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Abstract. This article concerns the existence of positive solutions for p-
Laplacian boundary value problem on time scales. By applying fixed point
index we obtain the existence of solutions. Emphasis is put on the fact that
the nonlinear term is allowed to change sign. An example illustrates our re-
sults.

1. Introduction

The development of the theory of time scales was initiated by Stefan Hilger
in his Ph.D. thesis in 1988 [12] as a means of a unifying structure for the study
of differential equations in the continuous case and the study of finite difference
equations in the discrete case. The study of time scales has led to several important
applications, e.g., in the study of insect population models, heat transfer, neural
networks, phytoremediation of metals, wound healing, and epidemic models [5, 14,
19, 24].

In the last few years, there is much attention focused on the existence of positive
solutions for second-order boundary value problems (BVPs) on time scales [2, 6,
7, 8, 11, 16, 17, 18, 21, 22, 23, 25, 27]. But for the existence of positive solutions
for p-Laplacian BVPs with sign changing nonlinearity on time scales [20, 28], few
works were done as far as we know.

Agarwal, Lü and O’Regan [1] studied the singular BVP

\[(\varphi_p(y'))' + q(t)f(t,y) = 0, \quad 0 < t < 1,\]
\[y(0) = y(1) = 0,\]

by means of the upper and lower solution method, where the nonlinearity \(f\) is
allowed to change sign.

Anderson, Avery and Henderson [3] studied the BVP

\[(g(u^\Delta))' + c(t)f(u) = 0, \quad a < t < b,\]
\[u(a) - B_0(u^\Delta(\nu)) = 0, \quad u^\Delta(b) = 0,\]
where \( g(z) = |z|^{p-2}z, \ p > 1, \ \nu \in (a,b), \ f \in C_{id}([0, +\infty), [0, +\infty]), \ c(t) \in C_{id}([a, b], [0, +\infty]) \) and \( K_m \leq B_0(x) \leq K_M \) for some positive constants \( K_m, K_M \). They proved the existence of at least one positive solution by using a fixed point theorem of cone expansion and compression of functional type.

Ji, Feng and Ge [13] considered the existence of multiple positive solutions to BVP for the one-dimensional \( p \)-Laplacian equation

\[
(\phi_p(u'(t)))' + q(t)f(t, u(t)) = 0, \quad t \in (0, 1),
\]

\[
u(t) = \sum_{i=1}^{n} \alpha_i u(\xi_i), \quad \nu(1) = \sum_{i=1}^{n} \beta_i u(\xi_i),
\]

where \( 0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1, \alpha_i, \beta_i \in [0, \infty) \) satisfy \( 0 < \sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \beta_i < 1 \). The nonlinear term \( f \) may change sign. By applying a fixed-point theorem for operators on a cone, they provided sufficient conditions for the existence of multiple positive solutions to the BVP.

In [18], Sang, Su and Xu studied the existence of positive solutions of the following dynamic equations on time scales:

\[
(\phi(u^\Delta))_t + a(t)f(t, u(t)) = 0, \quad t \in [0, T],
\]

\[
\phi(u^\Delta(0)) = \sum_{i=1}^{m-2} a_i \phi(u^\Delta(\xi_i)), \quad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i),
\]

where \( \phi : R \rightarrow R \) is an increasing homeomorphism and homomorphism and \( \phi(0) = 0 \). They established several existence theorems of positive solutions for nonlinear \( m \)-point BVP. The nonlinearity \( f \) is allowed to change sign.

Su, Li and Sun [20] investigated the following singular \( m \)-point \( p \)-Laplacian BVP with the sign changing nonlinearity on time scales:

\[
(\varphi_p(u^\Delta(t)))_t + q(t)f(t, u(t)) = 0, \quad t \in (0, T),
\]

\[
u(0) = 0, \quad \nu(T) - \sum_{i=1}^{m-2} \psi_i(u(\xi_i)) = 0,
\]

where \( \varphi_p(u) = |u|^{p-2}u, \ p > 1, \ 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(T), \ q(t) \in C_{id}((0, T), (0, +\infty)), \ f \in C_{id}((0, T) \times (0, +\infty), (-\infty, +\infty)) \). They presented some new existence criteria for positive solutions of the problem by using the well-known Schauder fixed point theorem and upper and lower solutions method.

Wang and Hou [24] considered the multipoint BVP

\[
(\phi_p(u'(t)))' + f(t, u(t)) = 0, \quad t \in (0, 1),
\]

\[
\phi_p(u'(0)) = \sum_{i=1}^{n-2} a_i \phi_p(u'(\xi_i)), \quad u(1) = \sum_{i=1}^{n-2} b_i u(\xi_i),
\]

where \( \phi_p(u) = |u|^{p-2}u, \ p > 1, \xi_i \in (0, 1) \) with \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1 \), and \( a_i, b_i \) satisfy \( a_i, b_i \in [0, \infty), \ 0 < \sum_{i=1}^{n-2} a_i < 1, \) and \( 0 < \sum_{i=1}^{n-2} b_i < 1 \). Using a fixed point theorem for operators on a cone, they provided sufficient conditions for the existence of at least three positive solutions to the above BVP.

Motivated by works mentioned above, in this paper, we are concerned with the existence of multiple positive solutions for \( p \)-Laplacian multipoint BVP on time
defines two cones:

\[ \phi_p(u(t)) = 0, \quad t \in [0, T]_{\mathbb{T}}, \]  
\[ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi_p(u(T)) = \sum_{i=1}^{m-2} b_i \phi_p(u(\xi_i)), \]  

where \( \phi_p(u) \) is \( p \)-Laplacian operator, i.e., \( \phi_p(u) = |u|^{p-2}u \), for \( p > 1 \) with \( (\phi_p)^{-1} = \phi_q \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). The usual notation and terminology for time scales as can be found in [1, 5, 9], will be used here.

We need the following hypotheses:

(H1) \( a_i, b_i \in [0, +\infty) \) satisfy \( \sum_{i=1}^{m-2} a_i < 1 \) and \( 0 < \sum_{i=1}^{m-2} b_i < 1 \);

(H2) \( w \in C_{id}([0, T]_{\mathbb{T}}, [0, +\infty)) \) with \( \int_{0}^{\xi_i} w(\tau) \Delta \tau > 0 \);

(H3) \( f : [0, T]_{\mathbb{T}} \times [0, +\infty) \to \mathbb{R} \) is continuous.

We find some new results on the existence of at least two positive solutions to the BVP (1.1) and (1.2) by using fixed point index. The interesting point of this article is that the nonlinear term \( f \) is allowed to change sign.

2. Preliminaries

Let the Banach space \( E = C_{id}([0, T]_{\mathbb{T}}, \mathbb{R}) \) with norm \( \|u\| = \sup_{t \in [0, T]_{\mathbb{T}}} |u(t)| \), and define two cones:

\[ P = \{ u : u \in E, u(t) \geq 0, t \in [0, T]_{\mathbb{T}} \}, \]
\[ P' = \{ u : u \in E, u \text{ is concave, nonnegative and increasing on } [0, T]_{\mathbb{T}} \} \].

Lemma 2.1. If \( 1 - \sum_{i=1}^{m-2} a_i \neq 0 \) and \( 1 - \sum_{i=1}^{m-2} b_i \neq 0 \), then for \( h \in C_{id}([0, T]_{\mathbb{T}}, \mathbb{R}) \), the BVP

\[ (\phi_p(u(t)))^\nabla + h(t) = 0, \quad t \in (0, T]_{\mathbb{T}}, \]  
\[ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi_p(u(T)) = \sum_{i=1}^{m-2} b_i \phi_p(u(\xi_i)) \]  

has a unique solution for which the following representation holds

\[ u(t) = \int_{t}^{T} \phi_q \left( \int_{0}^{s} h(\tau) \Delta \tau - C \right) \Delta s \]
\[ + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_{0}^{\xi_i} \phi_q \left( \int_{0}^{s} h(\tau) \Delta \tau - C \right) \Delta s \right. \]
\[ - \left. \int_{0}^{T} \phi_q \left( \int_{0}^{s} h(\tau) \Delta \tau - C \right) \Delta s \right], \]  

where

\[ C = \frac{-1}{1 - \sum_{i=1}^{m-2} b_i} \left[ \sum_{i=1}^{m-2} b_i \int_{0}^{\xi_i} h(\tau) \Delta \tau - \int_{0}^{T} h(\tau) \Delta \tau \right]. \]

Proof. Integrating (2.1) from 0 to \( t \), one gets

\[ \phi_p(u(t)) - \phi_p(u(0)) = - \int_{0}^{t} h(\tau) \Delta \tau. \]
Let $C = \phi_p(u^\Delta(0))$, then we can write
\[ \phi_p(u^\Delta(t)) = -\int_0^t h(\tau) \nabla \tau + C, \] (2.4)
i.e.
\[ u^\Delta(t) = \phi_q \left( -\int_0^t h(\tau) \nabla \tau + C \right). \] (2.5)
Integrating the dynamic equation $u^\Delta(t) = \phi_q \left( -\int_0^t h(\tau) \nabla \tau + C \right)$ from $t$ to $T$, we have
\[ u(T) - u(t) = \int_t^T \phi_q \left( -\int_0^s h(\tau) \nabla \tau + C \right) \Delta s. \] (2.6)

Applying the first boundary condition, one gets
\[ u(0) = u(T) + \int_0^T \phi_q \left( -\int_0^s h(\tau) \nabla \tau - C \right) \Delta s \]
\[ = \sum_{i=1}^{m-2} a_i u(\xi_i) = \sum_{i=1}^{m-2} a_i \left[ u(T) + \int_{\xi_i}^T \phi_q \left( -\int_0^s h(\tau) \nabla \tau - C \right) \Delta s \right], \]
i.e.,
\[ u(T) = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi_q \left( -\int_0^s h(\tau) \nabla \tau - C \right) \Delta s \right. \]
\[ - \int_0^T \phi_q \left( -\int_0^s h(\tau) \nabla \tau - C \right) \Delta s \] (2.7)
Therefore, by (2.6) and (2.7), we have
\[ u(t) = u(T) + \int_t^T \phi_q \left( -\int_0^s h(\tau) \nabla \tau - C \right) \Delta s \]
\[ = \sum_{i=1}^{m-2} b_i \phi_p(u^\Delta(\xi_i)) = \sum_{i=1}^{m-2} b_i \left( -\int_{\xi_i}^T \phi_q \left( -\int_0^s h(\tau) \nabla \tau - C \right) \Delta s \right. \]
\[ - \int_0^T \phi_q \left( -\int_0^s h(\tau) \nabla \tau - C \right) \Delta s \] (2.8)
Applying the second boundary condition in (2.4), we have
\[ \phi_p(u^\Delta(T)) = -\int_0^T h(\tau) \nabla \tau + C \]
\[ = \sum_{i=1}^{m-2} b_i \phi_p(u^\Delta(\xi_i)) = \sum_{i=1}^{m-2} b_i \left( -\int_{\xi_i}^T h(\tau) \nabla \tau - \int_0^T h(\tau) \nabla \tau \right). \]
Solving the above equation, we obtain
\[ C = \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[ \sum_{i=1}^{m-2} b_i \left( -\int_{\xi_i}^T h(\tau) \nabla \tau - \int_0^T h(\tau) \nabla \tau \right) \right]. \]
Let $u_1$ and $u_2$ be two solutions of problem (2.1), (2.2). Then, using representation (2.3), we obtain $u_1(t) - u_2(t) = 0, t \in [0, T]$. Thus $u$ in (2.3) is the unique solution of (2.1) and (2.2). The proof is complete. □
Lemma 2.2. Suppose (H1) holds, for $h \in C_{ld}([0,T]_{\tau},R)$ and $h(t) \geq 0, t \in [0,T]$. Then, the unique solution $u$ of (2.1) and (2.2) satisfies $u(t) \geq 0$, for $t \in [0,T]_{\tau}$.

Proof. Let

$$\varphi_0(s) = \phi_q \left( \int_0^s h(\tau) \nabla \tau - C \right).$$

Since

$$\int_0^s h(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} h(\tau) \nabla \tau - \int_0^T h(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} a_i} = \int_0^s h(\tau) \nabla \tau + \sum_{i=1}^{m-2} a_i \left( \int_0^T h(\tau) \nabla \tau + \int_0^T h(\tau) \nabla \tau \right) - \int_0^T h(\tau) \nabla \tau$$

$$= \int_0^s h(\tau) \nabla \tau - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} h(\tau) \nabla \tau - \int_0^T h(\tau) \nabla \tau - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} h(\tau) \nabla \tau$$

$$= \int_0^s h(\tau) \nabla \tau - \int_0^T h(\tau) \nabla \tau - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} h(\tau) \nabla \tau$$

it follows that $\varphi_0(s) \leq 0$. According to Lemma 2.1, we obtain

$$u(0) = \int_0^T \varphi_0(s) \Delta s + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s - \int_0^T \varphi_0(s) \Delta s \right]$$

$$= -\sum_{i=1}^{m-2} a_i \int_0^T \varphi_0(s) \Delta s + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s$$

$$= -\sum_{i=1}^{m-2} a_i \left[ \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s + \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s \right] + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s$$

$$= -\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s \geq 0$$

and

$$u(T) = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s - \int_0^T \varphi_0(s) \Delta s \right]$$

$$= \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s - \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s$$

$$= -\left( 1 - \sum_{i=1}^{m-2} a_i \right) \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s - \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s$$

$$= -\left( 1 - \sum_{i=1}^{m-2} a_i \right) \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s - \int_{\xi_i}^{\xi_i} \varphi_0(s) \Delta s$$
Lemma 2.3. If \( u \in P' \) and it satisfies (1.2), then
\[
\inf_{t \in [0,T]_T} u(t) \geq \gamma_1 \|u\|,
\]
where \( \gamma_1 = \sum_{i=1}^{m-2} a_i \xi_i \), \( \|u\| = \max_{t \in [0,T]_T} |u(t)| \).

Proof. Since \( u^{\Delta \nabla} (t) \leq 0 \), it follows that \( u^{\Delta} (t) \) is nonincreasing. Thus, for \( 0 < t < T \),
\[
u(t) - u(0) = \int_0^t u^{\Delta} (s) \Delta s \geq tu^\Delta (t)
\]
and
\[
u(T) - u(t) = \int_t^T u^{\Delta} (s) \Delta s \leq (T-t)u^\Delta (t)
\]
from which we have
\[
u(t) \geq tu(T) + (T-t)u(0) \geq \frac{t}{T} u(T) = \frac{t}{T} \|u\|,
\]
and so \( u(\xi_i) \geq \frac{\xi_i}{T} \|u\| \).

With the boundary condition \( u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i) \), we obtain
\[
\inf_{t \in [0,T]_T} u(t) = u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq \sum_{i=1}^{m-2} a_i \frac{\xi_i}{T} \|u\| = \gamma_1 \|u\|.
\]
This completes the proof.

Let

\[ K = \left\{ u : u \in E, u \text{ is nonnegative and increasing on } [0,T], \quad \min_{t \in [0,T]} u(t) \geq \gamma ||u|| \right\}, \]

where \( \gamma = \gamma_1 \gamma_2, \)

\[
\gamma_2 = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_s^T w(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_0^T w(\tau) \nabla \tau}{1-\sum_{i=1}^{m-2} b_i} \right) \Delta s}{\int_0^T \phi_q \left( \int_s^T w(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_0^T w(\tau) \nabla \tau}{1-\sum_{i=1}^{m-2} b_i} \right) \Delta s}. \]

Note that \( u \) is a solution of BVP (1.1) and (1.2) if and only if

\[
u(t) = \int_t^T \phi_q \left( \int_0^s w(\tau) f(\tau, u(\tau)) \nabla \tau - C \right) \Delta s \]

\[+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_s^T w(\tau) f(\tau, u(\tau)) \nabla \tau - C \right) \Delta s \]

\[- \int_0^T \phi_q \left( \int_0^s w(\tau) f(\tau, u(\tau)) \nabla \tau - C \right) \Delta s], \]

where

\[ C = \frac{-1}{1-\sum_{i=1}^{m-2} b_i} \left[ \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} w(\tau) f(\tau, u(\tau)) \nabla \tau - \int_0^T w(\tau) f(\tau, u(\tau)) \nabla \tau \right]. \]

We define the operators \( A : P \to E \) and \( F : K \to E \) as follows

\[(Au)(t) = \int_t^T \phi_q \left( \int_0^s w(\tau) f(\tau, u(\tau)) \nabla \tau - C \right) \Delta s \]

\[+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_s^T w(\tau) f(\tau, u(\tau)) \nabla \tau - C \right) \Delta s \]

\[- \int_0^T \phi_q \left( \int_0^s w(\tau) f(\tau, u(\tau)) \nabla \tau - C \right) \Delta s \quad (2.9) \]

where

\[ C = \frac{-1}{1-\sum_{i=1}^{m-2} b_i} \left[ \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} w(\tau) f(\tau, u(\tau)) \nabla \tau - \int_0^T w(\tau) f(\tau, u(\tau)) \nabla \tau \right], \]

\[(Fu)(t) = \int_t^T \phi_q \left( \int_0^s w(\tau) f^+(\tau, u(\tau)) \nabla \tau - \hat{C} \right) \Delta s \]

\[+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_s^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau - \hat{C} \right) \Delta s \]

\[- \int_0^T \phi_q \left( \int_0^s w(\tau) f^+(\tau, u(\tau)) \nabla \tau - \hat{C} \right) \Delta s], \]

(2.10)

where \( \hat{C} = \frac{-1}{1-\sum_{i=1}^{m-2} b_i} \left[ \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} w(\tau) f^+(\tau, u(\tau)) \nabla \tau - \int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau \right], \)
Let $D$ be an open bounded subset of $X$ with $D_K = D \cap K \neq \emptyset$ and $\overline{D_K} \neq K$. Suppose that $F : \overline{D_K} \to \overline{D_K}$ is completely continuous.
Lemma 3.1. Let $\Omega = \{u \in K : \|u\| < \rho\}$ and $\Omega_\rho = \{u \in K : \min_{t \in [0,T]} u(t) < \gamma \rho\}$. Then $\Omega_\rho$ has the following properties:

(a) $\Omega_\rho$ is open relative to $K$;
(b) $K_{\gamma \rho} \subset \Omega_\rho \subset K_{\rho}$;
(c) $u \in \partial \Omega_\rho$ if and only if $\min_{t \in [0,T]} u(t) = \gamma \rho$;
(d) $u \in \partial \Omega_\rho$, then $\gamma \rho \leq u(t) \leq \rho$ for $t \in [0,T]$.

Now, for convenience, we introduce the following notation. Let

$$
\begin{align*}
\frac{1}{\Lambda_1} &= \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^T \phi_q \left( \int_0^T w(\tau) \nabla \tau + \sum_{i=1}^{m-2} b_i \int_0^T w(\tau) \nabla \tau \right) \Delta s,
\frac{1}{\Lambda_2} &= \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_0^T \phi_q \left( \int_0^T w(\tau) \nabla \tau + \sum_{i=1}^{m-2} b_i \int_0^T w(\tau) \nabla \tau \right) \Delta s \right].
\end{align*}
$$

We remark that from (H1) it follows that $0 < \Lambda_1, \Lambda_2 < +\infty$, $\Lambda_2 \gamma = \Lambda_2 \gamma_1 \gamma_2 = \Lambda_1 \gamma_1 < \Lambda_1$.

Lemma 3.2. If $f$ satisfies the conditions

$$f(t,u) \leq \phi_p (\Lambda_1 \rho) \quad \text{and} \quad u \neq Fu, \text{ for } u \in \partial K_{\rho}, \ (t,u) \in [0,T] \times [0,\rho],$$

then $i_K(F,K_{\rho}) = 1$.

Proof. If $u \in \partial K_{\rho}$, then

$$
\begin{align*}
&\int_0^s w(\tau)f^+(\tau,u(\tau)) \nabla \tau - \tilde{C} \\
&= \int_0^s w(\tau)f^+(\tau,u(\tau)) \nabla \tau \\
&\quad + \sum_{i=1}^{m-2} b_i \int_0^s w(\tau)f^+(\tau,u(\tau)) \nabla \tau - \int_0^s w(\tau)f^+(\tau,u(\tau)) \nabla \tau \\
&\quad = \int_0^s w(\tau)f^+(\tau,u(\tau)) \nabla \tau \\
&\quad + \sum_{i=1}^{m-2} b_i \left( \int_0^T w(\tau)f^+(\tau,u(\tau)) \nabla \tau + \int_0^s w(\tau)f^+(\tau,u(\tau)) \nabla \tau \right) \\
&\quad - \int_0^s w(\tau)f^+(\tau,u(\tau)) \nabla \tau \\
&\quad = \sum_{i=1}^{m-2} b_i \left( \int_0^T w(\tau)f^+(\tau,u(\tau)) \nabla \tau + \int_0^s w(\tau)f^+(\tau,u(\tau)) \nabla \tau \right) \\
&\quad \times \frac{1}{1 - \sum_{i=1}^{m-2} b_i}.
\end{align*}
$$
\[
\begin{align*}
&\int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau \left/ \left( 1 - \sum_{i=1}^{m-2} b_i \right) \right. \\
&= \int_0^s w(\tau) f^+(\tau, u(\tau)) \nabla \tau \\
&\quad - \left( \int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau - \sum_{i=1}^{m-2} b_i \int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau \right) \\
&\quad - \sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau \\
&\quad \left/ \left( 1 - \sum_{i=1}^{m-2} b_i \right) \right. \\
&= \int_0^s w(\tau) f^+(\tau, u(\tau)) \nabla \tau - \left( \int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau \\
&\quad + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau \right) \\
&\quad \left/ \left( 1 - \sum_{i=1}^{m-2} b_i \right) \right. \\
&= - \int_s^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau - \sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) f^+(\tau, u(\tau)) \nabla \tau \\
&\quad \left/ \left( 1 - \sum_{i=1}^{m-2} b_i \right) \right. \\
&\geq -\phi_\rho(\rho \Lambda_1) \left[ \int_s^T w(\tau) \nabla \tau + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) \nabla \tau \right] \\
&\mathrm{so\ that} \\
&\phi(s) = \phi_\rho \left( \int_0^s w(\tau) f^+(\tau, u(\tau)) \nabla \tau - \tilde{C} \right) \\
&\geq -\rho \Lambda_1 \phi_\rho \left( \int_0^s w(\tau) \nabla \tau + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) \nabla \tau \right) \\
\end{align*}
\]

Therefore,
\[
\|Fu\| \leq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi(s) \Delta s - \int_0^T \varphi(s) \Delta s \right] \\
\leq -\frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^T \varphi(s) \Delta s \\
\leq \rho \Lambda_1 \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^T \phi_\rho \left( \int_0^s w(\tau) \nabla \tau + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) \nabla \tau \right) \Delta s \\
= \rho = \|u\|.
\]
This implies \( \|Fu\| \leq \|u\|, \ u \in \partial K_\rho \). By Theorem 2.5 (i), we have \( i_K(F, K_\rho) = 1 \).

**Lemma 3.3.** If \( f \) satisfies the conditions
\[
f(t, u) \geq \phi_p(\Lambda_2 \gamma \rho) \quad \text{and} \quad u \neq Fu \text{ for } u \in \partial \Omega_\rho, \quad (t, u) \in [0, T] \times [\gamma \rho, \rho],
\]
then \( i_K(F, \Omega_\rho) = 0 \).

**Proof.** Let \( e(t) \equiv 1 \) for \( t \in [0, T] \), then \( e \in \partial K_1 \). We claim that \( u \neq Fu + \lambda e, \ u \in \partial \Omega_\rho, \) and \( \lambda > 0 \). In fact, if not, there exist \( u_0 \in \partial \Omega_\rho \), and \( \lambda_0 > 0 \) such that
\[
0 = Fu_0 = Fu + \lambda e.
\]
Then we find that
\[
\begin{align*}
\int_0^T w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau - C &= \int_0^T w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left( \sum_{i=1}^{m-2} b_i \int_0^\xi_i w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau \right) \\
&= \int_0^T w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left( \sum_{i=1}^{m-2} b_i \int_0^\xi_i w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau \right) - \sum_{i=1}^{m-2} b_i \int_0^T w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau \\
&= \int_0^S w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau - \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left( \sum_{i=1}^{m-2} b_i \int_0^\xi_i w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau \right) \\
&= \int_0^T w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau + \int_0^T w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau \\
&= \int_0^T w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left( \sum_{i=1}^{m-2} b_i \int_0^\xi_i w(t) \frac{\partial K}{\partial f} (\tau, u_0(\tau)) \nabla \tau \right) \\
&\leq -\phi_p(\Lambda_2 \gamma \rho) \left( \int_0^T w(t) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_0^\xi_i w(t) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right)
\end{align*}
\]
so that
\[
\tilde{\varphi}(s) = \phi_q \left( \int_0^s w(\tau) f^+(\tau, u_0(\tau)) \nabla \tau - \tilde{C} \right)
\]
\[
\leq -\Delta_2 \gamma \rho \phi_q \left( \int_s^T w(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right).
\]

From (2.10), we have
\[
u_0(t) = F(u_0(t) + \lambda_0 \epsilon(t))
\]
\[
\geq \int_0^T \tilde{\varphi}(s) \Delta s + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \tilde{\varphi}(s) \Delta s - \int_0^T \tilde{\varphi}(s) \Delta s \right] + \lambda_0
\]
\[
= -\sum_{i=1}^{m-2} a_i \int_0^T \tilde{\varphi}(s) \Delta s + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \tilde{\varphi}(s) \Delta s
\]
\[
\int_0^T \tilde{\varphi}(s) \Delta s - \int_0^T \tilde{\varphi}(s) \Delta s \right] + \lambda_0
\]
\[
= -\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \tilde{\varphi}(s) \Delta s
\]
\[
\int_0^T \tilde{\varphi}(s) \Delta s + \lambda_0
\]
\[
\geq \gamma \rho \Lambda_2 \frac{\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \tilde{\varphi}(s) \Delta s}{1 - \sum_{i=1}^{m-2} a_i} \phi_q \left( \int_s^T w(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right) \Delta s + \lambda_0
\]
\[
= \gamma \rho + \lambda_0,
\]
which implies \( \lambda_0 \leq 0 \), it is a contradiction. Hence by Theorem 2.5 (ii), it follows that \( i_K(F, \Omega_\rho) = 0 \). □

Now, we set up and verify our existence results for the BVP (1.1) and (1.2).

**Theorem 3.4.** Suppose that one of the following conditions holds.

(C1) There exist \( \rho_1, \rho_2 \) and \( \rho_3 \in (0, +\infty) \) with \( \rho_1 < \gamma \rho_2 \), and \( \rho_2 < \rho_3 \) such that

(i) \( f(t, u) \geq \phi_p(\lambda_2 \gamma \rho_1), (t, u) \in [0, T] \times [0, \rho_1] \);

(ii) \( f(t, u) \geq 0, (t, u) \in [0, T] \times [\gamma \rho_1, \rho_3] \), in addition, \( f(t, u) \geq \phi_p(\lambda_2 \gamma \rho_2), (t, u) \in [0, T] \times [\gamma \rho_2, \rho_2], u \neq Fu, u \in \partial \Omega_\rho; \)

(iii) \( f(t, u) \leq \phi_p(\lambda_1 \rho_3), (t, u) \in [0, T] \times [0, \rho_3] \).

(C2) There exist \( \rho_1, \rho_2 \) and \( \rho_3 \in (0, +\infty) \) with \( \rho_1 < \rho_2 < \rho_3 \), such that

(i) \( f(t, u) \geq \phi_p(\lambda_2 \gamma \rho_1), (t, u) \in [0, T] \times [\gamma \rho_1, \rho_2] \);

(ii) \( f(t, u) \leq \phi_p(\lambda_1 \rho_2), (t, u) \in [0, T] \times [0, \rho_2], u \neq Fu, u \in \partial \Omega_\rho; \)

(iii) \( f(t, u) \geq 0, (t, u) \in [0, T] \times [\gamma \rho_2, \rho_3] \), in addition, \( f(t, u) \geq \phi_p(\lambda_2 \gamma \rho_3), (t, u) \in [0, T] \times [\gamma \rho_2, \rho_3] \).

Then BVP (1.1) and (1.2) has at least two positive solutions \( u_1 \) and \( u_2 \).

**Proof.** Suppose (C1) holds, we verify that \( F \) has a fixed point \( u_1 \) either in \( \partial K_{\rho_1} \) or \( u_1 \) in \( \Omega_{\rho_2} \setminus K_{\rho_1} \). If \( u \neq Fu, u \in \partial K_{\rho_1} \cup \partial K_{\rho_3} \), by Lemmas 3.2 and 3.3 we have
\[
i_K(F, K_{\rho_1}) = 1, \quad i_K(F, K_{\rho_2}) = 0, \quad i_K(F, K_{\rho_3}) = 1.
\]
By Lemma 3.1 (b) and \( \rho_1 < \gamma \rho_2 \), we have \( K_{\rho_1} \subseteq K_{\gamma \rho_2} \subseteq \Omega_{\rho_2} \). From Theorem 2.5 (iii), we obtain \( F \) has a fixed point \( u_1 \in \Omega_{\rho_2} \setminus K_{\rho_1} \). Similarly, \( F \) has a fixed point
Assume that one of the following conditions holds.

(i) \( f(t, u(t)) \leq \phi_p(A_1 \rho_1), (t, u) \in [0, T_T] \times [0, \rho_1]; \)

(ii) \( f(t, u) \geq 0, (t, u) \in [0, T_T] \times [\gamma \rho_1, \rho_2], \) in addition, \( f(t, u) \geq \phi_p(A_2 \gamma \rho_2), \)
\( (t, u) \in [0, T_T] \times [\gamma \rho_2, \rho_3]. \)

Then BVP (1.1) and (1.2) has at least two positive solutions \( u_1 \) and \( u_2. \)

The proof is similar when (C2) holds. The proof is complete. \( \square \)

By a similar argument to that of Theorem 3.4, we can find the following new results on existence of at least one positive solution of the BVP (1.1) and (1.2).

**Theorem 3.5.** Assume that one of the following conditions holds.

(C3) There exist \( \rho_1, \rho_2 \in (0, +\infty) \) with \( \rho_1 < \gamma \rho_2 \) such that

(i) \( f(t, u(t)) \leq \phi_p(A_1 \rho_1), (t, u) \in [0, T_T] \times [0, \rho_1]; \)

(ii) \( f(t, u(t)) \geq 0, (t, u) \in [0, T_T] \times [\gamma \rho_1, \rho_2], \) in addition, \( f(t, u) \geq \phi_p(A_2 \gamma \rho_2), \)
\( (t, u) \in [0, T_T] \times [\gamma \rho_2, \rho_3]. \)

(C4) There exist \( \rho_1, \rho_2 \in (0, +\infty) \) with \( \rho_1 < \rho_2 \) such that

(i) \( f(t, u(t)) \geq \phi_p(A_2 \gamma \rho_1), (t, u) \in [0, T_T] \times [\gamma^2 \rho_1, \rho_2]; \)

(ii) \( f(t, u(t)) \leq \phi_p(A_1 \rho_2), (t, u) \in [0, T_T] \times [0, \rho_2]. \)

Then BVP (1.1) and (1.2) has at least one positive solution.
It is easy to see that \( f : [0,1] \times [0, +\infty) \to (-\infty, +\infty) \) is continuous. In this case \( p = q = 2, w(t) \equiv 1, T = 1, a_1 = \frac{2}{3}, a_2 = \frac{1}{2}, b_1 = \frac{1}{3}, b_2 = \frac{1}{2}, \xi_1 = \frac{1}{5}, \xi_2 = \frac{1}{2} \), it follows from a direct calculation that

\[
\frac{1}{\Lambda_1} = \frac{1}{1 - (a_1 + a_2)} \int_0^T \phi_q \left( \int_s^T \nabla \tau + \frac{b_1 \int_{\xi_1}^T \nabla \tau + b_2 \int_{\xi_2}^T \nabla \tau}{1 - (b_1 + b_2)} \right) \Delta s
\]

\[
= \frac{1}{1 - (\frac{2}{3} + \frac{1}{2})} \int_0^1 \left( 1 - s + \frac{\frac{1}{3} \left( 1 - \frac{1}{3} \right) + \frac{1}{2} \left( 1 - \frac{1}{2} \right)}{1 - \left( \frac{2}{3} + \frac{1}{2} \right)} \right) \Delta s = 40,
\]

\[
\frac{1}{\Lambda_2} = \frac{1}{1 - (a_1 + a_2)} \left[ a_1 \int_0^{\xi_1} \phi_q \left( \int_s^T \nabla \tau + \frac{b_1 \int_{\xi_1}^T \nabla \tau + b_2 \int_{\xi_2}^T \nabla \tau}{1 - (b_1 + b_2)} \right) \Delta s 
\]

\[
+ a_2 \int_0^{\xi_2} \phi_q \left( \int_s^T \nabla \tau + \frac{b_1 \int_{\xi_1}^T \nabla \tau + b_2 \int_{\xi_2}^T \nabla \tau}{1 - (b_1 + b_2)} \right) \Delta s \right]\]

\[
= \frac{1}{1 - (\frac{2}{3} + \frac{1}{2})} \left[ \frac{2}{3} \int_0^{1/3} \left( 1 - s + \frac{\frac{1}{3} \left( 1 - \frac{1}{3} \right) + \frac{1}{2} \left( 1 - \frac{1}{2} \right)}{1 - \left( \frac{2}{3} + \frac{1}{2} \right)} \right) \Delta s \right] = 1091 \frac{1}{72},
\]

\[
\gamma_1 = a_1 \frac{\xi_1}{T} + a_2 \frac{\xi_2}{T} = 2 \cdot \frac{1}{3} \cdot 3 + 1 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{25}{72},
\]

Let

\[
N = a_1 \int_0^{\xi_1} \phi_q \left( \int_s^T \nabla \tau + \frac{b_1 \int_{\xi_1}^T \nabla \tau + b_2 \int_{\xi_2}^T \nabla \tau}{1 - (b_1 + b_2)} \right) \Delta s
\]

\[
+ a_2 \int_0^{\xi_2} \phi_q \left( \int_s^T \nabla \tau + \frac{b_1 \int_{\xi_1}^T \nabla \tau + b_2 \int_{\xi_2}^T \nabla \tau}{1 - (b_1 + b_2)} \right) \Delta s
\]

\[
= \frac{2}{3} \int_0^{1/3} \left( 1 - s + \frac{\frac{1}{3} \left( 1 - \frac{1}{3} \right) + \frac{1}{2} \left( 1 - \frac{1}{2} \right)}{1 - \left( \frac{2}{3} + \frac{1}{2} \right)} \right) \Delta s
\]

\[
+ \frac{1}{4} \int_0^{1/2} \left( 1 - s + \frac{\frac{1}{3} \left( 1 - \frac{1}{3} \right) + \frac{1}{2} \left( 1 - \frac{1}{2} \right)}{1 - \left( \frac{2}{3} + \frac{1}{2} \right)} \right) \Delta s = \frac{1091}{864}
\]

and

\[
D = \int_0^T \phi_q \left( \int_s^T \nabla \tau + \frac{b_1 \int_{\xi_1}^T \nabla \tau + b_2 \int_{\xi_2}^T \nabla \tau}{1 - (b_1 + b_2)} \right) \Delta s
\]

\[
= \int_0^1 \left( 1 - s + \frac{\frac{1}{3} \left( 1 - \frac{1}{3} \right) + \frac{1}{2} \left( 1 - \frac{1}{2} \right)}{1 - \left( \frac{2}{3} + \frac{1}{2} \right)} \right) \Delta s = \frac{10}{3}.
\]

Then

\[
\frac{\gamma_2}{D} = \frac{1091}{864} \cdot \frac{3}{10} = \frac{1091}{2880}, \quad \gamma = \gamma_1 \gamma_2 = 5455 \frac{1}{41472}.
\]
Choose \( \rho_1 = 1, \rho_2 = 10, \rho_3 = 20 \), it is easy to check that \( \gamma \rho_1 < \rho_1 < \gamma \rho_2 < \rho_2 < \rho_3 \).

By (2.10), we obtain

\[
(Fu)(t) = \int_t^T \phi_q \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau - \tilde{C} \right) ds \\
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[ \sum_{i=1}^{m-2} b_i \int_0^T \phi_q \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau - \tilde{C} \right) ds \right] \\
- \int_0^T \phi_q \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau - \tilde{C} \right) ds \\
= \int_1^t \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau - \tilde{C} \right) ds + 8 \int_{1/3}^1 \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau - \tilde{C} \right) ds \\
+ 3 \int_{1/2}^1 \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau - \tilde{C} \right) ds \\
- 12 \int_0^1 \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau - \tilde{C} \right) ds + \frac{31}{6} \tilde{C},
\]

where

\[
\tilde{C} = -\frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[ \sum_{i=1}^{m-2} b_i \int_0^T f^+(\tau, u(\tau)) \nabla \tau - \int_0^T f^+(\tau, u(\tau)) \nabla \tau \right] \\
= -2 \int_{1/3}^1 f^+(\tau, u(\tau)) \nabla \tau - 3 \int_{1/2}^1 f^+(\tau, u(\tau)) \nabla \tau + 6 \int_0^1 f^+(\tau, u(\tau)) \nabla \tau.
\]

Since \( f(t, u) \leq \frac{13}{80}, t \in [0, T], u \in [0, 10], \) for \( u \in \partial K_{10} \), we have

\[
\|Fu\| = \max_{t \in [0, T]} Fu(t) \\
= 8 \int_{1/3}^1 \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau \right) ds + 3 \int_{1/2}^1 \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau \right) ds \\
- 12 \int_0^1 \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau \right) ds + \frac{31}{6} \tilde{C} \\
= 8 \int_{1/3}^1 \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau \right) ds + 3 \int_{1/2}^1 \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau \right) ds \\
- 12 \int_0^1 \left( \int_0^s f^+(\tau, u(\tau)) \nabla \tau \right) ds + \frac{31}{6} \left[ -2 \int_{1/3}^1 f^+(\tau, u(\tau)) \nabla \tau \\
- 3 \int_{1/2}^1 f^+(\tau, u(\tau)) \nabla \tau + 6 \int_0^1 f^+(\tau, u(\tau)) \nabla \tau \right] \\
\leq \frac{13}{4} < 10 = \|u\|.
\]

This implies \( Fu \neq u \), for \( u \in \partial K_{10} \).

As a result, \( f \) satisfies the following conditions:

(i) \( f(t, u) \leq \phi_p(\Lambda_1 \rho_1) = \frac{1}{80}, \) \((t, u) \in [0, T] \times [0, 1] ; \)
(ii) \( f(t, u) \geq 0, \ (t, u) \in [0, T] \times \left[ \frac{5455}{4172}, 20 \right] \); in addition, \( f(t, u) \geq \phi_p(\Lambda_2 \gamma \rho_2) = \frac{72}{1091} \cdot \frac{5455}{4172} \cdot 4 = \frac{5}{144}, \ (t, u) \in [0, T] \times \left[ \frac{5455}{4172}, 20 \right] \);

(iii) \( f(t, u) \leq \phi_p(\Lambda_1 \rho_3) = \frac{7}{30}, \ (t, u) \in [0, T] \times [0, 20] \).

Therefore, (C1) of Theorem 3.4 is satisfied, then problem (4.1) and (4.2) has two positive solutions \( u_1, u_2 \) satisfying
\[
\|u_1\| \leq 10, \quad \|u_2\| > 10.
\]

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