*Electronic Journal of Differential Equations*, Vol. 2018 (2018), No. 37, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# EXISTENCE OF ASYMPTOTICALLY ALMOST AUTOMORPHIC MILD SOLUTIONS FOR NONAUTONOMOUS SEMILINEAR EVOLUTION EQUATIONS

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*Communicated by Jerome A. Goldstein*

Abstract. In this article, we study the existence of asymptotically almost automorphic mild solutions for a class of nonautonomous semilinear evolution equations and extend some related results in this direction. The working tools are based on the Krasnoselskii's fixed point theorem and compactness criterion. Finally, an example is presented to illustrate the main findings.

## 1. Introduction

The concept of almost automorphy, which is a generalization of almost periodicity, has been introduced in the literature by Bochner in relation to some aspects of differential geometry [8, 9, 7, 10]. Since then, the theory of almost automorphic functions has found several developments and applications in the theory of various ordinary differential equations, partial differential equations, functional differential equations, integro-differential equations, fractional differential equations as well as stochastic differential equations (see for instance [1, 6, 12, 13, 14, 17, 23, 26, 25, 28, 29, 30, 32, 35, 39] and the references therein).

As a natural extension of almost automorphy, the concept of asymptotic almost automorphy, which is the central issue to be discussed in this paper, was introduced in the literature in the early eighties by N'Guérékata [34]. Since then, this notion has found several developments and has been generalized into different directions. In particular, the existence of asymptotically almost automorphic solutions to various ordinary differential equations, partial differential equations, functional differential equations, integro-differential equations, fractional differential equations as well as stochastic differential equations has been investigated in many papers (see, e.g. [11, 15, 16, 19, 20, 24, 34, 44, 45, 47] and references therein), and it has become an attractive topic in the qualitative theory of differential equations due to their significance and applications in physics, mathematical biology, control theory and so on. We refer the reader to the monographs of N'Guérékata [33] for the recently theory and applications of asymptotically almost automorphic functions.

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2010 *Mathematics Subject Classification.* 34K14, 35K90.
*Key words and phrases.* Nonautonomous semilinear evolution equation;
asymptotic almost automorphy; compact semigroup.

Submitted November 1, 2017. Published January 30, 2018.

When dealing with parabolic evolution equation, it is usually assumed that the partial differential operator in the linear part (possibly unbounded) depends on time (i.e., it is the case of equations being nonautonomous), motivated by the fact that this class of operators appears very often in the applications (see, e.g., [5, 36]). Stimulated by the works above, the main purpose of this paper is to establish a new existence theorem of asymptotically almost automorphic mild solutions to the following nonautonomous semilinear evolution equations

$$x'(t) = A(t)x(t) + F(t, x(t)), \quad t \in \mathbb{R}. \tag{1.1}$$

In our result, the nonlinearity $F(t, x)$ does not have to satisfy a (locally) Lipschitz condition with respect to $x$ (see Remark 3.1). However, in many papers (for instance [15, 16, 19, 20, 24, 44, 45, 47]) on asymptotically almost automorphic solutions, to be able to apply the well known Banach contraction principle, a (locally) Lipschitz condition for the nonlinearity of corresponding differential equations is needed. Here we weaken the assumptions on nonlinearity $F(t, x)$ and deal with the existence of asymptotically almost automorphic solutions of (1.1) by Krasnoselskii's fixed point theorem. To the best our knowledge, few papers using this theorem to solve related problem and this is one of the key motivations of study.

The rest of this paper is organized as follows. In Section 2, some concepts, the related notations and some useful lemmas are introduced. In Section 3, we present some criteria ensuring the existence of asymptotically almost automorphic mild solutions. An example is given to illustrate our result in Section 4.

## 2. Preliminaries

This section is concerned with some notations, definitions, lemmas and preliminary facts which are used in what follows.

From now on, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ stand for the set of real numbers, nonnegative real numbers and complex numbers respectively, let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces, $L(X, Y)$ denotes the space of all bounded linear operators from $X$ to $Y$, we abbreviate $L(X, Y)$ to $L(X)$ when $X = Y$, and $\rho(A)$, $D(A)$, $R(A)$ stand for the resolvent, domain and range of operator $A$ respectively. $BC(\mathbb{R}, X)$ (resp., $BC(\mathbb{R} \times Y, X)$) is the space of all $X$-valued bounded continuous functions (resp., jointly bounded continuous functions $F : \mathbb{R} \times Y \to X$). Furthermore, $C_0(\mathbb{R}, X)$ (resp., $C_0(\mathbb{R} \times Y, X)$) is the closed subspace of $BC(\mathbb{R}, X)$ (resp., $BC(\mathbb{R} \times Y, X)$) consisting of functions vanishing at infinity (vanishing at infinity uniformly in any compact subset of $Y$, in other words,

$$\lim_{|t|\to+\infty} \|g(t, x)\| = 0 \quad \text{uniformly for } x \in \mathbb{K},$$

where $\mathbb{K}$ is an any compact subset of $Y$).

Now, we recall some basic definitions and results on almost automorphic functions and asymptotically almost automorphic functions.

**Definition 2.1** (Bochner [8], N'Guérékata [35]). A continuous function $F : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that

$$\Theta(t) = \lim_{n\to\infty} F(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \Theta(t - s_n) = F(t) \quad \text{for each } t \in \mathbb{R}.$$

Denote by $AA(\mathbb{R}, X)$ the set of all such functions.

**Remark 2.2** ([35]). By the pointwise convergence, the function $\Theta(t)$ in Definition 2.1 is measurable, but not necessarily continuous. Moreover, if $\Theta(t)$ is continuous, then $F(t)$ is uniformly continuous (cf., e.g., [30, Theorem 2.6]), and if the convergence in Definition 2.1 is uniform on $\mathbb{R}$, one gets almost periodicity (in the sense of Bochner and von Neumann). Almost automorphy is thus a more general concept than almost periodicity. There exists an almost automorphic function which is not almost periodic. The function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = \sin\Big(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\Big)$$

is an example of such function [18].

**Lemma 2.3** ([29]). *$AA(\mathbb{R}, X)$ is a Banach space with the supremum norm*

$$\|F\|_\infty = \sup_{t\in\mathbb{R}} \|F(t)\|.$$

**Definition 2.4** ([35]). A continuous function $F : \mathbb{R} \times Y \to X$ is said to be almost automorphic in $t \in \mathbb{R}$ for each $x \in Y$ if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that

$$\begin{gathered}
\lim_{n\to\infty} F(t + s_n, x) = \Theta(t, x) \quad \text{exists for each } t \in \mathbb{R} \text{ and each } x \in Y, \\
\lim_{n\to\infty} \Theta(t - s_n, x) = F(t, x) \quad \text{exists for each } t \in \mathbb{R} \text{ and each } x \in Y.
\end{gathered}$$

The collection of those functions is denoted by $AA(\mathbb{R} \times Y, X)$.

**Remark 2.5** ([46]). The function $F : \mathbb{R} \times X \to X$ given by

$$F(t, x) = \sin\Big(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\Big)\cos x$$

is almost automorphic in $t \in \mathbb{R}$ for each $x \in X$, where $X = L^2[0, 1]$.

**Lemma 2.6** ([35]). *Let $F : \mathbb{R}\times X \to X$ be almost automorphic in $t$ for each $x \in X$ and assume that $F(t, x)$ satisfies a Lipschitz condition in $x$ uniformly in $t \in \mathbb{R}$, i.e., for each pair $x, y \in X$,*

$$\|F(t, x) - F(t, y)\| \le L\|x - y\|$$

*uniformly in $t \in \mathbb{R}$, where $L > 0$ is called the Lipschitz constant for the function $F(t, x)$. Let $\gamma : \mathbb{R} \to X$ be almost automorphic. Then the function $\Upsilon : \mathbb{R} \to X$ defined by*

$$\Upsilon(t) = F(t, \gamma(t))$$

*is almost automorphic.*

**Definition 2.7** ([35]). A continuous function $F : \mathbb{R} \to X$ is said to be asymptotically almost automorphic if it can be decomposed as $F(t) = G(t) + \Phi(t)$, where

$$G(t) \in AA(\mathbb{R}, X),\ \ \Phi(t) \in C_0(\mathbb{R}, X).$$

Denote by $AAA(\mathbb{R}, X)$ the set of all such functions.

**Remark 2.8.** The function $F:\mathbb{R}\to\mathbb{R}$ defined by

$$F(t)=G(t)+\Phi(t)=\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)+e^{-|t|}$$

is an asymptotically almost automorphic function with

$$G(t)=\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)\in AA(\mathbb{R},\mathbb{R}),\quad \Phi(t)=e^{-|t|}\in C_0(\mathbb{R},\mathbb{R}).$$

**Lemma 2.9** ([35])**.** *$AAA(\mathbb{R},X)$ is also a Banach space with the supremum norm $\|\cdot\|_\infty$.*

**Definition 2.10** ([35])**.** A continuous function $F:\mathbb{R}\times Y\to X$ is said to be asymptotically almost automorphic if it can be decomposed as

$$F(t,x)=G(t,x)+\Phi(t,x),$$

where

$$G(t,x)\in AA(\mathbb{R}\times Y,X),\quad \Phi(t,x)\in C_0(\mathbb{R}\times Y,X).$$

Denote by $AAA(\mathbb{R}\times Y,X)$ the set of all such functions.

**Remark 2.11.** The function $F:\mathbb{R}\times X\to X$ given by

$$F(t,x)=G(t,x)+\Phi(t,x)=\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)\cos x+e^{-|t|}x\sin x^2$$

is asymptotically almost automorphic in $t\in\mathbb{R}$ for each $x\in X$, where $X=L^2[0,1]$ and

$$G(t,x)=\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)\cos x\in AA(\mathbb{R}\times X,X),$$
$$\Phi(t,x)=e^{-|t|}x\sin x^2\in C_0(\mathbb{R}\times X,X).$$

Next, let us recall the definition of bi-almost automorphic functions, which was introduced originally by Xiao, Zhu and Liang [40], it will be used to obtain our result.

**Definition 2.12** ([40])**.** A continuous function $F(t,s):\mathbb{R}\times\mathbb{R}\to X$ is called bi-almost automorphic if for every sequence of real numbers $\{\tau_n'\}$, we can extract a subsequence $\{\tau_n\}$ such that

$$G(t,s)=\lim_{n\to\infty}F(t+\tau_n,s+\tau_n)$$

is well defined in $t,s\in\mathbb{R}$, and

$$\lim_{n\to\infty}G(t-\tau_n,s-\tau_n)=G(t,s)$$

for each $t,s\in\mathbb{R}$. Let $bAA(\mathbb{R}\times\mathbb{R},X)$ stand for the set of all such functions.

**Remark 2.13** ([40])**.** If $F(t,s)\in C(\mathbb{R}\times\mathbb{R},X)$ and $F(t,s)=G(t-s)$ for some $G(t)\in C(\mathbb{R},X)$, then $F(t,s)\in bAA(\mathbb{R}\times\mathbb{R},X)$. On the other hand, the concept of bi-almost automorphic function is a natural generalization of the function $F(t,s)$ having the same period in the two arguments, that is

$$F(t+T,s+T)=F(t,s)\quad\text{for all } t,s\in\mathbb{R} \text{ for some } T\in\mathbb{R}/\{0\}.$$

**Remark 2.14** ([40])**.** $F(t,s)=\sin t\cos s$ is a bi-almost automorphic function from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ as

$$F(t+2\pi,s+2\pi)=F(t,s)\quad\text{for all } t,s\in\mathbb{R}.$$

We also need to recall the basic definitions and results on evolution family. Let

$$R(\lambda, L) := (\lambda I - L)^{-1} \quad \text{for all } \lambda \in \rho(L), \quad \Sigma_\theta := \{\lambda \in \mathbb{C}/\{0\} : |\arg \lambda| \le \theta\}.$$

Throughout, assumed that $A(t)$ (usually unbounded) for each $t \in \mathbb{R}$ is a closed and densely defined linear operator on $D = D(A(t))$ satisfying the so-called Acquistapace and Terreni conditions:

(AT1) There are constants $\lambda_0 \ge 0$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and $K_1 \ge 0$ such that $\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0)$ and for all $\lambda \in \Sigma_\theta \cup \{0\}$, $t \in \mathbb{R}$,

$$\|R(\lambda, A(t) - \lambda_0)\| \le \frac{K_1}{1 + |\lambda|}.$$

(AT2) There are constants $K_2 \ge 0$ and $\alpha, \beta \in (0, 1]$ with $\alpha + \beta > 1$ such that for all $\lambda \in \Sigma_\theta$ and $t, s \in \mathbb{R}$

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \le \frac{K_2|t - s|^\alpha}{|\lambda|^\beta}.$$

It should mentioned that conditions (AT1) and (AT2), which are initiated by Acquistapace and Terreni [2, 3] for $\lambda_0 = 0$, are well understood and widely used in the literature.

**Definition 2.15** ([21])**.** A two parameter family of bounded linear operators $\{U(t, s)\}_{t \ge s}$ on $X$ is called an evolution family if

1. (1) $U(t, r)U(r, s) = U(t, s)$ and $U(t, t) = I$ for all $t \ge r \ge s$ and $t, r, s \in \mathbb{R}$,
2. (2) the map $(t, s) \to U(t, s)x$ is continuous for all $x \in X$, $t \ge s$ and $t, s \in \mathbb{R}$.

It is worth pointing out that "evolution family" as a basic concept in the theory of nonautonomous evolution equations is also called evolution system, evolution operator, evolution process, propagator or fundamental solution. More details can be found in, e.g., [21, 36, 43].

By an obvious rescaling from [4, Theorem 2.3] and [42, Theorem 2.1], the Acquistapace and Terreni conditions (AT1) and (AT2) ensure that there exists a unique evolution family $\{U(t, s)\}_{t \ge s}$ on $X$ such that

(I) $U(\cdot, s) \in C^1((s, +\infty), L(X))$, $\frac{\partial U(t,s)}{\partial t} = A(t)U(t, s)$ for $t > s$, moreover there exists a constant $C > 0$ such that

$$\|A(t)^k U(t, s)\| \le C(t - s)^{-k} \text{ for } 0 < t - s \le 1, \ k = 0, 1;$$

(II) $\frac{\partial^+ U(t,s)x}{\partial s} = -U(t, s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

In this case we say that $(A(t))_{t \in \mathbb{R}}$ generate the evolution family $\{U(t, s)\}_{t \ge s}$.

It should also be mentioned that the above mentioned proprieties were mainly established in [4, Theorem 2.3] and [42, Theorem 2.1], see also [3, 41].

Note that in the particular case when $A(t)$ has a constant domain $D = D(A(t))$, it is wellknown [36, 5] that condition (AT2) can be replaced with the following condition: there exist constants $K_2 \ge 0$, $0 < \mu \le 1$ such that

$$\|(A(t) - A(s))R(\lambda_0, A(r))\| \le K_2|t - s|^\mu \quad \text{for all } s, t, r \in \mathbb{R}.$$

One says that an evolution family $\{U(t, s)\}_{t \ge s}$ has exponential stability if there exist constants $M > 0$, $\delta > 0$ such that

$$\|U(t, s)\| \le Me^{-\delta(t-s)} \quad \text{for all } t \ge s; \ t, s \in \mathbb{R}.$$

**Definition 2.16** ([21]). *An evolution family $\{U(t,s)\}_{t\geq s}$ is said to be compact if for all $t>s$, $U(t,s)$ is continuous and maps bounded subsets of $X$ into precompact subsets of $X$.*

**Remark 2.17** ([22]). Let us note that if for each $t\in\mathbb{R}$ and some $\lambda\in\rho(A(t))$, the resolvent $R(\lambda,A(t))$ is a compact operator, then $U(t,s)$ is a compact operator whenever $t>s$.

Similar to one-parameter semigroups, $\{U(t,s)\}_{t\geq s}$ satisfies the following property.

**Lemma 2.18** ([21]). *Let $\{U(t,s)\}_{t\geq s}$ be a compact evolution family on $X$. Then for each $s\in\mathbb{R}$, the function $t\to U(t,s)$ is continuous on $(s,+\infty)$ in the uniform operator topology.*

In the following, we present the following compactness criterion, which is a special case of the general compactness result in [37, Theorem 2.1].

**Lemma 2.19** ([37]). *A set $D\subset C_0(\mathbb{R},X)$ is relatively compact if*
1. *$D$ is equicontinuous;*
2. *$\lim_{|t|\to+\infty}x(t)=0$ uniformly for $x\in D$;*
3. *the set $D(t):=\{x(t):x\in D\}$ is relatively compact in $X$ for every $t\in\mathbb{R}$.*

The following Krasnoselskii's fixed point theorem plays a key role in the proofs of our main results, which can be found in many books.

**Lemma 2.20** ([38]). *Let $B$ be a bounded closed and convex subset of $X$, and $J_1,J_2$ be maps of $B$ into $X$ such that*

$$J_1x+J_2y\in B\quad \text{for } x,y\in B.$$

*If $J_1$ is a contraction and $J_2$ is completely continuous, then the equation*

$$J_1x+J_2x=x$$

*has a solution on $B$.*

## 3. Main result

In this section, we study the existence of asymptotically almost automorphic mild solutions for the Cauchy problem consisting in the standard nonautonomous parabolic evolution equation of the form

$$x'(t)=A(t)x(t)+F(t,x(t)),\quad t\in\mathbb{R} \tag{3.1}$$

in the Banach space $X$. Here, $A(t)$ for each $t$, which has domain $D(A(t))$ and satisfies the so-called Acquistapace and Terreni conditions (AT1) and (AT2), is a closed and densely defined linear operator on $X$ and is the generator of a compact as well as exponentially stable evolution family $\{U(t,s)\}_{t\geq s}$, $F:\mathbb{R}\times X\to X$ is a given function satisfying the following assumption:

(H1) $F(t,x)=F_1(t,x)+F_2(t,x)\in AAA(\mathbb{R}\times X,X)$ with

$$F_1(t,x)\in AA(\mathbb{R}\times X,X),\quad F_2(t,x)\in C_0(\mathbb{R}\times X,X)$$

and there exists a constant $L>0$ such that

$$\|F_1(t,x)-F_1(t,y)\|\leq L\|x-y\|\quad \text{for all } t\in\mathbb{R},\ x,y\in X. \tag{3.2}$$

Moreover, there exist a function $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ and a nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in \mathbb{R}$ and $x \in X$ with $\|x\| \leq r$,

$$\|F_2(t,x)\| \leq \beta(t)\Phi(r) \quad \text{and} \quad \liminf_{r\to+\infty} \frac{\Phi(r)}{r} = \rho_1. \tag{3.3}$$

In addition, we suppose that

(H2) The evolutionary family $\{U(t,s)\}_{t\geq s}$ is exponentially stable.

(H3) $U(t,s)x \in bAA(\mathbb{R}\times\mathbb{R}, X)$ uniformly for all $x$ in any bounded subset of $X$.

**Remark 3.1.** Assuming that $F(t,x)$ satisfies the assumption (H1), it is noted that $F(t,x)$ does not have to meet the Lipschitz continuity with respect to $x$. Such class of asymptotically almost automorphic functions $F(t,x)$ are more complicated than those with Lipschitz continuity and little is known about them.

In the proof of our result, we need the following auxiliary results concerning asymptotically almost automorphic functions.

**Lemma 3.2.** *Given*

$$F(t,x) = F_1(t,x) + F_2(t,x) \in AAA(\mathbb{R}\times X, X)$$

*with* $F_1(t,x) \in AA(\mathbb{R}\times X, X)$*,* $F_2(t,x) \in C_0(\mathbb{R}\times X, X)$ *satisfying the assumption* (H1) *and*

$$X(t) = Y(t) + Z(t) \in AAA(\mathbb{R}, X)$$

*with* $Y(t) \in AA(\mathbb{R}, X)$*,* $Z(t) \in C_0(\mathbb{R}, X)$*. Then* $F(t, X(t)) \in AAA(\mathbb{R}, X)$ *with*

$$F_1(t, Y(t)) \in AA(\mathbb{R}, X), \quad F_1(t, X(t)) - F_1(t, Y(t)) \in C_0(\mathbb{R}, X),$$
$$F_2(t, X(t)) \in C_0(\mathbb{R}, X).$$

*Proof.* Since

$$\begin{aligned} F(t, X(t)) =& F_1(t, Y(t)) + [F(t, X(t)) - F_1(t, Y(t))] \\ =& F_1(t, Y(t)) + [F_1(t, X(t)) - F_1(t, Y(t))] + F_2(t, X(t)). \end{aligned}$$

By Lemma 2.6, together with (3.2), one has $F_1(t, Y(t)) \in AA(\mathbb{R}, X)$. Obviously $F_2(t, X(t)) \in C_0(\mathbb{R}, X)$. Moreover (3.2) implies that

$$F_1(t, X(t)) - F_1(t, Y(t)) \in C_0(\mathbb{R}, X) \quad \text{as } Z(t) \in C_0(\mathbb{R}, X).$$

□

**Lemma 3.3.** *Let* (H2), (H3) *be satisfied. Given* $Y(t) \in AA(\mathbb{R}, X)$*. Let*

$$\Phi_1(t) := \int_{-\infty}^{t} U(t,s)Y(s)\mathrm{d}s, \; t \in \mathbb{R}.$$

*Then* $\Phi_1(t) \in AA(\mathbb{R}, X)$*.*

*Proof.* From (H2) it is clear that $\Phi_1(t)$ is well-defined and continuous on $\mathbb{R}$. Choose a bounded subset $K$ of $X$ such that $Y(t) \in K$ for all $t \in \mathbb{R}$. From $Y(t) \in AA(\mathbb{R}, X)$ and (H3) it follows that for every sequence of real numbers $\{\tau_n'\}$, we can extract a subsequence $\{\tau_n\}$ such that

(1) $\lim_{n\to\infty} Y(t+\tau_n) = \widetilde{Y}(t)$ for each $t \in \mathbb{R}4$,
(2) $\lim_{n\to\infty} \widetilde{Y}(t-\tau_n) = Y(t)$ for each $t \in \mathbb{R}$,
(3) $\lim_{n\to\infty} U(t+\tau_n, s+\tau_n)x = \widetilde{U}(t,s)x$ for each $t,s \in \mathbb{R}$, $x \in K$,
(4) $\lim_{n\to\infty} \widetilde{U}(t-\tau_n, s-\tau_n)x = U(t,s)x$ for each $t,s \in \mathbb{R}$, $x \in K$.

Write

$$\widetilde{\Phi_1}(t) := \int_{-\infty}^{t} \widetilde{U}(t,s)\widetilde{Y}(s)\mathrm{d}s, \quad t \in \mathbb{R}.$$

Then

$$\begin{aligned}
\|\Phi_1(t+\tau_n) - \widetilde{\Phi_1}(t)\| &= \Big\| \int_{-\infty}^{t+\tau_n} U(t+\tau_n, s)Y(s)\mathrm{d}s - \int_{-\infty}^{t} \widetilde{U}(t,s)\tilde{Y}(s)\mathrm{d}s \Big\| \\
&= \Big\| \int_{-\infty}^{t} U(t+\tau_n, s+\tau_n)Y(s+\tau_n)\mathrm{d}s - \int_{-\infty}^{t} \widetilde{U}(t,s)\tilde{Y}(s)\mathrm{d}s \Big\| \\
&\le \Big\| \int_{-\infty}^{t} U(t+\tau_n, s+\tau_n)[Y(s+\tau_n) - \widetilde{Y}(s)]\mathrm{d}s \Big\| \\
&\quad + \Big\| \int_{-\infty}^{t} [U(t+\tau_n, s+\tau_n) - \widetilde{U}(t,s)]\widetilde{Y}(s)\mathrm{d}s \Big\|.
\end{aligned}$$

From (H2) together with the Lebesgue dominated convergence theorem and (1), (3), it follows that

$$\lim_{n\to\infty} \Phi_1(t+\tau_n) = \widetilde{\Phi_1}(t), \quad t \in \mathbb{R}.$$

Similarly by (2) and (4) we can prove that

$$\lim_{n\to\infty} \widetilde{\Phi_1}(t-\tau_n) = \Phi_1(t), \; t \in \mathbb{R}.$$

Hence $\Phi_1(t) \in AP(\mathbb{R}, X)$. □

**Lemma 3.4.** *Let* (H2) *be satisfied. Given* $Z(t) \in C_0(\mathbb{R}, X)$. *Let*

$$\Phi_2(t) := \int_{-\infty}^{t} T(t-s)Z(s)\mathrm{d}s, \quad t \in \mathbb{R}.$$

*Then* $\Phi_2(t) \in C_0(\mathbb{R}, X)$.

*Proof.* From (H2) it is clear that $\Phi_2(t)$ are well-defined and continuous on $\mathbb{R}$. Since $Z(t) \in C_0(\mathbb{R}, X)$, one can choose a $T > 0$ such that

$$\|Z(t)\| < \varepsilon \quad \text{for all } t > T.$$

This enables us to conclude that for all $t > T$,

$$\|\Phi_2(t)\| \le \Big\| \int_{-\infty}^{T} T(t-s)Z(s)\mathrm{d}s \Big\| + \Big\| \int_{T}^{t} T(t-s)Z(s)\mathrm{d}s \Big\| \le \frac{Me^{-\delta(t-T)}}{\delta}\|Z\|_\infty + \frac{M\varepsilon}{\delta},$$

which implies $\lim_{t\to+\infty} \|\Phi_2(t)\| = 0$.

By a similar argument one can obtain $\lim_{t\to-\infty} \|\Phi_2(t)\| = 0$. □

**Definition 3.5.** A continuous function $x : \mathbb{R} \to X$ is called an asymptotically almost automorphic mild solution to equation (3.1) on $\mathbb{R}$ if $x \in AAA(\mathbb{R}, X)$ and satisfies the integral equation

$$x(t) = U(t,\tau)x(\tau) + \int_{\tau}^{t} U(t,s)F(s,x(s))\mathrm{d}s \quad \text{for all } t > \tau.$$

Let $\beta(t)$ be the function involved in assumption (H1). Define

$$\sigma(t) := \int_{-\infty}^{t} e^{-\delta(t-s)}\beta(s)\mathrm{d}s, \quad t \in \mathbb{R}.$$

Then $\sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$. Put $\rho_2 := \sup_{t\in\mathbb{R}} \sigma(t)$. Now we are in a position to present our existence result.

**Theorem 3.6.** *Under hypotheses* (H1)–(H3)*, equation* (3.1) *has at least one asymptotically almost automorphic mild solution provided that*

$$ML\delta^{-1} + M\rho_1\rho_2 < 1. \tag{3.4}$$

*Proof.* The proof is divided into the following six steps.

**Step 1.** Define a mapping $\Lambda$ on $AA(\mathbb{R}, X)$ by

$$(\Lambda v)(t) = \int_{-\infty}^{t} U(t,s)F_1(s, v(s))\mathrm{d}s, \quad t \in \mathbb{R}.$$

and prove $\Lambda$ has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$. Firstly, from $F_1(t, x) \in AA(\mathbb{R} \times X, X)$ satisfying (3.2) and Lemma 2.6 it follows that

$$F_1(\cdot, v(\cdot)) \in AA(\mathbb{R}, X) \quad \text{for every } v(\cdot) \in AA(\mathbb{R}, X).$$

This, together with Lemma 3.3, implies that $\Lambda$ is well defined and maps $AA(\mathbb{R}, X)$ into itself. On the other hand, for any $v_1(t), v_2(t) \in AA(\mathbb{R}, X)$, by (3.2) one has

$$\|(\Lambda v_1)(t) - (\Lambda v_2)(t)\| \le ML \int_{-\infty}^{t} e^{-\delta(t-s)}\|v_1(s) - v_2(s)\|\mathrm{d}s \le \frac{ML}{\delta}\|v_1 - v_2\|_\infty.$$

As a result

$$\|\Lambda v_1 - \Lambda v_2\|_\infty \le \frac{ML}{\delta}\|v_1 - v_2\|_\infty.$$

Together with (3.4), this proves that $\Lambda$ is a contraction on $AA(\mathbb{R}, X)$. Thus, the Banach's fixed point theorem implies that $\Lambda$ has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

**Step 2.** Set

$$\Omega_r := \{\omega(t) \in C_0(\mathbb{R}, X) : \|\omega\|_\infty \le r\}.$$

For the above $v(t)$, define $\Gamma := \Gamma^1 + \Gamma^2$ on $C_0(\mathbb{R}, X)$ as

$$(\Gamma^1\omega)(t) = \int_{-\infty}^{t} U(t,s)[F_1(s, v(s) + \omega(s)) - F_1(s, v(s))]\mathrm{d}s, \quad t \in \mathbb{R},$$

$$(\Gamma^2\omega)(t) = \int_{-\infty}^{t} U(t,s)F_2(s, v(s) + \omega(s))\mathrm{d}s, \quad t \in \mathbb{R},$$

and prove that $\Gamma$ maps $\Omega_{k_0}$ into itself, where $k_0$ is a given constant. Firstly, from hypothesis (H1) it follows that

$$\|F_1(s, v(s) + \omega(s)) - F_1(s, v(s))\| \le L\|\omega(s)\| \quad \text{for all } s \in \mathbb{R},\ \omega(s) \in X$$

and

$$\|F_2(s, v(s) + \omega(s))\| \le \beta(s)\Phi\Big(r + \sup_{s\in\mathbb{R}}\|v(s)\|\Big) \tag{3.5}$$

for all $s \in \mathbb{R}$ and $\omega(s) \in X$ with $\|\omega(s)\| \le r$, which implies

$$F_1(\cdot, v(\cdot) + \omega(\cdot)) - F_1(\cdot, v(\cdot)) \in C_0(\mathbb{R}, X) \quad \text{for every } \omega(\cdot) \in C_0(\mathbb{R}, X),$$
$$F_2(\cdot, v(\cdot) + \omega(\cdot)) \in C_0(\mathbb{R}, X) \quad \text{as } \beta(\cdot) \in C_0(\mathbb{R}, \mathbb{R}^+).$$

Those, together with Lemma 3.4, yields that $\Gamma$ is well-defined and maps $C_0(\mathbb{R}, X)$ into itself.

On the other hand, in view of (3.3) and (3.4) it is not difficult to see that there exists a constant $k_0 > 0$ such that

$$\frac{ML}{\delta}k_0 + M\rho_2\Phi\Big(k_0 + \sup_{s\in\mathbb{R}}\|v(s)\|\Big) \leq k_0.$$

This enables us to conclude that for any $t \in \mathbb{R}$ and $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$,

$$\begin{aligned}
\|(\Gamma^1\omega_1)(t) + (\Gamma^2\omega_2)(t)\| \leq & M\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\|F_1(s, v(s) + \omega_1(s)) - F_1(s, v(s))\|\mathrm{d}s\Big) \\
& + M\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\|F_2(s, v(s) + \omega_2(s))\|\mathrm{d}s\Big) \\
\leq & \frac{ML}{\delta}\|\omega_1\|_\infty + M\rho_2\Phi\Big(\|\omega_2\|_\infty + \sup_{s\in\mathbb{R}}\|v(s)\|\Big) \\
\leq & \frac{ML}{\delta}k_0 + M\rho_2\Phi\Big(k_0 + \sup_{s\in\mathbb{R}}\|v(s)\|\Big) \leq k_0,
\end{aligned}$$

which implies that $(\Gamma^1\omega_1)(t) + (\Gamma^2\omega_2)(t) \in \Omega_{k_0}$. Thus $\Gamma$ maps $\Omega_{k_0}$ into itself.

**Step 3.** Show that $\Gamma^1$ is a contraction on $\Omega_{k_0}$. In fact, for any $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$ and $t \in \mathbb{R}$, from (3.2) it follows that

$$\begin{aligned}
&\|[F_1(s, v(s) + \omega_1(s)) - F_1(s, v(s))] - [F_1(s, v(s) + \omega_2(s)) - F_1(s, v(s))]\| \\
&\leq L\|\omega_1(s) - \omega_2(s)\| \quad \text{for all } s \in \mathbb{R},\ \omega_1(s), \omega_2(s) \in X.
\end{aligned}$$

Thus

$$\|(\Gamma^1\omega_1)(t) - (\Gamma^1\omega_2)(t)\| \leq ML\int_{-\infty}^{t} e^{-\delta(t-s)}\|\omega_1(s) - \omega_2(s)\|\mathrm{d}s \leq \frac{ML}{\delta}\|\omega_1 - \omega_2\|_\infty.$$

As a result

$$\|\Gamma^1\omega_1 - \Gamma^1\omega_2\|_\infty \leq \frac{ML}{\delta}\|\omega_1 - \omega_2\|_\infty.$$

Thus, in view of (3.4), one obtains the conclusion.

**Step 4.** Show the set $\{(\Gamma^2\omega)(t) : \omega(t) \in \Omega_{k_0}\}$ is relatively compact in $X$ for each $t \in \mathbb{R}$. Firstly, from our assumption it is clear that $\Gamma^2$ is a continuous mapping from $\Omega_{k_0}$ to $\Omega_{k_0}$. Moveover, for all $\omega(t) \in \Omega_{k_0}$ and $t \in \mathbb{R}$,

$$\|(\Gamma^2\omega)(t)\| \leq \Big\|\int_{-\infty}^{t} U(t,s)F_2(s, v(s) + \omega(s))\mathrm{d}s\Big\| \leq M\sigma(t)\Phi\Big(k_0 + \sup_{s\in\mathbb{R}}\|v(s)\|\Big),$$

in view of $\sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$, one concludes that

$$\lim_{|t|\to+\infty}(\Gamma^2\omega)(t) = 0 \quad \text{uniformly for } \omega(t) \in \Omega_{k_0}.$$

Let $t \in \mathbb{R}$ be fixed. For given $\varepsilon_0 > 0$, from (3.5) it follows that

$$(\Gamma^2_{\varepsilon_0}\omega)(t) = \int_{-\infty}^{t-\varepsilon_0} U(t-\varepsilon_0, s)F_2(s, v(s) + \omega(s))\mathrm{d}s$$

is uniformly bounded for $\omega(t) \in \Omega_{k_0}$. This, and the compactness of $U(t, t-\varepsilon_0)$ yield that the set

$$\{U(t, t-\varepsilon_0)(\Gamma^2_{\varepsilon_0}\omega)(t) : \omega(t) \in \Omega_{k_0}\}$$

is relatively compact in $X$. On the other hand

$$\|(\Gamma^2\omega)(t) - U(t, t-\varepsilon_0)(\Gamma^2_{\varepsilon_0}\omega)(t)\| \leq \Big\|\int_{t-\varepsilon_0}^{t} U(t,s)F_2(s, v(s) + \omega(s))\mathrm{d}s\Big\|$$

$$\leq M \int_{t-\varepsilon_0}^{t} e^{-\delta(t-s)} \|F_2(s, v(s) + \omega(s))\| \mathrm{d}s \to 0$$

as $\varepsilon_0 \to 0^+$, this and the total boundedness yield that the set

$$\{(\Gamma^2\omega)(t) : \omega(t) \in \Omega_{k_0}\}$$

is relatively compact in $X$ for each $t \in \mathbb{R}$.

**Step 5.** Consider the equicontinuity of the set $\{(\Gamma^2\omega)(t) : \omega(t) \in \Omega_{k_0}\}$. Given $\varepsilon_1 > 0$. In view of (3.5) there exists an $\eta > 0$ such that for all $\omega(t) \in \Omega_{k_0}$ and $t_2 \geq t_1$ with $t_2 - t_1 < \eta$,

$$\Big\| \int_{t_1}^{t_2} U(t_2, s) F_2(s, v(s) + \omega(s)) \mathrm{d}s \Big\| < \frac{\varepsilon_1}{5},$$

$$\Big\| \int_{t_1-\eta}^{t_1} [U(t_2, s) - U(t_1, s)] F_2(s, v(s) + \omega(s)) \mathrm{d}s \Big\| < \frac{2\varepsilon_1}{5}.$$

Also, one can choose a $k > 0$ such that

$$\frac{2M}{\delta} \Phi\Big(k_0 + \sup_{s\in\mathbb{R}} \|v(s)\|\Big) e^{-\delta k} \sup_{s\in\mathbb{R}} \beta(s) < \frac{\varepsilon_1}{5},$$

which implies that for all $\omega(t) \in \Omega_{k_0}$,

$$\Big\| \int_{-\infty}^{t_1-k} [U(t_2, s) - U(t_1, s)] F_2(s, v(s) + \omega(s)) \mathrm{d}s \Big\| < \frac{\varepsilon_1}{5}.$$

In addition, from the fact that $\{U(t, s)\}_{t\geq s}$ is compact implies its norm continuity, it follows that there exists an $\eta' \in (0, \eta)$ such that for every $\omega(t) \in \Omega_{k_0}$ and $t_2 \geq t_1$ with $t_2 - t_1 < \eta'$,

$$\Big\| \int_{t_1-k}^{t_1-\eta'} [U(t_2, s) - U(t_1, s)] F_2(s, v(s) + \omega(s)) \mathrm{d}s \Big\| < \frac{\varepsilon_1}{5}.$$

Thus for every $\omega(t) \in \Omega_{k_0}$ and $t_2 \geq t_1$ with $t_2 - t_1 < \eta'$,

$$\begin{aligned}
&\|(\Gamma^2\omega)(t_2) - (\Gamma^2\omega)(t_1)\| \\
&= \Big\| \int_{-\infty}^{t_2} U(t_2, s) F_2(s, v(s) + \omega(s)) \mathrm{d}s - \int_{-\infty}^{t_1} U(t_1, s) F_2(s, v(s) + \omega(s)) \mathrm{d}s \Big\| \\
&\leq \Big\| \int_{t_1}^{t_2} U(t_2, s) F_2(s, v(s) + \omega(s)) \mathrm{d}s \Big\| \\
&\quad + \Big\| \int_{t_1-\eta'}^{t_1} [U(t_2, s) - U(t_1, s)] F_2(s, v(s) + \omega(s)) \mathrm{d}s \Big\| \\
&\quad + \Big\| \int_{-\infty}^{t_1-k} [U(t_2, s) - U(t_1, s)] F_2(s, v(s) + \omega(s)) \mathrm{d}s \Big\| \\
&\quad + \Big\| \int_{t_1-k}^{t_1-\eta'} [U(t_2, s) - U(t_1, s)] F_2(s, v(s) + \omega(s)) \mathrm{d}s \Big\| < \varepsilon_1,
\end{aligned}$$

which implies the equicontinuity of the set $\{(\Gamma^2\omega) : \omega(t) \in \Omega_{k_0}\}$.

**Step 6.** Show that equation (3.1) has at least one asymptotically almost automorphic mild solution. Firstly, the results of step 4 and step 5, together with Lemma 2.19, yields that $\Gamma^2$ is compact on $\Omega_{k_0}$. This, together with the results of step 2 and step 3 as well as Lemma 2.20, yields that $\Gamma$ has at least one fixed point

$\omega(t) \in \Omega_{k_0}$, furthermore $\omega(t) \in C_0(\mathbb{R}, X)$. Then, consider the coupled system of integral equations

$$\begin{aligned} v(t) &= \int_{-\infty}^{t} U(t,s)F_1(s, v(s))\mathrm{d}s, \quad t \in \mathbb{R}, \\ \omega(t) &= \int_{-\infty}^{t} U(t,s)[F_1(s, v(s)+\omega(s)) - F_1(s, v(s))]\mathrm{d}s \\ &\quad + \int_{-\infty}^{t} U(t,s)F_2(s, v(s)+\omega(s))\mathrm{d}s, \quad t \in \mathbb{R}. \end{aligned} \tag{3.6}$$

From the result of step 1, together with the above fixed point $\omega(t) \in C_0(\mathbb{R}, X)$, it follows that $(v(t), \omega(t)) \in AA(\mathbb{R}, X) \times C_0(\mathbb{R}, X)$ is a solution to system (3.6). Thus

$$x(t) := v(t) + \omega(t) \in AAA(\mathbb{R}, X)$$

and it is a solution to the integral equation

$$x(t) = \int_{-\infty}^{t} U(t,s)F(s, x(s))\mathrm{d}s, \quad t \in \mathbb{R}.$$

Since $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$, let

$$x(\tau) = \int_{-\infty}^{\tau} U(\tau,s)F(s, x(s))\mathrm{d}s,$$

then

$$U(t,\tau)x(\tau) = \int_{-\infty}^{\tau} U(t,s)F(s, x(s))\mathrm{d}s.$$

Furthermore for $t \geq \tau$,

$$\begin{aligned} \int_{\tau}^{t} U(t,s)F(s,x(s))\mathrm{d}s &= \int_{-\infty}^{t} U(t,s)F(s,x(s))\mathrm{d}s - \int_{-\infty}^{\tau} U(t,s)F(s,x(s))\mathrm{d}s \\ &= x(t) - U(t,\tau)x(\tau). \end{aligned}$$

So that

$$x(t) = U(t,\tau)x(\tau) + \int_{\tau}^{t} U(t,s)F(s,x(s))\mathrm{d}s \quad \text{for all } t > \tau,$$

which implies that $x(t)$ is an asymptotically almost automorphic mild solution to equation (3.1). $\square$

**Remark 3.7.** Note that the condition (3.2) in (H1) of Theorem 3.6 can be easily extended to the case of $F_1(t,x)$ being locally Lipschitz continuous:

$$\|F_1(t,x) - F_1(t,y)\| \leq L(r)\|X - Y\|$$

for all $t \in \mathbb{R}$ and $x, y \in X$ satisfying $\|x\|, \|y\| \leq r$.

## 4. Applications

In this section, an example illustrates the usefulness of the theoretical result established in the preceding section. Consider the partial differential equation with Dirichlet boundary conditions

$$\begin{aligned} \frac{\partial u(t,\xi)}{\partial t} &= \frac{\partial^2}{\partial \xi^2} u(t,\xi) - 2u(t,\xi) + \sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)u(t,\xi) \\ &\quad + \mu \sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)\cos u(t,\xi) \\ &\quad + \nu e^{-|t|}u(t,\xi)\sin u^2(t,\xi), \quad t\in\mathbb{R},\ \xi\in[0,\pi], \\ &\qquad u(t,0)=u(t,\pi)=0, \quad t\in\mathbb{R}, \end{aligned} \tag{4.1}$$

where $\mu$ and $\nu$ are two constants.

Take $X = L^2[0,\pi]$ with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)_2$. And define $A : D(A) \subset X \to X$ given by

$$Ax = \frac{\partial^2 x(\xi)}{\partial \xi^2} - 2x$$

with domain

$$\begin{aligned} D(A) = \Big\{ & x(\cdot)\in X : x''\in X, x'\in X \text{ is absolutely continuous on} \\ & [0,\pi], x(0)=x(\pi)=0\Big\}. \end{aligned}$$

It is well known that $A$ is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $X$ satisfying

$$\|T(t)\| \leq e^{-3t} \quad \text{for } t>0,$$

see [27]. Moreover

$$T(t)x = \sum_{n=1}^{+\infty} e^{(-n^2+2)t}(x,y_n)_2 y_n,\ t\geq 0,\ x\in X,$$

where $y_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Define a family of linear operators $A(t)$ by $D(A(t)) = D(A)$,

$$A(t)x(\xi) = \Big(A + \sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)\Big)x(\xi), \quad \forall \xi\in[0,\pi],\ x\in D(A).$$

Then, the system

$$\begin{gathered} x'(t) = A(t)x(t), \quad t>s, \\ x(s) = x \in X \end{gathered}$$

has an associated evolution family $\{U(t,s)\}_{t\geq s}$ on $X$, which can be explicitly express by

$$U(t,s)x = \Big(T(t-s)e^{\int_s^t \sin\big(\frac{1}{2+\cos\tau+\cos\sqrt{2}\tau}\big)\,\mathrm{d}\tau}\Big)x.$$

Moreover,

$$\|U(t,s)\| \leq e^{-2(t-s)} \quad \text{for } t\geq s.$$

Note that for each $t > s$, the operator $U(t,s)$ is a nuclear operator, which yields the compactness of $U(t,s)$ for $t > s$. Note also that

$$\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big) \in AA(\mathbb{R},\mathbb{R})$$

and it is not difficult to verify that $A(t)$ satisfies the Acquistapace-Terreni conditions (AT1), (AT2), and the assumptions (H2), (H3) hold with $M = 1$, $\delta = 2$. Let

$$F_1(t,x(\xi)) := \mu\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)\cos x(\xi),$$
$$F_2(t,x(\xi)) := \nu e^{-|t|}x(\xi)\sin x^2(\xi).$$

Then it is easy to verify that $F_1, F_2 : \mathbb{R}\times X \to X$ are continuous, $F_1(t,x) \in AA(\mathbb{R}\times X, X)$ and

$$\|F_1(t,x) - F_1(t,y)\| \le \mu\|x-y\|, \quad \|F_2(t,x)\| \le \nu e^{-|t|}\|x\| \quad \text{for all } t\in\mathbb{R},\ x,y\in X,$$

which implies $F_2(t,x) \in C_0(\mathbb{R}\times X, X)$ and

$$F(t,x) = F_1(t,x) + F_2(t,x) \in AAA(\mathbb{R}\times X, X).$$

Thus, (4.1) can be reformulated as the abstract problem (3.1) and assumption (H1) holds with

$$L = \mu, \quad \Phi(r) = r, \quad \beta(t) = \nu e^{-|t|}, \quad \rho_1 = 1, \quad \rho_2 \le \frac{\nu}{2}.$$

Then from Theorem 3.6 it follows that equation (4.1) at least has one asymptotically almost automorphic mild solution whenever $\mu + \nu < 2$.

**Acknowledgements.** This research was supported by the NNSF of China (No. 11561009) and (No. 41665006), by the Guangdong Province Natural Science Foundation (No. 2015A030313896), by the Characteristic Innovation Project (Natural Science) of Guangdong Province (No. 2016KTSCX094), by the Science and Technology Program Project of Guangzhou (No. 201707010230), and by the Guangxi Province Natural Science Foundation (No. 2016GXNSFAA380240).

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