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DIRICHLET BOUNDARY VALUE PROBLEM FOR A SYSTEM OF n SECOND ORDER ASYMPTOTICALLY ASYMMETRIC DIFFERENTIAL EQUATIONS

ARMANDS GRITSANS, FELIX SADYRBAEV, INARA YERMACHENKO

Abstract. We consider systems of the form

$$\begin{aligned} x_1'' + g_1(x_1) &= h_1(x_1, x_2, \dots, x_n), \\ x_2'' + g_2(x_2) &= h_2(x_1, x_2, \dots, x_n), \\ &\dots \\ x_n'' + g_n(x_n) &= h_n(x_1, x_2, \dots, x_n) \end{aligned} \tag{*}$$

along with the boundary conditions

$$x_1(0) = x_2(0) = \dots = x_n(0) = 0 = x_1(1) = x_2(1) = \dots = x_n(1)\,.$$

We assume that right sides are bounded continuous functions, and satisfy $h_i(0, 0, \dots, 0) = 0$. Also we assume that $g_i(x_i)$ are asymptotically asymmetric functions. By using vector field rotation theory, we provide the existence of solutions.

1. Introduction

This article concerns the existence results for asymptotically positively homogeneous systems of the form

$$\mathbf{x}'' = \mathbf{f}(\mathbf{x}), \tag{1.1}$$

satisfying the Dirichlet boundary conditions

$$\mathbf{x}(0) = \mathbf{0} = \mathbf{x}(1). \tag{1.2}$$

The nonlinearity is supposed to present a linear behaviour near zero, and to satisfy asymmetric assumptions at infinity. In particular, the problem is assumed to be autonomous and uncoupled in a neighbourhood of infinity. We prove the existence of at least one nontrivial solution to the problem (1.1), (1.2) when suitable indexes associated with the linearized problem at zero and the asymptotic problem at infinity are different.

This article has been motivated by the articles [11], [23], dealing with asymptotically linear systems, with the aim of extending the existence results obtained in the above mentioned papers to an asymmetric context. The present article and the articles [11], [23] follow an analogous approach based on vector fields rotation theory (Brouwer degree theory). The difference between the present article and the articles

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[11], [23] consists in the use of the notion of the Fučík spectrum for the scalar second order equation to study the positively homogeneous, autonomous, uncoupled problem at infinity. The main result Theorem 7.1 of the present article generalizes the main result Theorem 1.2 of the article [23] to an asymmetric n-dimensional setting. In both articles nonlinearity is supposed to satisfy autonomous uncoupled assumptions at infinity. If we use the notations of Section 4 of the present article, then the asymptotic at infinity system in [23] has the form

$$\begin{aligned} z_1'' &= -\lambda_1 z_1, \\ z_2'' &= -\lambda_2 z_2 \end{aligned}$$

with $\lambda_1 = \mu_1 = k^2$ and $\lambda_2 = \mu_2 = \ell^2$, where k and ℓ are notations from [23] and $k, \ell > 0$. The pairs $(\lambda_1, \mu_1) = (k^2, k^2)$ and $(\lambda_2, \mu_2) = (\ell^2, \ell^2)$ under the nonresonance condition $k, \ell \notin \{\pi j : j \in \mathbb{N}\}$ in [23] are located on the intersection of the set D with the bisectrix of the positive quadrant Q, where the sets D and Q are considered in Section 4 of the present article. Hence, the index at infinity in [23] belongs to the set $\{-1, 1\}$, while in the present article it can attain zero value also.

The analysis of existence and multiplicity of solutions for linear boundary value problems naturally leads to the study of the respective eigenvalue problems. The behavior at the zero solution is extended to infinity by superposition principle. In contrast, if the problem $x'' = -g(x)$, $x(0) = 0 = x(1)$ is considered, where a function $g(x)$ is linear as k^2x in some vicinity of zero (and $g(0) = 0$ in order the trivial solution to exist) and, at the same time, it is linear as m^2x for large in modulus values of x (and k and m essentially differ), then a number of solutions appear when passing from solutions of the Cauchy problem $x'' = -g(x)$, $x(0) = 0$, $x'(0) = \alpha$ with small α to solutions with large α. This is essentially nonlinear phenomenon and it was widely used in the studies of existence and multiplicity of solutions for nonlinear problems.

The idea of investigation of a two-point boundary value problem by comparing the behaviors of solutions near zero and at infinity was used previously in the paper by A.I. Perov [19]. The estimates of the number of solutions from below were obtained for the second order scalar nonlinear differential equations. A number of papers based on the same idea have appeared afterwards.

In the seminal work [1] by H. Amann and E. Zehnder the problem of the existence of solutions was studied for the equation $Au = F(u)$, where A is self-adjoint operator and F is the nonlinearity interacting in some way of the spectrum of A. The reduction to a variational problem was made and the critical points of a functional were studied. It was noticed that "the basic idea is to compare the behavior near zero to its asymptotic behavior at infinity". Proofs used the generalized Morse index theory as developed by C. Conley [5]. A similar technique was used in the work by C. Conley and E. Zehnder [6] when studying the existence of T-periodic solutions for time-dependent Hamiltonian systems. In [1] and [5] existence of nontrivial solutions is ensured when the Morse-type indexes at zero and at infinity are different. In the papers [7], [8], [15], [22] and references therein further generalizations of the classical existence results concerning asymptotically linear Hamiltonian systems were obtained by developing Morse and Maslov-type index theory.

The problem of existence and multiplicity of solutions for asymptotically linear systems in a non-Hamiltonian context has not yet been fully explored in the literature. In the non-Hamiltonian setting, let us mention, among others, the works [4], [17] and [16]. Remark that the first two papers focus on asymptotically linear problems whose linearizations at zero and at infinity present the form

$$\mathbf{u}''(t) + A(t)\mathbf{u}(t) = \mathbf{0},$$

where $A(\cdot)$ is a path of $n \times n$ symmetric matrices. The symmetric structure allows the authors of [4] and [17] to associate the Maslov and Morse index with the linearized problems. On the other hand, the work [16] develops a new index theory which guarantees existence results for planar first order systems, whose linearizations at zero and at infinity do not need a symmetry assumptions. In the authors papers [11] and [23] as well as in the present article neither Hamiltonian structure nor symmetry assumptions are required, due to the use of the Brouwer degree.

After considering asymptotically linear cases it is natural to pass to positively homogeneous equations and systems. The famous Fučík equation is not linear but possesses the important property of linear equations, that is, the positive homogeneity. The function $h : \mathbb{R} \to \mathbb{R}$ is positively homogeneous if $h(cx) = ch(x)$ for all positive c and every $x \in \mathbb{R}$. This is the case for the right side of the Fučík equation $x'' = -\lambda x^+ + \mu x^-$, where x^+ and x^- are respectively the positive and negative parts of x and λ and μ are positive coefficients. There are numerous papers dealing with Fučík type scalar equations. The so called "jumping-nonlinearity" studies fall into this class. There are fewer papers considering systems of Fučík type and, more generally, asymptotically asymmetric systems. Let us mention the papers [3], [18], [26], [27] dealing with asymptotically positively homogeneous systems. The article [3] focuses on multiplicity results for weakly coupled systems satisfying Dirichlet boundary conditions, while [18], [26], [27] are concerned with existence of periodic solutions. In the authors papers [10], [20] and [21] scalar asymptotically asymmetric problems with Dirichlet and nonlocal boundary conditions were considered.

In this article, we consider the problem (1.1), (1.2), where $\mathbf{f} = -\boldsymbol{g}+\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{g}(\mathbf{x}) = \big(g_1(x_1), \dots, g_n(x_n)\big)^T$, $\boldsymbol{h}(\mathbf{x}) = \big(h_1(\mathbf{x}), \dots, h_n(\mathbf{x})\big)^T$, $\mathbf{0} = (\underbrace{0, \dots, 0}_{n})^T \in \mathbb{R}^n$.

Suppose that the following conditions are fulfilled.

- (A1) The functions $g_i : \mathbb{R} \to \mathbb{R}$, $h_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, 2, \dots, n)$ are continuously differentiable.
- (A2) The functions h_i $(i = 1, 2, \dots, n)$ are bounded.
- (A3) $g_i(0) = 0$, $h_i(0, 0, \dots, 0) = 0$ $(i = 1, 2, \dots, n)$, hence the system (1.1) has the trivial solution $\mathbf{x} = \mathbf{0}$.
- (A4) There exist the limits:

$$\lim_{x_i \to +\infty} \frac{g_i(x_i)}{x_i} = \lambda_i > 0, \quad \lim_{x_i \to -\infty} \frac{g_i(x_i)}{x_i} = \mu_i > 0 \quad (i = 1, 2, \dots, n). \tag{1.3}$$

In Section 2, we introduce the vector field $\boldsymbol{\phi} : \mathbb{R}^n \to \mathbb{R}^n$,

$$\boldsymbol{\phi}(\boldsymbol{\beta}) = \mathbf{x}(1; \boldsymbol{\beta}), \quad \forall \boldsymbol{\beta} \in \mathbb{R}^n, \tag{1.4}$$

where $\mathbf{x}(t; \boldsymbol{\beta})$ is the solution of the Cauchy problem (1.1),

$$\mathbf{x}(0) = \mathbf{0}, \quad \mathbf{x}'(0) = \boldsymbol{\beta}. \tag{1.5}$$

The vector field $\boldsymbol{\phi}$ plays a crucial role in our considerations, since $\boldsymbol{\phi}(\boldsymbol{\beta}) = \mathbf{0}$ if and only if $\mathbf{x}(t;\boldsymbol{\beta})$ solves the problem (1.1), (1.2).

In Section 3, we consider the vector field $\boldsymbol{\phi}_0 : \mathbb{R}^n \to \mathbb{R}^n$, associated with the asymptotic at zero problem (3.1), (3.2). The assumptions (A1)-(A4) combined with the nonresonance at zero condition (A5) ensure that $\operatorname{ind}(\mathbf{0},\boldsymbol{\phi}) = \operatorname{ind}(\mathbf{0},\boldsymbol{\phi}_0)$.

In Section 4, we explore the vector field $\phi_\infty : \mathbb{R} \to \mathbb{R}$ associated with the scalar Fučík problem (4.1).

In Section 5, we introduce the vector field $\boldsymbol{\phi}_\infty : \mathbb{R}^n \to \mathbb{R}^n$ associated with the asymptotic at infinity problem (5.1), (5.2). In contrast to the analysis at zero when the index $\operatorname{ind}(\mathbf{0},\boldsymbol{\phi}_0) \in \{-1,1\}$, due to asymmetric character of limiting Fučík type system (5.1) the index $\operatorname{ind}(\mathbf{0},\boldsymbol{\phi}_\infty)$ attains values in a broader set $\{-1,0,1\}$.

In Section 6, we study the vector field $\boldsymbol{\phi}$ at infinity. The assumptions (A1)-(A4) coupled with the nonresonance at infinity condition (A6) provide that $\operatorname{ind}(\infty,\boldsymbol{\phi}) = \operatorname{ind}(\mathbf{0},\boldsymbol{\phi}_\infty)$.

In Section 7, we prove the main result of the paper. The assumptions (A1)-(A4) combined with asymptotic nonresonance conditions (A5), (A6) ensure the existence of a nontrivial solution to problem (1.1), (1.2), whenever $\operatorname{ind}(\mathbf{0},\boldsymbol{\phi}) \neq \operatorname{ind}(\infty,\boldsymbol{\phi})$. No Hamiltonian structure of the system is required and no symmetry assumptions are needed to prove the main result beyond the conditions (A1) to (A6).

The examples at the end of the article illustrate the main result.

2. Vector field $\boldsymbol{\phi}$ associated with the Dirichlet boundary value problem (1.1), (1.2)

Proposition 2.1. *Suppose that conditions* (A1), (A2), (A4) *are fulfilled. Then the vector field* $\mathbf{f}$ *is linearly bounded, that is, there exist* $a, b > 0$ *such that* $\|\mathbf{f}(\mathbf{x})\| \le a + b\,\|\mathbf{x}\|$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

Proof. It follows from the conditions (A1), (A2) and (A4) that for every $i = 1, 2, \dots, n$ there exist $M_i, q_i, N_i > 0$ such that

$$\big|g_i(x_i)\big| < M_i + q_i|x_i|, \quad \forall x_i \in \mathbb{R}, \tag{2.1}$$

$$\big|h_i(\mathbf{x})\big| \le N_i, \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{2.2}$$

It follows from (2.1) and (2.2) that for any $\mathbf{x} \in \mathbb{R}^n$ we have $\|\mathbf{f}(\mathbf{x})\| \le a + b\,\|\mathbf{x}\|$, where $a = \sum_{i=1}^n (M_i + N_i) > 0$, $b = \sqrt{n}\max_{1\le i\le n}|q_i| > 0$; $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. □

We rewrite (1.1) in the equivalent form $\mathbf{w}' = \mathbf{F}(\mathbf{w})$, where $\mathbf{F}(\mathbf{w}) = \big(\mathbf{v}, \mathbf{f}(\mathbf{x})\big)^T$, $\mathbf{q} = (\mathbf{x},\mathbf{v})^T \in \mathbb{R}^N$, $\mathbf{v} = \mathbf{x}'$, $N = 2n$.

Proposition 2.2. *Suppose that conditions* (A1)–(A4) *are fulfilled. Then the vector field* $\mathbf{F}$ *has the following properties.*

1. $\mathbf{F} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$.
2. $\mathbf{F}(\mathbf{o}) = \mathbf{o} \in \mathbb{R}^N$, *where* $\mathbf{o} = (\mathbf{0},\mathbf{0})^T$.
3. *The vector field* $\mathbf{F}$ *is linearly bounded, that is, there exist* $A, B > 0$ *such that*

$$\big\|\mathbf{F}(\mathbf{w})\big\|_N \le A + B\,\|\mathbf{w}\|_N, \quad \forall \mathbf{z} \in \mathbb{R}^N. \tag{2.3}$$

Proof. Properties 1. and 2. are immediate consequences of assumptions (A1) and (A3).

3. A direct application of Proposition 2.1 guarantees the validity of (2.3) with $A = a > 0$, $B = \sqrt{1+b^2} > 0$; $\|\cdot\|_N$ is the Euclidean norm in $\mathbb{R}^N$. $\square$

Suppose that conditions (A1)–(A4) hold. Denote by $\mathbf{w}(t;\boldsymbol{\gamma})$ the solution of the Cauchy problem

$$\mathbf{w}' = \mathbf{F}(\mathbf{w}), \quad \mathbf{w}(0) = \boldsymbol{\gamma}.$$

Denote by $\boldsymbol{\Phi}^t(\boldsymbol{\gamma}) := \mathbf{w}(t;\boldsymbol{\gamma})$ the flow of the vector field $\mathbf{F}$. Since the vector field $\mathbf{F}$ ir linearly bounded, then [2], [25] its flow $\boldsymbol{\Phi}^t(\boldsymbol{\gamma})$ is complete and belongs to $C^1(\mathbb{R}^N,\mathbb{R}^N)$. Therefore, the vector field $\boldsymbol{\phi}$, defined by (1.4), belongs to $C^1(\mathbb{R}^n,\mathbb{R}^n)$.

A point $\boldsymbol{\beta} \in \mathbb{R}^n$ is called a singular point of the vector field $\boldsymbol{\phi}$ if $\boldsymbol{\phi}(\boldsymbol{\beta}) = \mathbf{0}$. The singular points of the vector field $\boldsymbol{\phi}$ are in one-to-one correspondence with the solutions to the Dirichlet boundary value problem (1.1), (1.2). Any singular point $\boldsymbol{\beta} \neq \mathbf{0}$ of the vector field $\boldsymbol{\phi}$ generates a nontrivial solution to the problem (1.1), (1.2).

Consider a bounded open set $\Omega \subset \mathbb{R}^n$. Denote by $\mathcal{F}(\Omega)$ the set of all continuous vector fields $\boldsymbol{\phi} : \overline{\Omega} \to \mathbb{R}^n$ which are nonsingular on the boundary $\partial\Omega$, that is, $\boldsymbol{\phi}(\boldsymbol{\beta}) \neq \mathbf{0}$ for all $\boldsymbol{\beta} \in \partial\Omega$. If $\boldsymbol{\phi} \in \mathcal{F}(\Omega)$, then [13], [24] there is an integer $\gamma(\boldsymbol{\phi},\Omega)$ associated with the vector field $\boldsymbol{\phi}$ and called the rotation of $\boldsymbol{\phi}$ on $\partial\Omega$ or the Brouwer degree of $\boldsymbol{\phi}$ on Ω with respect to $\mathbf{0}$. For definitions of isolated singular points of vector fields and their indexes one may consult the last two references.

3. Vector field $\boldsymbol{\phi}$ near zero

Now, we recall briefly the study of the vector field $\boldsymbol{\phi}$ near zero developed in [11], where analogous assumptions at zero have been considered. Suppose that conditions (A1) and (A3) hold. Then there exists the derivative $\mathbf{f}'(\mathbf{0})$ (the Jacobian matrix) of the nonlinearity $\mathbf{f}$ at zero $\mathbf{x} = \mathbf{0}$. Consider the linearized system at zero

$$\mathbf{v}'' = \mathbf{f}'(\mathbf{0})\,\mathbf{v} \tag{3.1}$$

together with the Dirichlet boundary conditions

$$\mathbf{v}(0) = \mathbf{0} = \mathbf{v}(1). \tag{3.2}$$

If $\mathbf{v}(t;\boldsymbol{\beta})$ is the solution to the Cauchy problem: (3.1), $\mathbf{v}(0) = \mathbf{0}$, $\mathbf{v}'(0) = \boldsymbol{\beta}$, then we can define the linear vector field $\boldsymbol{\phi}_0 : \mathbb{R}^n \to \mathbb{R}^n$,

$$\boldsymbol{\phi}_0(\boldsymbol{\beta}) = \mathbf{v}(1;\boldsymbol{\beta}), \quad \forall \boldsymbol{\beta} \in \mathbb{R}^n.$$

Let us consider the following condition.

(A5) The linearized system at zero (3.1) is *nonresonant* with respect to the boundary conditions (3.2), that is, the boundary value problem (3.1), (3.2) has only the trivial solution.

Remark 3.1. It was shown in [11] that the condition (A5) is equivalent to each of the following conditions: 1) $\boldsymbol{\beta} = \mathbf{0}$ is the unique singular point of the vector field $\boldsymbol{\phi}_0$, 2) no eigenvalue of the matrix $\mathbf{f}'(\mathbf{0})$ belongs to the spectrum $\sigma_D = \{-(j\,\pi)^2 : j \in \mathbb{N}\}$ of the scalar Dirichlet boundary value problem $x'' = \lambda\, x$, $x(0) = 0 = x(1)$.

The next two statements are essentially [11, Proposition 3] and [11, Theorem 4].

Proposition 3.2. *Suppose that condition* (A5) *holds. If the matrix* $\mathbf{f}'(\mathbf{0})$ *has not negative eigenvalues with odd algebraic multiplicities, then* $\operatorname{ind}(\mathbf{0},\boldsymbol{\phi}_0) = 1$*. If the matrix* $\mathbf{f}'(\mathbf{0})$ *has* k $(1 \le k \le n)$ *different negative eigenvalues* ξ_j $(1 \le j \le k)$ *with odd algebraic multiplicities, then* $\operatorname{ind}(\mathbf{0},\boldsymbol{\phi}_0) = \operatorname{sgn}\Big(\prod_{j=1}^k \sin\sqrt{|\xi_j|}\Big)$.

Theorem 3.3. *Suppose that conditions* (A1)–(A5) *hold. Then* $\boldsymbol{\beta} = \mathbf{0}$ *is an isolated singular point of the vector field* $\boldsymbol{\phi}$ *and* $\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}) = \operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}_0)$.

4. Scalar vector field ϕ_∞

In what follows, we need some properties of the Fučík spectrum [9]. Consider the scalar Fučík problem

$$z'' = -\lambda z^+ + \mu z^-, \quad z(0) = 0 = z(1), \tag{4.1}$$

where $\lambda, \mu > 0$, $z^+ = \max\{z, 0\}$, $z^- = \max\{-z, 0\}$. The spectrum of the problem (4.1) is the subset

$$\Sigma = \big\{(\lambda, \mu) : \text{problem (4.1) has a nontrivial solution}\big\}$$

of the positive quadrant $Q = \big\{(\lambda, \mu) : \lambda > 0,\ \mu > 0\big\}$ of the plane.

Next, we split the set Q into some specific subsets with respect to the Fučík spectrum Σ, namely we consider the subsets of the set $Q \setminus \Sigma$:

$$D(k) = \Big\{(\lambda, \mu) : \frac{\pi m}{\sqrt{\lambda}} + \frac{\pi(m+1)}{\sqrt{\mu}} > 1,\ \frac{\pi(m+1)}{\sqrt{\lambda}} + \frac{\pi m}{\sqrt{\mu}} > 1,\ \frac{\pi m}{\sqrt{\lambda}} + \frac{\pi m}{\sqrt{\mu}} < 1\Big\}$$
$$(k = 2m;\ m = 0, 1, 2, \dots), \tag{4.2}$$

$$D(k) = \Big\{(\lambda, \mu) : \frac{\pi(m-1)}{\sqrt{\lambda}} + \frac{\pi m}{\sqrt{\mu}} < 1,\ \frac{\pi m}{\sqrt{\lambda}} + \frac{\pi(m-1)}{\sqrt{\mu}} < 1,\ \frac{\pi m}{\sqrt{\lambda}} + \frac{\pi m}{\sqrt{\mu}} > 1\Big\}$$
$$(k = 2m-1;\ m = 1, 2, 3, \dots), \tag{4.3}$$

$$E^+(2m) = \Big\{(\lambda, \mu) : \frac{\pi m}{\sqrt{\lambda}} + \frac{\pi(m+1)}{\sqrt{\mu}} < 1,\ \frac{\pi(m+1)}{\sqrt{\lambda}} + \frac{\pi m}{\sqrt{\mu}} > 1\Big\} \tag{4.4}$$
$$(m = 0, 1, 2, \dots), \tag{4.5}$$

$$E^-(2m) = \Big\{(\lambda, \mu) : \frac{\pi m}{\sqrt{\lambda}} + \frac{\pi(m+1)}{\sqrt{\mu}} > 1,\ \frac{\pi(m+1)}{\sqrt{\lambda}} + \frac{\pi m}{\sqrt{\mu}} < 1\Big\}$$
$$(m = 0, 1, 2, \dots), \tag{4.6}$$

$$D = \cup_{k=0}^{\infty} D(k), \quad E = Q \setminus (\Sigma \cup D) = \cup_{m=0}^{\infty}\big(E^+(2m) \cup E^-(2m)\big).$$

Denote by $z(t;\beta)$ the solution of the scalar Cauchy problem

$$z'' = -\lambda z^+ + \mu z^-, \quad z(0) = 0, \quad z'(0) = \beta$$

and define the vector field $\phi_\infty : \mathbb{R} \to \mathbb{R}$,

$$\phi_\infty(\beta) = z(1;\beta), \quad \forall \beta \in \mathbb{R}.$$

Proposition 4.1. *Consider* $\alpha > 0$.

(1) *If* $(\lambda, \mu) \in D(k)$, *then* $\phi_\infty(-\alpha)\phi_\infty(\alpha) < 0$, *more precisely*
  (a) *if* $k = 2m$ $(m = 0, 1, \dots)$, *then* $\phi_\infty(-\alpha) < 0$, $\phi_\infty(\alpha) > 0$;
  (b) *if* $k = 2m-1$ $(m = 1, 2, \dots)$, *then* $\phi_\infty(-\alpha) > 0$, $\phi_\infty(\alpha) < 0$.

(2) *If* $(\lambda, \mu) \in E$, *then* $\phi_\infty(-\alpha)\phi_\infty(\alpha) > 0$, *more precisely*
  (a) *if* $(\lambda, \mu) \in E^+(2m)$ $(m = 0, 1, 2, \dots)$, *then* $\phi_\infty(-\alpha) > 0$, $\phi_\infty(\alpha) > 0$;
  (b) *if* $(\lambda, \mu) \in E^-(2m)$ $(m = 0, 1, 2, \dots)$, *then* $\phi_\infty(-\alpha) < 0$, $\phi_\infty(\alpha) < 0$.

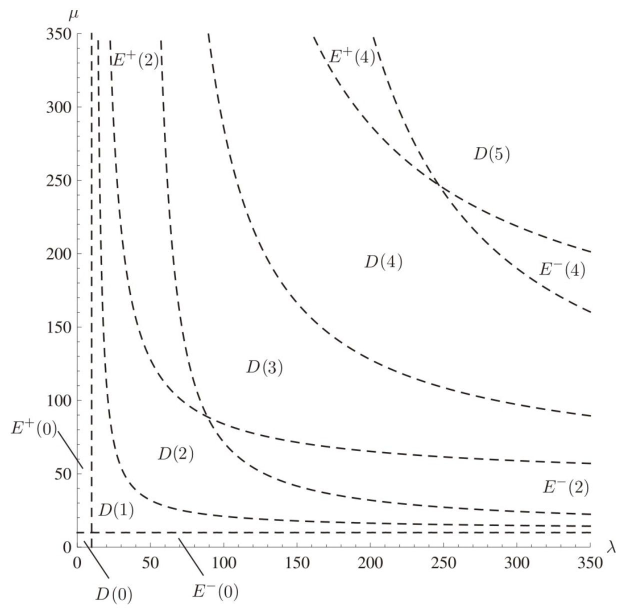

FIGURE 1. Subsets $D(k)$ $(k = 0, 1, 2, 3, 4, 5)$ and $E^{\pm}(2m)$ $(m = 0, 1, 2)$ of the positive quadrant Q.

Proof. If $(\lambda, \mu) \in D(k)$ and $k = 2m$ $(m = 0, 1, 2, \dots)$, then

$$\phi_\infty(\beta) = \begin{cases} \frac{\beta}{\sqrt{\lambda}} \sin\Big[\sqrt{\lambda}\Big(1 - \frac{\pi m}{\sqrt{\lambda}} - \frac{\pi m}{\sqrt{\mu}}\Big)\Big] > 0, & \text{if } \beta > 0, \\ 0, & \text{if } \beta = 0, \\ \frac{\beta}{\sqrt{\mu}} \sin\Big[\sqrt{\mu}\Big(1 - \frac{\pi m}{\sqrt{\lambda}} - \frac{\pi m}{\sqrt{\mu}}\Big)\Big] < 0, & \text{if } \beta < 0; \end{cases} \tag{4.7}$$

if $(\lambda, \mu) \in D(k)$ and $k = 2m - 1$ $(m = 1, 2, 3, \dots)$, then

$$\phi_\infty(\beta) = \begin{cases} -\frac{\beta}{\sqrt{\mu}} \sin\Big[\sqrt{\mu}\Big(1 - \frac{\pi m}{\sqrt{\lambda}} - \frac{\pi(m-1)}{\sqrt{\mu}}\Big)\Big] < 0, & \text{if } \beta > 0, \\ 0, & \text{if } \beta = 0, \\ -\frac{\beta}{\sqrt{\lambda}} \sin\Big[\sqrt{\lambda}\Big(1 - \frac{\pi(m-1)}{\sqrt{\lambda}} - \frac{\pi m}{\sqrt{\mu}}\Big)\Big] > 0, & \text{if } \beta < 0; \end{cases} \tag{4.8}$$

if $(\lambda, \mu) \in E^+(2m)$ $(m = 0, 1, 2, \dots)$, then

$$\phi_\infty(\beta) = \begin{cases} \frac{\beta}{\sqrt{\lambda}} \sin\Big[\sqrt{\lambda}\Big(1 - \frac{\pi m}{\sqrt{\lambda}} - \frac{\pi m}{\sqrt{\mu}}\Big)\Big] > 0, & \text{if } \beta > 0, \\ 0, & \text{if } \beta = 0, \\ \frac{\beta}{\sqrt{\lambda}} \sin\Big[\sqrt{\lambda}\Big(1 - \frac{\pi(m+1)}{\sqrt{\lambda}} - \frac{\pi(m+1)}{\sqrt{\mu}}\Big)\Big] > 0, & \text{if } \beta < 0; \end{cases} \tag{4.9}$$

if $(\lambda, \mu) \in E^-(2m)$ $(m = 0, 1, 2, \dots)$, then

$$\phi_\infty(\beta) = \begin{cases} \frac{\beta}{\sqrt{\mu}} \sin\Big[\sqrt{\mu}\Big(1 - \frac{\pi(m+1)}{\sqrt{\lambda}} - \frac{\pi(m+1)}{\sqrt{\mu}}\Big)\Big] < 0, & \text{if } \beta > 0, \\ 0, & \text{if } \beta = 0, \\ \frac{\beta}{\sqrt{\mu}} \sin\Big[\sqrt{\mu}\Big(1 - \frac{\pi m}{\sqrt{\lambda}} - \frac{\pi m}{\sqrt{\mu}}\Big)\Big] < 0, & \text{if } \beta < 0. \end{cases} \tag{4.10}$$

1. (a) Suppose that $(\lambda,\mu) \in D(k)$ and $k = 2m$ $(m = 0, 1, \dots)$. It follows from (4.2) that $0 < \sqrt{\lambda}\big(1 - \frac{\pi m}{\sqrt{\lambda}} - \frac{\pi m}{\sqrt{\mu}}\big) < \pi$. Hence, $\sin\big[\sqrt{\lambda}\big(1 - \frac{\pi m}{\sqrt{\lambda}} - \frac{\pi m}{\sqrt{\mu}}\big)\big] > 0$. If $\beta > 0$, then $\phi_\infty(\beta) = \frac{\beta}{\sqrt{\lambda}} \sin\big[\sqrt{\lambda}\big(1 - \frac{\pi m}{\sqrt{\lambda}} - \frac{\pi m}{\sqrt{\mu}}\big)\big] > 0$. Similarly, if $\beta < 0$, then $\phi_\infty(\beta) < 0$. Thus, $\phi_\infty(-\alpha)\phi_\infty(\alpha) < 0$.

The other cases can be considered similarly. □

Corollary 4.2. *The vector field ϕ_∞ is continuous.*

The proof follows from (4.7)-(4.10).

5. Vector field $\boldsymbol{\phi}_\infty$

Consider the uncoupled system of n Fučík equations

$$\begin{gathered} z_1'' = -\lambda_1 z_1^+ + \mu_1 z_1^-, \\ \dots \\ z_n'' = -\lambda_n z_n^+ + \mu_n z_n^- \end{gathered} \tag{5.1}$$

with respect to the Dirichlet boundary conditions

$$\mathbf{z}(0) = \mathbf{0} = \mathbf{z}(1), \tag{5.2}$$

where $\lambda_i, \mu_i > 0$ $(i = 1, 2, \dots, n)$ and $\mathbf{z} = (z_1, \dots, z_n)^T$.

Denote by $z_i(t; \beta_i)$ the solution to the scalar Cauchy problem

$$z_i'' = -\lambda_i z_i^+ + \mu_i z_i^-, \quad z_i(0) = 0, \quad z_i'(0) = \beta_i. \tag{5.3}$$

Then $\mathbf{z}(t; \boldsymbol{\beta}) = \big(z_1(t; \beta_1), z_2(t; \beta_2), \dots, z_n(t; \beta_n)\big)^T$ solves the system (5.1) with respect to the initial conditions

$$\mathbf{z}(0) = \mathbf{0}, \quad \mathbf{z}'(0) = \boldsymbol{\beta} := (\beta_1, \dots, \beta_n). \tag{5.4}$$

Define the vector fields $\phi_{\infty,i} : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, \dots, n)$,

$$\phi_{\infty,i}(\beta_i) = z_i(1; \beta_i), \quad \forall \beta_i \in \mathbb{R}.$$

Note that $\beta_i = 0$ is a singular point of the vector field $\phi_{\infty,i}$. Define the vector field $\boldsymbol{\phi}_\infty : \mathbb{R}^n \to \mathbb{R}^n$,

$$\boldsymbol{\phi}_\infty(\boldsymbol{\beta}) = \mathbf{z}(1; \boldsymbol{\beta}), \quad \forall \boldsymbol{\beta} \in \mathbb{R}^n.$$

Note that $\boldsymbol{\beta} = \mathbf{0}$ is a singular point of the vector field $\boldsymbol{\phi}_\infty$.

Let us consider the following condition.

(A6) System (5.1) is *nonresonant* with respect to the boundary conditions (5.2), that is, the boundary value problem (5.1), (5.2) has only the trivial solution.

Proposition 5.1. *The following three statements are equivalent:*

1. *Condition (A6) holds.*
2. $(\lambda_i, \mu_i) \notin \Sigma$ *for every* $i = 1, 2, \dots, n$.
3. $\boldsymbol{\beta} = \mathbf{0}$ *is the unique singular point of the vector field* $\boldsymbol{\phi}_\infty$.

Proof. Taking into account that the system (5.1) is uncoupled, the proof follows from the properties of solutions of the Fučík problem (4.1). □

Consider the one-dimensional subspaces of $\mathbb{R}^n$:

$$L_1 = \big\{(\beta_1, 0, \dots, 0) \in \mathbb{R}^n : \beta_1 \in \mathbb{R}\big\}, \dots, L_n = \big\{(0, 0, \dots, \beta_n) \in \mathbb{R}^n : \beta_n \in \mathbb{R}\big\}.$$

Then $\mathbb{R}^n = \oplus_{i=1}^n L_i$, that is, the space $\mathbb{R}^n$ is the direct sum of its subspaces L_i ($i = 1, 2, \dots, n$). Thus, every $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ can be expressed in the unique way as $\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \dots + \boldsymbol{\beta}_n$, where $\boldsymbol{\beta}_1 = (\beta_1, 0, \dots, 0) \in L_1, \dots, \boldsymbol{\beta}_n = (0, 0, \dots, \beta_n) \in L_n$.

Consider the one-dimensional vector fields $\boldsymbol{\phi}_{\infty,i} : L_i \to L_i$ ($i = 1, 2, \dots, n$):

$$\boldsymbol{\phi}_{\infty,1}(\boldsymbol{\beta}_1) = \big(z_1(1; \beta_1), 0, \dots, 0\big), \dots, \boldsymbol{\phi}_{\infty,n}(\boldsymbol{\beta}_n) = \big(0, 0, \dots, z_n(1; \beta_n)\big).$$

Since $\boldsymbol{\phi}_\infty = \sum_{i=1}^n \boldsymbol{\phi}_{\infty,i}$, according to the notation in [13], $\boldsymbol{\phi}_\infty = \oplus_{i=1}^n \boldsymbol{\phi}_{\infty,i}$.

For $\alpha > 0$ and for every $i = 1, 2, \dots, n$ consider the set

$$\begin{aligned} \Omega_{\alpha,i} = \Big\{\boldsymbol{\beta} \in \mathbb{R}^n : \beta_1 = \dots = \beta_{i-1} = 0,\ -\alpha < \beta_i < \alpha, \\ \beta_{i+1} = \dots = \beta_n = 0\Big\} \subset L_i. \end{aligned}$$

The n-dimensional cube $\Omega_\alpha = \big\{\boldsymbol{\beta} \in \mathbb{R}^n : -\alpha < \beta_i < \alpha;\ i = 1, 2, \dots, n\big\}$ is equal to the cartesian product $\Omega_{\alpha,1} \times \dots \times \Omega_{\alpha,n}$.

Suppose that condition (A6) holds. It follows from Proposition 5.1, coupled with condition (A6), that $\boldsymbol{\beta} = \mathbf{0}$ is the unique singular point of the vector field $\boldsymbol{\phi}_\infty$. Therefore

$$\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}_\infty) = \gamma(\boldsymbol{\phi}_\infty, \Omega_\alpha), \quad \alpha > 0. \tag{5.5}$$

By [13, Theorem 7.4, p. 20],

$$\gamma(\boldsymbol{\phi}_\infty, \Omega_\alpha) = \gamma\Big(\oplus_{i=1}^n \boldsymbol{\phi}_{\infty,i}, \Omega_\alpha\Big) = \prod_{i=1}^n \gamma\big(\boldsymbol{\phi}_{\infty,i}, \Omega_{\alpha,i}\big). \tag{5.6}$$

For every $i = 1, 2, \dots, n$ we identify the space L_i with $\mathbb{R}$, the set $\Omega_{\alpha,i}$ with the open interval $I_\alpha = (-\alpha, \alpha) \subset \mathbb{R}$ and the vector field $\boldsymbol{\phi}_{\infty,i}$ with the vector field $\phi_{\infty,i}$. Then, it follows from (5.5) and (5.6) that

$$\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}_\infty) = \prod_{i=1}^n \gamma\big(\phi_{\infty,i}, I_\alpha\big). \tag{5.7}$$

In the previous section we had split the set $Q \setminus \Sigma$ into the subsets $D(k)$ ($k = 0, 1, 2, \dots$) and E.

Proposition 5.2. *Suppose that condition* (A6) *holds.*

1. *If* $(\lambda_i, \mu_i) \in E$ *for some* $i \in \{1, 2, \dots, n\}$, *then* $\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}_\infty) = 0$.
2. *If* $(\lambda_i, \mu_i) \in D(k_i)$ *for every* $i \in \{1, 2, \dots, n\}$, *then*
$$\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}_\infty) = (-1)^{k_1 + k_2 + \dots + k_n}. \tag{5.8}$$

Proof. 1. Suppose that $(\lambda_i, \mu_i) \in E = \bigcup_{m=0}^\infty \big(E^+(2m) \cup E^-(2m)\big)$ for some $i \in \{1, 2, \dots, n\}$. Then, by Proposition 4.1, $\phi_{\infty,i}(-\alpha) > 0$ and $\phi_{\infty,i}(\alpha) > 0$, if $(\lambda_i, \mu_i) \in E^+_{2m}$, and, $\phi_{\infty,i}(-\alpha) < 0$ and $\phi_{\infty,i}(\alpha) < 0$, if $(\lambda_i, \mu_i) \in E^-_{2m}$, that is, the vector field $\phi_{\infty,i}$ at both endpoints of the interval $I_\alpha = (-\alpha, \alpha)$ points in the same direction. Therefore, [13, p. 6], the rotation of the one-dimensional vector field $\phi_{\infty,i} : \overline{I}_\alpha :\to \mathbb{R}$ on $\partial I_\alpha = \{\pm\alpha\}$ is equal to the zero, that is, $\gamma\big(\phi_{\infty,i}, I_\alpha\big) = 0$. It follows from (5.7) that $\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}_\infty) = 0$.

2. Let $(\lambda_i, \mu_i) \in D(k_i)$ for every $i \in \{1, 2, \dots, n\}$.

(a) If $k_i = 2m_i$ ($m_i = 0, 1, 2, \dots$), then, by Proposition 4.1, $\phi_{\infty,i}(-\alpha) < 0$ and $\phi_{\infty,i}(\alpha) > 0$, that is, the vector field $\phi_{\infty,i}$ at both endpoints of the interval

$I_\alpha = (-\alpha,\alpha)$ points to the exterior of the interval I_α. Therefore, [13, p. 6], $\gamma\big(\phi_{\infty,i}, I_\alpha\big) = 1 = (-1)^{2m_i} = (-1)^{k_i}$.

(b) If $k_i = 2m_i - 1$ ($m_i = 1,2,\dots$), then, by Proposition 4.1, $\phi_{\infty,i}(-\alpha) > 0$ and $\phi_{\infty,i}(\alpha) < 0$, that is, the vector field $\phi_{\infty,i}$ at both endpoints of the interval $I_\alpha = (-\alpha,\alpha)$ points to the interior of the interval Ω_i. Therefore, [13, p. 6], $\gamma\big(\phi_{\infty,i}, I_\alpha\big) = -1 = (-1)^{2m_i-1} = (-1)^{k_i}$.

Formula (5.8) follows from (a), (b) and (5.7). □

Proposition 5.3. $\boldsymbol{\phi}_\infty(\boldsymbol{\beta}) = \|\boldsymbol{\beta}\|\,\boldsymbol{\phi}_\infty\left(\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\right)$ *for all* $\boldsymbol{\beta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

Proof. The proof follows from the positive homogeneity of the system (5.1); see also (4.7)-(4.10). □

Proposition 5.4. *Suppose that the condition (A6) holds. Then, there exists* $c > 0$ *such that* $\big\|\boldsymbol{\phi}_\infty(\boldsymbol{\beta})\big\| \ge c\,\|\boldsymbol{\beta}\|$ *for all* $\boldsymbol{\beta} \in \mathbb{R}^n$.

Proof. From Proposition 5.1, since condition (A6) holds, we have that $(\lambda_i,\mu_i) \notin \Sigma$ ($i = 1,2,\dots,n$), therefore, $(\lambda_i,\mu_i) \in (D \cup E)$ ($i = 1,2,\dots,n$). For every $i = 1,2,\dots,n$ the vector field $\phi_{\infty,i}$, taking into account (4.7) to (4.10), can be represented in the form $\phi_{\infty,i}(\beta_i) = \beta_i\, p_i(\lambda_i,\mu_i)$, where $p_i(\lambda_i,\mu_i) \ne 0$. Hence, $\big\|\boldsymbol{\phi}_\infty(\boldsymbol{\beta})\big\| \ge c\,\|\boldsymbol{\beta}\|$ for all $\boldsymbol{\beta} \in \mathbb{R}^n$, where $c = \min_{1\le i\le n}\big|p_i(\lambda_i,\mu_i)\big| > 0$. □

6. Vector field $\boldsymbol{\phi}$ at infinity

Consider the function $\boldsymbol{y}(t;\boldsymbol{\beta}) = \frac{1}{\|\boldsymbol{\beta}\|}\,\mathbf{x}(t;\boldsymbol{\beta}) - \mathbf{z}\big(t;\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\big)$, $0 \le t \le 1$, $\boldsymbol{\beta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, where $\mathbf{x}(t;\boldsymbol{\beta})$ is the solution to the Cauchy problem (*), (1.5) and $\mathbf{z}\big(t;\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\big)$ is the solution to the Cauchy problem (5.1), $\mathbf{z}(0) = \mathbf{0}$, $\mathbf{z}'(0) = \frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}$.

Proposition 6.1. *Suppose that conditions* (A1)–(A4), (A6) *hold. Then*
$$\lim_{\|\boldsymbol{\beta}\|\to\infty} \big\|\boldsymbol{y}(1;\boldsymbol{\beta})\big\| = 0. \tag{6.1}$$

Proof. Step 1. The purpose of this step is to introduce the change of variables $\mathbf{u} := \frac{\mathbf{x}}{\boldsymbol{\beta}}$, rewriting system (*) in terms of $\mathbf{u}$. For every $i = 1,2,\dots,n$ let us introduce the functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ such that
$$g_i(x_i) = \lambda_i x_i^+ - \mu_i x_i^- + \varphi_i(x_i), \tag{6.2}$$
where $x_i^+ = \max\{x_i, 0\}$, $x_i^- = \max\{-x_i, 0\}$. For every $i = 1,2,\dots,n$ it follows from the conditions (A1), (A3) and (A4) that $\varphi_i \in C(\mathbb{R},\mathbb{R})$, $\varphi_i(0) = 0$ and
$$\lim_{|x_i|\to+\infty} \frac{\varphi_i(x_i)}{x_i} = 0. \tag{6.3}$$
Taking into account (6.2) and the positive homogeneity of the operations $^+$ and $^-$, we can conclude that the functions $u_1 = \frac{1}{\|\boldsymbol{\beta}\|}x_1, \dots, u_n = \frac{1}{\|\boldsymbol{\beta}\|}x_n$ solve the system
$$\begin{gathered} u_1'' + \lambda_1 u_1^+ - \mu_1 u_1^- = \omega_1(\mathbf{u};\boldsymbol{\beta}), \\ \dots \\ u_n'' + \lambda_n u_n^+ - \mu_n u_n^- = \omega_n(\mathbf{u};\boldsymbol{\beta}) \end{gathered} \tag{6.4}$$
and satisfy the initial conditions
$$u_1(0) = \dots = u_n(0) = 0, \dots u_1'(0) = \frac{\beta_1}{\|\boldsymbol{\beta}\|}, \dots, u_n'(0) = \frac{\beta_n}{\|\boldsymbol{\beta}\|}, \tag{6.5}$$

where $\mathbf{u} = (u_1, \dots, u_n)^T$ and $\omega_i(\mathbf{u};\boldsymbol{\beta}) = \frac{1}{\|\boldsymbol{\beta}\|}\Big[-\varphi_i\big(\|\boldsymbol{\beta}\|u_i\big) + h_i\big(\|\boldsymbol{\beta}\|\mathbf{u}\big)\Big]$, $i \in \{1, 2, \dots, n\}$.

Step 2. In this step we will prove that $\big\|\boldsymbol{\omega}(\mathbf{u}(t);\boldsymbol{\beta})\big\| \to 0$, uniformly in $t \in [0,1]$, as $\|\boldsymbol{\beta}\| \to \infty$. Let $\varepsilon > 0$ be arbitrary. By (2.2) and (6.3), for every $i = 1, 2, \dots, n$ there exists $M_i(\varepsilon) > 0$ such that

$$\big|\omega_i(\mathbf{u};\boldsymbol{\beta})\big| \le \frac{1}{\|\boldsymbol{\beta}\|}\Big[N_i + M_i(\varepsilon) + \varepsilon\,\big|x_i(t)\big|\Big], \quad 0 \le t \le 1. \tag{6.6}$$

Consider the Cauchy problem $\mathbf{w}'(t) = \mathbf{F}\big(\mathbf{z}(t)\big)$, $\mathbf{w}(0) = \mathbf{w}_0 = (\mathbf{0}, \boldsymbol{\beta})^T$ and the equivalent integral equation $\mathbf{w}(t) = \mathbf{w}_0 + \int_0^t \mathbf{F}\big(\mathbf{w}(s)\big)ds$. Taking into account (2.3), we obtain

$$\begin{aligned}
\big\|\mathbf{w}(t)\big\|_N &\le \|\mathbf{w}_0\|_N + \Big\|\int_0^t \mathbf{F}\big(\mathbf{w}(s)\big)ds\Big\|_N \le \|\boldsymbol{\beta}\| + \int_0^t \Big\|\mathbf{F}\big(\mathbf{w}(s)\big)\Big\|_N ds\\
&\le \|\boldsymbol{\beta}\| + \int_0^t \Big[A + B\big\|\mathbf{w}(s)\big\|_N\Big]ds = \|\boldsymbol{\beta}\| + At + B\int_0^t \big\|\mathbf{w}(s)\big\|_N ds,
\end{aligned}$$

for $0 \le t \le 1$. By the Grönwall's inequality, we have

$$\big|x_i(t)\big| \le \big\|\mathbf{x}(t)\big\| \le \big\|\mathbf{w}(t)\big\|_N \le A\,e^B + e^B\|\boldsymbol{\beta}\|, \quad 0 \le t \le 1, \quad i = 1, 2, \dots, n. \tag{6.7}$$

It follows from (6.6) and (6.7) that

$$\big|\omega_i(\mathbf{u};\boldsymbol{\beta})\big| \le \frac{N_i + M_i(\varepsilon) + \varepsilon\,A\,e^B}{\|\boldsymbol{\beta}\|} + \varepsilon\,e^B, \quad 0 \le t \le 1, \quad i = 1, 2, \dots, n.$$

Then

$$\big\|\boldsymbol{\omega}(\mathbf{u};\boldsymbol{\beta})\big\| \le \varepsilon\,n\,e^B + \frac{1}{\|\boldsymbol{\beta}\|}\sum_{i=1}^n \Big(N_i + M_i(\varepsilon) + \varepsilon\,A\,e^B\Big), \quad 0 \le t \le 1,$$

where $\boldsymbol{\omega}(\mathbf{u};\boldsymbol{\beta}) = \big(\omega_1(\mathbf{u};\boldsymbol{\beta}), \dots, \omega_n(\mathbf{u};\boldsymbol{\beta})\big)^T$. Since $\varepsilon > 0$ can be arbitrary,

$$\lim_{\|\boldsymbol{\beta}\| \to \infty} \big\|\boldsymbol{\omega}(\mathbf{u};\boldsymbol{\beta})\big\| = 0, \quad 0 \le t \le 1. \tag{6.8}$$

Step 3. In this step we will prove (6.1). Let us rewrite the system (5.1) in the form

$$\mathbf{z}'' = \boldsymbol{P}(\mathbf{z}),$$

where $\boldsymbol{P}(\mathbf{z}) = \big(P_1(z_1), \dots, P_n(z_n)\big)$, $P_i(z_i) = -\lambda_i z_i^+ + \mu_i z_i^-$ $(1 \le i \le n)$. Then

$$\big\|\boldsymbol{P}(\mathbf{z}) - \boldsymbol{P}(\hat{\mathbf{z}})\big\| \le L\,\|\mathbf{z} - \hat{\mathbf{z}}\,\|, \quad \forall \mathbf{z}, \hat{\mathbf{z}} \in \mathbb{R}^n, \tag{6.9}$$

where $L = \frac{\sqrt{n}}{2}\max_{1 \le i \le n}\{\lambda_i, \mu_i\} > 0$. We can rewrite the system (6.3) in the form

$$\mathbf{u}'' = \boldsymbol{P}(\mathbf{u}) + \boldsymbol{\omega}(\mathbf{u};\boldsymbol{\beta}).$$

The function $\boldsymbol{y}(t;\boldsymbol{\beta}) = \mathbf{u}\Big(t;\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\Big) - \mathbf{z}\Big(t;\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\Big)$, where $\mathbf{u}\Big(t;\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\Big) = \frac{1}{\|\boldsymbol{\beta}\|}\,\mathbf{x}(t;\boldsymbol{\beta})$, has the following properties:

$$\boldsymbol{y}'' = \boldsymbol{P}(\mathbf{u}) - \boldsymbol{P}(\mathbf{z}) + \boldsymbol{\omega}(\mathbf{u};\boldsymbol{\beta}), \quad \boldsymbol{y}(0) = \mathbf{0}, \quad \boldsymbol{y}'(0) = \mathbf{0},$$

where, for brevity, we write $\boldsymbol{y}(t) = \boldsymbol{y}(t;\boldsymbol{\beta})$, $\mathbf{u}(t) = \mathbf{u}\Big(t;\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\Big)$, $\mathbf{z}(t) = \mathbf{z}\Big(t;\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\Big)$.

Let $\varepsilon > 0$ be arbitrary. It follows from (6.8) that there exists $\rho = \rho(\varepsilon) > 0$ such that for all $\boldsymbol{\beta} \in \mathbb{R}^n$, $\|\boldsymbol{\beta}\| > \rho$, we have

$$\big\|\boldsymbol{\omega}\big(\mathbf{u}(t);\boldsymbol{\beta}\big)\big\| < \frac{\varepsilon}{2\cosh(\sqrt{L})} = \varepsilon_1, \quad t \in [0,1]. \tag{6.10}$$

It follows from $\boldsymbol{y}'(t) = \int_0^t \boldsymbol{y}''(s)ds$ coupled with (6.9) and (6.10) that

$$\begin{aligned}\big\|\boldsymbol{y}'(t)\big\| &\le \int_0^t \big\|\boldsymbol{P}\big(\mathbf{u}(s)\big) - \boldsymbol{P}\big(\mathbf{z}(s)\big)\big\|ds + \int_0^t \big\|\boldsymbol{\omega}\big(\mathbf{u}(s);\boldsymbol{\beta}\big)\big\|ds \\ &\le \int_0^t L\,\big\|\mathbf{u}(s) - \mathbf{z}(s)\big\|ds + \int_0^t \varepsilon_1 ds = L\int_0^t \big\|\boldsymbol{y}(s)\big\|ds + \varepsilon_1\, t \\ &\le L\,\theta(t) + \varepsilon_1, \quad t \in [0,1],\end{aligned} \tag{6.11}$$

where $\theta(t) := \int_0^t \big\|\boldsymbol{y}(s)\big\|ds$. By $\boldsymbol{y}(t) = \int_0^t \boldsymbol{y}'(s)ds$ and (6.11),

$$\theta'(t) = \big\|\boldsymbol{y}(t)\big\| \le \int_0^t \big\|\boldsymbol{y}'(s)\big\|ds \le L\,\psi(t) + \varepsilon_1, \quad t \in [0,1], \tag{6.12}$$

where $\psi(t) := \int_0^t \theta(s)ds$. Hence,

$$\psi''(t) \le L\,\psi(t) + \varepsilon_1, \quad t \in [0,1], \quad \psi(0) = 0, \quad \psi'(0) = 0. \tag{6.13}$$

The Cauchy problem

$$q''(t) = L\,q(t) + \varepsilon_1, \quad q(0) = 0, \quad q'(0) = 0 \tag{6.14}$$

has the solution $q^*(t) = \frac{\varepsilon_1}{2L}\, e^{-\sqrt{L}\,t}\left(e^{\sqrt{L}\,t} - 1\right)^2$. Let $\chi(t) := q^*(t) - \psi(t)$. It follows from (6.13), (6.14) that

$$\chi''(t) \ge L\,\chi(t), \quad t \in [0,1], \quad \chi(0) = 0, \quad \chi'(0) = 0.$$

Consider the function $\eta(t) := \chi''(t) - L\,\chi(t) \ge 0$, $t \in [0,1]$. Since $\chi(t)$ solves the Cauchy problem

$$\chi''(t) = L\,\chi(t) + \eta(t), \quad \chi(0) = 0, \quad \chi'(0) = 0$$

in the interval $[0,1]$, then, by the variation of constants formula, we have

$$\chi(t) = \int_0^t q(t,s)\eta(s)\,ds, \quad t \in [0,1], \tag{6.15}$$

where

$$q(t,s) = \frac{\sinh\left(\sqrt{L}\,t - \sqrt{L}\,s\right)}{\sqrt{L}}$$

is the Cauchy function [12, p. 199] for the linear homogeneous equation $q''(t) = L\,q(t)$. Since $q(t,s) \ge 0$ in the triangle $0 \le s \le t \le 1$ and $\eta(s) \ge 0$ in the interval $0 \le s \le t$, $t \in [0,1]$, then it follows from (6.15) that $\chi(t) \ge 0$, $t \in [0,1]$. Therefore,

$$\psi(t) \le q^*(t), \quad t \in [0,1]. \tag{6.16}$$

By (6.12) and (6.16),

$$\big\|\boldsymbol{y}(1;\boldsymbol{\beta})\big\| = \big\|\boldsymbol{y}(1)\big\| \le \varepsilon_1 + L\,q^*(1) = \varepsilon_1\,\cosh(\sqrt{L}) = \frac{\varepsilon}{2} < \varepsilon.$$

Thus, (6.1) fulfills. $\square$

Theorem 6.2. *Suppose that conditions* (A1)–(A4), (A6) *hold. Then* ∞ *is an isolated singular point of the vector field* $\boldsymbol{\phi}$ *and* $\operatorname{ind}(\infty,\boldsymbol{\phi}) = \operatorname{ind}(\mathbf{0},\boldsymbol{\phi}_\infty)$.

Proof. By Proposition 5.3, for all $\boldsymbol{\beta} \neq \mathbf{0}$ we have

$$\begin{aligned} \big\|\boldsymbol{\phi}(\boldsymbol{\beta}) - \boldsymbol{\phi}_\infty(\boldsymbol{\beta})\big\| &= \big\|\boldsymbol{\phi}(\boldsymbol{\beta}) - \|\boldsymbol{\beta}\|\,\boldsymbol{\phi}_\infty\big(\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\big)\big\| \\ &= \big\|\mathbf{x}(1;\boldsymbol{\beta}) - \|\boldsymbol{\beta}\|\,\mathbf{z}\big(1;\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\big)\big\| \\ &= \|\boldsymbol{\beta}\|\big\|\frac{1}{\|\boldsymbol{\beta}\|}\,\mathbf{x}(1;\boldsymbol{\beta}) - \mathbf{z}\big(1;\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}\big)\big\| \\ &= \|\boldsymbol{\beta}\|\,\big\|\boldsymbol{y}(1;\boldsymbol{\beta})\big\|. \end{aligned} \tag{6.17}$$

Let us take the constant $c > 0$ from Proposition 5.4. Using Proposition 6.1, there exists $R > 0$ such that

$$\big\|\boldsymbol{y}(1;\boldsymbol{\beta})\big\| \leq \frac{c}{2} \tag{6.18}$$

when $\|\boldsymbol{\beta}\| \geq R$. Taking into account assumption (A6) and Proposition 5.4 coupled with (6.17) and (6.18), for all $\|\boldsymbol{\beta}\| \geq R$ we have

$$\begin{aligned} \big\|\boldsymbol{\phi}(\boldsymbol{\beta}) - \boldsymbol{\phi}_\infty(\boldsymbol{\beta})\big\| &= \|\boldsymbol{\beta}\|\,\big\|\boldsymbol{y}(1;\boldsymbol{\beta})\big\| \leq \|\boldsymbol{\beta}\|\,\frac{c}{2} \\ &\leq \big(\frac{1}{c}\,\big\|\boldsymbol{\phi}_\infty(\boldsymbol{\beta})\big\|\big)\frac{c}{2} = \frac{1}{2}\,\big\|\boldsymbol{\phi}_\infty(\boldsymbol{\beta})\big\| \\ &< \big\|\boldsymbol{\phi}_\infty(\boldsymbol{\beta})\big\|. \end{aligned}$$

Consider the set $B_R(\mathbf{0}) = \big\{\boldsymbol{\beta} \in \mathbb{R}^n : \|\boldsymbol{\beta}\| < R\big\}$. The Rouché theorem [24] ensures that $\gamma\big(\boldsymbol{\phi}, B_R(\mathbf{0})\big) = \gamma\big(\boldsymbol{\phi}_\infty, B_R(\mathbf{0})\big) = \gamma(\boldsymbol{\phi}_\infty, \Omega_\alpha)$, which completes the proof. □

7. Main Theorem

Theorem 7.1. *Suppose that conditions* (A1)–(A6) *hold. Then the points* $\boldsymbol{\beta} = \mathbf{0}$ *and* $\boldsymbol{\beta} = \infty$ *are isolated singular points of the vector field* $\boldsymbol{\phi}$. *If* $\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}) \neq \operatorname{ind}(\infty, \boldsymbol{\phi})$, *then the boundary value problem* (1.1), (1.2) *has a nontrivial solution.*

Proof. The proof is the same as the proof of the main result in [11]. We sketch it briefly. By Theorems 3.3 and 6.2, the points $\boldsymbol{\beta} = \mathbf{0}$ and ∞ are isolated singular points of the vector field $\boldsymbol{\phi}$. Hence, we can find positive r, R $(r < R)$ such that the sets

$$\overline{B}_r(\mathbf{0}) \setminus \{\mathbf{0}\} = \big\{\boldsymbol{\beta} \in \mathbb{R}^n : 0 < \|\boldsymbol{\beta}\| \leq r\big\}, \quad \overline{B}_R(\infty) = \big\{\boldsymbol{\beta} \in \mathbb{R}^n : \|\boldsymbol{\beta}\| \geq R\big\}$$

contain no singular points of the vector field $\boldsymbol{\phi}$. Since the rotations on the spheres $S_r(\mathbf{0}) = \partial B_r(\mathbf{0})$ and $S_R(\mathbf{0}) = \partial B_R(\mathbf{0})$ are different:

$$\gamma\big(\boldsymbol{\phi}, B_r(\mathbf{0})\big) = \operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}) \neq \operatorname{ind}(\infty, \boldsymbol{\phi}) = \gamma\big(\boldsymbol{\phi}, B_R(\mathbf{0})\big),$$

then, by [24, Theorem 2], we can conclude that the n-dimensional annulus

$$Ann(r, R) = \big\{\boldsymbol{\beta} \in \mathbb{R}^n : r < \|\boldsymbol{\beta}\| < R\big\}$$

contains a singular point $\boldsymbol{\beta}_0 \neq \mathbf{0}$ of the vector field $\boldsymbol{\phi}$, which generates a nontrivial solution to the Dirichlet boundary value problem (1.1), (1.2). □

Corollary 7.2. *Suppose that* (A1)–(A6) *hold. If for some* $i \in \{1, 2, \dots, n\}$ *the pair* (λ_i, μ_i) *belongs to the set* E, *then the boundary value problem* (1.1), (1.2) *has a nontrivial solution.*

Proof. It follows from Propositions 3.2, 5.2 and Theorems 3.3, 6.2 that $\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}) \in \{-1, 1\}$, while $\operatorname{ind}(\infty, \boldsymbol{\phi}) = 0$. Theorem 7.1 completes the proof. □

8. Examples

Example 8.1. Consider the system

$$
\begin{aligned}
& x_1'' + 200x_1 \frac{1}{\pi}\Big(\arctan x_1 + \frac{\pi}{2}\Big) - 50x_1 \frac{1}{\pi}\Big(\arctan x_1 - \frac{\pi}{2}\Big) \\
& = 20\sin(x_1 + x_2), \\
& x_2'' + 100x_2 \frac{1}{\pi}\Big(\arctan x_2 + \frac{\pi}{2}\Big) - 200x_2 \frac{1}{\pi}\Big(\arctan x_2 - \frac{\pi}{2}\Big) \\
& = -30\sin(x_1 - x_2)
\end{aligned}
\tag{8.1}
$$

together with the boundary conditions

$$
x_1(0) = x_2(0) = 0 = x_1(1) = x_2(1). \tag{8.2}
$$

Note that $(\lambda_1, \mu_1) = (200, 50) \in E^-(2)$, $(\lambda_2, \mu_2) = (100, 200) \in D(3)$. Thus, conditions (A1)–(A4), (A6) hold. The Jacobi matrix

$$
\mathbf{f}'(\mathbf{0}) = \begin{pmatrix} -105 & 20 \\ -30 & -120 \end{pmatrix}
$$

has eigenvalues $\xi_{1,2} = \frac{5}{2}\left(-45 \pm i\sqrt{87}\right) \notin \sigma_D$. Using Remark 3.1, we can affirm that the condition (A5) fulfills also. By Corollary 7.2, the boundary value problem (8.1), (8.2) has a nontrivial solution. In Figure 2 a numerical nontrivial solution of the boundary value problem (8.1), (8.2) is depicted.

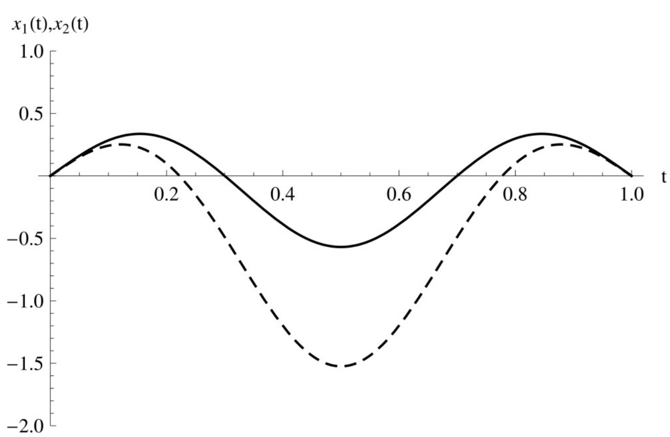

Figure 2. A solution $\mathbf{x} = (x_1, x_2)^T$ of the boundary value problem (8.1), (8.2): x_1 (solid), x_2 (dashed), with initial data $\boldsymbol{\beta} = (3.349695, 3.204575)^T$.

Example 8.2. Let us explore [23, Example 1] considering the system

$$
\begin{aligned}
x_1'' + 50\,x_1 &= 16\sin(x_2 + 3x_1^2), \\
x_2'' + 22\,x_2 &= -12\arctan x_1
\end{aligned}
\tag{8.3}
$$

together with the boundary conditions (8.2). Note that $(\lambda_1, \mu_1) = (50, 50) \in D(2)$, $(\lambda_2, \mu_2) = (22, 22) \in D(1)$. Thus, conditions (A1)–(A4) and (A6) hold. By Proposition 5.2 and Theorem 6.2, $\operatorname{ind}(\infty, \mathbf{f}) = (-1)^{2+1} = -1$. The Jacobi matrix

$$\mathbf{f}'(\mathbf{0}) = \begin{pmatrix} -50 & 16 \\ -12 & -22 \end{pmatrix}$$

has eigenvalues $\xi_1 = -38 \notin \sigma_D$ and $\xi_2 = -34 \notin \sigma_D$. Thus, condition (A5) holds also. By Proposition 3.2 and Theorem 3.3,

$$\operatorname{ind}(\mathbf{0}, \mathbf{f}) = \operatorname{sgn}\Big(\sin\sqrt{|\xi_1|}\,\sin\sqrt{|\xi_2|}\Big) = 1.$$

By Theorem 7.1, we come to the same conclusion as in [23] that the boundary value problem (8.3), (8.2) has a nontrivial solution.

Conclusions. We give precise description of the solvability for the case of multiple equations that are asymptotically asymmetric and which are coupled through the right sides (functions h_i). The analysis is made by studying the system at zero and at infinity. Any possible cases of interrelation of the spectrum of the matrix $\mathbf{f}'(\mathbf{0})$ with the limits $\lim_{x_i\to-\infty} \frac{g_i(x_i)}{x_i}$, $\lim_{x_i\to\infty} \frac{g_i(x_i)}{x_i}$ $(i = 1, 2, \dots, n)$ are covered by the Main Theorem.

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Armands Gritsans
Institute of Life Sciences and Technologies, Daugavpils University, Parades iela 1^a, Daugavpils LV 5400, Latvia
E-mail address: arminge@inbox.lv

Felix Sadyrbaev
Institute of Life Sciences and Technologies, Daugavpils University, Parades iela 1^a, Daugavpils LV 5400, Latvia.
Institute of Mathematics and Computer Science, University of Latvia, Raina bulv. 29, Riga LV-1459, Latvia
E-mail address: felix@latnet.lv

Inara Yermachenko
Institute of Life Sciences and Technologies, Daugavpils University, Parades iela 1^a, Daugavpils LV 5400, Latvia
E-mail address: inara.jermacenko@du.lv