equivalent to the problem. The main result is given in the end of Section 3, but first we give several auxiliary lemmas concerning properties of the kernel and refer to some (known) topological properties of the space $\mathcal{L}^1(I)$. The paper closes with a simple application of the existence result.

2. Conditions and reformulation of the BVP problem

Before we present our conditions, we make the following convention: Assume that X, Y are two normed spaces with norms $\|\cdot\|_X, \|\cdot\|_Y$ respectively. Then, the phrase J is $[\|\cdot\|_X - \|\cdot\|_Y]$-continuous will be used to emphasize the continuity of a function $J : X \to Y$. To proceed we need to present our basic conditions:

Hypothesis 2.1. The function g satisfies (1.3), as well as the following: There are $\tau_1, \tau_2, \tau_3 \in (0, 1)$ with $\tau_1 < \tau_2 < \tau_3$ and $\tau_2 + \tau_3 < 1$, such that

$$0 = g(\tau_1) < g(\tau_2) < g(\tau_3) \leq g(1) = 1.$$

Example 2.2. The function g defined by

$$g(s) := \frac{1}{5}\chi_{[\frac{1}{5},\frac{2}{5})}(s) + \frac{2}{5}\chi_{[\frac{2}{5},\frac{3}{5})}(s) + \chi_{[\frac{3}{5},1]}(s),$$

satisfies Hypothesis 2.1, with $\tau_1 = \frac{1}{10}$, $\tau_2 := \frac{1}{5}$ and $\tau_3 := \frac{1}{2}$.

Note that Hypothesis 2.1 implies the following useful inequality:

$$\begin{aligned} 1 - \int_0^1 r^2 dg(r) = \int_0^1 (1 - r^2) dg(r) &\geq \int_{\tau_2}^{\tau_3} (1 - r^2) dg(r) \\ &\geq (1 - \tau_3^2)(g(\tau_3) - g(\tau_2)) > 0. \end{aligned} \tag{2.1}$$

As we said in the beginning, the function p maps the set $I \times \mathcal{L}^1(I)$ into the positive real numbers. Assume that $p(t, 0) = 0$, for all $t \in I$ and moreover it satisfies the following conditions:

Hypothesis 2.3. $p(\cdot, u)$ is continuous in $t \in I$ for all u and $p(t, \cdot)$ is $[\|\cdot\|_{\mathcal{L}^1}\| - |\cdot|]$-continuous uniformly for all t.

Hypothesis 2.4. The mapping $u \to p(t, u) : \mathcal{L}^1(I) \to [0, +\infty)$ sends bounded sets into bounded sets, uniformly for $t \in I$.

Hypothesis 2.5. There exists an interval $(\alpha, \beta) \subseteq [\tau_2, \tau_3]$ with the following properties: Given any $R > 0$ there is some $N > 0$ such that for any $u \in {\mathcal{L}^1}_+(I)$ with $u(s) \geq N$, for a.a. $s \in [\tau_2, \tau_3]$, it holds $p(t, u) \geq R$, for all $t \in [\alpha, \beta]$.

Because of Hypothesis 2.5, the dependence of p on the argument u cannot be point-wise. Our requirement is that p depends on the values of u at least on an open subinterval of $[\tau_2, \tau_3]$. Thus its dependence would be, for example, through an integral. Here is such an example (which will be used in an application given at the end of this paper):

Example 2.6. Consider the function

$$p(t, u) := (1 + t) \int_0^t a(t, s) u(s) ds,$$

where $a(t, s)$ is continuous in $t, s \in I$. If there is a constant k_1 such that $0 < k_1 \leq a(t, s)$, then p satisfies the Conditions 2.3, 2.4, 2.5.

Proof. Hypothesis 2.3 is obvious. Hypothesis 2.4 follows from the fact that

$$p(t,u) \le 2k_2\|u\|_{\mathcal{L}^1(I)},$$

where $k_2 := \max_{t,s\in I} a(t,s)$. Now choose an open interval $(\alpha,\beta) \subseteq [\tau_2,\tau_3]$ by taking, for instance, the constants

$$\alpha := \frac{2}{3}\tau_2 + \frac{1}{3}\tau_3 \quad \text{and} \quad \beta := \frac{1}{3}\tau_2 + \frac{2}{3}\tau_3. \tag{2.2}$$

Then Hypothesis 2.5 is implied by the inequality

$$p(t,u) \ge (1+t)k_1 \int_{\tau_2}^{\alpha} u(s)ds \ge k_1 N(\alpha-\tau_2) = \frac{1}{3}k_1 N(\tau_3-\tau_2) \ge R.$$

The last inequality holds if we choose $N \ge 3R/k_1(\tau_3-\tau_2)$. □

Next we proceed to the formulation of the problem. We shall elaborate a little on the equation and the boundary conditions, in order to built an integral form of the problem.

Fix a positive parameter λ, which will be defined later in Section 3, and consider the auxiliary equation

$$u''(t) + \lambda u'(t) = \lambda u'(t) - p(t,u)u(t), \quad \text{a.a. } t \in [0,1],$$

associated with the initial value $u(0) = 0$ and the boundary value condition (1.3). Multiple both sides with $e^{\lambda t}$ and integrate to obtain

$$u'(t) - \lambda u(t) = e^{-\lambda t}u'(0) - e^{-\lambda t}\int_0^t e^{\lambda s}F(s,u)ds,$$

where F is the function

$$F(s,u) := \lambda^2 u(s) + p(s,u)u(s). \tag{2.3}$$

(According to Hypothesis 2.3 the function F is integrable.) Again, multiplying both sides with $e^{-\lambda t}$ and integrating we get

$$u(t) = \frac{u'(0)}{\lambda}\sinh(\lambda t) - \int_0^t e^{\lambda(t-2r)}\int_0^r e^{\lambda s}F(s,u)\,ds\,dr.$$

From this it follows that

$$u'(t) = u'(0)\cosh(\lambda t) - \lambda\int_0^t e^{\lambda(t-2r)}\int_0^r e^{\lambda s}F(s,u)\,ds\,dr - e^{-\lambda t}\int_0^t e^{\lambda s}F(s,u)ds.$$

To eliminate the factor $u'(0)$ we use (1.2). Thus we must have

$$\begin{aligned}
&u'(0)\cosh(\lambda) - \lambda\int_0^1 e^{\lambda(1-2r)}\int_0^r e^{\lambda s}F(s,u)\,ds\,dr - e^{-\lambda}\int_0^1 e^{\lambda s}F(s,u)ds\\
&= u'(0)\int_0^1 \cosh(\lambda t)dg(t) - \lambda\int_0^1\int_0^t e^{\lambda(t-2r)}\int_0^r e^{\lambda s}F(s,u)\,ds\,dr dg(t)\\
&\quad - \int_0^1 e^{-\lambda t}\int_0^t e^{\lambda s}F(s,u)\,ds\,dg(t)
\end{aligned} \tag{2.4}$$

Next, we apply Fubini's theorem and get the following:

$$\lambda\int_0^1 e^{\lambda(1-2r)}\int_0^r e^{\lambda s}F(s,u)\,ds\,dr = \int_0^1 \sinh(\lambda(1-s))F(s,u)ds,$$

$$
\begin{aligned}
\lambda & \int_0^1 \int_0^t e^{\lambda(t-2r)} \int_0^r e^{\lambda s} F(s,u)\,ds\,dr\,dg(t) \\
&= \int_0^1 \int_0^t \sinh(\lambda(t-s)) F(s,u)\,ds\,dg(t) \\
&= \int_0^1 \int_s^1 \sinh(\lambda(r-s)) dg(r) F(s,u) ds,
\end{aligned}
$$

$$
\int_0^1 e^{-\lambda r} \int_0^r e^{\lambda s} F(s,u)\,ds\,dg(r) = \int_0^1 \Big(\int_s^1 e^{-\lambda(r-s)} dg(r) \Big) F(s,u) ds.
$$

Hence (2.4) gives

$$
u'(0) = \frac{1}{h_\lambda(0)} \int_0^1 h_\lambda(s) F(s,u) ds,
$$

where the function h is defined by

$$
h_\lambda(s) := \cosh(\lambda(1-s)) - \int_s^1 \cosh(\lambda(r-s)) dg(r), \quad s \in I.
$$

Therefore the solution u satisfies the integral equation

$$
u(t) = \frac{1}{\lambda h_\lambda(0)} \sinh(\lambda t) \int_0^1 h_\lambda(s) F(s,u) ds - \int_0^t \int_0^r e^{\lambda(t+s-2r)} F(s,u)\,ds\,dr,
$$

which can be written as

$$
u(t) = \int_0^1 G_\lambda(t,s) F(s,u) ds, \tag{2.5}
$$

where the (one parameter) Green's function G_λ is defined by

$$
G_\lambda(t,s) := \frac{\sinh(\lambda t)}{\lambda h_\lambda(0)} h_\lambda(s) - \frac{1}{\lambda} \sinh(\lambda(t-s)) \chi_{[0,t]}(s).
$$

Function h_λ plays an important role in all the sequel. Here we must notice that, due to Hypothesis 2.1, it holds $dg(s) = 0$ for all s near zero. Hence, in such a neighborhood, h_λ can be differentiated:

$$
h'_\lambda(s) = -\lambda \sinh(\lambda(1-s)) + \lambda \int_s^1 \sinh(\lambda(r-s)) dg(r), \quad s \in (0, \tau_1). \tag{2.6}
$$

Claim: Function h_λ is positive and it stays away form zero.

To prove it observe that for all $s \in [0,1]$ it holds

$$
\begin{aligned}
h_\lambda(s) &= \cosh(\lambda(1-s)) - \int_s^1 \cosh(\lambda(r-s)) dg(r) \\
&= \cosh(\lambda(1-s)) g(s) + \int_s^1 [\cosh(\lambda(1-s)) - \cosh(\lambda(r-s))] dg(r).
\end{aligned}
$$

Let $s \le \tau_2$. Then we have

$$
\begin{aligned}
h_\lambda(s) &\ge \int_s^1 [\cosh(\lambda(1-s)) - \cosh(\lambda(r-s))] dg(r) \\
&\ge \int_{\tau_2}^{\tau_3} [\cosh(\lambda(1-s)) - \cosh(\lambda(r-s))] dg(r) \\
&\ge [\cosh(\lambda(1-\tau_2)) - \cosh(\lambda \tau_3)](g(\tau_3) - g(\tau_2)).
\end{aligned} \tag{2.7}
$$

Also, for $s \geq \tau_2$ we have

$$h_\lambda(s) \geq \cosh(\lambda(1-s))g(s) \geq g(\tau_2).$$

Hence for all $s \in [0,1]$ we have

$$b \leq h_\lambda(s) \leq \cosh(\lambda),$$

where b is the real number given by

$$b := \min\{g(\tau_2), [\cosh((\lambda(1-\tau_2)) - \cosh(\lambda\tau_3)][g(\tau_3) - g(\tau_2)]\}, \tag{2.8}$$

which is positive, and the Claim is proved.

Now we need to give some properties of G_λ. Clearly, for $s > t$ it becomes

$$G_\lambda(t,s) = \frac{\sinh(\lambda t)}{\lambda h_\lambda(0)} h_\lambda(s)$$

and for $s \leq t$,

$$G_\lambda(t,s) = \frac{1}{\lambda h_\lambda(0)}[\sinh(\lambda t)h_\lambda(s) - h_\lambda(0)\sinh(\lambda(t-s))]. \tag{2.9}$$

We rewrite the quantity in the brackets as

$$\begin{aligned}
&\sinh(\lambda t)\big[\cosh(\lambda(1-s)) - \int_s^1 \cosh(\lambda(r-s))dg(r)\big] \\
&- [\cosh(\lambda) - \int_0^1 \cosh(\lambda r)dg(r)]\sinh(\lambda(t-s)) \\
&= \sinh(\lambda t)\cosh(\lambda(1-s)) - \cosh(\lambda)\sinh(\lambda(t-s)) \\
&\quad + \int_s^1 \Big[\sinh(\lambda(t-s))\cosh(\lambda r) - \sinh(\lambda t)\cosh(\lambda(r-s))\Big]dg(r) \\
&\quad + \sinh(\lambda(t-s))\int_0^s \cosh(\lambda r)dg(r) \\
&= \frac{1}{4}[(e^{\lambda t} - e^{-\lambda t})(e^{\lambda(1-s)} + e^{-\lambda(1-s)}) - (e^{\lambda} + e^{-\lambda})(e^{\lambda(t-s)} - e^{-\lambda(t-s)})] \\
&\quad + \frac{1}{4}\int_s^1 \Big[[(e^{\lambda(t-s)} - e^{-\lambda(t-s)})(e^{\lambda r} + e^{-\lambda r}) \\
&\quad - (e^{\lambda t} - e^{-\lambda t})(e^{\lambda(r-s)} + e^{-\lambda(r-s)})\Big]dg(r) + \sinh(\lambda(t-s))\int_0^s \cosh(\lambda r)dg(r)
\end{aligned}$$

which is equal to

$$\begin{aligned}
&\sinh(\lambda s)\cosh(\lambda(1-t))g(s) + \int_0^s \sinh(\lambda(t-s))\cosh(\lambda r)dg(r) \\
&+ \int_s^1 \sinh(\lambda s)[\cosh((\lambda(1-t)) - \cosh(\lambda(r-t))]dg(r).
\end{aligned}$$

Therefore the Green's function, for $s \leq t$, becomes

$$\begin{aligned}
&G_\lambda(t,s) \\
&= \frac{1}{\lambda h_\lambda(0)}\Big[\sinh(\lambda s)\cosh(\lambda(1-t))g(s) + \int_0^s \sinh(\lambda(t-s))\cosh(\lambda r)dg(r) \\
&\quad + \int_s^1 \sinh(\lambda s)[\cosh((\lambda(1-t)) - \cosh(\lambda(r-t))]dg(r)\Big].
\end{aligned} \tag{2.10}$$

The previous presentation idicates that the Green's function is strictly positive for all $t, s \in (0, 1]$.

Lemma 2.7. *Let T_λ be the operator defined on the space $\mathcal{L}_1(I)$ by*

$$(T_\lambda u)(t) = \int_0^1 G_\lambda(t, s)F(s, u)ds. \tag{2.11}$$

Then a function $u \in \mathcal{L}^1_+(I)$ solves problem (1.1)-(1.3) *if and only if it is (a differentiable function which is) a fixed point of the operator equation $u = T_\lambda u$.*

Proof. The "if" part was proved in the lines above. To show the "only if" part, write $u = T_\lambda u$ in the form (2.5). Then observe first that $(T_\lambda u)(0) = 0$ and u is differentiable, satisfying

$$u'(t) = \frac{\cosh(\lambda t)}{h_\lambda(0)} \int_0^1 h_\lambda(s)F(s, u)ds - \int_0^t \cosh(\lambda(t - s))F(s, u)ds.$$

This formula implies that u is a C^2 function. For $t = 0$, we have

$$u'(0) = \frac{1}{h_\lambda(0)} \int_0^1 h_\lambda(s)F(s, u)ds, \tag{2.12}$$

and so u satisfies

$$u'(t) = \cosh(\lambda t)u'(0) - \int_0^t \cosh(\lambda(t - s))F(s, u)ds.$$

This gives

$$u'(t) - \lambda u(t) = e^{-\lambda t}u'(0) - \int_0^t e^{-\lambda(t-s)}F(s, u)ds,$$

and so

$$e^{\lambda t}[u'(t) - \lambda u(t)] = u'(0) - \int_0^t e^{\lambda s}(\lambda^2 u(s) + p(s, u)u(s))ds.$$

By differentiation we see that u solves equation (1.1).

To show that condition in (1.2) is satisfied, we must prove that it holds

$$\begin{aligned}
&\cosh(\lambda)u'(0) - \int_0^1 \cosh(\lambda(1 - s))F(s, u)ds \\
&= u'(0) \int_0^1 \cosh(\lambda r)dg(r) - \int_0^1 \int_0^r \cosh(\lambda(r - s))F(s, u)\, ds\, dg(r).
\end{aligned}$$

This relation is equivalent to

$$\begin{aligned}
u'(0)h_\lambda(0) &= u'(0)[\cosh(\lambda) - \int_0^1 \cosh(\lambda r)dg(r)] \\
&= \int_0^1 \cosh(\lambda(1 - s))F(s, u)ds - \int_0^1 \int_0^r \cosh(\lambda(r - s))F(s, u)\, ds\, dg(r) \\
&= \int_0^1 \cosh(\lambda(1 - s))F(s, u)ds - \int_0^1 \int_s^1 \cosh(\lambda(r - s))dg(r)F(s, u)ds \\
&= \int_0^1 [\cosh(\lambda(1 - s)) - \int_s^1 \cosh(\lambda(r - s))dg(r)]F(s, u)ds.
\end{aligned}$$

namely (2.12). Thus (1.2) is true. $\square$

3. Main results

From now on we shall assume that the parameter λ satisfies the following condition:

Hypothesis 3.1. There exist positive real numbers λ, σ, δ such that

$$\lambda^2 \sup_{s\in I} \int_0^1 G_\lambda(t,s)dt \le \sigma < 1, \tag{3.1}$$

$$\inf_{t\in[\alpha,\beta]} P_\lambda(t) \ge \delta > 0, \tag{3.2}$$

where the function P_λ is defined by

$$P_\lambda(t) := \cosh(\lambda(1-t)) - \int_0^1 \cosh(\lambda(r-t))dg(r). \tag{3.3}$$

For instance the function g defined in Example 2.2 satisfies Hypothesis 3.1, for all $\lambda \in (0, 0.85]$ and α, β defined, for example, as in (2.2) (This fact can be justified by applying a standard numerical process). With $\lambda = 0.85$, the constant δ can take any value in the interval $(0, 0.78]$. From now on we shall use such a λ.

Lemma 3.2. *For each $t > 0$ it holds*

$$\lim_{s\to 0} \frac{G_\lambda(t,s)}{G_\lambda(s,s)} = \frac{P_\lambda(t)}{h_\lambda(0)}. \tag{3.4}$$

Proof. Fix any $t \in I$. Since $G_\lambda(t,0) = 0 = G_\lambda(0,0)$ we shall apply L' Hospital's rule to find the desired limit To this end we need to know the limit of $\frac{d}{ds}G_\lambda(t,s)$ as $s \to 0$. First we observe that for all $s \in (0, \tau_1]$ it holds

$$\begin{aligned}
&\frac{1}{s}\Big[\int_s^1 \cosh(\lambda(r-s)dgr - \int_0^1 \cosh(\lambda r)dg(r)\Big] \\
&= \frac{1}{s}\Big[\int_0^1 [\cosh(\lambda(r-s)dgr - \int_0^1 \cosh(\lambda r)]dg(r)\Big] \\
&\to -\lambda \int_0^1 \sinh(\lambda r)dg(r),
\end{aligned}$$

as $s \to 0$. Hence we have

$$\begin{aligned}
\frac{d}{ds}G_\lambda(t,0) &= \frac{1}{h_\lambda(0)}\Big(\Big[\sinh(\lambda t)\big(-\sinh(\lambda) + \int_0^1 \sinh(\lambda r)dg(r)\big)\Big] \\
&\quad + \Big[\cosh(\lambda) - \int_0^1 \cosh(\lambda r)dg(r)\Big]\cosh(\lambda t)\Big) \\
&= \frac{P_\lambda(t)}{h_\lambda(0)},
\end{aligned}$$

where P_λ is defined in relation (3.3). On the other hand it holds

$$\begin{aligned}
\frac{d}{ds}G(s,s)\Big|_{s=0} &= \frac{\cosh(\lambda s)h_\lambda(s)}{h_\lambda(0)} + \frac{\sinh(\lambda s)}{h_\lambda(0)\lambda}\Big[-\lambda\sinh(\lambda(1-s)) \\
&\quad + \lambda\int_s^1 \sinh(\lambda(r-s))dg(r)\Big]\Big|_{s=0} = 1.
\end{aligned}$$

(Here we took into account relation (2.6).) These two relations imply the result. $\square$

Lemma 3.3. *There exists a positive constant μ, such that*

$$\mu G_\lambda(s,s) \le G_\lambda(t,s), \quad s\in[0,1], \quad \alpha\le t\le\beta. \tag{3.5}$$

Proof. From Lemma 3.2 and the inequality

$$0<\frac{\delta}{2\cosh(\lambda)}<\frac{\delta}{2h_\lambda(0)}\le\frac{P_\lambda(t)}{h_\lambda(0)},$$

we conclude that there is $s_1\in(0,\tau_1]$ such that

$$\frac{\delta}{2\cosh(\lambda)}=:\mu_0\le\frac{G_\lambda(t,s)}{G_\lambda(s,s)}, \quad s\in[0,s_1], \quad t\in[\alpha,\beta].$$

Now, fix any $t\in[\alpha,\beta]$ and take into account (3.2). Let $s\ge s_1$. If $s\ge t$, it holds

$$\frac{G_\lambda(t,s)}{G_\lambda(s,s)}=\frac{\sinh(\lambda t)}{\sinh(\lambda s)}\ge\frac{\sinh(\lambda\alpha)}{\sinh(\lambda)}.$$

Let $s_1\le s\le t$. If $s\le\tau_2$, then from (2.10) we see that

$$\begin{aligned}
&G_\lambda(t,s)\\
&\ge\frac{1}{\lambda h_\lambda(0)}\Big[\int_s^1\sinh(\lambda s)[\cosh(\lambda(1-t))-\cosh(\lambda(r-t))]dg(r)\Big]\\
&\ge\frac{1}{\lambda h_\lambda(0)}\Big[\int_{\tau_2}^{\tau_3}\sinh(\lambda s_1)[\cosh(\lambda(1-t))-\cosh(\lambda(r-t))]dg(r)\Big]\\
&\ge\frac{1}{\lambda h_\lambda(0)}\Big[\sinh(\lambda s_1)[\cosh(\lambda(1-\tau_2))-\cosh(\lambda(\tau_3-\tau_2))]\Big](g(\tau_3)-g(\tau_2)).
\end{aligned}$$

The quantity in the brackets is strictly positive.

If $\tau_2\le s\le t$, again, from (2.10) we get

$$G_\lambda(t,s)\ge\frac{1}{\lambda h_\lambda(0)}\sinh(\lambda s)\cosh(\lambda(1-t))g(s)\ge\frac{1}{\lambda\cosh(\lambda)}\sinh(\lambda\tau_2)g(\tau_2).$$

Finally, if $t\le s$ then

$$G_\lambda(t,s)=\frac{\sinh(\lambda t)}{\lambda h_\lambda(0)}h_\lambda(s)\ge\frac{\sinh(\lambda\alpha)}{\lambda\cosh(\lambda)}b.$$

From the previous arguments and the fact that, for all $s\in I$, it holds

$$G_\lambda(s,s)=\frac{1}{\lambda h_\lambda(0)}\sinh(\lambda s)h_\lambda(s)\le\frac{\sinh(\lambda)}{\lambda h_\lambda(0)}\cosh(\lambda)\le\frac{\sinh(2\lambda)}{2\lambda b},$$

we, finally, conclude that

$$\min_{t\in[\alpha,\beta],s\in I}\frac{G_\lambda(t,s)}{G_\lambda(s,s)}=\mu$$

exists and it is a positive real number. The proof is complete. $\square$

Lemma 3.4. *There exists a positive constant M, such that*

$$G_\lambda(t,s)\le MG_\lambda(s,s), \quad s,t\in[0,1].$$

Proof. From relation (3.4) and the fact that

$$\frac{P_\lambda(t)}{h_\lambda(0)}\le\frac{\cosh(\lambda)}{b},$$

it follows that there is a certain $s_2 \in (0,1)$, such that

$$\frac{G_\lambda(t,s)}{G_\lambda(s,s)} \le \frac{2\cosh(\lambda)}{b}, \quad s \in [0,s_2], \quad t \in I.$$

Also, for all $s \in [s_2, t]$ we have

$$\frac{G_\lambda(t,s)}{G_\lambda(s,s)} \le \frac{\sinh(\lambda t)}{\sinh(\lambda s)} \le \frac{\sinh(\lambda)}{\sinh(\lambda s_2)},$$

and for $s \ge t$,

$$\frac{G_\lambda(t,s)}{G_\lambda(s,s)} = \frac{\sinh(\lambda t)}{\sinh(\lambda s)} \le \frac{\sinh(\lambda t)}{\sinh(\lambda t)} = 1.$$

Now set

$$M := \max\{\frac{2\cosh(\lambda)}{b}, \frac{\sinh(\lambda)}{\sinh(\lambda s_2)}, 1\} = \max\{\frac{2\cosh(\lambda)}{b}, \frac{\sinh(\lambda)}{\sinh(\lambda s_2)}\},$$

and the proof is complete. $\square$

Lemma 3.5. *The operator T_λ, defined in* (2.11)*, is completely continuous, i.e., it is $[\|\cdot\|_{\mathcal{L}^1} - \|\cdot\|_{\mathcal{L}^1}]$-continuous and it maps bounded subsets of $\mathcal{L}^1(I)$ into totally bounded sets of $\mathcal{L}^1(I)$.*

Proof. To prove the continuity of T_λ consider a sequence (u_n) and a function $v \in \mathcal{L}^1$, such that $\|\cdot\|_{\mathcal{L}^1}$-$\lim u_n = v$. We observe that

$$\begin{aligned}
&\|Tu_n - Tv\|_{\mathcal{L}^1} \\
&= \Big| \int_0^1 \int_0^1 G_\lambda(t,s) F(s,u_n)\,ds\,dt - \int_0^1 \int_0^1 G_\lambda(t,s) F(s,v)\,ds\,dt \Big| \\
&= \Big| \int_0^1 \int_0^1 G_\lambda(t,s)[F(s,u_n) - F(s,v)]\,ds\,dt \Big| \\
&= \Big| \int_0^1 \Big(\int_0^1 G_\lambda(t,s)dt \Big) [F(s,u_n) - F(s,v)] ds \Big| \\
&\le \int_0^1 \Big(\int_0^1 G_\lambda(t,s)dt \Big) |F(s,u_n) - F(s,v)| ds \\
&\le \sup_{s\in I} \Big(\int_0^1 G_\lambda(t,s)dt \Big) \int_0^1 |F(s,u_n) - F(s,v)| ds \\
&\le \frac{\sigma}{\lambda^2} \Big(\lambda^2 \|u_n - v\|_{\mathcal{L}^1} + \sup_{s\in I} |p(s,u_n) - p(s,v)| \|u_n\|_{\mathcal{L}^1} + \sup_{s\in I} p(s,v) \|u_n - v\|_{\mathcal{L}^1} \Big).
\end{aligned}$$

This quantity converges to 0.

To prove the totally boundedness step, we shall apply Theorem 1.2. So let $\mathcal{F}$ be a bounded subset of $\mathcal{L}^1(I)$. This guarantees the existence of some $\kappa > 0$ such that $\|u\|_{\mathcal{L}^1} \le \kappa$, for all $u \in \mathcal{F}$. According to Hypothesis 2.4, to this constant there corresponds a certain $N_\kappa > 0$ such that

$$|p(s,u)| \le N_\kappa, \quad s \in I, \ u \in \mathcal{F}. \tag{3.6}$$

Then we have

$$\begin{aligned}
\|T_\lambda u\|_{\mathcal{L}^1} &= \int_0^1 \Big| \int_0^1 G_\lambda(t,s) F(s,u)\,ds \Big| \,dt \\
&\le \int_0^1 \Big(\int_0^1 G_\lambda(t,s)dt \Big) |F(s,u)| ds
\end{aligned}$$

$$\begin{aligned}
&\le \sup_{s\in I}\Big(\int_0^1 G_\lambda(t,s)dt\Big)\int_0^1 |F(s,u)|ds\\
&\le \sup_{s\in I}\Big(\int_0^1 G_\lambda(t,s)dt\Big)\int_0^1 (\lambda^2+p(s,u))|u(s)|ds\\
&\le \frac{\sigma}{\lambda^2}(\lambda^2+N_\kappa)\kappa
\end{aligned}$$

which means that the image $T_\lambda(\mathcal{F})$ of $\mathcal{F}$ is bounded.

Next take any $u\in\mathcal{F}$, $t\in[0,1)$ and $\eta>0$ such that $t+\eta\in I$. Then we have

$$\begin{aligned}
&\|(T_\lambda u)(\cdot+\eta)-(T_\lambda u)(\cdot)\|_{\mathcal{L}^1}\\
&=\int_0^1 |(T_\lambda u)(t+\eta)-(T_\lambda u)(t)|dt\\
&\le \int_0^1 \Big|\int_0^1 [G_\lambda(t+\eta,s)-G_\lambda(t,s)]\Big||F(s,u)|ds\,dt\,.
\end{aligned}$$

Write the second integral in the form

$$\int_0^t + \int_t^{t+\eta} + \int_{t+\eta}^1$$

and one can easily see that it is smaller than or equal to the quantity

$$\Big[\cosh(\lambda)(\frac{\cosh(\lambda)}{b}+1)+\frac{1}{\lambda}\sinh(\lambda)\Big]\eta.$$

Therefore, taking into account relation (3.6), we get

$$\begin{aligned}
&\|(T_\lambda u)(\cdot+\eta)-(T_\lambda u)(\cdot)\|_{\mathcal{L}^1}\\
&\le \Big[\cosh(\lambda)(\frac{\cosh(\lambda)}{b}+1)+\frac{1}{\lambda}\sinh(\lambda)\Big](\lambda^2+N_\kappa)\kappa\eta.
\end{aligned}$$

The same holds for any $t\in(0,1]$ and $\eta<0$ with $t+\eta\in I$. Therefore the family $T_\lambda\mathcal{F}$ is equicontinuous in mean. This completes the proof of the lemma. □

Before giving the main existence theorem, we need to define the set

$$\mathcal{K} := \{u\in {\mathcal{L}^1}_+(I) : u(t)\ge \frac{\mu}{M}\|u\|_{\mathcal{L}^1}, \text{a.a. } t\in[\alpha,\beta]\},$$

where the constants μ, M are given in Lemmas 3.3 and 3.4.

Lemma 3.6. *The set $\mathcal{K}$ is a cone in $\mathcal{L}^1(I)$.*

Proof. We have to show that $\mathcal{K}$ is a subset of the Banach space $\mathcal{L}^1(I)$ and it satisfies the properties:

(a) If $u\in\mathcal{K}$, then $ku\in\mathcal{K}$ for any real number $k\ge 0$.
(b) If $u,v\in\mathcal{K}$, then $u+v\in\mathcal{K}$.
(c) If both u and $-u$ belong to $\mathcal{K}$, then $u=0$.
(d) It is a closed subset of $\mathcal{L}^1(I)$.

The first algebraic properties (a), (b), (c) are obvious. So we have to prove that $\mathcal{K}$ is closed. To this end we assume that (u_n) is a sequence in $\mathcal{K}$ converging to some $v\in\mathcal{L}^1(I)$ in the $\|\cdot\|_{\mathcal{L}^1}$- norm, namely such that $\lim\|u_n-v\|_{\mathcal{L}^1}=0$. We have to show that v is a point in $\mathcal{K}$.

Although, from the classical analysis it is known that $\mathcal{L}^1$-convergence implies the existence of a subsequence (u_{n_k}) converging to v pointwise, almost everywhere in

I, for completeness of this paper we shall give the proof. And, first we shall remind the reader that this convergence implies convergence in measure. Denote by m the Lebesgue measure.

Get any $\epsilon > 0$ and define the set $E_n := \{t \in I : |u_n(t) - v(t)| \geq \epsilon\}$. Then we have

$$\|u_n - v\|_{\mathcal{L}^1} = \int_{E_n} |u_n(t) - v(t)|dt + \int_{I\setminus E_n} |u_n(t) - v(t)|dt$$

and so

$$\|u_n - v\|_{\mathcal{L}^1} \geq \int_{E_n} |u_n(t) - v(t)|dt \geq \epsilon m(E_n),$$

This implies that $m(E_n) \to 0$, namely $u_n \to v$ in measure.

Next we claim that there is a subsequence (u_{n_k}) converging to v pointwise, a.e. To do that we use the convergence in measure. Thus, for any $k = 1, 2, \cdots$ there is some n_k such that

$$n \geq n_k \implies m(\{t \in I : |u_{n_k}(t) - v(t)| > \frac{1}{k}\}) \leq \frac{1}{2^k}.$$

Put $Z_k := \{t \in I : |u_{n_k}(t) - v(t)| > \frac{1}{k}\}$ and $H_l := \cup_{l=k}^{+\infty} Z_l$. Then we have

$$m(Z_k) \leq \frac{1}{2^k} \quad \text{and} \quad m(H_l) \leq \sum_{i=l}^{+\infty} m(Z_i) \leq \sum_{i=l}^{+\infty} \frac{1}{2^k} = \frac{1}{2^{l-1}}.$$

Set $Z := \cap_{l=1}^{+\infty} H_l$, for which it holds that

$$m(Z) \leq m(H_l) \leq \frac{1}{2^{l-1}}.$$

Thus $m(Z) = 0$.

Now let $t \in I \setminus Z$. Then $t \notin H_l$, for some l. Hence $t \notin Z_k$ for all $k \geq l$. The latter implies that

$$|u_{n_k}(t) - v(t)| \leq \frac{1}{k}, \quad k \geq l.$$

Therefore $\lim u_{n_k}(t) = v(t)$, for all $t \in I \setminus Z$, which proves the claim.

Finally, define the set

$$S := Z \cup \Big(\cup_{n=1}^{+\infty} \{t \in [\alpha, \beta] : u_n(t) < \frac{\mu}{M} \|u_n\|_{\mathcal{L}^1}\} \Big)$$

which has measure 0. Observe that for any $t \in [\alpha, \beta] \setminus S$ it holds

$$v(t) = \lim u_{n_k}(t) \geq \frac{\mu}{M} \lim \|u_{n_k}\|_{\mathcal{L}^1} = \frac{\mu}{M} \|v\|_{\mathcal{L}^1},$$

because $|\|u_{n_k}\|_{\mathcal{L}^1} - \|v\|_{\mathcal{L}^1}| \leq \|u_{n_k} - v\|_{\mathcal{L}^1} \to 0$. Therefore we have $v \in \mathcal{K}$ and the proof is complete. □

Theorem 3.7. *Assume that Conditions* (1.3) *and 2.1–3.1 are satisfied. Then boundary value problem* (1.1), (1.3) *admits a positive solution.*

Proof. By Lemma 2.7, it is sufficient to prove that the operator T_λ defined by the type (2.11), where F is given in (2.3), admits a fixed point in $\mathcal{K}$.

From Lemma 3.5 we know that the operator T_λ is completely continuous. Now, let $u \in \mathcal{K}$. Then on one hand we have

$$\|T_\lambda u\|_{\sup} = \sup_{t\in I} \int_0^1 G_\lambda(t, s)F(s, u)ds \leq M \int_0^1 G_\lambda(s, s)F(s, u)ds$$

and on the other hand

$$(T_\lambda u)(t) \geq \mu \int_0^1 G_\lambda(s,s)F(s,u)ds,$$

for any $t \in [\alpha, \beta]$. Thus it holds

$$(T_\lambda u)(t) \geq \frac{\mu}{M}\|T_\lambda u\|_{\sup} \geq \frac{\mu}{M}\|T_\lambda u\|_{\mathcal{L}^1}, \quad t \in [\alpha, \beta]. \tag{3.7}$$

Also, for any $u \in \mathcal{K}$ we have

$$\begin{aligned}
\|T_\lambda u\|_{\mathcal{L}^1} &= \int_0^1 \int_0^1 G_\lambda(t,s)F(s,u)\,ds\,dt \\
&= \int_0^1 \Big(\int_0^1 G_\lambda(t,s)dt\Big)F(s,u)ds \\
&\leq \sup_{s\in I}\Big(\int_0^1 G_\lambda(t,s)dt\Big)\int_0^1 F(s,u)ds \\
&\leq \frac{\sigma}{\lambda^2}(\lambda^2 + \sup_{s\in I} p(s,u))\|u\|_{\mathcal{L}^1} < +\infty.
\end{aligned} \tag{3.8}$$

Inequalities (3.7) and (3.8) together with the fact that $(T_\lambda u)(t) \geq 0$, for all $u \in \mathcal{K}$, imply that the operator T_λ maps the cone $\mathcal{K}$ into itself.

Next, because of Hypothesis 3.1, we can choose $\rho > 0$ with $\rho < \frac{1-\sigma}{\sigma}\lambda^2$. From the $[\|\cdot\|_{\mathcal{L}^1} - |\cdot|]$-continuity of $p(t,\cdot)$ at 0, uniformly in t, (Hypothesis 2.3), it follows that, to this ρ there corresponds a certain $R_1 > 0$ such that, for any $u \in \mathcal{K}$, with $\|u\|_{\mathcal{L}^1} \leq R_1$ and $t \in I$, it holds $0 \leq p(t,u) \leq \rho$.

Take any $u \in \mathcal{K}$, such that $\|u\|_{\mathcal{L}^1} = R_1$. Then from (3.8) we get

$$\|T_\lambda u\|_{\mathcal{L}^1} \leq \sup_{s\in I}\Big(\int_0^1 G_\lambda(t,s)dt\Big)(\lambda^2+\rho)\|u\|_{\mathcal{L}^1} \leq \|u\|_{\mathcal{L}^1}.$$

The latter holds because of the choice of ρ.

By using the sup norm on the space $C(I)$ such an inequality is not satisfied for any $\lambda > 0$. Indeed, one can prove that for any positive $u \in C(I)$ the function $A(\lambda) := \frac{\|T_\lambda u\|_{sup}}{\|u\|_{sup}}$, $\lambda > 0$, satisfies $\lim_{\lambda\to 0} A(\lambda) = 1$ and, furthermore, $\lambda \to A(\lambda)$ is increasing. Therefore there does not exist any $\lambda > 0$ satisfying $A(\lambda) \leq 1$. Notice that because of the continuous dependence of the problem on parameters, for $\lambda = 0$ the integral form of the problem becomes a trivial identity. However, this is the point which motivated us to use the idea of working on $\mathcal{L}^1(I)$ instead of $C(I)$. This idea relies on the obvious fact that there are continuous functions having $\|\cdot\|_{sup}$-value greater than 1, and $\|\cdot\|_{\mathcal{L}^1}$ value less than 1.

Next fix any real number $R > 0$ such that

$$\mu^2(\beta-\alpha)(\lambda^2 + R)\int_{\tau_2}^{\tau_3} G_\lambda(s,s)ds \geq M.$$

From Hypothesis 2.5, there exists $N > 0$ such that $p(t,u) \geq R$, for all u in $\mathcal{K}$ with $u(s) \geq N$, $s \in [\tau_2, \tau_3]$ and $t \in [\alpha, \beta]$. Clearly, we can assume that $R_2 := \frac{M}{\mu}N > R_1$.

Now, fix a point $u \in \mathcal{K}$ such that $\|u\|_{\mathcal{L}^1} = R_2$. Then, for all $s \in [\tau_2, \tau_3]$, it holds

$$u(s) \geq \frac{\mu}{M}\|u\|_{\mathcal{L}^1} = N$$

and therefore

$$\begin{aligned}\|T_\lambda u\|_{\mathcal{L}^1} &= \int_0^1 \int_0^1 G_\lambda(t,s)F(s,u)\,ds\,dt = \int_0^1 \Big(\int_0^1 G_\lambda(t,s)dt\Big)F(s,u)ds \\ &\geq \int_{\tau_2}^{\tau_3} \Big(\int_\alpha^\beta G_\lambda(t,s)dt\Big)F(s,u)ds,\end{aligned}$$

and therefore

$$\begin{aligned}\|T_\lambda u\|_{\mathcal{L}^1} &\geq \mu \int_{\tau_2}^{\tau_3} \Big(\int_\alpha^\beta G_\lambda(s,s)dt\Big)(\lambda^2+R)u(s)ds \\ &\geq \frac{\mu^2}{M}(\beta-\alpha)\int_{\tau_2}^{\tau_3} G_\lambda(s,s)ds(\lambda^2+R)\|u\|_{\mathcal{L}^1} \geq \|u\|_{\mathcal{L}^1}.\end{aligned}$$

The previous arguments show that Theorem 1.1 applies, and so we conclude that a fixed point of T_λ exists in $\mathcal{K}\cap(\overline{\Omega_2}\setminus\Omega_1)$, where Ω_1,Ω_2 are the balls in $C(I)$ with center the origin and radius R_1 and R_2, respectively. The proof of the theorem is complete. □

4. An application

Consider the integro-differential equation

$$u'' + (1+t)u(t)\int_0^t u(s)ds = 0, \quad t\in I, \tag{4.1}$$

with $u(0) = 0$. Associate this equation with the nonlocal condition (1.2), where the function g is defined in Example 2.2. From the fact that $0 = g(0 < g(1) = 1$, the problem is at resonance. Taking into account Example 2.6 we can see that the conditions of Theorem 3.7 are satisfying, with α,β defined as in (2.2). Thus there exists a solution u of equation (4.1) satisfying condition (1.2) and $u(t)\geq u(0)=0$, $t\in I$ and being such that $\|u\|_{\mathcal{L}^1} > 0$.

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