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GRADIENT ESTIMATES FOR TRANSMISSION PROBLEMS WITH NONSMOOTH INTERNAL BOUNDARIES

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Abstract. In this paper we obtain an interior gradient estimate for a weak solution of a transmission problem with nonsmooth internal boundaries. The coefficients are assumed to be merely measurable in one variable and have small BMO semi-norms in the other variables on each subdomain whose boundary satisfies the so-called δ-Reifenberg flat condition. Under these assumptions, we prove a Calderón-Zygmund type estimate.

1. Introduction and statement of main results

In this study, we are interested in the regularity result for transmission problems. Transmission problems are related to inhomogeneities of conditions and regularity theory for transmission problems has been developed in various ways, see [2, 3, 8, 12, 14, 15, 16, 22, 23, 27] and references therein.

To study these problems, let Ω be a bounded connected open set in $\mathbb{R}^n$ with $n \geq 2$ and nonempty connected components Ω^+ and Ω^- of Ω be disjoint open subsets of Ω satisfying

$$\partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega,$$
$$\Omega = \Omega^+ \cup \Omega^- \cup (\partial\Omega^+ \cap \Omega).$$

We set

$$A_{ij}^{\alpha\beta}(x) = A_{ij}^{\alpha\beta,+}(x) \cdot \chi_{\Omega^+}(x) + A_{ij}^{\alpha\beta,-}(x) \cdot \chi_{\Omega^-}(x),$$

where $\chi_{\Omega^\pm}$ is the indicator function of $\Omega^\pm$ and $A_{ij}^{\alpha\beta,\pm} : \mathbb{R}^n \to \mathbb{R}$ for $1 \leq \alpha, \beta \leq n$ and $1 \leq i, j \leq m$ with $m \geq 2$. With these notation we consider the following Dirichlet problem for an elliptic system in divergence form:

$$D_\alpha\big(A_{ij}^{\alpha\beta}(x) D_\beta u^j(x)\big) = D_\alpha F_\alpha^i(x) \quad \text{in } \Omega, \tag{1.1}$$

for each $i = 1, \dots, m$, where the inhomogeneous term $F = \{F_\alpha^i\}$ is a given matrix valued function. The tensor coefficients $A(x) = \{A_{ij}^{\alpha\beta}(x)\}$ is assumed to be uniformly elliptic and uniformly bounded, namely, we assume that there exist positive constants ν and L such that

$$\nu|\xi|^2 \leq A_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \quad \text{and} \quad \|A_{ij}^{\alpha\beta}\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^{mn\times mn})} \leq L, \tag{1.2}$$

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for all matrix $\xi \in \mathbb{R}^{mn}$ and for almost every $x \in \mathbb{R}^n$. With these settings, we say that $u = (u^1, \dots, u^m) \in H^1(\Omega, \mathbb{R}^m)$ is a weak solution of (1.1) if

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \phi^i \, dx = \int_\Omega F_\alpha^i D_\alpha \phi^i \, dx, \quad \forall \phi = (\phi^1, \dots, \phi^m) \in H_0^1(\Omega, \mathbb{R}^m).$$

Now, we introduce some notation to be used throughout this paper.

- An open ball in $\mathbb{R}^n$ with center y and radius $r > 0$ is defined by
$$B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}.$$
- An open ball in $\mathbb{R}^{n-1}$ with center $y\prime$ and radius $r > 0$ is defined by
$$B'_r(y') = \{x' \in \mathbb{R}^{n-1} : |x' - y'| < r\}.$$
- An elliptic cylinder in $\mathbb{R}^n$ with center $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and size $r > 0$ is defined by
$$Q_r(y) = B'_r(y') \times (y_n - r, y_n + r).$$
If the center is the origin $0 = (0', 0)$, then we denote, for simplicity, $Q_r(0) = B'_r(0') \times (-r, r)$ by $Q_r = B'_r \times (-r, r)$.
- The integral average of $g \in L^1(U)$ over a bounded domain U in $\mathbb{R}^n$ is denoted by
$$\overline{g}_U = \fint_U g(x) \, dx = \frac{1}{|U|} \int_U g(x) \, dx.$$
- For each $x_n \in \mathbb{R}$ and for each bounded subset E' of $\mathbb{R}^{n-1}$ the integral average of $g(\cdot, x_n)$ over E' is denoted by
$$\overline{g}_{E'}(x_n) = \fint_{E'} g(x', x_n) \, dx' = \frac{1}{|E'|} \int_{E'} g(x', x_n) \, dx'.$$

In this work, we want to obtain the Calderón-Zygmund type regularity result for transmission problems with very rough internal boundaries, including Lipschitz continuous functions or even fractals. These problems are physically very natural and have many applications in multiple fields, such as electrochemisrty related to rough electrodes or transfer across irregular membranes, etc., see [1] and references therein. Because of the understanding of recent researches on the regularity results with respect to measurable coefficients, see [4, 5, 6, 7, 11, 13, 18, 20, 21, 25] and on the geometric properties of Reifenberg domains, see [19, 28], it is possible to prove the $W^{1,p}$ regularity for a weak solution of (1.1). For this, our main assumption is the following.

Definition 1.1. We say that $(A_{ij}^{\alpha\beta}, U)$ is (δ, R)-*vanishing of codimension 1* if for every point $x_0 \in U$ and for every number $r \in (0, 3R]$ with

$$\operatorname{dist}(x_0, \partial U) = \min_{x_1 \in \partial U} \operatorname{dist}(x_0, x_1) > \sqrt{2} r,$$

then there exists a coordinate system depending on x_0 and r, whose variables we still denote by $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$, so that in this new coordinate system

$$\fint_{Q_{\sqrt{2}r}} \left| A_{ij}^{\alpha\beta}(x', x_n) - \overline{A_{ij}^{\alpha\beta}}_{B'_{\sqrt{2}r}}(x_n) \right|^2 dx \le \delta^2, \tag{1.3}$$

while, for every point $x_0 \in U$ and for every number $r \in (0, 3R]$ with

$$\operatorname{dist}(x_0, \partial U) = \min_{x_1 \in \partial U} \operatorname{dist}(x_0, x_1) \le \sqrt{2} r,$$

there exists a coordinate system depending on x_0 and r, whose variables we still denote by $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$, so that in this new coordinate system

$$Q_{3r} \cap \{(x', x_n) : x_n > 3r\delta\} \subset Q_{3r} \cap U \subset Q_{3r} \cap \{(x', x_n) : x_n > -3r\delta\}, \tag{1.4}$$

$$\fint_{Q_{3r}} \big|A_{ij}^{\alpha\beta}(x', x_n) - \overline{A_{ij}^{\alpha\beta}}_{B'_{3r}}(x_n)\big|^2 \, dx \le \delta^2. \tag{1.5}$$

Remark 1.2. This means that if $(A_{ij}^{\alpha\beta}, U)$ is (δ, R)-vanishing of codimension 1, then at each point and at each scale $A_{ij}^{\alpha\beta}$ are allowed to be merely measurable in one variable while they have small BMO semi-norms in the other variables in some appropriate coordinates and at the same time U is (δ, R)-Reifenberg flat. Reifenberg flatness condition of U written in (1.4) is a generalization of Lipschitz domains with small Lipschitz constant and includes even fractal structures, so this definition is meaningful when $0 < \delta < 1/8$, see [5, 7, 26, 28]. In addition since (1.1) has a scaling invariance property, the constant R can be taken as 1 or any other constants greater than 1. However, the constant δ is a small positive constant which is still invariant under such scaling. This small number will be selected later.

The following is our main result in this article.

Theorem 1.3. *Suppose that $F \in L^p(\Omega, \mathbb{R}^{mn})$ for some $2 < p < \infty$, for $\hat{x} \in \Omega$, $Q_{150}(\hat{x}) \subset \Omega$ and $u \in H^1(\Omega, \mathbb{R}^m)$ is a weak solution of (1.1). Then there exists a small positive constant $\delta = \delta(\nu, L, m, n, p)$ such that if $(A_{ij}^{\alpha\beta,\pm}, \Omega^\pm)$ are $(\delta, 25)$-vanishing of codimension 1, then*

$$Du \in L^p(Q_1(\hat{x}), \mathbb{R}^{mn})$$

with the estimate

$$\int_{Q_1(\hat{x})} |Du|^p \, dx \le c \int_{Q_5(\hat{x})} |u|^p + |F|^p \, dx \tag{1.6}$$

where the constant c depends on ν, L, m, n, p.

Remark 1.4. In the case $p = 2$, estimate (1.6) a classical one. If we have estimate (1.6) in the case $2 < p < \infty$, then the estimate follows from a duality in the case $1 < p < 2$. For these reasons, we will consider the case $2 < p < \infty$.

It is well-known that with the basic structural conditions such as (1.2), $W^{1,p}$ regularity holds for only when p is close to 2, see [17]. However, in this study, we want to get estimate (1.6) for the full range $1 < p < \infty$, so we need some additional smoothness assumptions on both the coefficients and the boundaries of subdomains as Theorem 1.3. The concept of coefficients in Definition 1.1 was studied in some previous works, see [4, 5, 7, 13, 18, 20, 21] and related papers. However, in those works, they only considered the case that the coordinate system described in Definition 1.1 can be chosen in one fixed way at every point in the domain, while for our problem at some internal boundary point the coordinate systems with respect to Ω^+ and Ω^- may not coincide. For this reason, we additionally use geometric properties of δ-Reifenberg domains to obtain our main result. Finally, we note that our problem is not in the case of the counterexample in [24]. The counterexample in [24] says that the coefficients cannot be allowed to be measurable in two independent variables for the regularity theory considered in this direction. However, in our situation, even though we have to consider two measurable directions at the internal boundary point, because of such geometric properties of δ-Reifenberg domains, it is possible to prove Theorem 1.3, see Section 3 and Section 4.

2. Preliminaries

In this section, we introduce analytic and geometric tools which will be used later in the proof of main theorem. In a technical point of view, Our approach is based on the Hardy-Littlewood maximal function and Vitali type covering argument that is developed from [10, 29] and used in [6, 7].

We first recall the Hardy-Littlewood maximal function and its basic properties. Let g be a locally integrable function on $\mathbb{R}^n$. Then the Hardy-Littlewood maximal function is given by

$$(\mathcal{M}g)(x) = \sup_{r>0} \frac{1}{|Q_r(x)|} \int_{Q_r(x)} |g(y)|dy.$$

If g is defined only on a bounded subset of $\mathbb{R}^n$, we define as

$$\mathcal{M}g = \mathcal{M}\overline{g},$$

where $\overline{g}$ is the zero extension of g from a bounded set to $\mathbb{R}^n$. We also use the notation

$$\mathcal{M}_\Omega g = \mathcal{M}(\chi_\Omega g)$$

if g is not defined outside Ω. The Hardy-Littlewood maximal function has two basic properties that we will use in this paper: one is the weak 1-1 estimate and the other is the strong p-p estimate.

- (weak 1-1 estimate) For $g \in L^1(\mathbb{R}^n)$, there is a constant $c = c(n) > 0$ such that
$$|\{x \in \mathbb{R}^n : (\mathcal{M}g)(x) > t\}| \le \frac{c}{t}\|g\|_{L^1(\mathbb{R}^n)}, \quad \forall t > 0.$$
- (strong p-p estimate) For $g \in L^p(\mathbb{R}^n)$ for some $p \in (1,\infty)$, it holds $\mathcal{M}g \in L^p(\mathbb{R}^n)$ with the estimate
$$\frac{1}{c}\|g\|_{L^p(\mathbb{R}^n)} \le \|\mathcal{M}g\|_{L^p(\mathbb{R}^n)} \le c\|g\|_{L^p(\mathbb{R}^n)} \tag{2.1}$$
for some constant $c = c(n,p) > 0$.

We need the following classical measure theory.

Lemma 2.1 ([9]). *Assume that g is a nonnegative and measurable function defined on a bounded domain $\Omega \subset \mathbb{R}^n$. Let $\theta > 0$ and $\lambda > 1$ be constants. Then for $0 < q < \infty$,*

$$g \in L^q(\Omega) \iff S = \sum_{k\ge1} \lambda^{qk}|\{x \in \Omega : g(x) > \theta\lambda^k\}| < \infty$$

and

$$\frac{1}{c}S \le \|g\|^q_{L^q(\Omega)} \le c(|\Omega| + S), \tag{2.2}$$

where the positive constant c depending only on θ, λ, and q.

We will use the following version of Vitali covering lemma for the proof of our main theorem.

Lemma 2.2 ([29]). *Assume that C and D are measurable sets, $C \subset D \subset Q_1$, and that there exists a small $\epsilon > 0$ such that*

$$|C| < \epsilon |Q_1| \tag{2.3}$$

and for each $x \in Q_1$ and $r \in (0,1]$ with $|C \cap Q_r(x)| \geq \epsilon |Q_r(x)|$,

$$Q_r(x) \cap Q_1 \subset D. \tag{2.4}$$

Then $|C| \leq 2\sqrt{2}(10)^n \epsilon |D|$.

3. Comparison estimates

In this section, we use an approximation lemma which plays an important role in our perturbation argument. We start with a simple interior case, see [5, Lemma 3.3].

Lemma 3.1. *Assume that $Q_5 \subset \Omega^+$ or $Q_5 \subset \Omega^-$. Let $u \in H^1(Q_5, \mathbb{R}^m)$ be a weak solution of*

$$D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = D_\alpha F_\alpha^i \qquad \text{in } Q_5,$$

for $i = 1, \dots, m$, under the assumption

$$\fint_{Q_5} |Du|^2 \, dx \leq 1.$$

Then, there exists $n_1 = n_1(\nu, L, m, n) > 1$ so that for $0 < \epsilon < 1$ fixed, we can find a small $\delta_1 = \delta_1(\epsilon, \nu, L, m, n) > 0$ such that if

$$\fint_{Q_5} |A_{ij}^{\alpha\beta}(x', x_n) - \overline{A_{ij}^{\alpha\beta}}_{B_5'}(x_n)|^2 \, dx \leq \delta_1^2 \quad \text{and} \quad \fint_{Q_5} |F|^2 \, dx \leq \delta_1^2$$

hold for such a small δ_1, then there exists a weak solution $v \in H^1(Q_4, \mathbb{R}^m)$ of

$$D_\alpha\big(\overline{A_{ij}^{\alpha\beta}}_{B_5'}(x_n) D_\beta v^j\big) = 0 \quad \text{in } Q_4, \tag{3.1}$$

for $i = 1, \dots, m$, such that

$$\fint_{Q_2} |D(u - v)|^2 \, dx \leq \epsilon^2 \quad \text{and} \quad \|Dv\|^2_{L^\infty(Q_3)} \leq n_1^2.$$

For the case when two subdomains are involved, to construct our appropriate map, for simplicity we assume that $0 \in \partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega$ and then there exists an appropriate coordinate system depending on r, whose variables $x = (x_1, \dots, x_n)$, such that in this x-coordinate system the measurable direction of $A_{ij}^{\alpha\beta,-}$ is $(0, \dots, 0, 1)$ and

$$Q_{r,x} \cap \{x_n < -r\delta\} \subset \Omega^- \cap Q_{r,x} \subset Q_{r,x} \cap \{x_n < r\delta\}. \tag{3.2}$$

In addition, one can also find a coordinate system depending on r, whose variables $y = (y_1, \dots, y_n)$, such that in this y-coordinate system the measurable direction of $A_{ij}^{\alpha\beta,+}$ is $(0, \dots, 0, 1)$ and

$$Q_{r,y} \cap \{y_n > r\delta\} \subset \Omega^+ \cap Q_{r,y} \subset Q_{r,y} \cap \{y_n > -r\delta\}. \tag{3.3}$$

Here, we denote $Q_{\rho,z}$ as the Q_ρ cylinder with respect to z coordinate system. We observe that comparing two measurable directions of $A_{ij}^{\alpha\beta,-}$ and $A_{ij}^{\alpha\beta,+}$ at 0 is equivalent to comparing two straight lines. Therefore, we can further assume that the measurable direction $(0, \dots, 0, 0, 1)$ in the y-coordinate system is $(0, \dots, 0, -\sin\theta, \cos\theta)$ for some small $\theta > 0$ in the x-coordinate system. In fact, the special case $\theta = 0$, which means that x coordinate system coincides with y coordinate system, was previously treated in [5] with Lemma 3.1.

Next we define the "curved cylinder" $\widetilde{Q_r}$ in the z-chart with the notation

$$z = (z_1, \dots, z_{n-2}, z_{n-1}, z_n) = (z'', z_{n-1}, z_n) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R},$$

$$\begin{aligned}
\widetilde{Q_r} = \Big\{ & (z'', z_{n-1}, z_n) : -r \le z_i \le r \text{ for } i = 1, \dots, n-1 \text{ and} \\
& -r \le z_n \le -2r\tan(\frac{\theta}{2}) \Big\} \\
\cup \Big\{ & (z'', z_{n-1}\cos\theta - z_n \sin\theta, z_{n-1}\sin\theta + z_n\cos\theta) : \\
& -r \le z_i \le r \text{ for } i = 1, \dots, n-1 \text{ and } 2r\tan(\frac{\theta}{2}) \le z_n \le r \Big\} \\
\cup \Big\{ & \big(z'', -2r + (z_{n-1} + 2r)\cos\phi, (z_{n-1} + 2r)\sin\phi - 2r\tan(\frac{\theta}{2})\big) : \\
& -r \le z_i \le r \text{ for } i = 1, \dots, n-1 \text{ and } 0 < \phi < \theta \Big\}.
\end{aligned}$$

We also define $\widetilde{Q_r^{(a)}}$ for $a \in (2r\tan(\frac{\theta}{2}), r)$ by

$$\begin{aligned}
\widetilde{Q_r^{(a)}} = \Big\{ & (z'', z_{n-1}, z_n) : -a \le z_i \le a \text{ for } i = 1, \dots, n-1 \\
& \text{and } -a \le z_n \le -2r\tan(\frac{\theta}{2}) \Big\} \\
\cup \Big\{ & (z'', z_{n-1}\cos\theta - z_n \sin\theta, z_{n-1}\sin\theta + z_n\cos\theta) : \\
& -a \le z_i \le a \text{ for } i = 1, \dots, n-1 \text{ and } 2r\tan(\frac{\theta}{2}) \le z_n \le a \Big\} \\
\cup \Big\{ & \big(z'', -2r + (z_{n-1} + 2r)\cos\phi, (z_{n-1} + 2r)\sin\phi - 2r\tan(\frac{\theta}{2})\big) : \\
& -a \le z_i \le a \text{ for } i = 1, \dots, n-1 \text{ and } 0 < \phi < \theta \Big\}.
\end{aligned}$$

Now, we fix $r = 5$. Then we shall construct a Lipschitz map $\Phi : \widetilde{Q_5} \to Q_5$ with inverse $\Psi = \Phi^{-1} : Q_5 \to \widetilde{Q_5}$. To do this, we define Ψ as follows:

$$\begin{aligned}
&\Psi(z'', z_{n-1}, z_n) \\
&= \begin{cases}
(z'', z_{n-1}, z_n), & \text{if } z_n \le -10\tan(\frac{\theta}{2}); \\
(z'', z_{n-1}\cos\theta - z_n\sin\theta, z_{n-1}\sin\theta + z_n\cos\theta), & \text{if } z_n \ge 10\tan(\frac{\theta}{2}); \\
\Big(z'', -10 + (z_{n-1} + 10)\cos\frac{(z_n + 10\tan(\frac{\theta}{2}))\theta}{20\tan(\frac{\theta}{2})}, \\
-10\tan(\frac{\theta}{2}) + (z_{n-1} + 10)\sin\frac{(z_n + 10\tan(\frac{\theta}{2}))\theta}{20\tan(\frac{\theta}{2})}\Big), \\
\quad \text{if } -10\tan(\frac{\theta}{2}) < z_n < 10\tan(\frac{\theta}{2})
\end{cases}
\end{aligned}$$

and note that

$$\det D\Phi = \det D\Psi = 1 \quad \text{for } 10\tan(\frac{\theta}{2}) < |z_n| < 5,$$
$$\frac{1}{5} \le \det D\Phi \le 5 \quad \text{for } |z_n| < 10\tan(\frac{\theta}{2}).$$

Going back to (3.2)-(3.3) with $r = 5$, we now assume that

$$\begin{gathered}(x'', x_{n-1}, x_n) = (z'', z_{n-1}, z_n),\\ (y'', y_{n-1}, y_n) = \big(z'', z_{n-1}\cos\theta - z_n \sin\theta, z_{n-1}\sin\theta + z_n\cos\theta\big).\end{gathered} \tag{3.4}$$

Actually, in the above construction, x coordinate system and z coordinate system are same. However, to avoid confusion, we use x coordinate system and z coordinate system separately in context.

Remark 3.2. Under the settings (3.2) and (3.3), one can easily see that

$$\frac{\theta}{2} \le \tan(\frac{\theta}{2}) \le \delta, \quad \text{that is, } \theta \le 2\delta. \tag{3.5}$$

In fact, since

$$\begin{gathered}Q_{5,x} \cap \{x_n = -5\delta\} \subset \Omega^- \cap Q_{5,x},\\ Q_{5,y} \cap \{y_n = 5\delta\} \subset \Omega^+ \cap Q_{5,y},\\ \Omega^- \cap \Omega^+ = \emptyset,\end{gathered}$$

we observe that

$$(Q_{5,x} \cap \{x_n = -5\delta\}) \cap (Q_{5,y} \cap \{y_n = 5\delta\}) = \emptyset. \tag{3.6}$$

From (3.6), we know that the angle θ between x and y coordinate systems must be dependent on δ and we can derive from the geometry of $\widetilde{Q_5}$ that

$$5\tan(\frac{\theta}{2}) \le 5\delta.$$

This shows (3.5).

We next consider a mapping $\gamma : [-5, 5] \to \mathbb{R}^n$ defined by $\gamma(t) = \Psi(0, \dots, 0, t)$. Then since γ is a regular C^1 curve, the unit tangent vector of γ is well-defined. As a consequence, we see that for each $z \in \widetilde{Q_5}$ one can find a unique $t \in [-5, 5]$ such that z is on the $(n-1)$-dimensional hyperplane which is normal to the tangent vector of γ at t. We then let $P_{5,\gamma}(t)$ the $(n-1)$-dimensional sphere of radius 5 centered at $\gamma(t)$ in the $(n-1)$-dimensional hyperplane which is normal to the tangent vector of γ at t.

We now define

$$B_{ij}^{\sigma\tau}(z) = D_\alpha \Phi^\sigma(\Psi(z)) A_{ij}^{\alpha\beta}(\Psi(z)) D_\beta \Phi^\tau(\Psi(z)) \quad \text{for } z \in Q_5, \tag{3.7}$$

$$C_{ij}^{\alpha\beta}(w) = D_\sigma \Psi^\alpha(\Phi(w)) \overline{B_{ij}^{\sigma\tau}}_{B_5{}'}(\Phi(w)) D_\tau \Psi^\beta(\Phi(w)) \quad \text{for } w \in \widetilde{Q_5}. \tag{3.8}$$

Note that $\overline{B_{ij}^{\sigma\tau}}_{B_5{}'}(z) = \overline{B_{ij}^{\sigma\tau}}_{B_5{}'}(z_n)$ as a function of $z \in Q_5$ depending only on z_n.

Lemma 3.3. *Assume $\widetilde{Q_5} \subset \Omega$. We further assume that*

$$\frac{1}{|Q_5|} \int_{Q_{5,x} \cap \Omega^-} \big|A_{ij}^{\alpha\beta,-}(x', x_n) - \overline{A_{ij}^{\alpha\beta,-}}_{B'_{5,x}}(x_n)\big|^2 \, dx \le \delta^2, \tag{3.9}$$

$$\frac{1}{|Q_5|} \int_{Q_{5,y} \cap \Omega^+} \big|A_{ij}^{\alpha\beta,+}(y', y_n) - \overline{A_{ij}^{\alpha\beta,+}}_{B'_{5,y}}(y_n)\big|^2 dy \le \delta^2. \tag{3.10}$$

Then we have

$$\frac{1}{|\widetilde{Q_5}|} \int_{\widetilde{Q_5}} \big|A_{ij}^{\alpha\beta}(w) - C_{ij}^{\alpha\beta}(w)\big|^2 dw \le c\delta \tag{3.11}$$

for some positive constant $c = c(L, m, n)$.

Proof. We recall (3.5) in Remark 3.2 and we compute as follows:

$$
\begin{aligned}
&\frac{1}{|\widetilde{Q_5}|}\int_{\widetilde{Q_5}} \big|A_{ij}^{\alpha\beta}(w)-C_{ij}^{\alpha\beta}(w)\big|^2 dw\\
&=\frac{1}{|\widetilde{Q_5}|}\int_{\{w\in P_{5,\gamma}(t)|10\tan(\frac{\theta}{2})\le t\le 5\}} \big|A_{ij}^{\alpha\beta}(w)-C_{ij}^{\alpha\beta}(w)\big|^2 dw\\
&\quad+\frac{1}{|\widetilde{Q_5}|}\int_{\{w\in P_{5,\gamma}(t)|-10\tan(\frac{\theta}{2})<t<10\tan(\frac{\theta}{2})\}} \big|A_{ij}^{\alpha\beta}(w)-C_{ij}^{\alpha\beta}(w)\big|^2 dw\\
&\quad+\frac{1}{|\widetilde{Q_5}|}\int_{\{w\in P_{5,\gamma}(t)|-5\le t\le -10\tan(\frac{\theta}{2})\}} \big|A_{ij}^{\alpha\beta}(w)-C_{ij}^{\alpha\beta}(w)\big|^2 dw\\
&\le \frac{c}{|Q_5|}\int_{Q_{5,x}\cap\Omega^-} \big|A_{ij}^{\alpha\beta}(x',x_n)-\overline{A_{ij}^{\alpha\beta}}_{B'_{5,x}}(x_n)\big|^2\,dx\\
&\quad+\frac{1}{|\widetilde{Q_5}|}\int_{Q_5\cap\{-10\tan(\frac{\theta}{2})<z_n<10\tan(\frac{\theta}{2})\}} cL^2\,dw\\
&\quad+\frac{c}{|Q_5|}\int_{Q_{5,y}\cap\Omega^+} \big|A_{ij}^{\alpha\beta}(y',y_n)-\overline{A_{ij}^{\alpha\beta}}_{B'_{5,y}}(y_n)\big|^2 dy\\
&\le c\delta
\end{aligned}
$$

where $c=c(L,m,n)>0$. □

Remark 3.4. Different from the previous works as [5, 7], in our case we can only obtain that the left hand side of (3.11) is less than $c\delta$ instead of δ^2 because we consider the case that x coordinate system does not coincide with y coordinate system.

Now we are in a position to find an interior approximation lemma.

Lemma 3.5. *Let $u\in H^1(\widetilde{Q_5},\mathbb{R}^m)$ be a weak solution of*

$$D_\alpha(A_{ij}^{\alpha\beta}(w)D_\beta u^j(w))=D_\alpha F_\alpha^i(w)\quad in\ \widetilde{Q_5}\subset\Omega$$

under the assumption

$$\fint_{\widetilde{Q_5}} |Du(w)|^2\,dw\le 1. \tag{3.12}$$

There exists $n_2=n_2(\nu,L,m,n)>1$ so that for $0<\epsilon<1$ fixed, we can find a small $\delta=\delta(\epsilon,\nu,L,m,n)>0$ such that if (3.9), (3.10), and

$$\fint_{\widetilde{Q_5}} |F(w)|^2\,dw\le\delta^2 \tag{3.13}$$

hold for such a small δ, then there exists a weak solution $v\in H^1(\widetilde{Q_5^{(4)}},\mathbb{R}^m)$ of

$$D_\alpha\big(C_{ij}^{\alpha\beta}(w)D_\beta v^j(w)\big)=0\quad in\ \widetilde{Q_5^{(4)}} \tag{3.14}$$

for each $i=1,\dots,m$, such that

$$\fint_{\widetilde{Q_5^{(2)}}} |D(u-v)|^2\,dw\le\epsilon^2\quad and\quad \|Dv\|^2_{L^\infty(\widetilde{Q_5^{(3)}})}\le n_2^2. \tag{3.15}$$

Proof. Under the change of variables $w = \Psi(z)$, from (3.7) we see that

$$D_\sigma \left(B_{ij}^{\sigma\tau}(z) D_\tau {u'}^j(z) \right) = D_\sigma (F')^i_\sigma(z) \quad \text{in } Q_5$$

where $u'(z) = u(\Psi(z))$ and $(F')^i_\sigma(z) = D_\alpha \Phi^\sigma(\Psi(z)) F^i_\alpha(\Psi(z))$. Also, by (3.12) and (3.13), we have

$$\fint_{Q_5} |Du'(z)|^2 \, dz \le c \fint_{\widetilde{Q_5}} |Du(w)|^2 \, dw \le c,$$
$$\fint_{Q_5} |F'(z)|^2 \, dz \le c \fint_{\widetilde{Q_5}} |F(w)|^2 \, dw \le c\delta^2$$

for some constant c. Moreover, by (3.7), (3.8) and Lemma 3.3, we obtain

$$\fint_{Q_5} \left| B_{ij}^{\sigma\tau}(z) - \overline{B_{ij}^{\sigma\tau}}_{B_5{}'}(z_n) \right|^2 dz \le c \fint_{\widetilde{Q_5}} \left| A_{ij}^{\alpha\beta}(w) - C_{ij}^{\alpha\beta}(w) \right|^2 dw \le c\delta$$

for some constant $c = c(L, m, n)$.

Since our equation is invariant under normalization, we can apply Lemma 3.1 to our situation with small δ. That is, there exists a weak solution $v' \in H^1(Q_4, \mathbb{R}^m)$ of

$$D_\sigma \big(\overline{B_{ij}^{\sigma\tau}}_{B_5{}'}(z_n) D_\tau {v'}^j(z) \big) = 0 \quad \text{in } Q_4$$

such that

$$\fint_{Q_1} |D(u' - v')|^2 dz \le \epsilon^2$$

and we have an interior Lipschitz regularity as

$$\|Dv'\|^2_{L^\infty(Q_2)} \le c$$

where $c > 0$ is a positive constant independent from v', see [11].

Finally, we apply the change of variables $z = \Phi(w)$ then we obtain that $v \in H^1(\widetilde{Q_5^{(4)}}, \mathbb{R}^m)$ is a weak solution of

$$D_\alpha \big(C_{ij}^{\alpha\beta}(w) D_\beta v^j(w) \big) = 0 \quad \text{in } \widetilde{Q_5^{(4)}}$$

where $v(w) = v'(\Phi(w))$ satisfying (3.15). This completes the proof. □

4. $W^{1,p}$ ESTIMATES

In this section, we prove the main theorem, Theorem 1.3. Since our problem (1.1) is invariant under translation, without loss of generality, we prove Theorem 1.3 only for $\hat{x} = 0$.

Lemma 4.1. *Let $u \in H^1(\Omega, \mathbb{R}^m)$ be a weak solution of* (1.1) *and assume $Q_{150} \subset \Omega$. Then there exists a universal constant $N > 1$ so that for each $0 < \epsilon < 1$ fixed, one can select a small $\delta = \delta(\epsilon, \nu, L, m, n) > 0$ such that if $(A_{ij}^{\alpha\beta,-}, \Omega^-)$ and $(A_{ij}^{\alpha\beta,+}, \Omega^+)$ are $(\delta, 25)$-vanishing of codimension 1 for such δ and if for $0 < r \le 1$ and $x_* \in Q_1$, the cube $Q_r(x_*)$ satisfies*

$$|\{x \in Q_1 : \mathcal{M}(|Du|^2) > N^2\} \cap Q_r(x_*)| > \epsilon |Q_r(x_*)|, \tag{4.1}$$

then it holds

$$Q_r(x_*) \cap Q_1 \subset \{x \in Q_1 : \mathcal{M}(|Du|^2) > 1\} \cup \{x \in Q_1 : \mathcal{M}(|F|^2) > \delta^2\}. \tag{4.2}$$

Proof. We prove this lemma by contradiction. To do this, suppose that

$$Q_r(x_*) \cap Q_1 \nsubseteq \{x \in Q_1 : \mathcal{M}(|Du|^2) > 1\} \cup \{x \in Q_1 : \mathcal{M}(|F|^2) > \delta^2\}. \tag{4.3}$$

Then there is a point $x_1 \in Q_r(x_*) \cap Q_1$ such that

$$\frac{1}{|Q_\rho(x_1)|} \int_{Q_\rho(x_1)\cap\Omega} |Du|^2\,dx \le 1 \quad \text{and} \quad \frac{1}{|Q_\rho(x_1)|} \int_{Q_\rho(x_1)\cap\Omega} |F|^2\,dx \le \delta^2 \tag{4.4}$$

for all $\rho > 0$.

We first prove the simplest case, when $\operatorname{dist}(x_*, \partial\Omega^\pm) > 5\sqrt{2}r$, which means that $Q_{5\sqrt{2}r}(x_*) \subset \Omega^-$ or $Q_{5\sqrt{2}r}(x_*) \subset \Omega^+$. Then according to Definition 1.1, we may assume that $x_* = 0$ and

$$\fint_{Q_{5\sqrt{2}r}} \left|A_{ij}^{\alpha\beta}(z', z_n) - \overline{A_{ij}^{\alpha\beta}}_{B'_{5\sqrt{2}r}}(z_n)\right|^2 dz \le \delta^2.$$

Since $x_1 \in Q_r$, we observe that

$$Q_{5\sqrt{2}r} \subset Q_{(\sqrt{2}+10)r}(x_1) \subset Q_{10\sqrt{2}r}(x_1)$$

and then by (4.4) we obtain

$$\fint_{Q_{5\sqrt{2}r}} |Du|^2\,dx \le \frac{|Q_{10\sqrt{2}r}(x_1)|}{|Q_{5\sqrt{2}r}|} \fint_{Q_{10\sqrt{2}r}(x_1)} |Du|^2\,dx \le 2^n.$$

Similarly,

$$\fint_{Q_{5\sqrt{2}r}(y)} |F|^2\,dx \le 2^n\delta^2.$$

To apply Lemma 3.1, we define the rescaled maps

$$\tilde{u}(z) = \frac{u(\sqrt{2}rz)}{r\sqrt{2\cdot 2^n}}, \quad \tilde{F}(z) = \frac{F(\sqrt{2}rz)}{\sqrt{2^n}}, \quad \tilde{A_{ij}^{\alpha\beta}}(z) = A_{ij}^{\alpha\beta}(\sqrt{2}rz), \quad (z \in Q_5).$$

Then $\tilde{u} \in H^1(Q_5, \mathbb{R}^m)$ is a weak solution of

$$D_\alpha(\tilde{A_{ij}^{\alpha\beta}}(z) D_\beta \tilde{u}^j) = D_\alpha \tilde{F}_\alpha^i \quad \text{in } Q_5 \tag{4.5}$$

with

$$\fint_{Q_5} |D\tilde{u}(z)|^2 dz \le 1 \quad \text{and} \quad \fint_{Q_5} |\tilde{F}(z)|^2 dz \le \delta^2.$$

Then we are now in a position to apply Lemma 3.1 for (4.5), which implies that there exists $n_1 = n_1(\nu, L, m, n) > 1$ so that for any $0 < \eta < 1$ fixed, we find a small $\delta = \delta(\eta, \nu, L, m, n) > 0$ and a weak solution $\tilde{v}$ of

$$D_\alpha\big(\overline{A_{ij}^{\alpha\beta}}_{B'_5}(z_n) D_\beta \tilde{v}^j\big) = 0 \quad \text{in } Q_4$$

such that

$$\fint_{Q_2} |D(\tilde{u} - \tilde{v})|^2 dz \le \eta^2 \quad \text{and} \quad \|D\tilde{v}\|^2_{L^\infty(Q_3)} \le n_1^2.$$

We scale back and then there exists a function v defined in $Q_{3\sqrt{2}r}$ such that

$$\fint_{Q_{2\sqrt{2}r}} |D(u - v)|^2 dz \le 2^n\eta^2 \quad \text{and} \quad \|Dv\|^2_{L^\infty(Q_{3\sqrt{2}r})} \le 2^n n_1^2. \tag{4.6}$$

After letting $N_1^2 = 2^n n_1^2$, we now claim that

$$\{z \in Q_{\sqrt{2}r} : \mathcal{M}(|Du|^2) > N^2\} \subset \{z \in Q_{\sqrt{2}r} : \mathcal{M}_{Q_{2\sqrt{2}r}}(|D(u-v)|^2) > N_1^2\} \tag{4.7}$$

where $N^2 = \max\{4N_1^2, 3^n\}$. To do this, we suppose that

$$x_0 \in \{z \in Q_{\sqrt{2}r} : \mathcal{M}_{Q_{2\sqrt{2}r}}(|D(u-v)|^2)(z) \le N_1^2\}. \tag{4.8}$$

If $\rho \le \sqrt{2}r$, then from $Q_\rho(x_0) \subset Q_{2\sqrt{2}r}$, (4.6), and (4.8),

$$⨍_{Q_\rho(x_0)} |Du|^2 dz \le 2⨍_{Q_\rho(x_0)} \big[|D(u-v)|^2 + |Dv|^2\big]\, dz \le 4N_1^2$$

and if $\rho > \sqrt{2}r$, then $Q_\rho(x_0) \subset Q_{3\rho}(x_1)$

$$⨍_{Q_\rho(x_0)} |Du|^2 dz \le \frac{|Q_{3\rho}(x_1)|}{|Q_\rho(x_0)|}⨍_{Q_{3\rho}(x_1)} |Du|^2 dz \le 3^n.$$

Thus we have that

$$x_0 \in \{z \in Q_{\sqrt{2}r} : \mathcal{M}(|Du|^2)(z) \le N^2\}$$

and our claim (4.7) follows. Then we observe that $Q_r(x_*)$ in (4.3) is covered by $Q_{\sqrt{2}r}$ in $z = (z', z_n)$ coordinate system to find that

$$\begin{aligned}
&|\{x \in Q_r(x_*) : \mathcal{M}(|Du|^2)(x) > N^2\}| \\
&\le |\{z \in Q_{\sqrt{2}r} : \mathcal{M}(|Du|^2)(z) > N^2\}| \\
&\le |\{z \in Q_{\sqrt{2}r} : \mathcal{M}_{Q_{2\sqrt{2}r}}(|D(u-v)|^2)(z) > N_1^2\}| \\
&\le c\int_{Q_{2\sqrt{2}r}} |D(u-v)|^2 dz \\
&\le c\eta^2 |Q_{\sqrt{2}r}|
\end{aligned}$$

for some constant $c = c(\nu, L, m, n)$. By taking η small enough, we derive

$$|\{x \in Q_1 : \mathcal{M}(|Du|^2)(x) > N^2\} \cap Q_r(x_*)| \le \epsilon |Q_r(x_*)|$$

which is a contradiction to assumption (4.1).

We now consider the case $\operatorname{dist}(x_*, \partial\Omega^-) \le 5\sqrt{2}r$ or $\operatorname{dist}(x_*, \partial\Omega^+) \le 5\sqrt{2}r$. Without loss of generality, we assume that $x_* \in \Omega^-$. By using Definition 1.1 again, we can choose appropriate x coordinate system satisfying

$$Q_{75r,x} \cap \{x : x_n < -75r\delta\} \subset Q_{75r,x} \cap \Omega^- \subset Q_{75r,x} \cap \{x : x_n < 75r\delta\}, \tag{4.9}$$

$$⨍_{Q_{75r,x}\cap\Omega^-} \big|A_{ij}^{\alpha\beta,-}(x', x_n) - \overline{A_{ij}^{\alpha\beta,-}}_{B'_{75r,x}}(x_n)\big|^2\, dx \le \delta^2. \tag{4.10}$$

Note that $Q_{15r,x}$ contains $Q_{5\sqrt{2}r}(x_*)$ in this coordinate system. After fixing x coordinate system, we can take y coordinate system at the origin satisfying

$$Q_{75r,y} \cap \{y_n > 75r\delta\} \subset \Omega^+ \cap Q_{75r,y} \subset Q_{75r,y} \cap \{y_n > -75r\delta\}, \tag{4.11}$$

$$⨍_{Q_{75r,y}\cap\Omega^+} \big|A_{ij}^{\alpha\beta,+}(y', y_n) - \overline{A_{ij}^{\alpha\beta,+}}_{B'_{75r,y}}(y_n)\big|^2 dy \le \delta^2. \tag{4.12}$$

We let θ be the angle between x_n direction in x coordinate system and y_n direction in y coordinate system. Since

$$\big(Q_{75r,x} \cap \{x_n = -75r\delta\}\big) \cap \big(Q_{75r,y} \cap \{y_n = 75r\delta\}\big) = \emptyset, \tag{4.13}$$

with the same spirit in Remark 3.2 we can see that

$$\frac{\theta}{2} \le \tan(\frac{\theta}{2}) \le \delta\,.$$

For this θ, we define $\widetilde{Q_{75r}}$ as in Section 3 and note that $\widetilde{Q_{75r}} \subset Q_{150} \subset \Omega$. We recall from (4.4) that $x_1 \in Q_r(y) \cap \Omega$ to discover that

$$\widetilde{Q_{75r}} \subset Q_{150r}(x_1).$$

Consequently, we obtain

$$\frac{1}{|\widetilde{Q_{75r}}|} \int_{\widetilde{Q_{75r}}} |Du|^2 dw \le \frac{|Q_{150r}(x_1)|}{|\widetilde{Q_{75r}}|} \fint_{Q_{150r}(x_1)} |Du|^2 dw \le 5 \cdot 2^n.$$

Here we use the fact that $\frac{1}{5}|Q_r| \le |\widetilde{Q_r}| \le 5|Q_r|$. Similarly, we have

$$\frac{1}{|Q_{75r}|} \int_{Q_{75r} \cap \Omega} |F|^2 dz \le 5 \cdot 2^n \delta^2.$$

With the same scaling argument which is used for the previous case, we apply Lemma 3.5 to our case. Then for $0 < \eta < 1$ fixed, we can find a small $\delta = \delta(\eta, \nu, L, m, n)$ and a function v defined in $\widetilde{Q_{75r}^{(60r)}}$ such that

$$\fint_{\widetilde{Q_{75r}^{(30r)}}} |D(u-v)|^2 \, dw \le \eta^2 \quad \text{and} \quad \|Dv\|^2_{L^\infty(\widetilde{Q_{75r}^{(45r)}})} \le N_2^2 \tag{4.14}$$

where $N_2 = N_2(n, n_2)$ similar to (4.6).

Note that for small δ, we assume that

$$\widetilde{Q_{75r}^{(15r)}} \subset \widetilde{Q_{75r}^{(20r)}} \subset Q_{25r} \subset \widetilde{Q_{75r}^{(30r)}}.$$

Then, we claim that

$$\{w \in \widetilde{Q_{75r}^{(15r)}} : \mathcal{M}(|Du|^2) > N^2\} \subset \{w \in \widetilde{Q_{75r}^{(15r)}} : \mathcal{M}_{Q_{25r}}(|D(u-v)|^2) > N_2^2\}, \tag{4.15}$$

where $N^2 = \max\{4N_2^2, 6^n\}$. To do this, we suppose that

$$x_0 \in \{w \in \widetilde{Q_{75r}^{(15r)}} : \mathcal{M}_{Q_{25r}}(|D(u-v)|^2)(w) \le N_2^2\}. \tag{4.16}$$

If $\rho \le 5r$, then from $Q_\rho(x_0) \subset \widetilde{Q_{75r}^{(20r)}} \subset Q_{25r}$, (4.14), and (4.16),

$$\fint_{Q_\rho(x_0)} |Du|^2 dw \le 2 \fint_{Q_\rho(x_0)} [|D(u-v)|^2 + |Dv|^2] dw \le 4N_2^2$$

and if $\rho > 5r$, then $Q_\rho(x_0) \subset Q_{6\rho}(x_1)$

$$\fint_{Q_\rho(x_0)} |Du|^2 dw \le \frac{|Q_{6\rho}(x_1)|}{|Q_\rho(x_0)|} \fint_{Q_{6\rho}(x_1)} |Du|^2 dw \le 6^n.$$

Thus we have

$$x_0 \in \{w \in \widetilde{Q_{75r}^{(15r)}} : \mathcal{M}(|Du|^2)(w) \le N^2\}$$

and our claim (4.15) follows. Then we observe that $Q_r(x_*)$ in (4.3) is covered by $\widetilde{Q_{75r}^{(15r)}}$ to find that

$$\begin{aligned}
&|\{x \in Q_r(x_*) : \mathcal{M}(|Du|^2)(x) > N^2\}| \\
&\le |\{w \in \widetilde{Q_{75r}^{(15r)}} : \mathcal{M}(|Du|^2)(w) > N^2\}| \\
&\le |\{w \in \widetilde{Q_{75r}^{(15r)}} : \mathcal{M}_{Q_{25r}}(|D(u-v)|^2)(w) > N_2^2\}|
\end{aligned}$$

$$
\begin{aligned}
&\le c \int_{\widetilde{Q_{75r}^{(30r)}}} |D(u-v)|^2 dw \\
&\le c\eta^2 |\widetilde{Q_{75r}^{(15r)}}| \\
&\le c\eta^2 |Q_{15r}|
\end{aligned}
$$

for some constant $c = c(\nu, L, m, n)$. By taking η small enough, we derive

$$|\{x \in Q_1 : \mathcal{M}(|Du|^2)(x) > N^2\} \cap Q_r(x_*)| \le \epsilon |Q_r(x_*)|$$

which is a contradiction to assumption (4.1). □

Now, we are ready to prove the main Theorem.

Proof of Theorem 1.3. Let $u \in H_0^1(\Omega, \mathbb{R}^m)$ be the weak solution of (1.1) under the assumptions in Theorem 1.3. We first fix $p > 2$ and take $N > 1$ as in Lemma 4.1. We denote the letter c by the constant that can be explicitly computed in terms of known quantities, $\nu, L, m, n,$ and p. We assume that

$$\|u\|_{L^p(Q_5)} + \|F\|_{L^p(Q_5)} \le \delta \tag{4.17}$$

by replacing u and F by

$$\frac{u}{\frac{1}{\delta}(\|u\|_{L^p(Q_5)} + \|F\|_{L^p(Q_5)}) + \sigma} \quad \text{and} \quad \frac{F}{\frac{1}{\delta}(\|u\|_{L^p(Q_5)} + \|F\|_{L^p(Q_5)}) + \sigma}$$

for $\sigma > 0$, respectively. We want to show that

$$\|Du\|_{L^p(Q_1)} \le c$$

after letting $\sigma \to 0$. However, in view of (2.1), it suffices to show that

$$\|\mathcal{M}(|Du|^2)\|_{L^{p/2}(Q_1)} \le c.$$

To apply Lemma 2.2 we first define

$$
\begin{gathered}
C = \{x \in Q_1 : \mathcal{M}(|Du|^2) > N^2\}, \\
D = \{x \in Q_1 : \mathcal{M}(|Du|^2) > 1\} \cup \{x \in Q_1 : \mathcal{M}(|F|^2) > \delta^2\}.
\end{gathered}
$$

For $\epsilon \in (0,1)$ to be determined later, by weak 1-1 estimates, the standard L^2 estimates, and Hölder's inequality, we have

$$
\begin{aligned}
|C| &\le \frac{c}{N^2} \int_{Q_1} |Du|^2 \, dx \\
&\le \frac{c}{N^2} \int_{Q_5} |u|^2 + |F|^2 \, dx \\
&\le \frac{c}{N^2} (\|u\|_{L^p(Q_5)}^2 + \|F\|_{L^p(Q_5)}^2) \\
&\le \frac{c\delta^2}{N^2}.
\end{aligned}
$$

So we take $\delta > 0$ so small that

$$|C| \le \frac{c\delta^2}{N^2} < \epsilon |Q_1| \tag{4.18}$$

holds. This shows the first condition (2.3) of Lemma 2.2. Moreover, its second condition (2.4) is shown by Lemma 4.1. Then, by Lemma 2.2, we see that

$$|C| < \epsilon_1 |D| \quad \text{where } \epsilon_1 = 2\sqrt{2}(10)^n \epsilon.$$

Since our problem (1.1) is invariant under normalization, we can obtain the same results for $(\frac{u}{N}, \frac{F}{N})$, $(\frac{u}{N^2}, \frac{F}{N^2})$, $(\frac{u}{N^3}, \frac{F}{N^3})$, ..., inductively. From this iteration argument, see [6, Corollary 4.10], we have the following decay estimates of $\mathcal{M}(|Du|^2)$:

$$\begin{aligned}
&|\{x \in Q_1 : \mathcal{M}(|Du|^2) > N^{2k}\}| \\
&\le \epsilon_1^k |\{x \in Q_1 : \mathcal{M}(|Du|^2) > 1\}| + \sum_{i=1}^{k} \epsilon_1^i |\{x \in Q_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\}|.
\end{aligned}$$

Applying Lemma 2.1 to

$$g = \mathcal{M}(|Du|^2), \quad \lambda = N^2, \quad \theta = 1, \quad q = \frac{p}{2},$$

a direct computation yields

$$\begin{aligned}
&\|\mathcal{M}(|Du|^2)\|_{L^{p/2}(Q_1)}^{p/2} \\
&\le c\Big(1 + \sum_{k\ge1} N^{2k\frac{p}{2}} |\{x \in Q_1 : \mathcal{M}(|Du|^2) > N^{2k}\}|\Big) \\
&\le c(1 + \sum_{k\ge1} N^{kp} \epsilon_1^k |\{x \in Q_1 : \mathcal{M}(|Du|^2) > 1\}| \\
&\quad + \sum_{k\ge1} N^{kp} \sum_{i=1}^{k} \epsilon_1^i |\{x \in Q_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\}|\Big) \\
&=: S_1 + S_2.
\end{aligned}$$

We compute S_1 and S_2 in the following way:

$$S_1 \le c\Big(1 + \sum_{k\ge1} N^{kp} \epsilon_1^k |\{x \in Q_1 : \mathcal{M}(|Du|^2) > 1\}|\Big) \le c\Big(1 + \sum_{k\ge1} N^{kp} \epsilon_1^k\Big)$$

and

$$\begin{aligned}
S_2 &\le c \sum_{k\ge1} N^{kp} \sum_{i=1}^{k} \epsilon_1^i |\{x \in Q_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\}| \\
&= c \sum_{i\ge1} \sum_{k\ge i} N^{kp} \epsilon_1^i |\{x \in Q_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\}| \\
&= c \sum_{i\ge1} (N^p \epsilon_1)^i \sum_{k\ge i} (N^p)^{k-i} |\{x \in Q_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\}| \\
&= c \sum_{i\ge1} (N^p \epsilon_1)^i \sum_{j\ge0} (N^p)^j |\{x \in Q_1 : \mathcal{M}(|\frac{F}{\delta}|^2) > N^{2j}\}| \\
&\le c \sum_{i\ge1} (N^p \epsilon_1)^i \|\mathcal{M}(|\frac{F}{\delta}|^2)\|_{L^{p/2}(Q_1)} \\
&\le c \sum_{i\ge1} (N^p \epsilon_1)^i \frac{\|F\|_{L^p(Q_5)}^2}{\delta^2} \\
&\le c \sum_{i\ge1} (N^p \epsilon_1)^i.
\end{aligned}$$

Therefore we have

$$\|\mathcal{M}(|Du|^2)\|_{L^{p/2}(Q_1)}^{p/2} \leq c\Big(1 + \sum_{k\geq 1}(N^p\epsilon_1)^k\Big)$$

where $\epsilon_1 = 2\sqrt{2}(10)^n\epsilon$.

We first take $\epsilon > 0$ sufficiently small satisfying

$$N^p\epsilon_1 < 1.$$

Then one can select a corresponding small $\delta = \delta(\nu, L, m, n, p) > 0$ from Lemma 4.1. This completes the proof. □

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