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L_∞-ESTIMATE FOR THE ROBIN PROBLEM OF A SINGULAR VARIABLE p-LAPLACIAN EQUATION IN A CONICAL DOMAIN

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Abstract. We establish a bound for the modulus of the weak bounded solution to the Robin problem for an elliptic quasi-linear second-order equation with the variable $p(x)$-Laplacian.

1. Introduction

The aim of our article is to obtain an estimate for the modulus of weak bounded solutions to the Robin problem for quasi-linear elliptic second-order equations with the variable $p(x)$-Laplacian in a neighborhood of an angular or conical boundary point in a bounded domain. The Robin boundary conditions are related to Sturm-Liouville problems which are used in many contexts in science and engineering. For example, in electromagnetic problems, in heat transfer problems and for convection-diffusion equations (Fick's law of diffusion); a well as to study of reflected shocks in transonic flows.

Let $G \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with the boundary Γ. We suppose that Γ is a smooth surface everywhere except at the origin $\mathcal{O} \in \Gamma$, and near the point $\mathcal{O}$ it is a conical surface whose vertex is $\mathcal{O}$.

We consider the Robin problem

$$\begin{gathered} -\triangle_{p(x)} u + a_0(x) u|u|^{p(x)-1} + b(u, \nabla u) = f(x), \quad x \in G, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \overrightarrow{n}} + \frac{\gamma}{|x|^{p(x)-1}} u|u|^{p(x)-2} = g(x), \quad x \in \Gamma, \end{gathered} \tag{1.1}$$

where

$$\triangle_{p(x)} u \equiv \operatorname{div}\big(|\nabla u|^{p(x)-2} \nabla u\big). \tag{1.2}$$

We require that the following assumptions hold:

(i) $p(x) \in C^{(0)}(\overline{G})$ and $1 < p_- \leq p(x) \leq p_+ = p(0) < n$, $a_0(x) \geq a_0$, $a_0 = \text{const} > 0$ for all $x \in \overline{G}$, $\gamma = \text{const} > 0$;

(ii) the function $b(u, \xi)$ satisfies in $\mathfrak{M} = \mathbb{R} \times \mathbb{R}^n$ the inequality
$$|b(u, \xi)| \leq \mu |u|^{-1} |\xi|^{p(x)}, \quad 0 \leq \mu < 1, \ \forall x \in \overline{G};$$

(iii)
$$|f(x)| \leq f_0 |x|^{\beta(x)}, \quad \beta(x) \geq \beta_0 - \frac{n}{s}, \quad s > \frac{n}{p_-}, \quad f_0 \geq 0, \quad \beta_0 > 0, \quad \forall x \in \overline{G};$$

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$$|g(x)| \le g_0 |x|^{1-p(x)}, \quad g_0 \ge 0, \quad \forall x \in \Gamma.$$

The L_∞-regularity of weak solutions for quai-linear equations with $p(x)$-Laplacian was studied as follows:

- in [1] for $b(u,\xi) \equiv 0$ (the Dirichlet problem),
- in [2, 3] for $b(u,\xi)$ not depending on ξ (the Dirichlet and the Robin problems),
- in [8] for
$$|b(u,\xi)| \le c_1 |\xi|^{\alpha(x)} + c_2 |u|^{r(x)-1} + c_3,$$
$$\alpha(x) = \frac{r(x)-1}{r(x)} p(x), \quad p(x) \le r(x) < p^*(x),$$
where $p^*(x)$ is the Sobolev embedding exponent of $p(x)$ (the Dirichlet problem).

We define the functions class

$$\mathfrak{N}^{1,p(x)}_{-1,\infty}(G) = \Big\{ u(x) \in L_\infty(G) : \int_G \langle |x|^{-p(x)} |u|^{p(x)} + |u|^{-1} |\nabla u|^{p(x)} \} \rangle \, dx < \infty \Big\}.$$

It is obvious that $\mathfrak{N}^{1,p(x)}_{-1,\infty}(G) \subset W^{1,p(x)}(G)$.

Remark 1.1. If $p(x) > n$, by the Sobolev imbedding theorem, we have $u \in C^{1-\frac{n}{p(0)}}(G)$ (see [7]). Therefore we investigate only $p(x) \in (1,n)$ (see assumption (i)).

Definition 1.2. A function u is called a weak bounded solution of problem (1.1) provided that $u(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G)$ and u satisfies the integral identity

$$\begin{aligned} Q(u,\eta) :\equiv & \int_G \langle |\nabla u|^{p(x)-2} u_{x_i} \eta_{x_i} + a_0(x) u |u|^{p(x)-1} \eta(x) + b(u, \nabla u) \eta(x) \rangle \, dx \\ & + \gamma \int_\Gamma r^{1-p(x)} u |u|^{p(x)-2} \eta(x) ds \\ & = \int_\Gamma g(x) \eta(x) ds + \int_G f(x) \eta(x) \, dx. \end{aligned} \tag{1.3}$$

for all $\eta(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G)$.

Remark 1.3. It is easy to verify that the assumptions (i)–(iii) guarantee the existence of integrals over G and Γ. Therefore, $Q(u,\eta)$ well defined.

First we formulate well known lemmas.

Lemma 1.4 (see [6, Lemma 2.1] and [5, Lemma 1.60]). *Let us consider the function*

$$\eta(x) = \begin{cases} e^{\varkappa x} - 1, & x \ge 0, \\ -e^{-\varkappa x} + 1, & x \le 0, \end{cases}$$

where $\varkappa > 0$. Let a, b be positive constants, $m > 1$. If $\varkappa > (2b/a) + m$, then we have

$$a\eta'(x) - b\eta(x) \ge \frac{a}{2} e^{\varkappa x}, \quad \forall x \ge 0, \tag{1.4}$$

$$\eta(x) \ge [\eta(\frac{x}{m})]^m, \quad \forall x \ge 0. \tag{1.5}$$

Moreover, there exist a $d \ge 0$ *and an* $M > 0$ *such that*

$$\eta(x) \le M\big[\eta(\frac{x}{m})\big]^m \quad \text{and} \quad \eta'(x) \le M[\eta(\frac{x}{m})]^m, \quad \forall x \ge d; \tag{1.6}$$

$$|\eta(x)| \ge x, \quad \forall x \in \mathbb{R}. \tag{1.7}$$

Next we have Stampacchia's Lemma, see [9, Lemma 3.11] and [10].

Lemma 1.5. *Let* $\varphi : [k_0, \infty) \to \mathbb{R}$ *be a non-negative and non-increasing function which satisfies*

$$\varphi(l) \le \frac{C}{(l-k)^\alpha}[\varphi(k)]^\beta \quad \text{for } l > k > k_0, \tag{1.8}$$

where C, α, β *are positive constants with* $\beta > 1$. *Then*

$$\varphi(k_0 + \delta) = 0, \quad \text{where} \quad \delta^\alpha = C|\varphi(k_0)|^{\beta-1}2^{\alpha\beta/(\beta-1)}.$$

Our main result is the following.

Theorem 1.6. *Let* $u(x)$ *be a weak solution of* (1.1). *If assumptions* (i)–(iii) *hold, then there exists a constant* $M_0 > 0$ *depending only on* $\operatorname{meas} G$, n, $p_\pm$, s, μ, f_0, g_0, a_0, β_0, γ *and such that* $\|u\|_{L_\infty(G)} \le M_0$.

Proof. Let us define the set $A(k) = \{x \in \overline{G} : |u(x)| > k\}$ and let $\chi_{A(k)}$ be the characteristic function of the set $A(k)$. We observe that $A(k+d) \subseteq A(k)$ for all $d > 0$.

Putting $\eta((|u| - k)_+)\chi_{A(k)} \operatorname{sign} u$ as the test function in (1.3), where η is defined by Lemma 1.4 and $k \ge k_0$ (without loss of generality we can assume $k_0 \ge 1$), we obtain the inequality

$$\begin{aligned}
&\int_{A(k)} \Big\{|\nabla u|^{p(x)}\eta'((|u|-k)_+) + \langle a_0(x)|u|^{p(x)} \\
&+ b(u,\nabla u)\operatorname{sign} u\rangle \eta((|u|-k)_+)\Big\}\,dx + \gamma\int_{\Gamma\cap A(k)} \big(\frac{|u|}{r}\big)^{p(x)-1}\eta((|u|-k)_+)ds \\
&\le \int_{A(k)} |f(x)|\eta((|u|-k)_+)\,dx + \int_{\Gamma\cap A(k)} |g(x)|\eta((|u|-k)_+)\,ds.
\end{aligned} \tag{1.9}$$

By assumptions (i) and (iii), the inequality (1.9) implies that

$$\begin{aligned}
&\int_{A(k)} \Big\{|\nabla u|^{p(x)}\langle \eta'((|u|-k)_+) - \mu k_0^{-1}\eta((|u|-k)_+)\rangle \\
&+ a_0|u|^{p(x)}\eta((|u|-k)_+)\Big\}\,dx \\
&+ \int_{\Gamma\cap A(k)} \big(\gamma|u|^{p(x)-1} - g_0\big)r^{1-p(x)}\eta((|u|-k)_+)ds \\
&\le \int_{A(k)} |f(x)|\eta((|u|-k)_+)\,dx.
\end{aligned} \tag{1.10}$$

On the other hand, by assumption (i) and the definition of $A(k)$, we have

$$|u|^{p(x)} \ge k_0^{p_-}. \tag{1.11}$$

Therefore, the inequality (1.10) can be rewritten as

$$\begin{aligned}
&\int_{A(k)} \Big\{|\nabla u|^{p(x)}\langle \eta'((|u|-k)_+) - \mu k_0^{-1}\eta((|u|-k)_+)\rangle \\
&+ a_0|u|^{p(x)}\eta((|u|-k)_+)\Big\}\,dx \\
&+ \int_{\Gamma\cap A(k)} \big(\gamma k_0^{p_- -1} - g_0\big) r^{1-p(x)}\eta((|u|-k)_+)\,ds \\
&\le \int_{A(k)} |f(x)|\eta((|u|-k)_+)\,dx.
\end{aligned} \tag{1.12}$$

We take

$$k_0 \ge \big(\frac{g_0}{\gamma}\big)^{\frac{1}{p_- -1}} \tag{1.13}$$

and obtain

$$\begin{aligned}
&\int_{A(k)} \Big\{|\nabla u|^{p(x)}\langle \eta'((|u|-k)_+) - \mu k_0^{-1}\eta((|u|-k)_+)\rangle \\
&+ a_0|u|^{p(x)}\eta((|u|-k)_+)\Big\}\,dx \\
&\le \int_{A(k)} |f(x)|\eta((|u|-k)_+)\,dx.
\end{aligned} \tag{1.14}$$

Additionally, let us define the sets

$$\begin{aligned}
A_-(k) &= A(k) \cap \{|\nabla u| \le 1\}, \\
A_+(k) &= A(k) \cap \{|\nabla u| \ge 1\}.
\end{aligned} \tag{1.15}$$

Then $A(k) = A_-(k) \cup A_+(k)$. Also we define the functions

$$v_k(x) := \eta\Big(\frac{(|u|-k)_+}{p_-}\Big), \quad w_k(x) := \eta\Big(\frac{(|u|-k)_+}{p_+}\Big). \tag{1.16}$$

We note that the inequalities

$$|\nabla u|^{p_+} \le |\nabla u|^{p(x)} \le |\nabla u|^{p_-} \quad \text{on } A_-(k); \tag{1.17}$$

$$|\nabla u|^{p_-} \le |\nabla u|^{p(x)} \le |\nabla u|^{p_+} \quad \text{on } A_+(k) \tag{1.18}$$

hold by (i).

Direct calculations give

$$\begin{gathered}
|\nabla v_k| = \frac{1}{p_-}|\nabla u|\eta'\Big(\frac{(|u|-k)_+}{p_-}\Big) = \frac{\varkappa}{p_-}|\nabla u| \exp\Big(\varkappa\frac{(|u|-k)_+}{p_-}\Big), \quad \varkappa > 0 \\
\Longrightarrow |\nabla v_k|^{p_-} = \big(\frac{\varkappa}{p_-}\big)^{p_-}|\nabla u|^{p_-}e^{\varkappa(|u|-k)_+},
\end{gathered} \tag{1.19}$$

where η is given in Lemma 1.4. Choosing $\varkappa > p_- + \frac{2\mu}{k_0}$ according to (1.4), we have

$$\eta'((|u|-k)_+) - \mu k_0^{-1}\eta((|u|-k)_+) \ge \frac{1}{2}e^{\varkappa(|u|-k)_+}. \tag{1.20}$$

From (1.19) and (1.20) it follows that

$$|\nabla u|^{p_-}\langle \eta'((|u|-k)_+) - \mu k_0^{-1}\eta((|u|-k)_+)\rangle \ge \frac{1}{2}\big(\frac{p_-}{\varkappa}\big)^{p_-}|\nabla v_k|^{p_-}$$

which by (1.18) implies

$$
\begin{aligned}
&\int_{A_+(k)} |\nabla u|^{p(x)} \langle \eta'((|u|-k)_+) - \mu k_0^{-1}\eta((|u|-k)_+)\rangle\,dx \\
&\geq \int_{A_+(k)} |\nabla u|^{p_-} \langle \eta'((|u|-k)_+) - \mu k_0^{-1}\eta((|u|-k)_+)\rangle\,dx \\
&\geq \frac{1}{2}\big(\frac{p_-}{\varkappa}\big)^{p_-}\int_{A_+(k)} |\nabla v_k|^{p_-}\,dx.
\end{aligned} \tag{1.21}
$$

Similarly, choosing $\varkappa > p_+ + \frac{2\mu}{k_0}$ and taking into account (1.17), we obtain

$$
\begin{aligned}
&\int_{A_-(k)} |\nabla u|^{p(x)} \langle \eta'((|u|-k)_+) - \mu k_0^{-1}\eta((|u|-k)_+)\rangle\,dx \\
&\geq \frac{1}{2}\big(\frac{p_+}{\varkappa}\big)^{p_+}\int_{A_-(k)} |\nabla w_k|^{p_+}\,dx.
\end{aligned} \tag{1.22}
$$

Since $p_+ \geq p_-$, inequalities (1.21) and (1.22) hold for $\varkappa > p_+ + \frac{2\mu}{k_0}$. Therefore, adding inequalities (1.21) and (1.22) we obtain

$$
\begin{aligned}
&\frac{1}{2}\big(\frac{p_-}{\varkappa}\big)^{p_-}\int_{A_+(k)} |\nabla v_k|^{p_-}\,dx + \frac{1}{2}\big(\frac{p_+}{\varkappa}\big)^{p_+}\int_{A_-(k)} |\nabla w_k|^{p_+}\,dx \\
&\leq \int_{A(k)} |\nabla u|^{p(x)} \langle \eta'((|u|-k)_+) - \mu k_0^{-1}\eta((|u|-k)_+)\rangle\,dx
\end{aligned} \tag{1.23}
$$

by (1.15). Finally, from (1.14) and (1.23) we derive

$$
\begin{aligned}
&\frac{1}{2}\big(\frac{p_-}{\varkappa}\big)^{p_-}\int_{A_+(k)} |\nabla v_k|^{p_-}\,dx + \frac{1}{2}\big(\frac{p_+}{\varkappa}\big)^{p_+}\int_{A_-(k)} |\nabla w_k|^{p_+}\,dx \\
&+ a_0\int_{A(k)} |u|^{p(x)}\eta((|u|-k)_+)\,dx \\
&\leq \int_{A(k)} |f(x)|\eta((|u|-k)_+)\,dx.
\end{aligned}
$$

Since $\int_{A(k)} = \int_{A_+(k)} + \int_{A_-(k)}$, by (1.15) we have

$$
\begin{aligned}
&\frac{1}{2}\big(\frac{p_-}{\varkappa}\big)^{p_-}\int_{A_+(k)} |\nabla v_k|^{p_-}\,dx + \frac{1}{2}\big(\frac{p_+}{\varkappa}\big)^{p_+}\int_{A_-(k)} |\nabla w_k|^{p_+}\,dx \\
&+ a_0\int_{A_+(k)} |u|^{p(x)}\eta((|u|-k)_+)\,dx + a_0\int_{A_-(k)} |u|^{p(x)}\eta((|u|-k)_+)\,dx \\
&\leq \int_{A_+(k)} |f(x)|\eta((|u|-k)_+)\,dx + \int_{A_-(k)} |f(x)|\eta((|u|-k)_+)\,dx.
\end{aligned} \tag{1.24}
$$

Now, by (1.5), (1.11) and (1.16), we derive

$$
\begin{aligned}
&a_0\int_{A_+(k)} |u|^{p(x)}\eta((|u|-k)_+)\,dx + a_0\int_{A_-(k)} |u|^{p(x)}\eta((|u|-k)_+)\,dx \\
&\geq a_0 k_0^{p_-}\Big(\int_{A_+(k)} v_k^{p_-}\,dx + \int_{A_-(k)} w_k^{p_+}\,dx\Big).
\end{aligned} \tag{1.25}
$$

From (1.24) and (1.25) it follows that

$$\begin{aligned} &\frac{1}{2}\Big(\frac{p_-}{\varkappa}\Big)^{p_-}\int_{A_+(k)}|\nabla v_k|^{p_-}\,dx+\frac{1}{2}\Big(\frac{p_+}{\varkappa}\Big)^{p_+}\int_{A_-(k)}|\nabla w_k|^{p_+}\,dx\\ &+a_0k_0^{p_-}\Big(\int_{A_+(k)}v_k^{p_-}\,dx+\int_{A_-(k)}w_k^{p_+}\,dx\Big)\\ &\le\int_{A_+(k)}|f(x)|\eta((|u|-k)_+)\,dx+\int_{A_-(k)}|f(x)|\eta((|u|-k)_+)\,dx. \end{aligned} \tag{1.26}$$

Next, we have

$$\begin{aligned} &\int_{A_\pm(k)}|f(x)|\eta((|u|-k)_+)\,dx\\ &=\int_{A_\pm(k+d)}|f(x)|\eta((|u|-k)_+)\,dx\\ &\quad+\int_{A_\pm(k)\setminus A_\pm(k+d)}|f(x)|\eta((|u|-k)_+)\,dx,\quad\forall d>0. \end{aligned} \tag{1.27}$$

By (1.6), we obtain

$$\eta((|u|-k)_+)\Big|_{A_\pm(k+d)}\le M\Big[\eta\Big(\frac{(|u|-k)_+}{p_\mp}\Big)\Big]^{p_\mp}.$$

Then (1.16) implies

$$\int_{A_+(k+d)}|f(x)|\eta((|u|-k)_+)\,dx\le M\int_{A_+(k+d)}|f(x)|v_k^{p_-}\,dx; \tag{1.28}$$

$$\int_{A_-(k+d)}|f(x)|\eta((|u|-k)_+)\,dx\le M\int_{A_-(k+d)}|f(x)|w_k^{p_+}\,dx. \tag{1.29}$$

Using the definition of η from Lemma 1.4, we arrive to

$$\eta((|u|-k)_+)\Bigg|_{A_\pm(k)\setminus A_\pm(k+d)}\le e^{\varkappa d},\quad\forall d>0$$

which implies

$$\int_{A_\pm(k)\setminus A_\pm(k+d)}|f(x)|\eta((|u|-k)_+)\,dx\le e^{\varkappa d}\int_{A_\pm(k)\setminus A_\pm(k+d)}|f(x)|\,dx, \tag{1.30}$$

for all $d>0$. Now, we recall [4, formula (6.3.9) page 145]:

$$\begin{aligned} &\int_{A_+(k+d)}|f(x)|v_k^{p_-}\,dx\\ &\le\varepsilon(1-\theta_-)\Big(\int_{A_+(k)}v_k^{p_-^\sharp}\,dx\Big)^{\frac{p_-}{p_-^\sharp}}+\theta_-\varepsilon^{\frac{\theta_--1}{\theta_-}}\|f\|_{L_s(G)}^{\frac{1}{\theta_-}}\int_{A_+(k)}v_k^{p_-}\,dx,\\ &\int_{A_-(k+d)}|f(x)|w_k^{p_+}\,dx\\ &\le\varepsilon(1-\theta_+)\Big(\int_{A_-(k)}w_k^{p_+^\sharp}\,dx\Big)^{\frac{p_+}{p_+^\sharp}}+\theta_+\varepsilon^{\frac{\theta_+-1}{\theta_+}}\|f\|_{L_s(G)}^{\frac{1}{\theta_+}}\int_{A_-(k)}w_k^{p_+}\,dx,\\ &\forall\varepsilon>0,\ p_\mp^\sharp=\frac{np_\mp}{n-p_\mp},\ \theta_\mp=1-\frac{n}{sp_\mp},s>\max\{\frac{n}{p_-},\ \frac{n}{p_+}\}=\frac{n}{p_-}>1. \end{aligned} \tag{1.31}$$

Then applying (1.31) to (1.27)–(1.30), we obtain

$$
\begin{aligned}
&\int_{A_+(k)} |f(x)|\eta((|u|-k)_+)\,dx \\
&\le M\varepsilon(1-\theta_-)\Big(\int_{A_+(k)} v_k^{p_-^\sharp}\,dx\Big)^{\frac{p_-}{p_-^\sharp}} + e^{\varkappa d}\int_{A_+(k)}|f(x)|\,dx \\
&\quad + M\theta_-\varepsilon^{\frac{\theta_- -1}{\theta_-}}\|f\|_{L_s(G)}^{\frac{1}{\theta_-}}\int_{A_+(k)} v_k^{p_-}\,dx \\
&\int_{A_-(k)} |f(x)|\eta((|u|-k)_+)\,dx \\
&\le M\varepsilon(1-\theta_+)\Big(\int_{A_-(k)} w_k^{p_+^\sharp}\,dx\Big)^{\frac{p_+}{p_+^\sharp}} + e^{\varkappa d}\int_{A_-(k)}|f(x)|\,dx \\
&\quad + M\theta_+\varepsilon^{\frac{\theta_+ -1}{\theta_+}}\|f\|_{L_s(G)}^{\frac{1}{\theta_+}}\int_{A_-(k)} w_k^{p_+}\,dx
\end{aligned}
\tag{1.32}
$$

By well known the Sobolev embedding theorem and taking into account (1.31), we obtain

$$
\begin{aligned}
&\Big(\int_{A_+(k)} v_k^{p_-^\sharp}\,dx\Big)^{\frac{p_-}{p_-^\sharp}} \le c_-\int_{A_+(k)}(v_k^{p_-}+|\nabla v_k|^{p_-})\,dx; \\
&\Big(\int_{A_-(k)} w_k^{p_+^\sharp}\,dx\Big)^{\frac{p_+}{p_+^\sharp}} \le c_+\int_{A_-(k)}(w_k^{p_+}+|\nabla w_k|^{p_+})\,dx,
\end{aligned}
\tag{1.33}
$$

where $c_\mp$ are positive constants. Finally, (1.26)–(1.33) imply that

$$
\begin{aligned}
&\big[\frac{1}{2}\big(\frac{p_-}{\varkappa}\big)^{p_-} - Mc_-(1-\theta_-)\varepsilon\big]\int_{A_+(k)}|\nabla v_k|^{p_-}\,dx \\
&+\big[\frac{1}{2}\big(\frac{p_+}{\varkappa}\big)^{p_+} - Mc_+(1-\theta_+)\varepsilon\big]\int_{A_-(k)}|\nabla w_k|^{p_+}\,dx \\
&+\big[a_0k_0^{p_-} - Mc_-(1-\theta_-)\varepsilon - M\theta_-\varepsilon^{\frac{\theta_- -1}{\theta_-}}\|f\|_{L_s(G)}^{\frac{1}{\theta_-}}\big]\int_{A_+(k)} v_k^{p_-}\,dx \\
&+\big[a_0k_0^{p_-} - Mc_+(1-\theta_+)\varepsilon - M\theta_+\varepsilon^{\frac{\theta_+ -1}{\theta_+}}\|f\|_{L_s(G)}^{\frac{1}{\theta_+}}\big]\int_{A_-(k)} w_k^{p_+}\,dx \\
&\le e^{\varkappa d}\int_{A(k)}|f(x)|\,dx, \quad \forall\varepsilon>0.
\end{aligned}
\tag{1.34}
$$

Further, at first, we choose

$$
\varepsilon = \frac{1}{4M}\min\Big\{\frac{1}{c_-(1-\theta_-)}\big(\frac{p_-}{\varkappa}\big)^{p_-},\ \frac{1}{c_+(1-\theta_+)}\big(\frac{p_+}{\varkappa}\big)^{p_+}\Big\}
\tag{1.35}
$$

and next

$$
k_0 \ge \Big(\frac{2MF}{a_0}\Big)^{\frac{1}{p_-}},
\tag{1.36}
$$

where

$$
F = \max\Big\{c_-(1-\theta_-)\varepsilon + \theta_-\varepsilon^{\frac{\theta_- -1}{\theta_-}}\|f\|_{L_s(G)}^{\frac{1}{\theta_-}};\ c_+(1-\theta_+)\varepsilon + \theta_+\varepsilon^{\frac{\theta_+ -1}{\theta_+}}\|f\|_{L_s(G)}^{\frac{1}{\theta_+}}\Big\}.
$$

Thus, by the above arguments, we derive

$$\int_{A_+(k)} \big(|\nabla v_k|^{p_-} + v_k^{p_-}\big)\,dx + \int_{A_-(k)} \big(|\nabla w_k|^{p_+} + w_k^{p_+}\big)\,dx \le C\int_{A(k)} |f(x)|\,dx, \tag{1.37}$$

where $C = \mathrm{const}(n, p_-, p_+, a_0, k_0, \mu, s, \|f\|_{L_s(G)}) > 0$. The inequalities (1.33) and (1.37) give

$$\Big(\int_{A_+(k)} v_k^{p_-^\#}\,dx\Big)^{\frac{p_-}{p_-^\#}} + \Big(\int_{A_-(k)} w_k^{p_+^\#}\,dx\Big)^{\frac{p_+}{p_+^\#}} \le \max\{c_-, c_+\}C\int_{A(k)} |f(x)|\,dx, \tag{1.38}$$

for all $k \ge k_0$. At last, by the Hölder inequality, we have

$$\int_{A(k)} |f(x)|\,dx \le \|f(x)\|_{L_s(G)} \operatorname{meas}^{1-\frac{1}{s}} A(k); \quad s > \frac{n}{p_-} > 1.$$

Then from (1.38) it follows that

$$\begin{aligned} &\Big(\int_{A_+(k)} v_k^{p_-^\#}\,dx\Big)^{\frac{p_-}{p_-^\#}} + \Big(\int_{A_-(k)} w_k^{p_+^\#}\,dx\Big)^{\frac{p_+}{p_+^\#}} \\ &\le \max\{c_-, c_+\}C\|f(x)\|_{L_s(G)} \operatorname{meas}^{1-\frac{1}{s}} A(k), \quad s > \frac{n}{p_-} > 1,\ \forall k \ge k_0. \end{aligned} \tag{1.39}$$

Now, let $l > k > k_0$. By (1.7) and the definition of the functions $v_k(x)$, $w_k(x)$, we have $v_k \ge \frac{1}{p_-}(|u| - k)_+$, $w_k \ge \frac{1}{p_+}(|u| - k)_+$. Therefore,

$$\int_{A_+(l)} v_k^{p_-^\#}\,dx \ge \Big(\frac{l-k}{p_-}\Big)^{p_-^\#} \operatorname{meas} A_+(l), \quad \int_{A_-(l)} w_k^{p_+^\#}\,dx \ge \Big(\frac{l-k}{p_+}\Big)^{p_+^\#} \operatorname{meas} A_-(l).$$

Hence, (1.39) together with $A_\pm(l) \subseteq A_\pm(k)$ imply that

$$\begin{aligned} \operatorname{meas} A(l) &= \operatorname{meas}\big(A_+(l) \cup A_-(l)\big) \le \operatorname{meas} A_+(l) + \operatorname{meas} A_-(l) \\ &\le \Big(\frac{p_-}{l-k}\Big)^{p_-^\#} \int_{A_+(k)} v_k^{p_-^\#}\,dx + \Big(\frac{p_+}{l-k}\Big)^{p_+^\#} \int_{A_-(k)} w_k^{p_+^\#}\,dx \\ &\le C_-\Big(\frac{p_-}{l-k}\Big)^{p_-^\#} \|f(x)\|_{L_s(G)}^{\frac{p_-^\#}{p_-}} \operatorname{meas}^{\frac{p_-^\#}{p_-}(1-\frac{1}{s})} A(k) \\ &\quad + C_+\Big(\frac{p_+}{l-k}\Big)^{p_+^\#} \|f(x)\|_{L_s(G)}^{\frac{p_+^\#}{p_+}} \operatorname{meas}^{\frac{p_+^\#}{p_+}(1-\frac{1}{s})} A(k) \end{aligned} \tag{1.40}$$

for all $l > k \ge k_0$, where $C_\mp = \big(C\max\{c_-, c_+\}\big)^{p_\mp^\#/p_\mp}$. Since $\frac{p_-^\#}{p_-} \le \frac{p_+^\#}{p_+}$ (see (1.31)), we have

$$\operatorname{meas}^{\frac{p_-^\#}{p_-}(1-\frac{1}{s})} A(k) \ge \operatorname{meas}^{\frac{p_+^\#}{p_+}(1-\frac{1}{s})} A(k), \quad \text{if } \operatorname{meas} A(k) \le 1.$$

Moreover,

$$\frac{p_+^\#}{p_+}\Big(1 - \frac{1}{s}\Big) \ge \frac{p_-^\#}{p_-}\Big(1 - \frac{1}{s}\Big) > 1 \quad \text{for } s > \frac{n}{p_-} > 1.$$

Let us introduce $\psi(k) = \operatorname{meas} A(k)$. Then from (1.40) it follows that

$$\psi(l) \le 2\widetilde{C}\psi^\varsigma(k) \begin{cases} \frac{1}{(l-k)^{p_-^\#}} & \text{if } \ l - k \ge 1; \\ \frac{1}{(l-k)^{p_+^\#}} & \text{if } \ 0 < l - k < 1, \end{cases}$$

for all $l > k \geq k_0$, where $\zeta = (1 - \frac{1}{s})\frac{n}{n-p_-} > 1$,

$$\widetilde{C} = \mathrm{const}(n, p_-, p_+, a_0, k_0, \mu, s, \|f\|_{L_s(G)}) > 0.$$

By the Stampacchia Lemma, we have that $\psi(k_0 + \delta) = 0$ with δ depending only on the quantities given in Theorem 1.6. This fact means that $|u(x)| \leq k_0 + \delta$ for almost all $x \in G$. Thus, we derive $M_0 = k_0 + \delta$, where k_0 is defined by (1.13), (1.36) with (1.31) and (1.35). Then Theorem 1.6 is proved. $\square$

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