NONLINEAR ANISOTROPIC ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS AND DEGENERATE COERCIVITY

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ABSTRACT. In this article, we prove the existence and the regularity of distributional solutions for a class of nonlinear anisotropic elliptic equations with \( p_i(x) \) growth conditions, degenerate coercivity and \( L^m(\cdot) \) data, with \( m(\cdot) \) being small, in appropriate Lebesgue-Sobolev spaces with variable exponents. The obtained results extend some existing ones [8, 10].

1. Introduction

We consider the problem

\[
-\sum_{i=1}^{N} D_i\left(a_i(x,u)|D_i u|^{p_i(x)-2} D_i u\right) + |u|^{s(x)-1} u = f \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega,
\]

where \( \Omega \) is a bounded open domain in \( \mathbb{R}^N (N \geq 2) \) with Lipschitz boundary \( \partial \Omega \), and the right-hand side \( f \) in \( L^1(\Omega) \) (or \( L^{m(\cdot)}(\Omega) \)). We assume that the variable exponents \( s : \overline{\Omega} \to (0, +\infty) \), \( p_i : \overline{\Omega} \to (1, +\infty) \), \( i = 1, \ldots, N \) are continuous functions such that

\[
1 < p(x) \leq N \quad \text{where } \frac{1}{p(x)} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i(x)}, \quad \forall x \in \overline{\Omega}.
\]

Here, we suppose that \( a_i : \Omega \times \mathbb{R} \to \mathbb{R}, \ i = 1, \ldots, N \) are Carathéodory functions such that for a.e. \( x \in \Omega \), for every \( t \in \mathbb{R} \), we have

\[
\frac{\alpha}{(1 + |t|)^{\gamma_i(x)}} \leq a_i(x,t) \leq \beta, \quad i = 1, \ldots, N,
\]

where \( \alpha, \beta \) are strictly positive real numbers and \( \gamma_i \in C(\overline{\Omega}), \gamma_i(x) \geq 0 \) for all \( x \in \overline{\Omega} \) and \( i = 1, \ldots, N \).
The main difficulty in dealing with problem (1.1) is the fact that, because of assumption (1.3), the differential operator

\[ A(u) = - \sum_{i=1}^{N} D_i \left( a_i(x,u) |D_i u|^{p_i(x)-2} D_i u \right) \]

is well defined between \( \dot{W}^{1, s(\cdot)}(\Omega) \) and its dual but it fails to be coercive if \( u \) is large (see [17]). This shows that the classical methods for elliptic operators can’t be applied. To overcome this problem, we will proceed by approximation by means of truncatures in \( a(x,t) \) to get a coercive differential operator. We cite some papers that have dealt with the equation (1.1) or similar problems, see [1, 3, 5, 6, 7, 8, 9, 10, 14, 20] and the references therein. In case of a constant exponent \( p_i(x) = 2, s(x) = q \) and \( \gamma_i(x) = \gamma \) (resp. \( p_i(x) = p \)) similar results can be found in [8, 10]. The problem was also considered in [14] when \( p_i(x) = p(x) \) and \( \gamma_i(x) = \theta \) (resp. \( p_i(x) = p \)) \( \geq 0 \), where the authors supposed that \( a_i(x,u) = a(x,u) \leq \beta |u|^\theta \) with \( \beta : [0, +\infty) \to (0, +\infty) \) is a continuous function. The lack of growth condition on \( a(x,u) \) prompted them to consider only the renormalized and entropy solutions. The corresponding results in the isotropic case and without lower order term are developed in [1, 5, 6, 7, 20]. More general results are obtained in [1] in the constant case \( p_i(x) = p, \gamma_i(x) = \theta(p(x) - 1), \theta \in [0, 1] \). In the case \( p_i(x) = p(x) \) and \( \gamma_i(x) = \theta(p(x) - 1) \) where \( 0 \leq \theta \leq \frac{p-1}{p+1} \), the main results are collected in the paper [20]. Recently, the mathematical researchers paid attention to the anisotropic nonlinear problems with variable exponents. For instance, Problem (1.1) was investigated in [3, 4, 15, 16] under uniform ellipticity condition i.e \( \gamma_i(x) = 0 \). In this article we assume that the condition (1.3) holds and \( f \in L^{m(\cdot)}(\Omega) \) where \( p_i \) is assumed to be merely a continuous function, and we treat the regularity of distributional solution \( u \) depending simultaneously on \( s(\cdot) \) and \( m(\cdot) \).

2. Preliminaries

In this section we recall some facts on anisotropic spaces with variable exponents and we give some of their properties. For further details on the Lebesgue-Sobolev spaces with variable exponents, we refer to [2, 11, 12] and references therein. In this article we set

\[ C_+(\Omega) = \{ p \in C(\Omega) : p(x) > 1, \text{ for any } x \in \Omega \} \]

For any \( p \in C_+(\Omega) \), we denote

\[ p^+ = \max_{x \in \Omega} p(x) \quad \text{and} \quad p^- = \min_{x \in \Omega} p(x). \]

We define the Lebesgue space with variable exponent \( L^{p(\cdot)}(\Omega) \) as the set of all measurable functions \( u : \Omega \to \mathbb{R} \) for which the convex modular

\[ \rho_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} \, dx, \]

is finite. The expression

\[ \|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \rho_{p(\cdot)}(u) \lambda \leq 1 \}, \]
defines a norm on \( L^{p(\cdot)}(\Omega) \), called the Luxemburg norm. The space \( (L^{p(\cdot)}(\Omega), \|u\|_{p(\cdot)}) \) is a separable Banach space. Moreover, if \( 1 < p^- \leq p^+ < +\infty \), then \( L^{p(\cdot)}(\Omega) \) is uniformly convex, hence reflexive and its dual space is isomorphic to \( L^{p'(\cdot)}(\Omega) \) where \( \frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \). For all \( u \in L^{p(\cdot)}(\Omega) \) and \( v \in L^{p'(\cdot)}(\Omega) \), the Hölder type inequality
\[
\left| \int_{\Omega} u v \, dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{p'^-} \right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},
\]
holds. We define also the Banach space
\[
W^{1, p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},
\]
which is equipped with the norm
\[
\|u\|_{1, p(\cdot)} = \|u\|_{W^{1, p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.
\]
The space \( (W^{1, p(\cdot)}(\Omega), \|\cdot\|_{1, p(\cdot)}) \) is a Banach space. Next, we define \( W^{1, p(\cdot)}_0(\Omega) \) the Sobolev space with zero boundary values by
\[
W^{1, p(\cdot)}_0(\Omega) = \{u \in W^{1, p(\cdot)}(\Omega) : u = 0 \text{ on } \partial \Omega\},
\]
edowed with the norm \( \|\cdot\|_{1, p(\cdot)} \). The space \( W^{1, p(\cdot)}_0(\Omega) \) is separable and reflexive provided that \( 1 < p^- \leq p^+ < +\infty \). For \( u \in W^{1, p(\cdot)}_0(\Omega) \) with \( p \in C_+(\Omega) \), the Poincaré inequality holds
\[
\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}, \tag{2.1}
\]
for some \( C > 0 \) which depends on \( \Omega \) and \( p(\cdot) \). Therefore, \( \|\nabla u\|_{p(\cdot)} \) and \( \|u\|_{1, p(\cdot)} \) are equivalent norms.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular \( \rho_{p(\cdot)}(u) \) of the space \( L^{p(\cdot)}(\Omega) \). We have the following results.

**Proposition 2.1** \((2) \). If \( u_n, u \in L^{p(\cdot)}(\Omega) \) and \( p^+ < +\infty \), then the following properties hold:

- \( \|u\|_{p(\cdot)} < 1 \) (resp. \( 1 < 1, > 1 \) \( \iff \) \( \rho_{p(\cdot)}(u) < 1 \) (resp. \( 1 < 1, > 1 \)),
- \( \min \left( \rho_{p(\cdot)}(u) \rho_{p(\cdot)}(u)^{-\frac{1}{p^-}} \right) \leq \|u\|_{p(\cdot)} \leq \max \left( \rho_{p(\cdot)}(u) \rho_{p(\cdot)}(u)^{-\frac{1}{p^+}} \right) \),
- \( \min \left( \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \right) \leq \rho_{p(\cdot)}(u) \leq \max \left( \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \right) \),
- \( \|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1 \),
- \( \|u_n - u\|_{p(\cdot)} \to 0 \iff \rho_{p(\cdot)}(u_n - u) \to 0 \).

**Remark 2.2.** Note that the inequality
\[
\int_{\Omega} |f|^{p(x)} \, dx \leq C \int_{\Omega} |Df|^{p(x)} \, dx,
\]
in general does not hold (see [3]). But by Proposition 2.1 and 2.1, we have
\[
\int_{\Omega} |f|^{p(x)} \, dx \leq C \max\{\|Df\|_{p(\cdot)}^{p^+}, \|Df\|_{p(\cdot)}^{p^-}\}. \tag{2.2}
\]

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of problem \((1)\). First of all, let \( p_i(\cdot) : \overline{\Omega} \to [1, +\infty), i = 1, \ldots, N \) be continuous functions, we set \( \bar{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot)) \) and \( p_+(x) = \max_{1 \leq i \leq N} p_i(x) \), for all \( x \in \overline{\Omega} \). The anisotropic variable exponent Sobolev space \( W^{1, \bar{p}(\cdot)}(\Omega) \) is defined as
\[
W^{1, \bar{p}(\cdot)}(\Omega) = \{u \in L^{p_i(\cdot)}(\Omega) : D_i u \in L^{p_i(\cdot)}(\Omega), i = 1, \ldots, N\},
\]
Suppose that \( W^{1, p(\cdot)}(\Omega) \) is a Banach space with respect to the norm
\[
\|u\|_{W^{1, p(\cdot)}(\Omega)} = \|u\|_{p_+} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}.
\]
We denote by \( W^{1, p(\cdot)}_0(\Omega) \) the closure of \( C^\infty_0(\Omega) \) in \( W^{1, p(\cdot)}(\Omega) \), and we define
\[
\tilde{W}^{1, p(\cdot)}(\Omega) = W^{1, p(\cdot)}_0(\Omega) \cap W^{1, 1}_0(\Omega).
\]
If \( \Omega \) is a bounded open set with Lipschitz boundary \( \partial \Omega \), then
\[
\tilde{W}^{1, p(\cdot)}(\Omega) = \{ u \in W^{1, p(\cdot)}(\Omega) : u|_{\partial \Omega} = 0 \}.
\]
It is well-known that in the constant exponent case, that is, when \( \bar{p}(\cdot) = \bar{p} \in [1, +\infty)^N \), \( W^{1, \bar{p}}_0(\Omega) = W^{1, \bar{p}}(\Omega) \). However in the variable exponent case, in general \( W^{1, \bar{p}(\cdot)}_0(\Omega) \subset \tilde{W}^{1, \bar{p}(\cdot)}(\Omega) \) and the smooth functions are in general not dense in \( W^{1, \bar{p}(\cdot)}(\Omega) \), but if for each \( i = 1, \ldots, N \), \( p_i \) is log-Hölder continuous, that is, there exists a positive constant \( L \) such that
\[
|p_i(x) - p_i(y)| \leq \frac{L}{\ln |x - y|}, \quad \forall x, y \in \Omega, \ |x - y| < 1.
\]
Then \( C^\infty_0(\Omega) \) is dense in \( \tilde{W}^{1, \bar{p}(\cdot)}(\Omega) \), thus \( W^{1, \bar{p}(\cdot)}_0(\Omega) = \tilde{W}^{1, \bar{p}(\cdot)}(\Omega) \). We set for all \( x \in \Omega \)
\[
\bar{p}(x) = \sum_{i=1}^{N} \frac{1}{p_i(x)},
\]
and we define
\[
\bar{p}^*(x) = \begin{cases} \frac{N\bar{p}(x)}{N - \bar{p}(x)}, & \text{for } \bar{p}(x) < N, \\ +\infty, & \text{for } \bar{p}(x) \geq N. \end{cases}
\]
We have the following embedding results.

**Lemma 2.3** ([12]). Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain, and \( \bar{p}(\cdot) \in (C_+((\Omega)))^N \). If \( q \in C_+((\Omega)) \) and for all \( x \in \Omega \), \( q(x) < \max(p_+(x), \bar{p}^*(x)) \). Then the embedding
\[
\tilde{W}^{1, \bar{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega),
\]
is compact.

**Lemma 2.4** ([12]). Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain, and \( \bar{p}(\cdot) \in (C_+((\Omega)))^N \).
Suppose that
\[
\forall x \in \Omega, \ p_+(x) < \bar{p}^*(x), \tag{2.3}
\]
Then the following Poincaré-type inequality holds
\[
\|u\|_{L^{p_+}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i}(\Omega)}, \quad \forall u \in \tilde{W}^{1, \bar{p}(\cdot)}(\Omega), \tag{2.4}
\]
where \( C \) is a positive constant independent of \( u \). Thus \( \sum_{i=1}^{N} \|D_i u\|_{L^{p_i}(\Omega)} \) is an equivalent norm on \( \tilde{W}^{1, \bar{p}(\cdot)}(\Omega) \).

The following embedding results for the anisotropic constant exponent Sobolev space are well-known [18] [19].
Lemma 2.5. Let $\alpha_i \geq 1, \ i = 1, \ldots, N$, we pose $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$. Suppose $u \in W_0^{1,\bar{\alpha}}(\Omega)$, and set
\begin{align*}
\bar{\alpha} &= \frac{1}{\bar{\alpha}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\alpha_i}, \quad N = \left\{ \begin{array}{ll}
\frac{N^r}{N-r} & \text{if } \bar{\alpha} < N, \\
\text{any number in } [1, +\infty) & \text{if } \bar{\alpha} \geq N.
\end{array} \right.
\end{align*}
Then, there exists a constant $C$ depending on $N, p_1, \ldots, p_N$ if $\bar{\alpha} < N$ and also on $r$ and $|\Omega|$ if $\bar{\alpha} \geq N$, such that
\begin{equation}
\|u\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|D_i u\|_{L^{p_i}(\Omega)}^{1/N}. \tag{2.5}
\end{equation}

Lemma 2.6. Let $Q$ be a cube of $\mathbb{R}^N$ with faces parallel to the coordinate planes and $\alpha_i \geq 1, i = 1, \ldots, N$. Suppose $u \in W^{1,\bar{\alpha}}(Q)$, and set
\begin{align*}
r &= \bar{\alpha}' \quad \text{if } \bar{\alpha} < N, \\
r &= [1, +\infty) \quad \text{if } \bar{\alpha} \geq N.
\end{align*}
Then, there exists a constant $C$ depending on $N, \alpha_1, \ldots, \alpha_N$ if $\bar{\alpha} < N$ and also on $r$ and $|Q|$ if $\bar{\alpha} \geq N$, such that
\begin{equation}
\|u\|_{L^r(Q)} \leq C \prod_{i=1}^{N} \left( \|u\|_{L^{p_i}(Q)} + \|D_i u\|_{L^{p_i}(Q)} \right)^{1/N}. \tag{2.6}
\end{equation}
We will use through the paper, the truncation function $T_k$ at height $k$ ($k > 0$), that is $T_k(t) = \max\{-k, \min\{k, t\}\}$.

Proposition 2.7. If $u : \Omega \to \mathbb{R}$ is a measurable function such that $T_k(u) \in W^{1,\bar{\alpha}_k}(\Omega)$ for all $k > 0$, then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that
\begin{equation}
\nabla T_k(u) = v \chi_{\{|u| \leq k\}} \quad \text{a.e. in } \Omega, \tag{2.7}
\end{equation}
where $\chi_A$ denotes the characteristic function of a measurable set $A$. Moreover, if $u \in W^{1,1}_0(\Omega)$ then $v$ coincides with the standard distributional gradient of $u$.

A function $u$ such that $T_k(u) \in W^{1,\bar{\alpha}_k}(\Omega)$ for any $k > 0$, does not necessarily belong to $W^{1,1}_0(\Omega)$. However, according to the above proposition, it is possible to define its weak gradient, still denoted by $\nabla u$, as the unique function $v$ which satisfies (2.7).

Definition 2.8. For $0 < r < +\infty$, the set of all measurable functions $v : \Omega \to \mathbb{R}$ such that the functional $\|v\|_r = \sup_{k \geq 0} k \text{meas}\{x \in \Omega : |v(x)| > k\}^{1/r}$ is finite is called a Marcinkiewicz space and is denoted by $M^r(\Omega)$. If $|\Omega| < \infty$ and $0 < \epsilon < r-1$, we can show that $L^r(\Omega) \subset M^r(\Omega) \subset L^{r-\epsilon}(\Omega)$.

3. Statement of results

Definition 3.1. We say that $u$ is a distributional solution for problem (1.1) if $u \in W^{1,1}_0(\Omega)$, $|u|^{s(\cdot)} \in L^1(\Omega)$ and
\begin{equation}
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u)|D_i u|^{p_i(x)-2} D_i u D_i \varphi \, dx + \int_{\Omega} |u|^{s(x)-1} u \varphi \, dx = \int_{\Omega} f \varphi \, dx, \tag{3.1}
\end{equation}
for every $\varphi \in C_0^{\infty}(\Omega)$. 

Our main results are the following

**Theorem 3.2.** Let \( f \in L^1(\Omega), \ p_i : \overline{\Omega} \to (1, +\infty), \ s : \overline{\Omega} \to (0, +\infty) \) and \( \gamma_i : \overline{\Omega} \to [0, +\infty) \) be continuous functions such that for all \( x \in \overline{\Omega}, \)

\[
s(x) \geq p_i(x), \quad i = 1, \ldots, N,
\]

\[
\frac{\bar{p}(x)(N - 1 - \gamma_i^+)}{N(\bar{p}(x) - 1 - \gamma_i^+)} < p_i(x) < k(x),
\]

where

\[
k(x) = \begin{cases} \frac{\bar{p}(x)(N - 1 - \gamma_i^+)}{1 + \gamma_i^+ (N - \bar{p}(x))}, & \text{if } \bar{p}(x) < N, \\
\infty, & \text{if } \bar{p}(x) = N \end{cases}
\]

Let \( a_i \) be Carathéodory functions for \( i = 1, \ldots, N \) satisfying \([1.3]\). Then, problem \([1.1]\) has at least one distributional solution \( u \in W^{1, \bar{p}(\cdot)}(\Omega) \) where \( q_i(\cdot) \) are continuous functions on \( \overline{\Omega} \) satisfying

\[
1 \leq q_i(x) < \frac{N(\bar{p}(x) - 1 - \gamma_i^+)p_i(x)}{\bar{p}(x)(N - 1 - \gamma_i^+)}, \quad \forall x \in \overline{\Omega}, \quad i = 1, \ldots, N.
\]

**Theorem 3.3.** Let \( f \in L^1(\Omega), \ p_i : \overline{\Omega} \to (1, +\infty), \ s : \overline{\Omega} \to (0, +\infty) \) and \( \gamma_i : \overline{\Omega} \to [0, +\infty) \) be continuous functions such that \([2.3]\) holds and for all \( x \in \overline{\Omega}, \)

\[
s(x) > \max \left( 1 + \gamma_i(x) \right) \frac{1}{p_i(x)(1 + \gamma_i(x))}, \quad i = 1, \ldots, N.
\]

Let \( a_i \) be Carathéodory functions satisfying \([1.3]\). Then, problem \([1.1]\) has at least one distributional solution \( u \in W^{1, \bar{p}(\cdot)}(\Omega) \cap L^{K(\cdot)}(\Omega) \) where \( q_i(\cdot) \) are continuous functions on \( \overline{\Omega} \) satisfying

\[
1 < q_i(x) < \frac{p_i(x)s(x)}{s(x) + 1 + \gamma_i(x)}, \quad \forall x \in \overline{\Omega}, \quad i = 1, \ldots, N.
\]

**Theorem 3.4.** Let \( m : \overline{\Omega} \to (1, +\infty), \ p_i : \overline{\Omega} \to (1, +\infty), \gamma_i : \overline{\Omega} \to [0, +\infty) \) and \( s : \overline{\Omega} \to (0, +\infty) \) be continuous functions such that \([2.3]\) holds and for all \( x \in \overline{\Omega}, \)

\[
1 < m(x) < h(x), \quad \nabla m \in L^\infty(\Omega),
\]

where

\[
h(x) = \begin{cases} \frac{N[\bar{p}(x)]}{\bar{p}(x) - 1}, & \text{if } \bar{p}(x) < N \\
\infty, & \text{if } \bar{p}(x) = N \end{cases}
\]

and

\[
s(x) \geq \frac{1 + \gamma_i(x)}{m(x) - 1}, \quad \nabla s \in L^\infty(\Omega), \quad \nabla \gamma_i \in L^\infty(\Omega), \quad \gamma_i(x) = \max_{1 \leq i \leq N} \gamma_i(x).
\]

Let \( f \in L^{m(\cdot)}(\Omega) \) and let \( a_i \) be Carathéodory functions satisfying \([1.3]\). Then, problem \([1.1]\) has at least one distributional solution \( u \in W^{1, \bar{p}(\cdot)}(\Omega) \cap L^{K(\cdot)m(\cdot)}(\Omega) \).

**Theorem 3.5.** Let \( f \in L^{m(\cdot)}(\Omega) \) with \( m \) as in \([3.7]\), \( p_i : \overline{\Omega} \to (1, +\infty), \gamma_i : \overline{\Omega} \to [0, +\infty), \) and \( s : \overline{\Omega} \to (0, +\infty) \) be continuous functions. Assume \([2.3]\) and for all \( x \in \overline{\Omega}, \)

\[
\frac{1 + \gamma_i(x)}{m(x) - 1} > s(x) > \max \left( \frac{1 + \gamma_i(x)}{p_i(x)m(x) - 1}(1 + \gamma_i(x))(p_i(x) - 1) \right),
\]
\( \nabla s \in L^\infty(\Omega), \) and \( i = 1, \ldots, N, \) where \( \gamma_+ (\cdot) = \max_{1 \leq i \leq N} \gamma_i (\cdot) . \) Let \( a_i \) be Carathéodory functions satisfying \((1.3)\). Then, problem \((1.1)\) has at least one distributional solution \( u \) such that \( |u|^{m(x,s(x)} \in L^1(\Omega) \) and \( u \in W^{1,\vec{\gamma}(\cdot)}(\Omega) \) where \( q_i (\cdot) \) are continuous functions on \( \Omega \) satisfying

\[
1 < q_i (x) = \frac{p_i (x)m(x)s(x)}{s(x) + 1 + \gamma_i (x)}, \quad \forall x \in \Omega, \ i = 1, \ldots, N. \tag{3.10}
\]

**Remark 3.6.** In Theorem 3.2 it is clear that the conditions \((1.2)\) and \((3.3)\) imply that \((2.3)\) holds since we have

\[
k(x) \leq \vec{p}^*(x), \quad \forall x \in \Omega.
\]

**Remark 3.7.** Observe that the conditions \((3.7)\), \((3.8)\), and \((2.3)\) guarantee that

\[
s(x) > (1 + \gamma_+ (x)) (p_i (x) - 1), \quad \forall x \in \Omega, \ i = 1, \ldots, N.
\]

**Remark 3.8.** In Theorem 3.3 the conditions \((3.7)\) and \((2.3)\) imply that the assumption \((3.9)\) is not empty since we have

\[
\frac{1}{m(x) - 1} > p_i (x) - 1, \quad \forall x \in \Omega, \ i = 1, \ldots, N. \tag{3.11}
\]

**Remark 3.9.** Let \( f \in L^1(\Omega) \). Assume that for all \( x \in \Omega, \) \( \vec{p}(x) < N \) and \( s(x) > (1 + \gamma_+ (x)) N(p_i (x) - 1 - \gamma_+^i) \) for all \( i = 1, \ldots, N. \) Then, assumption \((3.3)\) implies \((3.5)\) and

\[
\frac{p_i (x)s(x)}{s(x) + 1 + \gamma_i (x)} > \frac{N(p_i (x) - 1 - \gamma_+^i)p_i (x)}{\vec{p}(x)(N - 1 - \gamma_+^i)}, \quad \forall x \in \Omega, \ i = 1, \ldots, N,
\]

so Theorem 3.3 improves Theorem 3.2 (and [3, Theorem 3.1]).

### 4. Approximate Equation

We will use the following approximating problem

\[
-\sum_{i=1}^{N} D_i \left( a_i (x, T_n(u_n))|D_i u_n|^{p_i (x)-2}D_i u_n \right) + |u_n|^{s(x)-1}u_n = T_n(f) \quad \text{in } \Omega, \tag{4.1}
\]

\[
u_n = 0 \quad \text{on } \partial \Omega.
\]

We are going to prove the existence of solution \( u_n \) to problem \((4.1)\).

**Lemma 4.1.** Let \( f \in L^1(\Omega) \) and let \( s : \Omega \to (0, +\infty), \) \( p_i : \Omega \to (1, +\infty), \ i = 1, \ldots, N \) be continuous functions. Assume that \((2.3)\) holds. Then, there exists at least one solution \( u_n \in W^{1,\vec{\gamma}(\cdot)}(\Omega) \) to problem \((4.1)\) in the sense that

\[
\sum_{i=1}^{N} \int a_i (x, T_n(u_n))|D_i u_n|^{p_i (x)-2}D_i u_n D_i \varphi \, dx + \int |u_n|^{s(x)-1}u_n \varphi \, dx = \int T_n(f) \varphi \, dx,
\]

for every \( \varphi \in W^{1,\vec{\gamma}(\cdot)}(\Omega) \cap L^\infty(\Omega) \). Moreover

\[
\int |u_n|^{s(x)} \, dx \leq \int |f| \, dx. \tag{4.3}
\]
Proof. Consider the problem

$$- \sum_{i=1}^{N} D_i \left( a_i(x, T_n(u_{n_k}))|D_i u_{n_k}|_{p_i(x)}^{-2} D_i u_{n_k} \right) + T_k \left( |u_{n_k}|^{s(x)-1} u_{n_k} \right) = T_n(f) \quad \text{in } \Omega, \quad u_{n_k} = 0 \quad \text{on } \partial \Omega. \quad (4.4)$$

It has been proved in [12] that there exists a solution $u_{n_k} \in \tilde{W}^{1,\tilde{p}(\cdot)}(\Omega)$ to problem (4.4), which satisfies

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_{n_k}))|D_i u_{n_k}|_{p_i(x)}^{-2} D_i u_{n_k} D_i \varphi \, dx + \int_{\Omega} T_k \left( |u_{n_k}|^{s(x)-1} u_{n_k} \right) \varphi \, dx = \int_{\Omega} T_n(f) \varphi \, dx, \quad \forall \varphi \in \tilde{W}^{1,\tilde{p}(\cdot)}(\Omega). \quad (4.5)$$

Choosing $\varphi = u_{n_k}$ in (4.5), by (1.3) and using that $T_k \left( |u_{n_k}|^{s(x)-1} u_{n_k} \right) \geq 0$, we have

$$\frac{\alpha}{n(1+n)^{\gamma}} \sum_{i=1}^{N} \int_{\Omega} |D_i u_{n_k}|_{p_i(x)} \, dx \leq \int_{\Omega} |u_{n_k}| \, dx.$$

Using Young’s inequality for all $\varepsilon > 0$, we obtain

$$\sum_{i=1}^{N} \int_{\Omega} |D_i u_{n_k}|_{p_i(x)} \, dx \leq \varepsilon \int_{\Omega} |u_{n_k}|^{p_-} \, dx + C_1$$

$$\leq \varepsilon C_2 \int_{\Omega} |D_i u_{n_k}|^{p_-} \, dx + C_1$$

$$\leq \varepsilon C_2 \sum_{i=1}^{N} \int_{\Omega} |D_i u_{n_k}|^{p_i(x)} \, dx + C_3,$$

where $C_1$, $C_2$, and $C_3$ are positive constants not depending on $k$. Now, we choose $\varepsilon = 1/(2C_2)$, then

$$\sum_{i=1}^{N} \int_{\Omega} |D_i u_{n_k}|_{p_i(x)} \, dx \leq C(n).$$

It follows that the sequence $\{u_{n_k}\}_k$ is bounded in $\tilde{W}^{1,\tilde{p}(\cdot)}(\Omega)$. So, there exists a function $u_n \in W^{1,\tilde{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by $u_{n_k}$) such that

$$u_{n_k} \rightharpoonup u_n \quad \text{weakly in } \tilde{W}^{1,\tilde{p}(\cdot)}(\Omega) \text{ and a.e in } \Omega. \quad (4.6)$$

Choosing $\varphi = u_{n_k} - u_n$ in (4.5) as a test function, we can easily prove that, for all $i = 1, \ldots, N$,

$$\int_{\Omega} a_i(x, T_n(u_{n_k})) \left[ |D_i u_{n_k}|_{p_i(x)}^{-2} D_i u_{n_k} - |D_i u_n|_{p_i(x)}^{-2} D_i u_n \right] D_i (u_{n_k} - u_n) \, dx \to 0$$

as $k \to +\infty$. By (1.3), we obtain

$$E_i(k) = \int_{\Omega} \left[ |D_i u_{n_k}|_{p_i(x)}^{-2} D_i u_{n_k} - |D_i u_n|_{p_i(x)}^{-2} D_i u_n \right] D_i (u_{n_k} - u_n) \, dx \to 0$$
as \( k \to +\infty \). We recall the following well-known inequalities, that hold for any two real vectors \( \xi, \eta \) and a real \( p > 1 \):

\[
(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq \begin{cases} 2^{2-p}|\xi - \eta|^p, & \text{if } p \geq 2, \\ (p-1)^{\frac{2}{p}}|\frac{\xi - \eta}{|\xi + \eta|}|^p, & \text{if } 1 < p < 2. \end{cases} \tag{4.7}
\]

Therefore,

\[
2^{2-p} \int_{\{x \in \Omega, p_i(x) \geq 2\}} |D_i(u_{nk} - u_n)|^{p_i(x)} \, dx \\
\leq \int_{\{x \in \Omega, p_i(x) \geq 2\}} \left[ |D_iu_{nk}|^{p_i(x)-2}D_iu_{nk} - |D_iu_n|^{p_i(x)-2}D_iu_n \right] D_i(u_{nk} - u_n) \, dx \\
\leq \int_\Omega \left[ |D_iu_{nk}|^{p_i(x)-2}D_iu_{nk} - |D_iu_n|^{p_i(x)-2}D_iu_n \right] D_i(u_{nk} - u_n) \, dx \\
= E_i(k). \tag{4.8}
\]

On the set \( \Omega_i = \{x \in \Omega, 1 < p_i(x) < 2\} \), we employ \([4.7]\) as follows

\[
\int_{\Omega_i} |D_iu_{nk} - D_iu_n|^{p_i(x)} \, dx \\
\leq \int \frac{|D_iu_{nk} - D_iu_n|^{p_i(x)}}{|(D_iu_{nk}) + |D_iu_n|^{p_i(x)-(2-p_i(x))}} \left[ |D_iu_{nk}| + |D_iu_n| \right]^{\frac{p_i(x)(2-p_i(x))}{2}} \, dx \\
\leq 2\| |D_iu_{nk} - D_iu_n|^{p_i(x)} \left( |D_iu_{nk}| + |D_iu_n| \right)^{\frac{p_i(x)(2-p_i(x))}{2}} \|_{L^{\frac{2}{p_i-(2-p_i)}}(\Omega_i)} \\
\times \| |D_iu_{nk}| + |D_iu_n| \|_{L^{\frac{2}{2-p_i}}(\Omega)}^{\frac{2}{p_i-(2-p_i)}} \leq 2 \max \left\{ \left( \int \frac{|D_iu_{nk} - D_iu_n|^{p_i(x)}}{|(D_iu_{nk}) + |D_iu_n|^{p_i(x)-(2-p_i(x))}} \, dx \right)^{\frac{p_i}{p_i-1}} \right\} \\
\times \max \left\{ \left( \int |D_iu_{nk}| + |D_iu_n| \right)^{\frac{2-p_i}{2}} \right\} \\
\times \left( \int |D_iu_{nk}| + |D_iu_n| \right)^{-\frac{2}{2-p_i}} \left( \int |D_iu_{nk}| + |D_iu_n| \right)^{\frac{2-p_i}{2}} \\
\leq 2 \max \left\{ (p_i^- - 1)^{-\frac{p_i^-}{2}} E_i(k)^{\frac{p_i^+}{2}}, (p_i^- - 1)^{-\frac{p_i^-}{2}} E_i(k)^{\frac{p_i^+}{2}} \right\} \\
\times \max \left\{ \left( \int |D_iu_{nk}| + |D_iu_n| \right)^{\frac{2-p_i}{2}} \right\} \\
\times \left( \int |D_iu_{nk}| + |D_iu_n| \right)^{-\frac{2}{2-p_i}} \left( \int |D_iu_{nk}| + |D_iu_n| \right)^{\frac{2-p_i}{2}}. \tag{4.9}
\]
Therefore, we can obtain (4.2) by passing to the limit in (4.5).

Then we deduce that the sequence

From (4.6) and Lebesgue’s dominated convergence theorem, we obtain

Since \( u_n \) is bounded in \( \hat{W}^{1, p(\cdot)}(\Omega) \) and \( u_n \in \hat{W}^{1, p(\cdot)}(\Omega) \), after letting \( k \to +\infty \) in (4.8) and (4.9), we find

\[
\lim_{k \to +\infty} \int_{\Omega} |D_i u_{n_k} - D_i u_n|^{p_i(x)} \, dx = 0,
\]

which implies, for all \( i = 1, \ldots, N \),

\[
D_i u_{n_k} \to D_i u_n \quad \text{strongly in } L^{p_i(\cdot)}(\Omega) \text{ and a.e. in } \Omega.
\]

We are going to prove (4.2) by passing to the limit in (4.5). By (4.10) we have

\[
|D_i u_{n_k}|^{p_i(x)-2}D_i u_{n_k} \to |D_i u_n|^{p_i(x)-2}D_i u_n \quad \text{weakly in } L^{p_i(\cdot)}(\Omega), \quad p_i'(\cdot) = \frac{p_i(\cdot)}{p_i(\cdot) - 1}.
\]

From (4.6) and Lebesgue’s dominated convergence theorem, we obtain

\[
a_i(x, T_n(u_{n_k}))D_i \varphi \to a_i(x, T_n(u_n))D_i \varphi \quad \text{strongly in } L^{p_i(\cdot)}(\Omega), \quad 1 \leq i \leq N.
\]

Let \( \rho_j(t) \) be an increasing, uniformly bounded Lipschitz function (or \( W^{1, \infty}(\Omega) \) function), such that \( \rho_j(\sigma) \to \chi_{\{|\sigma| > t\}} \text{sign}(\sigma) \), as \( j \to +\infty \). Taking \( \rho_j(u_{n_k}) \) as a test function in (4.5), we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} \rho_j'(u_{n_k})a_i(x, T_n(u_{n_k}))|D_i u_{n_k}|^{p_i(x)} \, dx + \int_{\Omega} T_k(|u_{n_k}|^{s(x)-1} u_{n_k}) \rho_j(u_{n_k}) \, dx
\]

\[
= \int_{\Omega} T_n(f) \rho_j(u_{n_k}) \, dx.
\]

As \( j \to +\infty \), we obtain

\[
\int_{\{\{u_{n_k} > t\}} \rho_j(|u_{n_k}|^{s(x)-1} u_{n_k}) \, dx \leq \int_{\{u_{n_k} > t\}} |f| \, dx.
\]

Let \( E \subset \Omega \) be any measurable set, using (4.11), we have

\[
\int_{E} |T_k(|u_{n_k}|^{s(x)-1} u_{n_k})| \, dx
\]

\[
= \int_{E \cap \{u_{n_k} \leq t\}} |T_k(|u_{n_k}|^{s(x)-1} u_{n_k})| \, dx + \int_{E \cap \{u_{n_k} > t\}} |T_k(|u_{n_k}|^{s(x)-1} u_{n_k})| \, dx
\]

\[
\leq (t^{s^+} + t^{s^-}) \text{meas}(E) + \int_{\{u_{n_k} > t\}} |f| \, dx.
\]

Then we deduce that the sequence \( \{T_k(|u_{n_k}|^{s(x)-1} u_{n_k})\} \) is equi-integrable in \( L^1(\Omega) \), and since \( T_k(|u_{n_k}|^{s(x)-1} u_{n_k}) \to |u_n|^{s(x)-1} u_n \) a.e. in \( \Omega \), Vitali’s theorem implies that

\[ T_k(|u_{n_k}|^{s(x)-1} u_{n_k}) \to |u_n|^{s(x)-1} u_n \quad \text{in } L^1(\Omega). \]

Therefore, we can obtain (4.2) by passing to the limit in (4.5).

To show (4.3), we choose \( \varphi = \frac{T_k(u_n)}{k} \) in (4.2) as a test function, we have

\[
\int_{\Omega} |u_n|^{s(x)-1} u_n T_k(u_n) \, dx \leq \int_{\Omega} T_n(f) \frac{T_k(u_n)}{k} \, dx \leq \int_{\Omega} |f| \, dx.
\]

Fatou’s lemma implies that estimate (4.3) holds as \( k \to 0 \).

In the rest of this paper, we will denote by \( C_i \) (or \( C \)) the positive constants depending only on the data of the problem, but not on \( n \).
5. Uniform estimates

In this section, we assume that $u_n$ is a solution of (4.1).

**Lemma 5.1.** Let $p_i : \Omega \to (1, \infty)$, $s : \Omega \to (0, \infty)$ and $\gamma_i : \Omega \to [0, \infty)$ be continuous functions. Then, there exists a constant $C > 0$ such that

$$\sum_{i=1}^{N} \int_{\Omega} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+\lambda}} \, dx \leq C, \quad \forall \lambda > 1, \quad (5.1)$$

$$\int_{\Omega} |D_i T_k(u_n)|^{p_i(x)} \, dx \leq \frac{k}{\alpha} (1+k) \gamma_i^+ \|f\|_{L^1(\Omega)}, \quad i = 1, \ldots, N. \quad (5.2)$$

**Proof.** We introduce the function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi_\lambda(t) = \int_{0}^{t} \frac{dx}{(1+|x|)^{\lambda}} = \frac{1}{1-\lambda} \left[ (1+|t|)^{1-\lambda} - 1 \right] \text{sign}(t), \quad \lambda > 1.$$ 

Note that $\psi_\lambda$ is a continuous function satisfies $\psi_\lambda(0) = 0$ and $|\psi_\lambda'(t)| \leq 1$. We take $\psi_\lambda(u_n)$ as a test function in (4.2) and we use the assumption (1.3), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+\lambda}} \, dx \leq C \int_{\Omega} |f| \, dx,$$

In particular, there exists $C_2 > 0$ such that

$$\int_{\Omega} \frac{|D_i u_n|^{p_i}}{(1+|u_n|)^{\gamma_i+\lambda}} \, dx \leq C_2, \quad \forall i = 1, \ldots, N. \quad (5.3)$$

We take $\varphi = T_k(u_n)$ in (4.2), we find

$$\int_{\Omega} \frac{|D_i T_k(u_n)|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)}} \, dx \leq \frac{k}{\alpha} \|f\|_{L^1(\Omega)}.$$ 

Hence

$$\int_{\Omega} |D_i T_k(u_n)|^{p_i(x)} \, dx = \int_{\Omega} \frac{|D_i T_k(u_n)|^{p_i(x)}}{(1+|T_k(u_n)|)^{\gamma_i(x)}} (1 + |T_k(u_n)|)^{\gamma_i(x)} \, dx \leq \frac{k}{\alpha} (1+k) \gamma_i^+ \|f\|_{L^1(\Omega)}.$$

Which yields (5.2). \hfill \square

**Lemma 5.2.** Assume that $s(\cdot)$, $p_i(\cdot)$ and $\gamma_i(\cdot)$ are restricted as in Theorem 3.3. Then, there exists a constant $C > 0$ such that for all continuous functions $q_i(\cdot)$, $i = 1, \ldots, N$ on $\Omega$ as in (3.4), we have

$$\|D_i u_n\|_{L^{p_i} (\Omega)} \leq C, \quad (5.4)$$

$$\|u_n\|_{L^{\gamma_i} (\Omega)} \leq C. \quad (5.5)$$

**Proof.** Firstly, for $p_i$ is defined in (3.3), we have

$$1 < \frac{N (\overline{p}(x) - 1 - \gamma_i^+) p_i(x)}{\overline{p}(x) (N - 1 - \gamma_i^+)}, \quad \forall x \in \Omega.$$ 

By (3.4) and (1.2), we deduce $q_i(x) < p_i(x)$ for all $x \in \Omega$, $i = 1, \ldots, N$. 

Case (a): In the first step, let \( q_i^+ \) be a constant satisfying
\[
q_i^+ < \frac{N(p^- - 1 - \gamma_+)p_i^-}{p^- (N - 1 - \gamma_+)} , \quad i = 1, \ldots, N, \quad \frac{1}{p^-} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i^-}.
\] (5.6)

We can assume that \( \frac{q_i^+}{p_i^-} = \frac{p_i^+}{p} \), where \( \frac{1}{p} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{q_i^+} \). If not, we set \( \theta = \max\{q_i^+ / p_i^-, i = 1, \ldots, N\} \) and replace \( q^+ \) by \( \theta p_i^- \). Observe that, since \( \theta p_i^- \geq q_i^+ \), the fact that \( (D_i u_n) \) remains in a bounded set of \( L^{p_i^-} (\Omega) \) implies the result.

From now on, we set \( q_i^+ = \theta p_i^- \), \( \theta = \frac{p_i^+}{p} \in (0, \frac{N(p^- - 1 - \gamma_+)}{p^- (N - 1 - \gamma_+)} ) \subset (0, 1) \). Then (5.6) is equivalent to
\[
(1 - \frac{\theta}{\theta}) q^+ - \gamma_+ > 1, \quad q^+ = \frac{N q_i^+}{N - q_i^+}.
\]

Hence there exists \( \lambda > 1 \) such that
\[
(\gamma_+ + \lambda) (\frac{\theta}{1 - \theta}) < q^+.
\] (5.7)

Using Hölder’s inequality and (5.3), we obtain
\[
\int_{\Omega} |D_i u_n|^{q_i^+} dx = \int_{\Omega} |D_i u_n|^{q_i^+} \frac{1}{(1 + |u_n|)^{(\gamma_+ + \lambda)\theta}} (1 + |u_n|)^{(\gamma_+ + \lambda)\theta} dx
\]
\[
\leq \left( \int_{\Omega} \frac{|D_i u_n|^{p_i^-}}{(1 + |u_n|)^{(\gamma_+ + \lambda)\theta}} dx \right)^\theta \left( \int_{\Omega} (1 + |u_n|)^{(\gamma_+ + \lambda)\frac{\theta}{1 - \theta}} \right)^{1 - \theta}
\]
\[
\leq C \left( \int_{\Omega} (1 + |u_n|)^{(\gamma_+ + \lambda)\frac{\theta}{1 - \theta}} \right)^{1 - \theta},
\]
so that
\[
\prod_{i=1}^{N} (\|D_i u_n\|_{L^{q_i^+} (\Omega)})^{1/N} \leq C \left( \int_{\Omega} (1 + |u_n|)^{(\gamma_+ + \lambda)\frac{\theta}{1 - \theta}} \right)^{1 - \theta}.
\]
(5.8)

Therefore, by (5.7) and Young’s inequality, we can write
\[
\prod_{i=1}^{N} (\|D_i u_n\|_{L^{q_i^+} (\Omega)})^{1/N} \leq C_1 (\varepsilon) + \varepsilon \left( \int_{\Omega} |u_n|^{-} \right)^{1 - \theta},
\]
In view of 2.5, with \( r = \tilde{q}^+ \), we obtain
\[
\|u_n\|_{L^{q^+} (\Omega)} \leq C_0 \prod_{i=1}^{N} (\|D_i u_n\|_{L^{q_i^+} (\Omega)})^{1/N} \leq C_2 (\varepsilon) + \varepsilon C_0 \left( \|u_n\|_{L^{q^+} (\Omega)} \right)^{N(1 - \theta)}.
\]
(5.9)

We choose \( \varepsilon = 1/(2C_0) \), then
\[
\|u_n\|_{L^{q^+} (\Omega)} \leq C_3 + \frac{1}{2} \|u_n\|_{L^{q^+} (\Omega)}^{\gamma}, \quad \eta = (1 - \theta) \frac{N}{N - \tilde{q}^+}.
\] (5.10)

The assumption 1.2 implies that \( \eta \in (0, 1) \). Hence, the estimate (5.10) implies (5.5), and by (5.8) we deduce that (5.4) holds. This completes the proof of the case (a).
Case (b): In the second, we suppose that (3.4) holds and
\[ \phi_i^+ \geq \frac{N(p_i^+ - 1 + \gamma_i^+)}{p_i^+(N - 1 + \gamma_i^+)} . \]

By the continuity of \( p_i(\cdot) \) and \( q_i(\cdot) \) on \( \Omega \), there exists a constant \( \delta > 0 \) such that
\[ \max_{t \in Q(x,\delta) \cap \Omega} q_i(t) < \min_{t \in Q(x,\delta) \cap \Omega} \frac{N(p_i(t) - 1 + \gamma_i^+)}{p_i(t)(N - 1 + \gamma_i^+)} \quad \forall x \in \Omega, \quad (5.11) \]
where \( Q(x,\delta) \) is a cube with center \( x \) and diameter \( \delta \). Note that \( \Omega \) is compact and therefore we can cover it with a finite number of cubes \( (Q_j)_{j=1,\ldots,k} \) with edges parallel to the coordinate axes. Moreover there exists a constant \( \nu > 0 \) such that
\[ \delta > |\Omega_j| = \text{meas}(\Omega_j) > \nu, \quad \Omega_j = Q_j \cap \Omega \quad \forall j = 1, \ldots, k. \]
We denote by \( q_{i,j}^+ \) the local maximum of \( q_i(\cdot) \) on \( \Omega_j \) (respectively \( p_{i,j}^- \) the local minimum of \( p_i(\cdot) \) on \( \Omega_j \)), such that
\[ q_{i,j}^+ < \frac{N(p_j^- - 1 + \gamma_j^+)}{p_j^-(N - 1 + \gamma_j^+)} \quad \text{for all } j = 1, \ldots, k, \quad \frac{1}{p_j^+} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_{i,j}^-}. \quad (5.12) \]

By (2.6), we have
\[ \|u_n\|_{L_{\sigma_0^+}^\gamma(\Omega_j)} \leq C_1 \prod_{i=1}^N \left( \|u_n\|_{L_{\sigma^+_{i,j}(\Omega_j)}} + \|D_i u_n\|_{L_{\sigma^+_{i,j}(\Omega_j)}} \right)^{1/N}. \quad (5.13) \]
We combine (3.2), (4.3), (5.13), and the fact that \( q_{i,j}^+ < p_{i,j}^- \leq s^- = \min_{x \in \Omega_j} s(x) \), we obtain
\[ \|u_n\|_{L_{\sigma_0^+}^\gamma(\Omega_j)} \leq C_2 \prod_{i=1}^N \left( 1 + \|D_i u_n\|_{L_{\sigma^+_{i,j}(\Omega_j)}} \right)^{1/N}. \quad (5.14) \]
Now, arguing locally as in (5.8) and (5.9), we obtain
\[ \|u_n\|_{L_{\sigma_0^+}^\gamma(\Omega_j)} \leq C_3 \prod_{i=1}^N \left( 1 + \|D_i u_n\|_{L_{\sigma^+_{i,j}(\Omega_j)}} \right)^{1/N} \leq C_3 + \frac{1}{2} \|u_n\|_{L_{\sigma_0^+}^\gamma(\Omega_j)}, \quad (5.15) \]
where
\[ \eta_j = \left( 1 - \frac{\bar{q}_j^+}{\bar{p}_j^+} \right) \frac{N}{N - \bar{q}_j^+}. \]
Thanks to (1.2), we have \( \eta_j \in (0,1] \). Hence, the estimate (5.15) implies
\[ \int_{\Omega_j} |u_n|^{\sigma^+_{i,j}} \, dx \leq C_4 \quad \text{for } j = 1, \ldots, k, \]
\[ \int_{\Omega_j} |D_i u_n|^{\sigma^+_{i,j}} \, dx \leq C_5 \quad \text{for } j = 1, \ldots, k. \quad (5.16) \]
Knowing that \( q_i(x) \leq q_{i,j}^+ \) and \( \bar{\sigma}^i(x) \leq \bar{\sigma}_j^+ \) for all \( x \in \Omega_j \) and for \( j = 1, \ldots, k \), we conclude that
\[ \int_{\Omega_j} |u_n|^{\sigma^i(x)} \, dx + \int_{\Omega_j} |D_i u_n|^{\eta_j(x)} \, dx \leq C_6, \]
which finally implies

\[
\int_{\Omega} |u_n|^{q(x)} \, dx + \int_{\Omega} |D_i u_n|^{q_i(x)} \, dx \leq \sum_{j=1}^{k} \left( \int_{\Omega_j} |u_n|^{\gamma_i(x)} \, dx + \int_{\Omega_j} |D_i u_n|^{q_i(x)} \, dx \right) \leq C.
\]

Where \( C \) is a constant independent of \( n \). This finishes the proof of lemma 5.2 \( \Box \)

**Lemma 5.3.** Let \( m, s, p_i \) and \( \gamma_i \) be restricted as in Theorem 3.5 Then, there exists a constant \( C > 0 \) such that

\[
\sum_{i=1}^{N} \int_{\Omega} \frac{|D_i u_n|^{p_i(x)}}{(1 + |u_n|^{\gamma_i(x)+1-[(m(x)-1)s(x)]})} \, dx + \int_{\Omega} |u_n|^{m(x)s(x)} \, dx \leq C. \tag{5.17}
\]

**Proof.** Taking \( \psi(x, u_n) = ((1 + |u_n|)^{(m(x)-1)s(x)} - 1) \, \text{sign}(u_n) \) in (4.1) as a test function, by (1.3) and the fact that for a.e. \( x \in \Omega \) and for all \( i = 1, \ldots, N \)

\[
D_i \psi(x, u_n) = (m(x) - 1)(1 + |u_n|)^{(m(x)-1)s(x)} \, \text{sign}(u_n) D_i s(x) \ln(1 + |u_n|) + (m(x) - 1)s(x) D_i u_n \frac{(m(x)-1)s(x)}{(1 + |u_n|)^{1-(m(x)-1)s(x)}} + D_i m(x)(1 + |u_n|)^{(m(x)-1)s(x)} \, \text{sign}(u_n) s(x) \ln(1 + |u_n|),
\]

we obtain

\[
\alpha s^{-}(m - 1) \sum_{i=1}^{N} \int_{\Omega} \frac{|D_i u_n|^{p_i(x)}}{(1 + |u_n|^{\gamma_i(x)+1-[(m(x)-1)s(x)]})} \, dx + \int_{\Omega} |u_n|^{s(x)} \left( (1 + |u_n|)^{(m(x)-1)s(x)} - 1 \right) \, dx \\
\leq \int_{\Omega} |f| \left( (1 + |u_n|)^{(m(x)-1)s(x)} - 1 \right) \, dx + C_1 \sum_{i=1}^{N} \int_{\Omega} (1 + |u_n|)^{(m(x)-1)s(x)} \ln(1 + |u_n|)|D_i u_n|^{p_i(x)-1} \, dx.
\]

Using that \( |u_n|^{s(x)} \geq \min\{1; 2^{1-s^+}\}(1 + |u_n|)^{s(x)} - 1 \), Proposition 2.1 and Young inequality, we have

\[
\sum_{i=1}^{N} \int_{\Omega} \frac{|D_i u_n|^{p_i(x)}}{(1 + |u_n|^{\gamma_i(x)+1-[(m(x)-1)s(x)]})} \, dx + \frac{1}{2} \int_{\Omega} (1 + |u_n|)^{m(x)s(x)} \, dx \\
\leq C_2 + C_3 \max \left( \|f\|_{L^{m(\cdot)}(\Omega)}^{p^+}, \|f\|_{L^{m(\cdot)}(\Omega)}^{p^-} \right) + C_4 \sum_{i=1}^{N} \int_{\Omega} (1 + |u_n|)^{(m(x)-1)s(x)} \ln(1 + |u_n|)|D_i u_n|^{p_i(x)-1} \, dx. \tag{5.18}
\]
We can estimate the last term in (5.18) by applying Young’s inequality
\[
\int_\Omega (1 + |u_n|)^{(m(x)-1)s(x) + (p_i(x)-1)(\gamma_i(x)+1)} \ln(1 + |u_n|) D_i u_n |^{p_i(x)-1} \, dx
\]
\[
= \int_\Omega (1 + |u_n|)^{(m(x)-1)s(x) + \frac{p_i(x)-1}{p_i(x)}(\gamma_i(x)+1)} \ln(1 + |u_n|)
\times \frac{|D_i u_n|^{p_i(x)-1}}{(1 + |u_n|)^{\frac{p_i(x)-1}{p_i(x)}}} \, dx
\]
\[
\leq C_5 \int_\Omega (1 + |u_n|)^{(m(x)-1)s(x) + (p_i(x)-1)(\gamma_i(x)+1)} \ln(1 + |u_n|)^{p_i(x)} \, dx
+ \frac{1}{4C_4} \int_\Omega \frac{|D_i u_n|^{p_i(x)}}{(1 + |u_n|)^{\gamma_i(x)+1-(m(x)-1)s(x)}} \, dx.
\]
We combine (5.18) and (5.19), we obtain
\[
\int_\Omega (1 + |u_n|)^{(m(x)-1)s(x) + (p_i(x)-1)(\gamma_i(x)+1)} \ln(1 + |u_n|) D_i u_n |^{p_i(x)-1} \, dx
\]
\[
\leq C_6 + C_7 \sum_{i=1}^N \int_\Omega (1 + |u_n|)^{(m(x)-1)s(x) + (p_i(x)-1)(\gamma_i(x)+1)} \ln(1 + |u_n|)^{p_i(x)} \, dx
\times \ln(1 + |u_n|)^{p_i(x)} \, dx = I.
\]
Since \( s(x) > (p_i(x) - 1)(\gamma_i(x) + 1) \), we have
\[
(p_i(x) - 1)(\gamma_i(x) + 1) - s(x) \leq ((p_i(x) - 1)(\gamma_i(x) + 1) - s(x))^+ \leq b_i < \frac{b_i}{2} < 0,
\]
and \( (1 + |t|)^{(p_i(x)-1)(\gamma_i(x)+1)-s(x)} = \frac{1}{2} \ln(1 + |t|)^{p_i(x)} \) is bounded for all \( x \in \Omega \) and \( t \in \mathbb{R} \). We conclude that
\[
(1 + |u_n|)^{(m(x)-1)s(x) + (p_i(x)-1)(\gamma_i(x)+1)} \ln(1 + |u_n|)^{p_i(x)} = (1 + |u_n|)^{(m(x)s(x) + \frac{b_i}{2})}
\times (1 + |u_n|)^{(p_i(x)-1)(\gamma_i(x)+1)-s(x)} - b_i \ln(1 + |u_n|)^{p_i(x)}
\leq C(1 + |u_n|)^{(m(x)s(x) + \frac{b_i}{2})}.
\]
By another application of Young inequality, we obtain
\[
I \leq \frac{1}{8} \int_\Omega (1 + |u_n|)^{m(x)s(x)} \, dx + C_8.
\]
Therefore, by (5.20) and (5.21) we obtain the estimation (5.17). \( \square \)

**Lemma 5.4.** Let \( p_i, s \) and \( \gamma_i \) be restricted as in Theorem 3.3. Then, there exists a positive constant \( C \) such that
\[
\| D_i u_n \|_{L^{p_i}(\Omega)} \leq C,
\]
for all continuous functions \( q_i \) on \( \Omega \) satisfying (3.6).

**Proof.** Note that, assumption (3.6) implies that \( q_i(x) < p_i(x) \) for all \( x \in \Omega, i = 1, \ldots, N \). We can write
\[
\int_\Omega |D_i u_n|^{q_i(x)} \, dx = \int_\Omega \frac{|D_i u_n|^{q_i(x)}}{(1 + |u_n|)^{\frac{q_i(x)}{p_i(x)}(\gamma_i(x)+\lambda)}} (1 + |u_n|)^{\frac{q_i(x)}{p_i(x)}(\gamma_i(x)+\lambda)} \, dx.
\]
Using Young inequality and (5.11), we obtain
\[ \int_\Omega |D_i u_n|^{q_i(x)} \, dx \leq C_1 + C_2 \int_\Omega (1 + |u_n|)^{\frac{q_i(x)(\gamma_i(x) + \lambda)}{p_i(x)}(1 + |u_n|)} \, dx. \] (5.23)

Then assumption (3.6) implies \( \frac{s(x)(p_i(x) - q_i(x))}{q_i(x)} - \gamma_i(x) > 1 \). Choosing
\[ \lambda = \min_{1 \leq i \leq N} \min_{x \in \Omega} \left( \frac{s(x)(p_i(x) - q_i(x))}{q_i(x)} - \gamma_i(x) \right) > 1. \]

Thanks to the choice of \( \lambda \) and (5.6), we have
\[ \frac{q_i(x)(\gamma_i(x) + \lambda)}{p_i(x) - q_i(x)} \leq s(x), \quad \forall x \in \overline{\Omega}, \forall i = 1, \ldots, N. \] (5.24)

Combining (5.24), (5.23), and (4.3) results (5.22).

**Lemma 5.5.** Let \( m, s, p_i, \) and \( \gamma_i \) be restricted as in Theorem 3.5. Then, there exists a constant \( C > 0 \) such that
\[ \|D_i u_n\|_{L^{q_i}((\cdot)^{\gamma_i})} \leq C, \] (5.25)

for all continuous functions \( q_i \) on \( \overline{\Omega} \) satisfying (3.10).

**Proof.** Note that \( s(x) < \frac{1 + \gamma_i(x)}{m(x) - 1} \) and (3.10) imply \( q_i(x) < p_i(x) \). Then by Young’s inequality, we have
\[
\int_\Omega |D_i u_n|^{q_i(x)} \, dx \\
= \int_\Omega \frac{|D_i u_n|^{q_i(x)}}{(1 + |u_n|)^{(\gamma_i(x) + 1 - [(m(x) - 1)s(x)])}} \frac{q_i(x)}{p_i(x)} \left( 1 + |u_n| \right)^{(\gamma_i(x) + 1 - [(m(x) - 1)s(x)])} \frac{q_i(x)}{p_i(x)} \, dx \\
\leq \int_\Omega \left( \frac{q_i(x)}{p_i(x)} \right) (1 + |u_n|)^{(\gamma_i(x) + 1 - [(m(x) - 1)s(x)])} \, dx \\
+ \int_\Omega \left( 1 - \frac{q_i(x)}{p_i(x)} \right) (1 + |u_n|)^{(\gamma_i(x) + 1 - [(m(x) - 1)s(x)])} \, dx,
\]

and by (3.10), we obtain
\[
\int_\Omega |D_i u_n|^{q_i(x)} \, dx \\
\leq C_1 \int_\Omega \left( 1 + |u_n| \right)^{(\gamma_i(x) + 1 - [(m(x) - 1)s(x)])} \, dx + C_2 \int_\Omega (1 + |u_n|)^{m(x)s(x)} \, dx.
\] (5.26)

Therefore, (5.26) and (5.17) imply the desired result.

**Lemma 5.6.** Let \( m, s, p_i, \) and \( \gamma_i \) be restricted as in Theorem 3.4. Then, there exists a constant \( C > 0 \) such that
\[ \sum_{i=1}^N \int_\Omega |D_i u_n|^{p_i(x)} \, dx + \int_\Omega |u_n|^{s(x) + 1 + \gamma_+} \, dx \leq C. \] (5.27)

**Proof.** Taking \( \psi(x, u_n) = \left( (1 + |u_n|)^{1 + \gamma_+(x)} - 1 \right) \text{sign}(u_n) \) in (4.1) as a test function, by (1.3) and the fact that for a.e. \( x \in \Omega \), for all \( i = 1, \ldots, N \),
\[
D_i \psi(x, u_n) = (1 + \gamma_+(x))(1 + |u_n|)^{\gamma_+(x)} D_i u_n \\
+ D_i [\gamma_+(x)(1 + |u_n|)^{1 + \gamma_+}] \ln(1 + |u_n|) \text{sign}(u_n),
\]

...
we set \( \gamma_+^* = \max_{1 \leq i \leq N} \min_{x \in \Pi} \gamma_i(x) \), we obtain
\[
\alpha \left( 1 + \gamma_+^* \right) \sum_{i=1}^{N} \int_{\Omega} |D_i u_n|^{p_i} dx + \int_{\Omega} |u_n|^{s(x)} \left( (1 + |u_n|)^{1 + \gamma_+} - 1 \right) dx
\leq \int_{\Omega} |f| \left( (1 + |u_n|)^{1 + \gamma_+} - 1 \right) dx
+ C_1 \sum_{i=1}^{N} \int_{\Omega} (1 + |u_n|)^{1 + \gamma_+} \ln(1 + |u_n|) |D_i u_n|^{p_i - 1} dx.
\]

Using that \( |u_n|^{s(x)} \geq \min\{1, 2^{1-s^*}\} (1 + |u_n|)^{s(x)} - 1 \), Proposition 2.1 and Young’s inequality and since \( m'(\cdot)(1 + \gamma_+(x)) \leq s(x) + 1 + \gamma_+(x) \), we have
\[
\sum_{i=1}^{N} \int_{\Omega} |D_i u_n|^{p_i} dx + \frac{1}{2} \int_{\Omega} (1 + |u_n|)^{s(x)+1+\gamma_+(x)} dx
\leq C_2 + C_3 \max \left( \| f \|_{L^{m^+}(\Omega)}, \| f \|_{L^{m^-}(\Omega)} \right)
+ C_4 \sum_{i=1}^{N} \int_{\Omega} (1 + |u_n|)^{1+\gamma_+(x)} \ln(1 + |u_n|) |D_i u_n|^{p_i - 1} dx.
\]

We can estimate the last term in (5.28) by applying Young’s inequality,
\[
\int_{\Omega} (1 + |u_n|)^{1+\gamma_+(x)} \ln(1 + |u_n|) |D_i u_n|^{p_i - 1} dx
\leq C_5 \int_{\Omega} (1 + |u_n|)^{p_i(1+\gamma_+(x))} \ln(1 + |u_n|)^{p_i(x)} + \frac{1}{2C_4} \int_{\Omega} |D_i u_n|^{p_i(x)} dx.
\]

We combine (5.28) and (5.29) to obtain
\[
\sum_{i=1}^{N} \int_{\Omega} |D_i u_n|^{p_i(x)} dx + \int_{\Omega} (1 + |u_n|)^{s(x)+1+\gamma_+(x)} dx
\leq C_6 + C_7 \sum_{i=1}^{N} \int_{\Omega} (1 + |u_n|)^{p_i(x)(\gamma_+(x)+1)} \ln(1 + |u_n|)^{p_i(x)} dx = J.
\]

Thanks to Remark 3.7 we have \( s(x) > (p_i(x) - 1)\gamma_+(x) + 1 \), so
\[
(p_i(x) - 1)(\gamma_+(x) + 1) - s(x) \leq ((p_i(x) - 1)\gamma_+(x) + 1) - s(x) = d_i < \frac{d_i}{2} < 0,
\]
and \( (1 + |t|)^{p_i(x)-1}(\gamma_+(x)+1)-s(x)-\frac{d_i}{2} \ln(1 + |t|)^{p_i(x)} \) is bounded for all \( x \in \Omega \) and \( t \in \mathbb{R} \). We write
\[
(1 + |u_n|)^{p_i(x)(\gamma_+(x)+1)} \ln(1 + |u_n|)^{p_i(x)}
= (1 + |u_n|)^{s(x)+\gamma_+(x)+1+\frac{d_i}{2}} (1 + |u_n|)^{\gamma_+(x)(x+1)(p_i(x)-1)-s(x)-\frac{d_i}{2}} \ln(1 + |u_n|)^{p_i(x)}.
\]

By another application of Young’s inequality, we obtain
\[
J \leq \frac{1}{4} \int_{\Omega} (1 + |u_n|)^{s(x)+1+\gamma_+(x)} dx + C_8.
\]

Using (5.30) and (5.31), we obtain (5.27).
Lemma 5.7. Let \( f_n \in L^\infty(\Omega) \) be a sequence of functions which is strongly convergent to some \( f \) in \( L^1(\Omega) \) and let \( u_n \) be a solution of the problem
\[
- \sum_{i=1}^{N} D_i \left( a_i(x, T_n(u_n)|D_i u_n|^{p_i(x)} - 2D_i u_n \right) = f_n \quad \text{in } \Omega, \tag{5.32}
\]
\[ u_n = 0 \quad \text{on } \partial \Omega. \]

Suppose that:

(i) \( u_n \) is such that \( T_k(u_n) \in \tilde{W}^{1, \tilde{r}^i(\cdot)}(\Omega) \) for all \( k > 0 \).

(ii) \( u_n \) converges almost everywhere in \( \Omega \) to some measurable function \( u \) which is finite almost everywhere, and such that \( T_k(u) \in \tilde{W}^{1, \tilde{r}^i(\cdot)}(\Omega) \) for all \( k > 0 \) (note that (i) and (ii) imply that \( T_k(u_n) \) weakly converges to \( T_k(u) \) in \( \tilde{W}^{1, \tilde{r}^i(\cdot)}(\Omega) \)).

(iii) \( u_n \) is bounded in \( M^{r_1}(\Omega) \) for some \( r_1 > 0 \) and \( u \in M^{r_1}(\Omega) \).

(iv) There exists \( \theta_i > 0 \), \( i = 1, \ldots, N \) such that \( |D_i u_n|^{\theta_i} \) is bounded in \( L^{r_2}(\Omega) \), for some \( r_2 > 1 \) and \( |D_i u|^{\theta_i} \in L^{r_2}(\Omega) \).

Then, up to a subsequence, \( D_i u_n \) converges to \( D_i u \) almost everywhere in \( \Omega \) for all \( i = 1, \ldots, N \).

Proof. It has been proved in [12] that there exists a solution \( u_n \in \tilde{W}^{1, \tilde{r}^i(\cdot)}(\Omega) \) to problem (5.32). We follow the technique in [11, 20] with some modifications, since our method depends on the anisotropic variable exponent. Define the vector-valued function \( \hat{a}(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \), where \( \hat{a}(x, s, \xi) = \{ \hat{a}_i(x, s, \xi) \}_{i=1}^{N} \) with \( \hat{a}_i(x, s, \xi) = a_i(x, s, \xi) \cdot |\xi|^{p_i(x) - 2} \cdot \xi_i \). Let \( \theta \) be a real number between 0 and 1, which will be chosen later, and
\[
I(n) = \int_{\Omega} \left\{ (\hat{a}(x, T_n(u_n), \nabla u_n) - \hat{a}(x, T_n(u_n), \nabla u)) \right\}^{\theta} dx.
\]

Note that \( I(n) \) is well defined and \( I(n) \geq 0 \). We fixe \( k > 0 \) and split the integral \( I(n) \) on the sets \( \{|u| \geq k \} \) and \( \{|u| < k \} \), obtaining
\[
I_1(n, k) = \int_{\{|u| \geq k \}} \left\{ (\hat{a}(x, T_n(u_n), \nabla u_n) - \hat{a}(x, T_n(u_n), \nabla u)) \right\}^{\theta} dx,
\]
\[
I_2(n, k) = \int_{\{|u| < k \}} \left\{ (\hat{a}(x, T_n(u_n), \nabla u_n) - \hat{a}(x, T_n(u_n), \nabla u)) \right\}^{\theta} dx.
\]

By condition (1.3) and Young’s inequality, we have
\[
I_1(n, k) \leq C_1 \int_{\{|u| \geq k \}} \left\{ \sum_{i=1}^{N} \left( |D_i u_n|^{p_i(x)} + |D_i u|^{p_i(x)} \right) \right\}^{\theta} dx,
\]
\[
\leq C_1 \sum_{i=1}^{N} \int_{\{|u| \geq k \}} \left( 2 + |D_i u_n|^{\theta p_i^+} + |D_i u|^{\theta p_i^+} \right) dx.
\]

We now choose \( \theta < 1 \) such that \( \theta p_i^+ < \theta_i \), \( i = 1, \ldots, N \). Using the Hölder inequality and (iv), we obtain
\[
I_1(n, k) \leq C_2 \sum_{i=1}^{N} \left( \left( \int_{\Omega} |D_i u_n|^{\theta r_2} dx \right)^{\frac{1}{\theta r_2}} + \left( \int_{\Omega} |D_i u|^{\theta r_2} dx \right)^{\frac{1}{\theta r_2}} \right) \left( \left\{|u| \geq k \} \right\}^{1-\frac{1}{\theta r_2}} + C_2 \left\{|u| \geq k \} \right\}^{1-\frac{1}{\theta r_2}} + C_2 \left\{|u| \geq k \} \right\}.
\]
By (ii), for any $k > 1$, we have

$$|\{ |u| \geq k \}| \leq |\{ |u| > k - 1 \}| \leq \frac{C}{(k-1)^r}.$$  

Using the above inequality, we obtain

$$\lim_{k \to +\infty} \limsup_{n \to +\infty} I_1(n, k) = 0. \quad (5.33)$$

As $\nabla u = \nabla T_k(u)$ on the set $\{|u| < k\}$, we obtain

$$I_2(n, k) = \int_{\{|u| < k\}} \{ (\hat{a}(x, T_n(u_n), \nabla u_n) - \hat{a}(x, T_n(u_n), \nabla T_k(u))) \nabla (u_n - T_k(u)) \}^\theta \, dx.$$  

Take $h > k + 1$ and split the integral $I_2(n, k)$ on the sets $\{|u_n - T_k(u)| \geq h\}$ and $\{|u_n - T_k(u)| < h\}$, obtaining

$$I_3(n, k, h) = \int_{\{|u_n - T_k(u)| \geq h\}} \{ (\hat{a}(x, T_n(u_n), \nabla u_n) - \hat{a}(x, T_n(u_n), \nabla T_k(u))) \nabla (u_n - T_k(u)) \}^\theta \, dx,$$

and

$$I_4(n, k, h) = \int_{\{|u_n - T_k(u)| < h\}} \{ (\hat{a}(x, T_n(u_n), \nabla u_n) - \hat{a}(x, T_n(u_n), \nabla T_k(u))) \nabla (u_n - T_k(u)) \}^\theta \, dx.$$  

As $|u_n| \geq h - k$ on the set $\{|u_n - T_k(u)| \geq h\}$, we obtain

$$|\{ |u_n - T_k(u)| \geq h \}| \leq |\{ |u_n| \geq h - k \}| \leq \frac{C}{(h - k - 1)^r}.$$  

Similarly to the discussion of $I_1(n, k)$ (with the same choice of $\theta$), we obtain

$$\lim_{k \to +\infty} \limsup_{n \to +\infty} I_3(n, k, h) = 0. \quad (5.34)$$

Since $\nabla (u_n - T_k(u)) = \nabla T_h(u_n - T_k(u))$ on the set $\{|u_n - T_k(u)| < h\}$, by Hölder inequality (with exponents $\frac{1}{\theta}$ and $\frac{1}{1 - \theta}$), we have

$$I_4(n, k, h) \leq |\Omega|^{1 - \theta} \left\{ \int_{\{|u_n - T_k(u)| < h\}} \left( \hat{a}(x, T_n(u_n), \nabla u_n) - \hat{a}(x, T_n(u_n), \nabla T_k(u))) \nabla T_h(u_n - T_k(u)) \, dx \right) \right\}^\theta.$$  

Define

$$I_5(n, k, h) = \int_{\{|u_n - T_k(u)| < h\}} \left( \hat{a}(x, T_n(u_n), \nabla u_n) - \hat{a}(x, T_n(u_n), \nabla T_k(u))) \nabla T_h(u_n - T_k(u)) \, dx, \right.$$  

which we split as the difference $I_6 - I_7$, where

$$I_6(n, k, h) = \int_{\{|u_n - T_k(u)| < h\}} \hat{a}(x, T_n(u_n), \nabla u_n) \nabla T_h(u_n - T_k(u)) \, dx,$$

$$I_7(n, k, h) = \int_{\{|u_n - T_k(u)| < h\}} \hat{a}(x, T_n(u_n), \nabla T_k(u)) \nabla T_h(u_n - T_k(u)) \, dx.$$
Take \( n \) sufficiently large such that \( n > h + k \). Since \( |u_n| \leq k + h \) on the set where \( \{|u_n - T_k(u)| \leq h\} \), we obtain

\[
I_7(n, k, h) = \int_\Omega \hat{a}(x, T_{h+k}(u_n), \nabla T_k(u)) \nabla T_h(u_n - T_k(u)) \, dx.
\]

According to condition 1.3, we have

\[
|\hat{a}(x, T_{h+k}(u_n), \nabla T_k(u))| \leq \beta |D_i T_k(u)|^{p_i(x) - 1}, \quad \forall i = 1, \ldots, N.
\]

Note that

\[
\hat{a}_i(x, T_{h+k}(u_n), \nabla T_k(u)) \to \hat{a}_i(x, u, \nabla T_k(u)) \quad \text{a.e. in } \Omega, \quad i = 1, \ldots, N,
\]

using Lebesgue dominated convergence theorem, we derive

\[
\hat{a}_i(x, T_{h+k}(u_n), \nabla T_k(u)) \to \hat{a}_i(x, u, \nabla T_k(u)) \quad \text{strongly in } L^{p_i(x)}(\Omega), \forall i = 1, \ldots, N.
\]

Using the weak convergence of \( D_i T_h(u_n - T_k(u)) \) to \( D_i T_h(u - T_k(u)) \) in \( L^{p_i(x)}(\Omega), i = 1, \ldots, N \) (a consequence of (i) and (ii)), we find

\[
\lim_{n \to +\infty} I_7(n, k, h) = \int_\Omega \sum_{i=1}^N (\hat{a}_i(x, u, \nabla T_k(u)) D_i T_h(u) - D_i T_h(u - T_k(u))) \, dx,
\]

so

\[
\lim_{k \to +\infty} \lim_{n \to +\infty} I_7(n, k, h) = 0. \tag{5.35}
\]

For \( I_6(n, k, h) \), by (5.32) we obtain

\[
I_6(n, k, h) = \int_\Omega f_n T_h(u_n - T_k(u)) \, dx,
\]

by the strong convergence of \( f_n \) in \( L^1(\Omega) \), we have

\[
\lim_{k \to +\infty} \lim_{n \to +\infty} I_6(n, k, h) = 0 \tag{5.36}
\]

Putting together (5.33), (5.34), (5.35), and (5.36), one thus has

\[
\lim_{n \to +\infty} I(n) = 0.
\]

As in [20], we obtain \( D_i u_n \to D_i u \) a.e. in \( \Omega, \quad i = 1, \ldots, N \). \( \square \)

**Lemma 5.8.** Let \( u_n \) be a solution to the equation (4.1), suppose that \( u_n \) converges to \( u \) almost everywhere in \( \Omega \). Then

\[
|u_n|^{s(x)-1} u_n \to |u|^{s(x)-1} u \quad \text{in } L^1(\Omega).
\]

**Proof.** Let \( \rho_j(t) \) be an increasing, uniformly bounded Lipschitz function such that \( \rho_j \to \chi_{\{|t|>k\}} \text{sign}(t) \) (\( k > 0 \)), as \( j \to +\infty \). Taking \( \rho_j(u_n) \) as a test function in (4.2), we obtain

\[
\sum_{i=1}^N \int_\Omega \rho_j^i(u_n) a_i(x, T_n(u_n)) |D_i u_n|^{p_i(x)} \, dx + \int_\Omega |u_n|^{s(x)-1} u_n \rho_j(u_n) \, dx
\]

\[
= \int_\Omega T_n(f) \rho_j(u_n) \, dx.
\]

As \( j \to +\infty \), we obtain

\[
\int_{\{|u_n|>k\}} |u_n|^{s(x)} \, dx \leq \int_{\{|u_n|>k\}} |f| \, dx. \tag{5.37}
\]
Let $E \subset \Omega$ be any measurable set, using (5.37). We have

$$\int_E |u_n|^{s(x)} \, dx = \int_{E \cap \{|u_n| \leq k\}} |u_n|^{s(x)} \, dx + \int_{E \cap \{|u_n| > k\}} |u_n|^{s(x)} \, dx$$

$$\leq (k^{s^+} + k^{s^-}) \text{meas}(E) + \int_{E \cap \{|u_n| > k\}} |f| \, dx.$$  

Then we deduce that $(|u_n|^{s(x)-1}u_n)$ is equi-integrable in $L^1(\Omega)$, and since $u_n \to u$ a.e. in $\Omega$, then Vitali’s theorem implies

$$|u_n|^{s(x)-1}u_n \to |u|^{s(x)-1}u \text{ in } L^1(\Omega).$$

(5.38)

6. Proof of main results

In this section, using the uniform estimates of Section 4 we prove Theorem 3.2, 3.3, 3.4 and 3.5.

6.1. Proof of theorems 3.2, 3.3. By Lemma 5.2, the sequence $(u_n)$ is bounded in $W^{1,q_i}((\Omega))$ where $q_i(\cdot)$ is defined as (3.4). Without loss of generality, we can therefore assume that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,q_i}((\Omega)),$$

$$u_n \to u \text{ strongly in } L^{q_i}(\Omega), \quad q_0 = \min_{1 \leq i \leq N} q_i(x),$$

(6.1)

$$u_n \to u \text{ a.e. in } \Omega.$$

It follows from (4.3) and Fatou’s lemma that

$$\int_{\Omega} |u|^{s(x)} \, dx \leq \liminf_{n \to +\infty} \int_{\Omega} |u_n|^{s(x)} \, dx \leq C,$$

thus $|u|^{s(x)} \in L^1(\Omega)$, furthermore, $u \in M^{s^-}(\Omega)$. Then, there exists $r_1 = s^- > 0$ such that

$$\|u_n\|_{M^{r_1}(\Omega)} \leq C \quad \text{and} \quad u \in M^{r_1}(\Omega).$$

(6.2)

Let $f_n = T_n(\cdot) - T_n(\{s(x)-1\}u_n) \in L^{\infty}(\Omega)$, where $u_n$ is a solution of (5.32). Then, from (5.2), (5.4), (6.2), (6.1), and Lemma 5.7 we can deduce that

$$D_i u_n \rightharpoonup D_i u \text{ a.e. in } \Omega, \text{ for all } i = 1, \ldots, N.$$

So, by (5.4), we have

$$|D_i u_n|^{p_i(x)-2}D_i u_n \to |D_i u|^{p_i(x)-2}D_i u \text{ weakly in } L^{\frac{q_i(\cdot)}{p_i(\cdot)-1}}(\Omega), \forall i = 1, \ldots, N,$$

(6.3)

where $q_i$ is defined as in (3.4). The choice of $\frac{q_i(\cdot)}{p_i(\cdot)-1} > 1$ is possible since we have (3.3). From (1.3) and (6.1), we obtain

$$a_i(x, T_n(u_n)) \rightharpoonup a_i(x, u) \text{ weak}^* \text{ in } L^{\infty}(\Omega).$$

(6.4)

For any given $\varphi \in C_0^{\infty}(\Omega), \text{ using } \varphi \text{ as a test function in (4.1), we have}$

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n))|D_i u_n|^{p_i(x)-2}D_i u_n D_i \varphi \, dx + \int_{\Omega} |u_n|^{s(x)-1}u_n \varphi \, dx = \int_{\Omega} T_n(f) \varphi \, dx,$$

(6.5)
Letting $n \to +\infty$ in (6.3), by (6.3), (6.4), and (5.38), we obtain
\[ \sum_{i=1}^{N} \int_{\Omega} a_i(x,u) D_i u |p_i(x)^{q_i} - 2 D_i u D_i \varphi \, dx + \int_{\Omega} |u|^{s(x)-1} u \varphi \, dx = \int_{\Omega} f \varphi \, dx. \]

For the proof of Theorem 3.3, we only replace (6.3) with
\[ |D_i u_n|^{p_i(x)^{q_i}} - 2 D_i u_n \to |D_i u|^{p_i(x)^{q_i}} - 2 D_i u \quad \text{weakly in } L^{\frac{q_i}{p_i(x)^{q_i}-1}}(\Omega), \quad \forall i = 1, \ldots, N. \]
where $q_i$ is defined as in (3.6). The choice of $\frac{q_i}{p_i(x)^{q_i}-1} > 1$ is possible since we have (3.5).

6.2. **Proof of theorem 3.4 3.5** Because the proof of Theorem 3.5 is similar to that of Theorem 3.2 here we only give the proof of Theorem 3.4. According to Lemma 5.6, the sequence $(u_n)$ is bounded in $W^{1,p(\cdot)}(\Omega)$. This implies that we can extract a subsequence (denote again by $(u_n)$), such that
\[ u_n \rightharpoonup u \quad \text{weakly in } W^{1,p(\cdot)}(\Omega), \]
\[ u_n \to u \quad \text{strongly in } L^{p_0}(\Omega), \quad p_0 = \min_{1 \leq i \leq N} \min_{x \in \Omega} p_i(x), \]
\[ u_n \to u \quad \text{a.e. in } \Omega. \]

Arguing as the proof of Theorem 3.2 by using (5.27), we conclude that
\[ |D_i u_n|^{p_i(x)^{q_i}} - 2 D_i u_n \to |D_i u|^{p_i(x)^{q_i}} - 2 D_i u \quad \text{weakly in } L^{q_i(\cdot)}(\Omega), \quad \forall i = 1, \ldots, N. \]
The proof of Theorem 3.4 is complete.

**Remark 6.1.** All the results in this work also hold if our problem is exchanged by a more general one,
\[- \text{div}(a(x,u,\nabla u)) + g(x,u) = f \quad \text{in } \Omega, \]
\[ u = 0 \quad \text{on } \partial \Omega, \]
where $a(x,t,\xi) = \{a_i(x,t,\xi)\}_{i=1,\ldots,N} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-valued function such that for a.e. $x \in \Omega$ and for every $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$, the following assumptions hold:
\[ a(x,t,\xi) \xi \geq \alpha \sum_{i=1}^{N} \frac{\xi_i |p_i(x)|}{(1 + |t|)^{\gamma_i(x)}}, \quad \alpha > 0, \]
\[ |a_i(x,t,\xi)| \leq \beta \left(1 + \sum_{j=1}^{N} \xi_j |p_j(x)|^{1 - \gamma_j(x)}\right)^{1 - \frac{1}{p_i(x)}}, \quad i = 1, \ldots, N, \quad \beta > 0, \]
\[ (a(x,s,\xi) - a(x,s,\xi'))(\xi - \xi') > 0, \quad \forall \xi \neq \xi'. \]
Assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying
\[ \sup_{|t| \leq k} |g(x,t)| = h_k(x) \in L^1(\Omega), \quad \forall k > 0, \]
\[ g(x,t) \text{sign}(t) \geq |t|^{s(x)}. \]

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