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NONLINEAR ANISOTROPIC ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS AND DEGENERATE COERCIVITY

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Abstract. In this article, we prove the existence and the regularity of distributional solutions for a class of nonlinear anisotropic elliptic equations with $p_i(x)$ growth conditions, degenerate coercivity and $L^{m(\cdot)}$ data, with $m(\cdot)$ being small, in appropriate Lebesgue-Sobolev spaces with variable exponents. The obtained results extend some existing ones [8, 10].

1. Introduction

We consider the problem

$$
\begin{gathered}
-\sum_{i=1}^{N} D_i\big(a_i(x,u)|D_iu|^{p_i(x)-2}D_iu\big)+|u|^{s(x)-1}u=f \quad \text{in } \Omega,\\
u=0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N\geq 2)$ with Lipschitz boundary $\partial\Omega$, and the right-hand side f in $L^1(\Omega)$ (or $L^{m(\cdot)}(\Omega)$). We assume that the variable exponents $s:\overline{\Omega}\to(0,+\infty)$, $p_i:\overline{\Omega}\to(1,+\infty)$, $i=1,\dots,N$ are continuous functions such that

$$
1<\overline{p}(x)\leq N \quad \text{where } \frac{1}{\overline{p}(x)}=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}, \quad \forall x\in\overline{\Omega}. \tag{1.2}
$$

Here, we suppose that $a_i:\Omega\times\mathbb{R}\to\mathbb{R}$, $i=1,\dots,N$ are Carathéodory functions such that for a.e. $x\in\Omega$, for every $t\in\mathbb{R}$, we have

$$
\frac{\alpha}{(1+|t|)^{\gamma_i(x)}}\leq a_i(x,t)\leq\beta, \quad i=1,\dots,N, \tag{1.3}
$$

where α,β are strictly positive real numbers and $\gamma_i\in C(\overline{\Omega})$, $\gamma_i(x)\geq 0$ for all $x\in\overline{\Omega}$ and $i=1,\dots,N$.

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The main difficulty in dealing with problem (1.1) is the fact that, because of assumption (1.3), the differential operator

$$A(u)=-\sum_{i=1}^{N}D_i\big(a_i(x,u)|D_iu|^{p_i(x)-2}D_iu\big)$$

is well defined between $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ and its dual but it fails to be coercive if u is large (see [17]). This shows that the classical methods for elliptic operators can't be applied. To overcome this problem, we will proceed by approximation by means of truncatures in $a(x,t)$ to get a coercive differential operator. We cite some papers that have dealt with the equation (1.1) or similar problems, see [1, 3, 5, 6, 7, 8, 9, 10, 14, 20] and the references therein. In case of a constant exponent $p_i(x)=2$, $s(x)=q$ and $\gamma_i(x)=\gamma$ (resp. $p_i(x)=p$) similar results can be found in [8, 10]. The problem was also considered in [14] when $p_i(x)=p(x)$ and $\gamma_i(x)=\gamma(x)\geq 0$, where the authors supposed that $a_i(x,u)=a(x,u)\leq\beta(|u|)$ with $\beta:[0,+\infty)\to(0,+\infty)$ is a continuous function. The lack of growth condition on $a(x,u)$ prompted them to consider only the renormalized and entropy solutions. The corresponding results in the isotropic case and without lower order term are developed in [1, 5, 6, 7, 20]. More general results are obtained in [1] in the constant case $p_i(x)=p$, $\gamma_i(x)=\theta(p-1)$, $\theta\in[0,1]$. In the case $p_i(x)=p(x)$ and $\gamma_i(x)=\theta(p(x)-1)$ where $0\leq\theta\leq\frac{p^--1}{p^+-1}$ the main results are collected in the paper [20]. Recently, the mathematical researchers paid attention to the anisotropic nonlinear problems with variable exponents. For instance, Problem (1.1) was investigated in [3, 4, 15, 16] under uniform ellipticity condition i.e $\gamma_i(x)=0$. In this article we assume that the condition (1.3) holds and $f\in L^{m(\cdot)}(\Omega)$ where p_i is assumed to be merely a continuous function, and we treat the regularity of distributional solution u depending simultaneously on $s(\cdot)$ and $m(\cdot)$.

2. Preliminaries

In this section we recall some facts on anisotropic spaces with variable exponents and we give some of their properties. For further details on the Lebesgue-Sobolev spaces with variable exponents, we refer to [2, 11, 12] and references therein. In this article we set

$$C_+(\overline{\Omega})=\{p\in C(\overline{\Omega}):p(x)>1,\ \text{for any } x \text{ in } \overline{\Omega}\}.$$

For any $p\in C_+(\overline{\Omega})$, we denote

$$p^+=\max_{x\in\overline{\Omega}}p(x)\quad\text{and}\quad p^-=\min_{x\in\overline{\Omega}}p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u:\Omega\to\mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u)=\int_\Omega|u|^{p(x)}\,dx,$$

is finite. The expression

$$\|u\|_{p(\cdot)}:=\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\big\{\lambda>0:\rho_{p(\cdot)}\big(\frac{u}{\lambda}\big)\leq 1\big\},$$

defines a norm on $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega),\|u\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1<p^-\le p^+<+\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. For all $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\Big|\int_\Omega uv\,dx\Big|\le\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}\le 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

holds. We define also the Banach space

$$W^{1,p(\cdot)}(\Omega)=\big\{u\in L^{p(\cdot)}(\Omega):|\nabla u|\in L^{p(\cdot)}(\Omega)\big\},$$

which is equipped with the norm

$$\|u\|_{1,p(\cdot)}=\|u\|_{W^{1,p(\cdot)}(\Omega)}=\|u\|_{p(\cdot)}+\|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega),\|u\|_{1,p(\cdot)})$ is a Banach space. Next, we define $W_0^{1,p(\cdot)}(\Omega)$ the Sobolev space with zero boundary values by

$$W_0^{1,p(\cdot)}(\Omega)=\big\{u\in W^{1,p(\cdot)}(\Omega):u=0\text{ on }\partial\Omega\big\},$$

endowed with the norm $\|\cdot\|_{1,p(\cdot)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that $1<p^-\le p^+<+\infty$. For $u\in W_0^{1,p(\cdot)}(\Omega)$ with $p\in C_+(\overline{\Omega})$, the Poincaré inequality holds

$$\|u\|_{p(\cdot)}\le C\|\nabla u\|_{p(\cdot)},\tag{2.1}$$

for some $C>0$ which depends on Ω and $p(\cdot)$. Therefore, $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\cdot)}(u)$ of the space $L^{p(\cdot)}(\Omega)$. We have the following results.

Proposition 2.1 ([2, 11])**.** *If $u_n,u\in L^{p(\cdot)}(\Omega)$ and $p^+<+\infty$, then the following properties hold:*

- $\|u\|_{p(\cdot)}<1$ *(resp.* $=1,>1$*)* $\Longleftrightarrow\rho_{p(\cdot)}(u)<1$ *(resp.* $=1,>1$*),*
- $\min\big(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}},\rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\big)\le\|u\|_{p(\cdot)}\le\max\big(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}},\rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\big)$,
- $\min\big(\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\big)\le\rho_{p(\cdot)}(u)\le\max\Big(\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\Big)$,
- $\|u\|_{p(\cdot)}\le\rho_{p(\cdot)}(u)+1$,
- $\|u_n-u\|_{p(\cdot)}\to0\Longleftrightarrow\rho_{p(\cdot)}(u_n-u)\to0$.

Remark 2.2. Note that the inequality

$$\int_\Omega|f|^{p(x)}\,dx\le C\int_\Omega|Df|^{p(x)}\,dx,$$

in general does not hold (see [13]). But by Proposition 2.1 and (2.1) we have

$$\int_\Omega|f|^{p(x)}\,dx\le C\max\{\|Df\|_{p(\cdot)}^{p^+},\|Df\|_{p(\cdot)}^{p^-}\}.\tag{2.2}$$

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of problem (1.1). First of all, let $p_i(\cdot):\overline{\Omega}\to[1,+\infty)$, $i=1,\dots,N$ be continuous functions, we set $\vec{p}(\cdot)=(p_1(\cdot),\dots,p_N(\cdot))$ and $p_+(x)=\max_{1\le i\le N}p_i(x)$, for all $x\in\overline{\Omega}$. The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as

$$W^{1,\vec{p}(\cdot)}(\Omega)=\big\{u\in L^{p_+(\cdot)}(\Omega):D_iu\in L^{p_i(\cdot)}(\Omega),\ i=1,\dots,N\big\},$$

which is Banach space with respect to the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}.$$

We denote by $W_0^{1,\vec{p}(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$, and we define

$$\mathring{W}^{1,\vec{p}(\cdot)}(\Omega) = W^{1,\vec{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

If Ω is a bounded open set with Lipschitz boundary $\partial\Omega$, then

$$\mathring{W}^{1,\vec{p}(\cdot)}(\Omega) = \big\{u \in W^{1,\vec{p}(\cdot)}(\Omega) : u|_{\partial\Omega} = 0\big\}.$$

It is well-known that in the constant exponent case, that is, when $\vec{p}(\cdot) = \vec{p} \in [1,+\infty)^N$, $W_0^{1,\vec{p}}(\Omega) = \mathring{W}^{1,\vec{p}}(\Omega)$. However in the variable exponent case, in general $W_0^{1,\vec{p}(\cdot)}(\Omega) \subset \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ and the smooth functions are in general not dense in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, but if for each $i = 1,\dots,N$, p_i is log-Hölder continuous, that is, there exists a positive constant L such that

$$|p_i(x) - p_i(y)| \le \frac{L}{-\ln|x-y|}, \quad \forall x, y \in \Omega,\ |x-y| < 1.$$

Then $C_0^\infty(\Omega)$ is dense in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, thus $W_0^{1,\vec{p}(\cdot)}(\Omega) = \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$. We set for all $x \in \overline{\Omega}$

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^N \frac{1}{p_i(x)}},$$

and we define

$$\overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N, \\ +\infty, & \text{for } \overline{p}(x) \ge N. \end{cases}$$

We have the following embedding results.

Lemma 2.3 ([12]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and $\vec{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. If $q \in C_+(\overline{\Omega})$ and for all $x \in \overline{\Omega}$, $q(x) < \max(p_+(x), \overline{p}^\star(x))$. Then the embedding*

$$\mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega),$$

is compact.

Lemma 2.4 ([12]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and $\vec{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x \in \overline{\Omega},\ p_+(x) < \overline{p}^\star(x). \tag{2.3}$$

Then the following Poincaré-type inequality holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \le C \sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \quad \forall u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega), \tag{2.4}$$

where C is a positive constant independent of u. Thus $\sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$.

The following embedding results for the anisotropic constant exponent Sobolev space are well-known [18, 19].

Lemma 2.5. *Let $\alpha_i \geq 1$, $i = 1, \dots, N$, we pose $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)$. Suppose $u \in W_0^{1,\vec{\alpha}}(\Omega)$, and set*

$$\frac{1}{\overline{\alpha}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\alpha_i}, \quad r = \begin{cases} \overline{\alpha}^\star = \frac{N\overline{\alpha}}{N-\overline{\alpha}} & \text{if } \overline{\alpha} < N, \\ \text{any number in } [1, +\infty) & \text{if } \overline{\alpha} \geq N. \end{cases}$$

Then, there exists a constant C depending on $N, p_1, \dots, p_N$ if $\overline{\alpha} < N$ and also on r and $|\Omega|$ if $\overline{\alpha} \geq N$, such that

$$\|u\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|D_i u\|_{L^{\alpha_i}(\Omega)}^{1/N}. \tag{2.5}$$

Lemma 2.6. *Let Q be a cube of $\mathbb{R}^N$ with faces parallel to the coordinate planes and $\alpha_i \geq 1$, $i = 1, \dots, N$. Suppose $u \in W^{1,\vec{\alpha}}(Q)$, and set*

$$r = \overline{\alpha}^\star \quad \text{if } \overline{\alpha} < N,$$
$$r \in [1, +\infty) \quad \text{if } \overline{\alpha} \geq N.$$

Then, there exists a constant C depending on $N, \alpha_1, \dots, \alpha_N$ if $\overline{\alpha} < N$ and also on r and $|Q|$ if $\overline{r} \geq N$, such that

$$\|u\|_{L^r(Q)} \leq C \prod_{i=1}^{N} \left(\|u\|_{L^{\alpha_i}(Q)} + \|D_i u\|_{L^{\alpha_i}(Q)}\right)^{1/N}. \tag{2.6}$$

We will use through the paper, the truncation function T_k at height k ($k > 0$), that is $T_k(t) = \max\{-k, \min\{k, t\}\}$.

Proposition 2.7. *If $u : \Omega \to \mathbb{R}$ is a measurable function such that $T_k(u) \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ for all $k > 0$, then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u| \leq k\}} \quad \text{a.e. in } \Omega, \tag{2.7}$$

where χ_A denotes the characteristic function of a measurable set A. Moreover, if $u \in W_0^{1,1}(\Omega)$ then v coincides with the standard distributional gradient of u.

A function u such that $T_k(u) \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ for any $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, according to the above proposition, it is possible to define its weak gradient, still denoted by ∇u, as the unique function v which satisfies (2.7).

Definition 2.8. For $0 < r < +\infty$, the set of all measurable functions $v : \Omega \to \mathbb{R}$ such that the functional $[u]_r = \sup_{k>0} k \operatorname{meas}\{x \in \Omega : |u(x)| > k\}^{1/r}$ is finite is called a Marcinkiewicz space and is denoted by $M^r(\Omega)$. If $|\Omega| < \infty$ and $0 < \epsilon < r-1$, we can show that $L^r(\Omega) \subset M^r(\Omega) \subset L^{r-\epsilon}(\Omega)$.

3. Statement of results

Definition 3.1. We say that u is a distributional solution for problem (1.1) if $u \in W_0^{1,1}(\Omega)$, $|u|^{s(\cdot)} \in L^1(\Omega)$ and

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u)|D_i u|^{p_i(x)-2} D_i u D_i \varphi \, dx + \int_\Omega |u|^{s(x)-1} u \varphi \, dx = \int_\Omega f \varphi \, dx, \tag{3.1}$$

for every $\varphi \in C_0^\infty(\Omega)$.

Our main results are the following

Theorem 3.2. *Let $f \in L^1(\Omega)$, $p_i : \overline{\Omega} \to (1,+\infty)$, $s : \overline{\Omega} \to (0,+\infty)$ and $\gamma_i : \overline{\Omega} \to [0,+\infty)$ be continuous functions such that for all $x \in \overline{\Omega}$,*

$$s(x) \geq p_i(x), \quad i = 1,\dots,N, \tag{3.2}$$

$$\frac{\overline{p}(x)(N-1-\gamma_+^+)}{N(\overline{p}(x)-1-\gamma_+^+)} < p_i(x) < k(x), \tag{3.3}$$

where

$$k(x) = \begin{cases} \frac{\overline{p}(x)(N-1-\gamma_+^+)}{(1+\gamma_+^+)(N-\overline{p}(x))}, & \text{if } \overline{p}(x) < N \\ +\infty, & \text{if } \overline{p}(x) = N \end{cases} \quad \text{and} \quad \gamma_+^+ = \max_{1 \leq i \leq N} \max_{x \in \overline{\Omega}} \gamma_i(x).$$

Let a_i be Carathéodory functions for $i = 1,\cdots,N$ satisfying (1.3)*. Then, problem* (1.1) *has at least one distributional solution $u \in \mathring{W}^{1,\vec{q}(\cdot)}(\Omega)$ where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying*

$$1 \leq q_i(x) < \frac{N(\overline{p}(x)-1-\gamma_+^+)p_i(x)}{\overline{p}(x)(N-1-\gamma_+^+)}, \quad \forall x \in \overline{\Omega},\ i = 1,\dots,N. \tag{3.4}$$

Theorem 3.3. *Let $f \in L^1(\Omega)$, $p_i : \overline{\Omega} \to (1,+\infty)$, $s : \overline{\Omega} \to (0,+\infty)$ and $\gamma_i : \overline{\Omega} \to [0,+\infty)$ be continuous functions such that* (2.3) *holds and for all $x \in \overline{\Omega}$,*

$$s(x) > \max\Big(\frac{1+\gamma_i(x)}{p_i(x)-1}; (1+\gamma_i(x))(p_i(x)-1)\Big), \quad i = 1,\dots,N. \tag{3.5}$$

Let a_i be Carathéodory functions satisfying (1.3)*. Then, problem* (1.1) *has at least one distributional solution $u \in \mathring{W}^{1,\vec{q}(\cdot)}(\Omega) \cap L^{s(\cdot)}(\Omega)$ where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying*

$$1 < q_i(x) < \frac{p_i(x)s(x)}{s(x)+1+\gamma_i(x)}, \quad \forall x \in \overline{\Omega},\ i = 1,\dots,N. \tag{3.6}$$

Theorem 3.4. *Let $m : \overline{\Omega} \to (1,+\infty)$, $p_i : \overline{\Omega} \to (1,+\infty)$, $\gamma_i : \overline{\Omega} \to [0,+\infty)$ and $s : \overline{\Omega} \to (0,+\infty)$ be continuous functions such that* (2.3) *holds and for all $x \in \overline{\Omega}$*

$$1 < m(x) < h(x), \quad \nabla m \in L^\infty(\Omega), \tag{3.7}$$

where

$$h(x) = \begin{cases} \frac{N\overline{p}(x)}{N\overline{p}(x)+\overline{p}(x)-N}, & \text{if } \overline{p}(x) < N \\ \frac{p_+(x)}{p_+(x)-1}, & \text{if } \overline{p}(x) = N, \end{cases}$$

and

$$s(x) \geq \frac{1+\gamma_+(x)}{m(x)-1}, \quad \nabla s \in L^\infty(\Omega), \quad \nabla \gamma_+ \in L^\infty(\Omega), \quad \gamma_+(x) = \max_{1 \leq i \leq N} \gamma_i(x). \tag{3.8}$$

Let $f \in L^{m(\cdot)}(\Omega)$ and let a_i be Carathéodory functions satisfying (1.3)*. Then, problem* (1.1) *has at least one distributional solution $u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \cap L^{s(\cdot)m(\cdot)}(\Omega)$.*

Theorem 3.5. *Let $f \in L^{m(\cdot)}(\Omega)$ with m as in* (3.7)*, $p_i : \overline{\Omega} \to (1,+\infty)$, $\gamma_i : \overline{\Omega} \to [0,+\infty)$, and $s : \overline{\Omega} \to (0,+\infty)$ be continuous functions. Assume* (2.3)*, and for all $x \in \overline{\Omega}$,*

$$\frac{1+\gamma_+(x)}{m(x)-1} > s(x) > \max\Big(\frac{1+\gamma_i(x)}{p_i(x)m(x)-1}; (1+\gamma_i(x))(p_i(x)-1)\Big), \tag{3.9}$$

$\nabla s \in L^\infty(\Omega)$, and $i = 1, \dots, N$, where $\gamma_+(\cdot) = \max_{1\le i\le N} \gamma_i(\cdot)$. Let a_i be Carathéodory functions satisfying (1.3). Then, problem (1.1) has at least one distributional solution u such that $|u|^{m(x)s(x)} \in L^1(\Omega)$ and $u \in \mathring{W}^{1,\vec{q}(\cdot)}(\Omega)$ where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying

$$1 < q_i(x) = \frac{p_i(x)m(x)s(x)}{s(x) + 1 + \gamma_i(x)}, \quad \forall x \in \overline{\Omega},\ i = 1, \dots, N. \tag{3.10}$$

Remark 3.6. In Theorem 3.2, it is clear that the conditions (1.2) and (3.3) imply that (2.3) holds since we have

$$k(x) \le \overline{p}^\star(x),\ \forall x \in \overline{\Omega}.$$

Remark 3.7. Observe that the conditions (3.7), (3.8), and (2.3) guarantee that

$$s(x) > (1 + \gamma_+(x))(p_i(x) - 1), \quad \forall x \in \overline{\Omega},\ i = 1, \dots, N.$$

Remark 3.8. In Theorem 3.5, the conditions (3.7) and (2.3) imply that the assumption (3.9) is not empty since we have

$$\frac{1}{m(x) - 1} > p_i(x) - 1, \quad \forall x \in \overline{\Omega},\ i = 1, \dots, N. \tag{3.11}$$

Remark 3.9. Let $f \in L^1(\Omega)$. Assume that for all $x \in \overline{\Omega}$, $\overline{p}(x) < N$ and $s(x) > \frac{(1+\gamma_i(x))N(\overline{p}(x)-1-\gamma_+^+)}{(1+\gamma_+^+)(N-\overline{p}(x))}$ for all $i = 1, \dots, N$. Then, assumption (3.3) implies (3.5) and

$$\frac{p_i(x)s(x)}{s(x) + 1 + \gamma_i(x)} > \frac{N(\overline{p}(x) - 1 - \gamma_+^+)p_i(x)}{\overline{p}(x)(N - 1 - \gamma_+^+)}, \quad \forall x \in \overline{\Omega},\ i = 1, \dots, N,$$

so Theorem 3.3 improves Theorem 3.2 (and [3, Theorem 3.1]).

4. Approximate equation

We will use the following approximating problem

$$\begin{gathered} -\sum_{i=1}^N D_i\Big(a_i(x, T_n(u_n))|D_i u_n|^{p_i(x)-2} D_i u_n\Big) + |u_n|^{s(x)-1}u_n = T_n(f) \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.1}$$

We are going to prove the existence of solution u_n to problem (4.1).

Lemma 4.1. *Let $f \in L^1(\Omega)$ and let $s : \overline{\Omega} \to (0, +\infty)$, $p_i : \overline{\Omega} \to (1, +\infty)$, $i = 1, \dots, N$ be continuous functions. Assume that (2.3) holds. Then, there exists at least one solution $u_n \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ to problem (4.1) in the sense that*

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n))|D_i u_n|^{p_i(x)-2} D_i u_n D_i\varphi \,dx + \int_\Omega |u_n|^{s(x)-1}u_n\varphi \,dx \\ &= \int_\Omega T_n(f)\varphi \,dx, \end{aligned} \tag{4.2}$$

for every $\varphi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Moreover

$$\int_\Omega |u_n|^{s(x)} \,dx \le \int_\Omega |f| \,dx. \tag{4.3}$$

Proof. Consider the problem

$$\begin{gathered}
-\sum_{i=1}^{N} D_i\Big(a_i(x,T_n(u_{n_k}))|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}\Big)+T_k\big(|u_{n_k}|^{s(x)-1}u_{n_k}\big)\\
=T_n(f)\quad\text{in }\Omega,\\
u_{n_k}=0\quad\text{on }\partial\Omega.
\end{gathered}\tag{4.4}$$

It has been proved in [12] that there exists a solution $u_{n_k}\in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ to problem (4.4), which satisfies

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_{n_k}))|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}D_i\varphi\,dx\\
&+\int_\Omega T_k\big(|u_{n_k}|^{s(x)-1}u_{n_k}\big)\varphi\,dx\\
&=\int_\Omega T_n(f)\varphi\,dx,\quad \forall\varphi\in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega).
\end{aligned}\tag{4.5}$$

Choosing $\varphi=u_{n_k}$ in (4.5), by (1.3) and using that $T_k(|u_{n_k}|^{s(x)-1})u_{n_k}\geq 0$, we have

$$\frac{\alpha}{n(1+n)^{\gamma_+^+}}\sum_{i=1}^{N}\int_\Omega |D_iu_{n_k}|^{p_i(x)}\,dx\leq\int_\Omega |u_{n_k}|\,dx.$$

Using Young's inequality for all $\varepsilon>0$, we obtain

$$\begin{aligned}
\sum_{i=1}^{N}\int_\Omega |D_iu_{n_k}|^{p_i(x)}\,dx&\leq\varepsilon\int_\Omega |u_{n_k}|^{p_-^-}\,dx+C_1\\
&\leq\varepsilon C_2\int_\Omega |D_iu_{n_k}|^{p_-^-}\,dx+C_1\\
&\leq\varepsilon C_2\sum_{i=1}^{N}\int_\Omega |D_iu_{n_k}|^{p_i(x)}\,dx+C_3,
\end{aligned}$$

where C_1, C_2, and C_3 are positive constants not depending on k. Now, we choose $\varepsilon=1/(2C_2)$, then

$$\sum_{i=1}^{N}\int_\Omega |D_iu_{n_k}|^{p_i(x)}\,dx\leq C(n).$$

It follows that the sequence $\{u_{n_k}\}_k$ is bounded in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$. So, there exists a function $u_n\in\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by u_{n_k}) such that

$$u_{n_k}\rightharpoonup u_n\quad\text{weakly in }\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)\text{ and a.e in }\Omega.\tag{4.6}$$

Choosing $\varphi=u_{n_k}-u_n$ in (4.5) as a test function, we can easily prove that, for all $i=1,\dots,N$,

$$\int_\Omega a_i(x,T_n(u_{n_k}))\Big[|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}-|D_iu_n|^{p_i(x)-2}D_iu_n\Big]D_i(u_{n_k}-u_n)\,dx\to 0$$

as $k\to+\infty$. By (1.3), we obtain

$$E_i(k)=\int_\Omega\big[|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}-|D_iu_n|^{p_i(x)-2}D_iu_n\big]D_i(u_{n_k}-u_n)\,dx\to 0$$

as $k \to +\infty$. We recall the following well-known inequalities, that hold for any two real vectors ξ, η and a real $p > 1$:

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq \begin{cases} 2^{2-p}|\xi - \eta|^p, & \text{if } p \geq 2, \\ (p-1)\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}}, & \text{if } 1 < p < 2. \end{cases} \tag{4.7}$$

Therefore,

$$\begin{aligned} &2^{2-p_i^+} \int_{\{x\in\Omega, p_i(x)\geq 2\}} |D_i(u_{n_k} - u_n)|^{p_i(x)}\,dx \\ &\leq \int_{\{x\in\Omega, p_i(x)\geq 2\}} \big[|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k} - |D_iu_n|^{p_i(x)-2}D_iu_n\big] D_i(u_{n_k} - u_n)\,dx \\ &\leq \int_\Omega \big[|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k} - |D_iu_n|^{p_i(x)-2}D_iu_n\big] D_i(u_{n_k} - u_n)\,dx \\ &= E_i(k). \end{aligned} \tag{4.8}$$

On the set $\Omega_i = \{x \in \Omega, 1 < p_i(x) < 2\}$, we employ (4.7) as follows

$$\begin{aligned} &\int_{\Omega_i} |D_iu_{n_k} - D_iu_n|^{p_i(x)}\,dx \\ &\leq \int_{\Omega_i} \frac{|D_iu_{n_k} - D_iu_n|^{p_i(x)}}{(|D_iu_{n_k}| + |D_iu_n|)^{\frac{p_i(x)(2-p_i(x))}{2}}} (|D_iu_{n_k}| + |D_iu_n|)^{\frac{p_i(x)(2-p_i(x))}{2}}\,dx \\ &\leq 2\Big\|\frac{|D_iu_{n_k} - D_iu_n|^{p_i(x)}}{(|D_iu_{n_k}| + |D_iu_n|)^{\frac{p_i(x)(2-p_i(x))}{2}}}\Big\|_{L^{\frac{2}{p_i(\cdot)}}(\Omega_i)} \\ &\quad\times \big\|(|D_iu_{n_k}| + |D_iu_n|)^{\frac{p_i(x)(2-p_i(x))}{2}}\big\|_{L^{\frac{2}{2-p_i(\cdot)}}(\Omega)} \\ &\leq 2\max\Big\{\Big(\int_{\Omega_i} \frac{|D_iu_{n_k} - D_iu_n|^{p_i(x)}}{(|D_iu_{n_k}| + |D_iu_n|)^{\frac{p_i(x)(2-p_i(x))}{2}}}\,dx\Big)^{\frac{p_i^-}{2}}, \\ &\quad\Big(\int_{\Omega_i} \frac{|D_iu_{n_k} - D_iu_n|^{p_i(x)}}{(|D_iu_{n_k}| + |D_iu_n|)^{\frac{p_i(x)(2-p_i(x))}{2}}}\,dx\Big)^{\frac{p_i^+}{2}}\Big\} \\ &\quad\times \max\Big\{\Big(\int_\Omega (|D_iu_{n_k}| + |D_iu_n|)^{p_i(x)}\,dx\Big)^{\frac{2-p_i^+}{2}}, \\ &\quad\Big(\int_\Omega (|D_iu_{n_k}| + |D_iu_n|)^{p_i(x)}\,dx\Big)^{\frac{2-p_i^-}{2}}\Big\} \\ &\leq 2\max\Big\{(p_i^- - 1)^{-\frac{p_i^+}{2}} E_i(k)^{\frac{p_i^+}{2}}, (p_i^- - 1)^{-\frac{p_i^-}{2}} E_i(k)^{\frac{p_i^-}{2}}\Big\} \\ &\quad\times \max\Big\{\Big(\int_\Omega (|D_iu_{n_k}| + |D_iu_n|)^{p_i(x)}\,dx\Big)^{\frac{2-p_i^+}{2}}, \\ &\quad\Big(\int_\Omega (|D_iu_{n_k}| + |D_iu_n|)^{p_i(x)}\,dx\Big)^{\frac{2-p_i^-}{2}}\Big\}. \end{aligned} \tag{4.9}$$

Since u_{n_k} is bounded in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ and $u_n \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, after letting $k \to +\infty$ in (4.8) and (4.9), we find

$$\lim_{k\to+\infty} \int_\Omega |D_i u_{n_k} - D_i u_n|^{p_i(x)}\,dx = 0,$$

which implies, for all $i = 1, \dots, N$,

$$D_i u_{n_k} \to D_i u_n \quad \text{strongly in } L^{p_i(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \tag{4.10}$$

We are going to prove (4.2) by passing to the limit in (4.5). By (4.10) we have

$$|D_i u_{n_k}|^{p_i(x)-2} D_i u_{n_k} \rightharpoonup |D_i u_n|^{p_i(x)-2} D_i u_n \quad \text{weakly in } L^{p_i'(\cdot)}(\Omega),\ p_i'(\cdot) = \frac{p_i(\cdot)}{p_i(\cdot)-1}.$$

From (4.6) and Lebesgue's dominated convergence theorem, we obtain

$$a_i(x, T_n(u_{n_k})) D_i\varphi \to a_i(x, T_n(u_n)) D_i\varphi \quad \text{strongly in } L^{p_i(\cdot)}(\Omega),\ 1 \le i \le N.$$

Let $\rho_j(t)$ be an increasing, uniformly bounded Lipschitz function [5] (or $W^{1,\infty}(\Omega)$ function), such that $\rho_j(\sigma) \to \chi_{\{|\sigma|>t\}} \operatorname{sign}(\sigma)$, as $j \to +\infty$. Taking $\rho_j(u_{n_k})$ as a test function in (4.5), we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \rho_j'(u_{n_k}) a_i(x, T_n(u_{n_k})) |D_i u_{n_k}|^{p_i(x)}\,dx + \int_\Omega T_k(|u_{n_k}|^{s(x)-1} u_{n_k}) \rho_j(u_{n_k})\,dx \\
&= \int_\Omega T_n(f) \rho_j(u_{n_k})\,dx.
\end{aligned}$$

As $j \to +\infty$, we obtain

$$\int_{\{|u_{n_k}|>t\}} |T_k(|u_{n_k}|^{s(x)-1} u_{n_k})|\,dx \le \int_{\{|u_{n_k}|>t\}} |f|\,dx. \tag{4.11}$$

Let $E \subset \Omega$ be any measurable set, using (4.11), we have

$$\begin{aligned}
&\int_E |T_k(|u_{n_k}|^{s(x)-1} u_{n_k})|\,dx \\
&= \int_{E\cap\{|u_{n_k}|\le t\}} |T_k(|u_{n_k}|^{s(x)-1} u_{n_k})|\,dx + \int_{E\cap\{|u_{n_k}|>t\}} |T_k(|u_{n_k}|^{s(x)-1} u_{n_k})|\,dx \\
&\le (t^{s^+} + t^{s^-}) \operatorname{meas}(E) + \int_{\{|u_{n_k}|>t\}} |f|\,dx.
\end{aligned}$$

Then we deduce that the sequence $\{T_k(|u_{n_k}|^{s(x)-1} u_{n_k})\}$ is equi-integrable in $L^1(\Omega)$, and since $T_k(|u_{n_k}|^{s(x)-1} u_{n_k}) \to |u_n|^{s(x)-1} u_n$ a.e. in Ω, Vitali's theorem implies that

$$T_k(|u_{n_k}|^{s(x)-1} u_{n_k}) \to |u_n|^{s(x)-1} u_n \quad \text{in } L^1(\Omega).$$

Therefore, we can obtain (4.2) by passing to the limit in (4.5).

To show (4.3), we choose $\varphi = \frac{T_k(u_n)}{k}$ in (4.2) as a test function, we have

$$\int_\Omega |u_n|^{s(x)-1} u_n \frac{T_k(u_n)}{k}\,dx \le \int_\Omega T_n(f) \frac{T_k(u_n)}{k}\,dx \le \int_\Omega |f|\,dx.$$

Fatou's lemma implies that estimate (4.3) holds as $k \to 0$. $\square$

In the rest of this paper, we will denote by C_i (or C) the positive constants depending only on the data of the problem, but not on n.

5. Uniform estimates

In this section, we assume that u_n is a solution of (4.1).

Lemma 5.1. *Let $p_i : \overline{\Omega} \to (1, \infty)$, $s : \overline{\Omega} \to (0, \infty)$ and $\gamma_i : \overline{\Omega} \to [0, \infty)$ be continuous functions. Then, there exists a constant $C > 0$ such that*

$$\sum_{i=1}^{N} \int_{\Omega} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+\lambda}}\, dx \le C, \quad \forall \lambda > 1, \tag{5.1}$$

$$\int_{\Omega} |D_i T_k(u_n)|^{p_i(x)}\, dx \le \frac{k}{\alpha}(1+k)^{\gamma_+^+} \|f\|_{L^1(\Omega)}, \quad i = 1, \dots, N. \tag{5.2}$$

Proof. We introduce the function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi_\lambda(t) = \int_0^t \frac{dx}{(1+|x|)^\lambda} = \frac{1}{1-\lambda}[(1+|t|)^{1-\lambda} - 1]\operatorname{sign}(t), \ \lambda > 1.$$

Note that ψ_λ is a continuous function satisfies $\psi_\lambda(0) = 0$ and $|\psi'_\lambda(\cdot)| \le 1$. We take $\psi_\lambda(u_n)$ as a test function in (4.2) and we use the assumption (1.3), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+\lambda}}\, dx \le C_1 \int_{\Omega} |f|\, dx,$$

In particular, there exists $C_2 > 0$ such that

$$\int_{\Omega} \frac{|D_i u_n|^{p_i^-}}{(1+|u_n|)^{\gamma_+^+ + \lambda}}\, dx \le C_2, \quad \forall i = 1, \dots, N. \tag{5.3}$$

We take $\varphi = T_k(u_n)$ in (4.2), we find

$$\int_{\Omega} \frac{|D_i T_k(u_n)|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)}}\, dx \le \frac{k}{\alpha} \|f\|_{L^1(\Omega)}.$$

Hence

$$\begin{aligned}
\int_{\Omega} |D_i T_k(u_n)|^{p_i(x)}\, dx &= \int_{\Omega} \frac{|D_i T_k(u_n)|^{p_i(x)}}{(1+|T_k(u_n)|)^{\gamma_i(x)}} (1+|T_k(u_n)|)^{\gamma_i(x)}\, dx \\
&\le \frac{k}{\alpha}(1+k)^{\gamma_+^+} \|f\|_{L^1(\Omega)}.
\end{aligned}$$

Which yields (5.2). $\square$

Lemma 5.2. *Assume that $s(\cdot)$, $p_i(\cdot)$ and $\gamma_i(\cdot)$ are restricted as in Theorem 3.2. Then, there exists a constant $C > 0$ such that for all continuous functions $q_i(\cdot)$, $i = 1, \dots, N$ on $\overline{\Omega}$ as in (3.4), we have*

$$\|D_i u_n\|_{L^{q_i(\cdot)}(\Omega)} \le C, \tag{5.4}$$

$$\|u_n\|_{L^{\overline{q}^\star(\cdot)}(\Omega)} \le C. \tag{5.5}$$

Proof. Firstly, for p_i is defined in (3.3), we have

$$1 < \frac{N(\overline{p}(x) - 1 - \gamma_+^+) p_i(x)}{\overline{p}(x)(N - 1 - \gamma_+^+)}, \quad \forall x \in \overline{\Omega}.$$

By (3.4) and (1.2), we deduce $q_i(x) < p_i(x)$ for all $x \in \overline{\Omega}$, $i = 1, \dots, N$.

Case (a): In the first step, let q_i^+ be a constant satisfying

$$q_i^+ < \frac{N(\overline{p}^- - 1 - \gamma_+^+)p_i^-}{\overline{p}^-(N-1-\gamma_+^+)}, \quad i = 1,\dots,N, \ \frac{1}{\overline{p}^-} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i^-}. \tag{5.6}$$

We can assume that $\frac{q_i^+}{p_i^-} = \frac{\overline{q}^+}{\overline{p}^-}$, where $\frac{1}{\overline{q}^+} = \frac{1}{N}\sum_{i=1}^N \frac{1}{q_i^+}$. If not, we set $\theta = \max\{q_i^+/p_i^-, \, i = 1,\dots,N\}$ and replace q_i^+ by θp_i^-. Observe that, since $\theta p_i^- \geq q_i^+$, the fact that $(D_i u_n)$ remains in a bounded set of $L^{\theta p_i^-}(\Omega)$ implies the result.

From now on, we set $q_i^+ = \theta p_i^-$, $\theta = \frac{\overline{q}^+}{\overline{p}^-} \in (0, \frac{N(\overline{p}^- - 1 - \gamma_+^+)}{\overline{p}^-(N-1-\gamma_+^+)}) \subseteq (0,1)$. Then (5.6) is equivalent to

$$\Big(\frac{1-\theta}{\theta}\Big)\overline{q}^{+\star} - \gamma_+^+ > 1, \quad \overline{q}^{+\star} = \frac{N\overline{q}^+}{N - \overline{q}^+}.$$

Hence there exists $\lambda > 1$ such that

$$\Big(\frac{1-\theta}{\theta}\Big)\overline{q}^{+\star} - \gamma_+^+ > \lambda > 1,$$

so,

$$(\gamma_+^+ + \lambda)\Big(\frac{\theta}{1-\theta}\Big) < \overline{q}^{+\star}. \tag{5.7}$$

Using Hölder's inequality and (5.3), we obtain

$$\begin{aligned}
\int_\Omega |D_i u_n|^{q_i^+}\,dx &= \int_\Omega \frac{|D_i u_n|^{q_i^+}}{(1+|u_n|)^{(\gamma_+^+ + \lambda)\theta}}(1+|u_n|)^{(\gamma_+^+ + \lambda)\theta}\,dx \\
&\leq \Big(\int_\Omega \frac{|D_i u_n|^{p_i^-}}{(1+|u_n|)^{\gamma_+^+ + \lambda}}\,dx\Big)^{\theta}\Big(\int_\Omega (1+|u_n|)^{(\gamma_+^+ + \lambda)\frac{\theta}{1-\theta}}\Big)^{1-\theta} \\
&\leq C\Big(\int_\Omega (1+|u_n|)^{(\gamma_+^+ + \lambda)\frac{\theta}{1-\theta}}\Big)^{1-\theta},
\end{aligned}$$

so that

$$\prod_{i=1}^{N}\Big(\|D_i u_n\|_{L^{q_i^+}(\Omega)}\Big)^{1/N} \leq C^{\frac{1}{\overline{q}^+}}\Big(\int_\Omega (1+|u_n|)^{(\gamma_+^+ + \lambda)\frac{\theta}{1-\theta}}\Big)^{\frac{1-\theta}{\overline{q}^+}}.$$

Therefore, by (5.7) and Young's inequality, we can write

$$\prod_{i=1}^{N}\Big(\|D_i u_n\|_{L^{q_i^+}(\Omega)}\Big)^{1/N} \leq C_1(\varepsilon) + \varepsilon\Big(\int_\Omega |u_n|^{\overline{q}^{+\star}}\Big)^{\frac{1-\theta}{\overline{q}^+}}. \tag{5.8}$$

In view of (2.5), with $r = \overline{q}^{+\star}$, we obtain

$$\|u_n\|_{L^{\overline{q}^{+\star}}(\Omega)} \leq C_0 \prod_{i=1}^{N}\Big(\|D_i u_n\|_{L^{q_i^+}(\Omega)}\Big)^{1/N} \leq C_2(\varepsilon) + \varepsilon C_0\big(\|u_n\|_{L^{\overline{q}^{+\star}}(\Omega)}\big)^{\frac{N(1-\theta)}{N-\overline{q}^+}}. \tag{5.9}$$

We choose $\varepsilon = 1/(2C_0)$, then

$$\|u_n\|_{L^{\overline{q}^{+\star}}(\Omega)} \leq C_3 + \frac{1}{2}\|u_n\|^{\eta}_{L^{\overline{q}^{+\star}}(\Omega)}, \quad \eta = (1-\theta)\frac{N}{N-\overline{q}^+}. \tag{5.10}$$

The assumption (1.2) implies that $\eta \in (0,1]$. Hence, the estimate (5.10) implies (5.5), and by (5.8) we deduce that (5.4) holds. This completes the proof of the case (a).

Case (b): In the second, we suppose that (3.4) holds and

$$q_i^+ \geq \frac{N(\overline{p}^- - 1 - \gamma_+^+)p_i^-}{\overline{p}^-(N - 1 - \gamma_+^+)}.$$

By the continuity of $p_i(\cdot)$ and $q_i(\cdot)$ on $\overline{\Omega}$, there exists a constant $\delta > 0$ such that

$$\max_{t \in Q(x,\delta)\cap\Omega} q_i(t) < \min_{t \in Q(x,\delta)\cap\Omega} \frac{N(\overline{p}(t) - 1 - \gamma_+^+)p_i(t)}{\overline{p}(t)(N - 1 - \gamma_+^+)}, \quad \forall\, x \in \Omega, \tag{5.11}$$

where $Q(x,\delta)$ is a cube with center x and diameter δ. Note that $\overline{\Omega}$ is compact and therefore we can cover it with a finite number of cubes $(Q_j)_{j=1,\dots,k}$ with edges parallel to the coordinate axes. Moreover there exists a constant $\nu > 0$ such that

$$\delta > |\Omega_j| = \operatorname{meas}(\Omega_j) > \nu, \quad \Omega_j = Q_j \cap \Omega \quad \text{for all } j = 1, \dots, k.$$

We denote by $q_{i,j}^+$ the local maximum of $q_i(\cdot)$ on $\overline{\Omega_j}$ (respectively $p_{i,j}^-$ the local minimum of $p_i(\cdot)$ on $\overline{\Omega_j}$), such that

$$q_{i,j}^+ < \frac{N(\overline{p}_j^- - 1 - \gamma_+^+)p_{i,j}^-}{\overline{p}_j^-(N - 1 - \gamma_+^+)} \quad \text{for all } j = 1, \dots, k,\ \frac{1}{\overline{p}_j^-} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_{i,j}^-}. \tag{5.12}$$

By (2.6), we have

$$\|u_n\|_{L^{\overline{q}_j^{+\star}}(\Omega_j)} \leq C_1 \prod_{i=1}^N \Big(\|u_n\|_{L^{q_{i,j}^+}(\Omega_j)} + \|D_i u_n\|_{L^{q_{i,j}^+}(\Omega_j)}\Big)^{1/N}. \tag{5.13}$$

We combine (3.2), (4.3), (5.13), and the fact that $q_{i,j}^+ < p_{i,j}^- \leq s_j^- = \min_{x \in \overline{\Omega_j}} s(x)$, we obtain

$$\|u_n\|_{L^{\overline{q}_j^{+\star}}(\Omega_j)} \leq C_2 \prod_{i=1}^N \Big(1 + \|D_i u_n\|_{L^{q_{i,j}^+}(\Omega_j)}\Big)^{1/N}. \tag{5.14}$$

Now, arguing locally as in (5.8) and (5.9), we obtain

$$\|u_n\|_{L^{\overline{q}_j^{+\star}}(\Omega_j)} \leq C_2 \prod_{i=1}^N \Big(1 + \|D_i u_n\|_{L^{q_{i,j}^+}(\Omega_j)}\Big)^{1/N} \leq C_3 + \frac{1}{2}\|u_n\|^{\eta_j}_{L^{\overline{q}_j^{+\star}}(\Omega_j)}, \tag{5.15}$$

where

$$\eta_j = \Big(1 - \frac{\overline{q}_j^+}{\overline{p}_j^-}\Big)\frac{N}{N - \overline{q}_j^+}.$$

Thanks to (1.2), we have $\eta_j \in (0,1]$. Hence, the estimate (5.15) implies

$$\begin{gathered}\int_{\Omega_j} |u_n|^{\overline{q}_j^{+\star}}\,dx \leq C_4 \quad \text{for } j = 1, \dots, k,\\ \int_{\Omega_j} |D_i u_n|^{q_{i,j}^+}\,dx \leq C_5 \quad \text{for } j = 1, \dots, k.\end{gathered} \tag{5.16}$$

Knowing that $q_i(x) \leq q_{i,j}^+$ and $\overline{q}^\star(x) \leq \overline{q}_j^{+\star}$ for all $x \in \overline{\Omega}_j$ and for $j = 1, \dots, k$, we conclude that

$$\int_{\Omega_j} |u_n|^{\overline{q}^\star(x)}\,dx + \int_{\Omega_j} |D_i u_n|^{q_i(x)}\,dx \leq C_6,$$

which finally implies

$$\int_\Omega |u_n|^{\overline{q}^\star(x)}\,dx+\int_\Omega |D_iu_n|^{q_i(x)}\,dx \le \sum_{j=1}^k \Big(\int_{\Omega_j}|u_n|^{\overline{q}^\star(x)}\,dx+\int_{\Omega_j}|D_iu_n|^{q_i(x)}\,dx\Big)\le C.$$

Where C is a constant independent of n. This finishes the proof of lemma 5.2. $\square$

Lemma 5.3. *Let m, s, p_i and γ_i be restricted as in Theorem 3.5. Then, there exists a constant $C > 0$ such that*

$$\sum_{i=1}^N \int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+1-[(m(x)-1)s(x)]}}\,dx+\int_\Omega |u_n|^{m(x)s(x)}\,dx \le C. \tag{5.17}$$

Proof. Taking $\psi(x,u_n) = \big((1+|u_n|)^{(m(x)-1)s(x)}-1\big)\operatorname{sign}(u_n)$ in (4.1) as a test function, by (1.3) and the fact that for a.e. $x\in\Omega$ and for all $i=1,\dots,N$

$$\begin{aligned}
D_i\psi(x,u_n) &= (m(x)-1)(1+|u_n|)^{(m(x)-1)s(x)}\operatorname{sign}(u_n)D_is(x)\ln(1+|u_n|)\\
&\quad+\frac{(m(x)-1)s(x)D_iu_n}{(1+|u_n|)^{1-(m(x)-1)s(x)}}\\
&\quad+D_im(x)(1+|u_n|)^{(m(x)-1)s(x)}\operatorname{sign}(u_n)s(x)\ln(1+|u_n|),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\alpha s^-(m^--1)\sum_{i=1}^N\int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+1-((m(x)-1)s(x))}}\,dx\\
&+\int_\Omega |u_n|^{s(x)}\Big((1+|u_n|)^{(m(x)-1)s(x)}-1\Big)\,dx\\
&\le \int_\Omega |f|\Big((1+|u_n|)^{(m(x)-1)s(x)}-1\Big)\,dx\\
&\quad+C_1\sum_{i=1}^N\int_\Omega (1+|u_n|)^{(m(x)-1)s(x)}\ln(1+|u_n|)|D_iu_n|^{p_i(x)-1}\,dx.
\end{aligned}$$

Using that $|u_n|^{s(x)}\ge \min\{1;2^{1-s^+}\}(1+|u_n|)^{s(x)}-1$, Proposition 2.1, and Young inequality, we have

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+1-((m(x)-1)s(x))}}\,dx+\frac12\int_\Omega (1+|u_n|)^{m(x)s(x)}\,dx\\
&\le C_2+C_3\max\big(\|f\|_{L^{m(\cdot)}(\Omega)}^{m^+},\|f\|_{L^{m(\cdot)}(\Omega)}^{m^-}\big)\\
&\quad+C_4\sum_{i=1}^N\int_\Omega (1+|u_n|)^{(m(x)-1)s(x)}\ln(1+|u_n|)|D_iu_n|^{p_i(x)-1}\,dx.
\end{aligned}\tag{5.18}$$

We can estimate the last term in (5.18) by applying Young's inequality

$$
\begin{aligned}
&\int_\Omega (1+|u_n|)^{(m(x)-1)s(x)} \ln(1+|u_n|)|D_iu_n|^{p_i(x)-1}\,dx \\
&= \int_\Omega (1+|u_n|)^{\frac{(m(x)-1)s(x)+(p_i(x)-1)(\gamma_i(x)+1)}{p_i(x)}} \ln(1+|u_n|) \\
&\quad\times \frac{|D_iu_n|^{p_i(x)-1}}{(1+|u_n|)^{\frac{\gamma_i(x)+1-(m(x)-1)s(x)}{p_i'(x)}}}\,dx \\
&\le C_5 \int_\Omega (1+|u_n|)^{(m(x)-1)s(x)+(p_i(x)-1)(\gamma_i(x)+1)} \left(\ln(1+|u_n|)\right)^{p_i(x)}\,dx \\
&\quad+ \frac{1}{4C_4}\int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+1-(m(x)-1)s(x)}}\,dx.
\end{aligned} \tag{5.19}
$$

We combine (5.18) and (5.19), we obtain

$$
\begin{aligned}
&\frac{3}{4}\sum_{i=1}^N \int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+1-(m(x)-1)s(x)}}\,dx + \frac{1}{2}\int_\Omega (1+|u_n|)^{m(x)s(x)}\,dx \\
&\le C_6 + C_7\sum_{i=1}^N \int_\Omega (1+|u_n|)^{(m(x)-1)s(x)+(p_i(x)-1)(\gamma_i(x)+1)} \\
&\quad\times \ln(1+|u_n|)^{p_i(x)}\,dx = I.
\end{aligned} \tag{5.20}
$$

Since $s(x) > (p_i(x)-1)(\gamma_i(x)+1)$, we have

$$
(p_i(x)-1)(\gamma_i(x)+1) - s(x) \le ((p_i(x)-1)(\gamma_i(x)+1)-s(x))^+ = b_i < \frac{b_i}{2} < 0,
$$

and $(1+|t|)^{(p_i(x)-1)(\gamma_i(x)+1)-s(x)-\frac{b_i}{2}} \ln(1+|t|)^{p_i(x)}$ is bounded for all $x \in \overline{\Omega}$ and $t \in \mathbb{R}$. We conclude that

$$
\begin{aligned}
&(1+|u_n|)^{(m(x)-1)s(x)+(p_i(x)-1)(\gamma_i(x)+1)} \ln(1+|u_n|)^{p_i(x)} = (1+|u_n|)^{m(x)s(x)+\frac{b_i}{2}} \\
&\times (1+|u_n|)^{(p_i(x)-1)(\gamma_i(x)+1)-s(x)-\frac{b_i}{2}} \ln(1+|u_n|)^{p_i(x)} \\
&\le C(1+|u_n|)^{m(x)s(x)+\frac{b_i}{2}}.
\end{aligned}
$$

By another application of Young inequality, we obtain

$$
I \le \frac{1}{8}\int_\Omega (1+|u_n|)^{m(x)s(x)}\,dx + C_8. \tag{5.21}
$$

Therefore, by (5.20) and (5.21) we obtain the estimation (5.17). □

Lemma 5.4. *Let p_i, s and γ_i be restricted as in Theorem 3.3. Then, there exists a positive constant C such that*

$$
\|D_iu_n\|_{L^{q_i(\cdot)}(\Omega)} \le C, \tag{5.22}
$$

for all continuous functions q_i on $\overline{\Omega}$ satisfying (3.6).

Proof. Note that, assumption (3.6) implies that $q_i(x) < p_i(x)$ for all $x \in \overline{\Omega}$, $i = 1,\dots,N$. We can write

$$
\int_\Omega |D_iu_n|^{q_i(x)}\,dx = \int_\Omega \frac{|D_iu_n|^{q_i(x)}}{(1+|u_n|)^{\frac{q_i(x)}{p_i(x)}(\gamma_i(x)+\lambda)}} (1+|u_n|)^{\frac{q_i(x)}{p_i(x)}(\gamma_i(x)+\lambda)}\,dx.
$$

Using Young inequality and (5.1), we obtain

$$\int_\Omega |D_i u_n|^{q_i(x)}\,dx \le C_1 + C_2\int_\Omega (1+|u_n|)^{\frac{q_i(x)(\gamma_i(x)+\lambda)}{p_i(x)-q_i(x)}}\,dx. \tag{5.23}$$

Then assumption (3.6) implies $\frac{s(x)(p_i(x)-q_i(x))}{q_i(x)} - \gamma_i(x) > 1$. Choosing

$$\lambda = \min_{1\le i\le N}\min_{x\in\overline{\Omega}}\Big(\frac{s(x)(p_i(x)-q_i(x))}{q_i(x)} - \gamma_i(x)\Big) > 1\,.$$

Thanks to the choice of λ and (3.6), we have

$$\frac{q_i(x)(\gamma_i(x)+\lambda)}{p_i(x)-q_i(x)} \le s(x), \quad \forall x\in\overline{\Omega},\ \forall i=1,\dots,N. \tag{5.24}$$

Combining (5.24), (5.23), and (4.3) results (5.22). $\square$

Lemma 5.5. *Let m, s, p_i, and γ_i be restricted as in Theorem 3.5. Then, there exists a constant $C>0$ such that*

$$\|D_i u_n\|_{L^{q_i(\cdot)}(\Omega)} \le C, \tag{5.25}$$

for all continuous functions q_i on $\overline{\Omega}$ satisfying (3.10).

Proof. Note that $s(x) < \frac{1+\gamma_+(x)}{m(x)-1}$ and (3.10) imply $q_i(x) < p_i(x)$. Then by Young's inequality, we have

$$\begin{aligned}
&\int_\Omega |D_i u_n|^{q_i(x)}\,dx\\
&= \int_\Omega \frac{|D_i u_n|^{q_i(x)}}{(1+|u_n|)^{(\gamma_i(x)+1-[(m(x)-1)s(x)])\frac{q_i(x)}{p_i(x)}}}(1+|u_n|)^{(\gamma_i(x)+1-[(m(x)-1)s(x)])\frac{q_i(x)}{p_i(x)}}\,dx\\
&\le \int_\Omega \Big(\frac{q_i(x)}{p_i(x)}\Big)\frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+1-[(m(x)-1)s(x)]}}\,dx\\
&\quad + \int_\Omega \Big(1-\frac{q_i(x)}{p_i(x)}\Big)(1+|u_n|)^{\frac{(\gamma_i(x)+1-[(m(x)-1)s(x)])q_i(x)}{p_i(x)-q_i(x)}}\,dx,
\end{aligned}$$

and by (3.10), we obtain

$$\begin{aligned}
&\int_\Omega |D_i u_n|^{q_i(x)}\,dx\\
&\le C_1\int_\Omega \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)+1-[(m(x)-1)s(x)]}}\,dx + C_2\int_\Omega (1+|u_n|)^{m(x)s(x)}\,dx.
\end{aligned} \tag{5.26}$$

Therefore, (5.26) and (5.17) imply the desired result. $\square$

Lemma 5.6. *Let m, s, p_i, and γ_i be restricted as in Theorem 3.4. Then, there exists a constant $C>0$ such that*

$$\sum_{i=1}^N \int_\Omega |D_i u_n|^{p_i(x)}\,dx + \int_\Omega |u_n|^{s(x)+1+\gamma_+(x)}\,dx \le C. \tag{5.27}$$

Proof. Taking $\psi(x,u_n) = \big((1+|u_n|)^{1+\gamma_+(x)} - 1\big)\operatorname{sign}(u_n)$ in (4.1) as a test function, by (1.3) and the fact that for a.e. $x\in\Omega$, for all $i=1,\dots,N$,

$$\begin{aligned}
D_i\psi(x,u_n) &= (1+\gamma_+(x))(1+|u_n|)^{\gamma_+(x)}D_i u_n\\
&\quad + D_i\gamma_+(x)(1+|u_n|)^{1+\gamma_+(x)}\ln(1+|u_n|)\operatorname{sign}(u_n),
\end{aligned}$$

we set $\gamma_+^- = \max_{1\le i\le N} \min_{x\in\overline{\Omega}} \gamma_i(x)$, we obtain

$$
\begin{aligned}
&\alpha(1+\gamma_+^-)\sum_{i=1}^{N}\int_\Omega |D_iu_n|^{p_i(x)}\,dx + \int_\Omega |u_n|^{s(x)}\Big((1+|u_n|)^{1+\gamma_+(x)}-1\Big)\,dx\\
&\le \int_\Omega |f|\Big((1+|u_n|)^{1+\gamma_+(x)}-1\Big)\,dx\\
&\quad + C_1\sum_{i=1}^{N}\int_\Omega (1+|u_n|)^{1+\gamma_+(x)}\ln(1+|u_n|)|D_iu_n|^{p_i(x)-1}\,dx.
\end{aligned}
$$

Using that $|u_n|^{s(x)} \ge \min\{1, 2^{1-s^+}\}(1+|u_n|)^{s(x)}-1$, Proposition 2.1 and Young's inequality and since $m'(\cdot)(1+\gamma_+(x)) \le s(x)+1+\gamma_+(x)$, we have

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega |D_iu_n|^{p_i(x)} + \frac{1}{2}\int_\Omega (1+|u_n|)^{s(x)+1+\gamma_+(x)}\,dx\\
&\le C_2 + C_3\max\Big(\|f\|_{L^{m(\cdot)}(\Omega)}^{m^+}, \|f\|_{L^{m(\cdot)}(\Omega)}^{m^-}\Big)\\
&\quad + C_4\sum_{i=1}^{N}\int_\Omega (1+|u_n|)^{1+\gamma_+(x)}\ln(1+|u_n|)|D_iu_n|^{p_i(x)-1}\,dx.
\end{aligned}
\tag{5.28}
$$

We can estimate the last term in (5.28) by applying Young's inequality,

$$
\begin{aligned}
&\int_\Omega (1+|u_n|)^{1+\gamma_+(x)}\ln(1+|u_n|)|D_iu_n|^{p_i(x)-1}\,dx\\
&\le C_5\int_\Omega (1+|u_n|)^{p_i(x)(1+\gamma_+(x))}\,(\ln(1+|u_n|))^{p_i(x)} + \frac{1}{2C_4}\int_\Omega |D_iu_n|^{p_i(x)}\,dx.
\end{aligned}
\tag{5.29}
$$

We combine (5.28) and (5.29) to obtain

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega |D_iu_n|^{p_i(x)}\,dx + \int_\Omega (1+|u_n|)^{s(x)+1+\gamma_+(x)}\,dx\\
&\le C_6 + C_7\sum_{i=1}^{N}\int_\Omega (1+|u_n|)^{p_i(x)(\gamma_i(x)+1)}\ln(1+|u_n|)^{p_i(x)}\,dx = J.
\end{aligned}
\tag{5.30}
$$

Thanks to Remark 3.7 we have $s(x) > (p_i(x)-1)(\gamma_+(x)+1)$, so

$$
(p_i(x)-1)(\gamma_+(x)+1)-s(x) \le ((p_i(x)-1)(\gamma_+(x)+1)-s(x))^+ = d_i < \frac{d_i}{2} < 0,
$$

and $(1+|t|)^{(p_i(x)-1)(\gamma_+(x)+1)-s(x)-\frac{d_i}{2}}\ln(1+|t|)^{p_i(x)}$ is bounded for all $x\in\overline{\Omega}$ and $t\in\mathbb{R}$. We write

$$
\begin{aligned}
&(1+|u_n|)^{p_i(x)(\gamma_+(x)+1)}\ln(1+|u_n|)^{p_i(x)}\\
&= (1+|u_n|)^{s(x)+\gamma_+(x)+1+\frac{d_i}{2}}(1+|u_n|)^{(\gamma_i(x)+1)(p_i(x)-1)-s(x)-\frac{d_i}{2}}\ln(1+|u_n|)^{p_i(x)}.
\end{aligned}
$$

By another application of Young's inequality, we obtain

$$
J \le \frac{1}{4}\int_\Omega (1+|u_n|)^{s(x)+1+\gamma_+(x)}\,dx + C_8. \tag{5.31}
$$

Using (5.30) and (5.31), we obtain (5.27). $\square$

Lemma 5.7. *Let $f_n \in L^\infty(\Omega)$ be a sequence of functions which is strongly convergent to some f in $L^1(\Omega)$ and let u_n be a solution of the problem*

$$\begin{gathered} -\sum_{i=1}^N D_i\Big(a_i(x,T_n(u_n))|D_iu_n|^{p_i(x)-2}D_iu_n\Big) = f_n \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{5.32}$$

Suppose that:

(i) *u_n is such that $T_k(u_n) \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ for all $k > 0$.*
(ii) *u_n converges almost everywhere in Ω to some measurable function u which is finite almost everywhere, and such that $T_k(u) \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ for all $k > 0$ (note that (i) and (ii) imply that $T_k(u_n)$ weakly converges to $T_k(u)$ in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$).*
(iii) *u_n is bounded in $M^{r_1}(\Omega)$ for some $r_1 > 0$ and $u \in M^{r_1}(\Omega)$.*
(iv) *There exists $\theta_i > 0$, $i = 1,\dots,N$ such that $|D_iu_n|^{\theta_i}$ is bounded in $L^{r_2}(\Omega)$, for some $r_2 > 1$ and $|D_iu|^{\theta_i} \in L^{r_2}(\Omega)$.*

Then, up to a subsequence, D_iu_n converges to D_iu almost everywhere in Ω for all $i = 1,\dots,N$.

Proof. It has been proved in [12] that there exists a solution $u_n \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ to problem (5.32). We follow the technique in [1, 20] with some modifications, since our method depends on the anisotropic variable exponent. Define the vector-valued function $\hat{a}(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, where $\hat{a}(x,s,\xi) = \{\hat{a}_i(x,s,\xi)\}_{i=1,\dots,N}$ with $\hat{a}_i(x,s,\xi) = a_i(x,s)|\xi_i|^{p_i(x)-2}\xi_i$. Let θ be a real number between 0 and 1, which will be chosen later, and

$$I(n) = \int_\Omega \big\{\big(\hat{a}(x,T_n(u_n),\nabla u_n) - \hat{a}(x,T_n(u_n),\nabla u)\big)\nabla(u_n-u)\big\}^\theta\,dx.$$

Note that $I(n)$ is well defined and $I(n) \ge 0$. We fixe $k > 0$ and split the integral $I(n)$ on the sets $\{|u| \ge k\}$ and $\{|u| < k\}$, obtaining

$$\begin{aligned} I_1(n,k) &= \int_{\{|u|\ge k\}} \big\{\big(\hat{a}(x,T_n(u_n),\nabla u_n) - \hat{a}(x,T_n(u_n),\nabla u)\big)\nabla(u_n-u)\big\}^\theta\,dx, \\ I_2(n,k) &= \int_{\{|u|< k\}} \big\{\big(\hat{a}(x,T_n(u_n),\nabla u_n) - \hat{a}(x,T_n(u_n),\nabla u)\big)\nabla(u_n-u)\big\}^\theta\,dx. \end{aligned}$$

By condition (1.3) and Young's inequality, we have

$$\begin{aligned} I_1(n,k) &\le C_1 \int_{\{|u|\ge k\}} \Big\{\sum_{i=1}^N \Big(|D_iu_n|^{p_i(x)} + |D_iu|^{p_i(x)}\Big)\Big\}^\theta\,dx, \\ &\le C_1 \sum_{i=1}^N \int_{\{|u|\ge k\}} \Big(2 + |D_iu_n|^{\theta p_i^+} + |D_iu|^{\theta p_i^+}\Big)\,dx. \end{aligned}$$

We now choose $\theta < 1$ such that $\theta p_i^+ < \theta_i$, $i = 1,\dots,N$. Using the Hölder inequality and (iv), we obtain

$$\begin{aligned} I_1(n,k) &\le C_2 \sum_{i=1}^N \Big(\Big(\int_\Omega |D_iu_n|^{\theta_i r_2}\,dx\Big)^{\frac{1}{r_2}} + \Big(\int_\Omega |D_iu|^{\theta_i r_2}\,dx\Big)^{\frac{1}{r_2}}\Big)|\,\{|u| \ge k\}\,|^{1-\frac{1}{r_2}} \\ &\quad + C_2|\{|u| \ge k\}| \le C|\{|u| \ge k\}|^{1-\frac{1}{r_2}} + C_2|\,\{|u| \ge k\}\,|. \end{aligned}$$

By (ii), for any $k > 1$, we have

$$|\{|u| \geq k\}| \leq |\{|u| > k-1\}| \leq \frac{C}{(k-1)^{r_1}}.$$

Using the above inequality, we obtain

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} I_1(n,k) = 0. \tag{5.33}$$

As $\nabla u = \nabla T_k(u)$ on the set $\{|u| < k\}$, we obtain

$$I_2(n,k) = \int_{\{|u|<k\}} \big\{\big(\hat{a}(x,T_n(u_n),\nabla u_n) - \hat{a}(x,T_n(u_n),\nabla T_k(u))\big)\nabla(u_n - T_k(u))\big\}^\theta \,dx.$$

Take $h > k+1$ and split the integral $I_2(n,k)$ on the sets $\{|u_n - T_k(u)| \geq h\}$ and $\{|u_n - T_k(u)| < h\}$, obtaining

$$\begin{aligned} &I_3(n,k,h)\\ &= \int_{\{|u_n-T_k(u)|\geq h\}} \big\{\big(\hat{a}(x,T_n(u_n),\nabla u_n) - \hat{a}(x,T_n(u_n),\nabla T_k(u))\big)\nabla(u_n - T_k(u))\big\}^\theta \,dx, \end{aligned}$$

and

$$\begin{aligned} I_4(n,k,h) = \int_{\{|u_n-T_k(u)|<h\}} \Big\{\Big(&\hat{a}(x,T_n(u_n),\nabla u_n)\\ &- \hat{a}(x,T_n(u_n),\nabla T_k(u))\Big)\nabla(u_n - T_k(u))\Big\}^\theta \,dx. \end{aligned}$$

As $|u_n| \geq h-k$ on the set $\{|u_n - T_k(u)| \geq h\}$, we obtain

$$|\{|u_n - T_k(u)| \geq h\}| \leq |\{|u_n| \geq h-k\}| \leq \frac{C}{(h-k-1)^{r_1}}.$$

Similarly to the discussion of $I_1(n,k)$ (with the same choice of θ), we obtain

$$\lim_{h\to+\infty} \limsup_{k\to+\infty} \limsup_{n\to+\infty} I_3(n,k,h) = 0. \tag{5.34}$$

Since $\nabla(u_n - T_k(u)) = \nabla T_h(u_n - T_k(u))$ on the set $\{|u_n - T_k(u)| < h\}$, by Hölder inequality (with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$), we have

$$\begin{aligned} I_4(n,k,h) \leq |\Omega|^{1-\theta}\Big\{\int_{\{|u_n-T_k(u)|<h\}} \Big(&\hat{a}(x,T_n(u_n),\nabla u_n)\\ &- \hat{a}(x,T_n(u_n),\nabla T_k(u))\Big)\nabla T_h(u_n - T_k(u))\,dx\Big\}^\theta. \end{aligned}$$

Define

$$\begin{aligned} &I_5(n,k,h)\\ &= \int_{\{|u_n-T_k(u)|<h\}} \big(\hat{a}(x,T_n(u_n),\nabla u_n) - \hat{a}(x,T_n(u_n),\nabla T_k(u))\big)\nabla T_h(u_n - T_k(u))\,dx, \end{aligned}$$

which we split as the difference $I_6 - I_7$, where

$$\begin{gathered} I_6(n,k,h) = \int_{\{|u_n-T_k(u)|<h\}} \hat{a}(x,T_n(u_n),\nabla u_n)\nabla T_h(u_n - T_k(u))\,dx,\\ I_7(n,k,h) = \int_{\{|u_n-T_k(u)|<h\}} \hat{a}(x,T_n(u_n),\nabla T_k(u))\nabla T_h(u_n - T_k(u))\,dx. \end{gathered}$$

Take n sufficiently large such that $n > h + k$. Since $|u_n| \le k + h$ on the set where $\{|u_n - T_k(u)| \le h\}$, we obtain

$$I_7(n,k,h) = \int_\Omega \hat{a}(x, T_{h+k}(u_n), \nabla T_k(u)) \nabla T_h(u_n - T_k(u))\, dx.$$

According to condition (1.3), we have

$$|\hat{a}_i(x, T_{h+k}(u_n), \nabla T_k(u))| \le \beta |D_i T_k(u))|^{p_i(x)-1}, \ \forall i = 1, \dots, N.$$

Note that

$$\hat{a}_i(x, T_{h+k}(u_n), \nabla T_k(u)) \to \hat{a}_i(x, u, \nabla T_k(u)) \quad \text{a.e. in } \Omega,\ i = 1, \dots, N,$$

using Lebesgue dominated convergence theorem, we derive

$$\hat{a}_i(x, T_{h+k}(u_n), \nabla T_k(u)) \to \hat{a}_i(x, u, \nabla T_k(u)) \quad \text{strongly in } L^{p_i'(\cdot)}(\Omega), \forall i = 1, \dots, N.$$

Using the weak convergence of $D_i T_h(u_n - T_k(u))$ to $D_i T_h(u - T_k(u))$ in $L^{p_i(\cdot)}(\Omega)$, $i = 1, \dots, N$ (a consequence of (i) and (ii)), we find

$$\lim_{n\to+\infty} I_7(n,k,h) = \int_\Omega \sum_{i=1}^N (\hat{a}_i(x, u, \nabla T_k(u)) D_i T_h(u - T_k(u)))\, dx,$$

so

$$\lim_{k\to+\infty} \lim_{n\to+\infty} I_7(n,k,h) = 0. \tag{5.35}$$

For $I_6(n,k,h)$, by (5.32) we obtain

$$I_6(n,k,h) = \int_\Omega f_n T_h(u_n - T_k(u))\, dx,$$

by the strong convergence of f_n in $L^1(\Omega)$, we have

$$\lim_{k\to+\infty} \lim_{n\to+\infty} I_6(n,k,h) = 0 \tag{5.36}$$

Putting together (5.33), (5.34), (5.35), and (5.36), one thus has

$$\lim_{n\to+\infty} I(n) = 0.$$

As in [20], we obtain $D_i u_n \to D_i u$ a.e. in Ω, $i = 1, \dots, N$. $\square$

Lemma 5.8. *Let u_n be a solution to the equation* (4.1)*, suppose that u_n converges to u almost everywhere in Ω. Then*

$$|u_n|^{s(x)-1} u_n \to |u|^{s(x)-1} u \ \text{in } L^1(\Omega).$$

Proof. Let $\rho_j(t)$ be an increasing, uniformly bounded Lipschitz function such that $\rho_j \to \chi_{\{|t|>k\}} \operatorname{sign}(t)$ $(k > 0)$, as $j \to +\infty$. Taking $\rho_j(u_n)$ as a test function in (4.2), we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega \rho_j'(u_n) a_i(x, T_n(u_n)) |D_i u_n|^{p_i(x)}\, dx + \int_\Omega |u_n|^{s(x)-1} u_n \rho_j(u_n)\, dx \\ &= \int_\Omega T_n(f) \rho_j(u_n)\, dx. \end{aligned}$$

As $j \to +\infty$, we obtain

$$\int_{\{|u_n|>k\}} |u_n|^{s(x)}\, dx \le \int_{\{|u_n|>k\}} |f|\, dx. \tag{5.37}$$

Let $E \subset \Omega$ be any measurable set, using (5.37). We have

$$\begin{aligned}\int_E |u_n|^{s(x)}\,dx &= \int_{E\cap\{|u_n|\le k\}} |u_n|^{s(x)}\,dx + \int_{E\cap\{|u_n|>k\}} |u_n|^{s(x)}\,dx \\ &\le (k^{s^+} + k^{s^-})\operatorname{meas}(E) + \int_{E\cap\{|u_n|>k\}} |f|\,dx.\end{aligned}$$

Then we deduce that $(|u_n|^{s(x)-1}u_n)$ is equi-integrable in $L^1(\Omega)$, and since $u_n \to u$ a.e. in Ω, then Vitali's theorem implies

$$|u_n|^{s(x)-1}u_n \to |u|^{s(x)-1}u \quad \text{in } L^1(\Omega). \tag{5.38}$$

□

6. Proof of main results

In this section, using the uniform estimates of Section 4, we prove Theorem 3.2, 3.3, 3.4 and 3.5.

6.1. **Proof of theorems 3.2, 3.3.** By Lemma 5.2 the sequence (u_n) is bounded in $\mathring{W}^{1,\vec{q}(\cdot)}(\Omega)$ where $q_i(\cdot)$ is defined as (3.4). Without loss of generality, we can therefore assume that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\vec{q}(\cdot)}(\Omega), \\ u_n \to u \quad \text{strongly in } L^{q_0}(\Omega), \quad q_0 = \min_{1\le i\le N}\min_{x\in\overline{\Omega}} q_i(x), \\ u_n \to u \quad \text{a.e. in} \quad \Omega. \end{gathered} \tag{6.1}$$

It follows from (4.3) and Fatou's lemma that

$$\int_\Omega |u|^{s(x)}\,dx \le \liminf_{n\to+\infty}\int_\Omega |u_n|^{s(x)}\,dx \le C,$$

thus $|u|^{s(x)} \in L^1(\Omega)$, furthermore, $u \in \mathcal{M}^{s^-}(\Omega)$. Then, there exists $r_1 = s^- > 0$ such that

$$\|u_n\|_{M^{r_1}(\Omega)} \le C \quad \text{and} \quad u \in M^{r_1}(\Omega). \tag{6.2}$$

Let $f_n = T_n(f) - T_n(|u_n|^{s(x)-1}u_n) \in L^\infty(\Omega)$, where u_n is a solution of (5.32). Then, from (5.2), (5.4), (6.2), (6.1), and lemma 5.7 we can deduce that

$$D_iu_n \to D_iu \quad \text{a.e. in } \Omega, \text{for all } i = 1,\dots,N.$$

So, by (5.4), we have

$$|D_iu_n|^{p_i(x)-2}D_iu_n \rightharpoonup |D_iu|^{p_i(x)-2}D_iu \quad \text{weakly in } L^{\frac{q_i(\cdot)}{p_i(\cdot)-1}}(\Omega),\ \forall i = 1,\dots,N, \tag{6.3}$$

where q_i is defined as in (3.4). The choice of $\frac{q_i(\cdot)}{p_i(\cdot)-1} > 1$ is possible since we have (3.3). From (1.3) and (6.1), we obtain

$$a_i(x, T_n(u_n)) \to a_i(x,u) \quad \text{weak}^* \text{ in } L^\infty(\Omega). \tag{6.4}$$

For any given $\varphi \in C_0^\infty(\Omega)$, using φ as a test function in (4.1), we have

$$\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n))|D_iu_n|^{p_i(x)-2}D_iu_nD_i\varphi\,dx + \int_\Omega |u_n|^{s(x)-1}u_n\varphi\,dx = \int_\Omega T_n(f)\varphi\,dx, \tag{6.5}$$

Letting $n \to +\infty$ in (6.5), by (6.3), (6.4), and (5.38), we obtain

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u)|D_i u|^{p_i(x)-2} D_i u D_i \varphi \, dx + \int_\Omega |u|^{s(x)-1} u \varphi \, dx = \int_\Omega f \varphi \, dx.$$

For the proof of Theorem 3.3, we only replace (6.3) with

$$|D_i u_n|^{p_i(x)-2} D_i u_n \rightharpoonup |D_i u|^{p_i(x)-2} D_i u \quad \text{weakly in } L^{\frac{q_i(\cdot)}{p_i(\cdot)-1}}(\Omega), \ \forall i = 1, \dots, N.$$

where q_i is defined as in (3.6). The choice of $\frac{q_i(\cdot)}{p_i(\cdot)-1} > 1$ is possible since we have (3.5).

6.2. **Proof of theorem 3.4, 3.5.** Because the proof of Theorem 3.5 is similar to that of Theorem 3.2, here we only give the proof of Theorem 3.4. According to Lemma 5.6, the sequence (u_n) is bounded in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$. This implies that we can extract a subsequence (denote again by (u_n)), such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega), \\ u_n \to u \quad \text{strongly in } L^{p_0}(\Omega), \quad p_0 = \min_{1 \le i \le N} \min_{x \in \overline{\Omega}} p_i(x), \\ u_n \to u \quad \text{a.e. in } \Omega. \end{gathered}$$

Arguing as the proof of Theorem 3.2, by using (5.27), we conclude that

$$|D_i u_n|^{p_i(x)-2} D_i u_n \rightharpoonup |D_i u|^{p_i(x)-2} D_i u \quad \text{weakly in } L^{p_i'(\cdot)}(\Omega), \ \forall i = 1, \dots, N.$$

The proof of Theorem 3.4 is complete.

Remark 6.1. All the results in this work also hold if our problem is exchanged by a more general one,

$$\begin{gathered} -\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $a(x,t,\xi) = \{a_i(x,t,\xi)\}_{i=1,\dots,N} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-valued function such that for a.e. $x \in \Omega$ and for every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$, the following assumptions hold:

$$\begin{gathered} a(x,t,\xi)\xi \ge \alpha \sum_{i=1}^{N} \frac{|\xi_i|^{p_i(x)}}{(1+|t|)^{\gamma_i(x)}}, \quad \alpha > 0, \\ |a_i(x,t,\xi)| \le \beta \Big(1 + \sum_{j=1}^{N} |\xi_j|^{p_j(x)}\Big)^{1-\frac{1}{p_i(x)}}, \quad i = 1, \dots, N, \ \beta > 0, \\ (a(x,s,\xi) - a(x,s,\xi'))(\xi - \xi') > 0, \quad \forall \xi \ne \xi'. \end{gathered}$$

Assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$\begin{gathered} \sup_{|t| \le k} |g(x,t)| = h_k(x) \in L^1(\Omega), \quad \forall k > 0, \\ g(x,t)\operatorname{sign}(t) \ge |t|^{s(x)}. \end{gathered}$$

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