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ALMOST ENTIRE SOLUTIONS OF THE BURGERS EQUATION

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Abstract. We consider Burgers equation on the whole $x-t$ plane. We require the solution to be classical everywhere, except possibly over a closed set S of potential singularities, which is
(a) a subset of a countable union of ordered graphs of differentiable functions,
(b) has one dimensional Hausdorff measure, $H^1(S)$, equal to zero.
We establish that under these conditions the solution is identically equal to a constant.

1. Introduction

In this note we establish a sort of rigidity theorem for solutions of the Burgers equation

$$h_t(x,t) + h(x,t)h_x(x,t) = 0 \tag{1.1}$$

in the plane $\mathbb{R}_x \times \mathbb{R}_t$. We consider functions $h(x,t)$ that solve (1.1) classically, pointwise, except perhaps on a closed set S of the $x-t$ plane as in the Abstract, and we show that h must be identically constant. We note that such a statement is false in the half plane $\mathbb{R}_x \times \mathbb{R}_t^+$ because of rarefaction waves. We also note that the conclusion of the theorem is relatively simple to recover for entropy solutions. Indeed if $u(x,t)$ is an $L^\infty(\mathbb{R}_x \times \mathbb{R}_t)$ entropy solution to

$$\begin{gathered} u_t + \frac{1}{2}(u^2)_x = 0 \\ u(x,0) = u_0(x) \end{gathered} \tag{1.2}$$

then we have the (well known) estimate

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t} \tag{1.3}$$

for every $a > 0$, $t > 0$ with E depending only on $\|u_0\|_{L^\infty} = M$ (see [4, Theorem 16-4] or [3, Lemma in 3.4.3]). By shifting the origin of time all the way to $t = -\infty$, and by uniqueness in the entropy class, we conclude via (1.3) that $x \to u(x,t)$ is nonincreasing for every t. Thus in particular u_0 is a nonincreasing L^∞ function, and if u_0 is not identically constant (a.e.) then the solution of (1.2) will have a shock. Thus, the hypothesis $H^1(S) = 0$ will force u_0 to be identically constant, and so also u.

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There is a similar result for the eikonal equation

$$\Big(\frac{\partial u}{\partial t}\Big)^2 + \Big(\frac{\partial u}{\partial x}\Big)^2 = 1 \tag{1.4}$$

by Caffarelli and Crandall [1] which states that if u solves (1.4) pointwise on $\mathbb{R}^2 \setminus \widehat{S}$, with $H^1(\widehat{S}) = 0$, then necessarily u is either affine or a double "cone function", $u(y) = a \pm |y - z|$, $y = (x, t)$, $z = (x_0, t_0)$. The point in [1] again is that u is not assumed a viscosity solution.

The proof of our result is based on a simple and explicit change of variables (see (2.2) below) that transforms (1.2) into (1.4), and actually establishes almost the equivalence of the two problems in $\mathbb{R}^2$. Note that for the set $\widehat{S}$ in [1] there is no extra hypothesis besides that $H^1(\widehat{S}) = 0$. Our only excuse for writing it down is that it concerns the Burgers equation, which in spite of its simplicity pervades the theory of hyperbolic conservation laws [2, 3].

2. Main result

Theorem 2.1. *Let $h(x,t)$ be a measurable function on $\mathbb{R}^2$ and suppose that S is closed and on $\mathbb{R}^2 \setminus S$ the following hold: $h(x,t)$ is continuous, $\frac{\partial h}{\partial t}$, $\frac{\partial h}{\partial x}$ exist, $x \to \frac{\partial h}{\partial x}(x,t)$ is $L^1_{\rm loc}$ and moreover*

$$h_t + hh_x = 0, \quad \text{on } \mathbb{R}^2 \setminus S. \tag{2.1}$$

If $H^1(S) = 0$ and $S \subset \cup_{i\in Z}\Gamma_i$, where $\Gamma_i := \{(x,t) : t = p_i(x),\ p_i$ differentiable, $x \in \mathbb{R}\}$,

$$\dots < p_{-n}(x) < \dots < p_{-1}(x) < p_1(x) < p_2(x) < \dots < p_n(x) < \cdots ,$$

then $h \equiv$ constant on $\mathbb{R}^2$, and $S = \emptyset$.

Notes. (1) The change of variables $h = c(v)$ converts $v_t + c(v)v_x = 0$ into Burgers' equation $h_t + hh_x = 0$, hence this more general equation is covered for differentiable c provided that $c' \neq 0$. Note that if we write the equation for v in divergence form $v_t + (C(v))_x = 0$, where $C' = c$, then the condition $c' \neq 0$ corresponds to $C'' \neq 0$ which is naturally weaker than the usual condition of genuine nonlinearity $C'' > 0$, since we do not require any orientation of the $x - t$ plane.

(2) The change of variables relating (1.2) to (1.4) is basically

$$u(x,t) = \int_0^t \frac{ds}{\sqrt{h^2(x,s) + 1}} + g(x) \tag{2.2}$$

where $g(x) = \int_0^x \frac{h(u,0)du}{\sqrt{h^2(u,0)+1}}$. Note that the projected characteristics of the corresponding equations coincide,

$$\begin{aligned} \frac{dx}{d\tau} &= h\frac{dx}{d\tau} = u_x = \frac{h}{\sqrt{h^2+1}} \\ \frac{dt}{d\tau} &= 1\frac{dt}{d\tau} = u_t = \frac{1}{\sqrt{h^2+1}}. \end{aligned}$$

The need for differentiating under the integral sign in (2.2) for obtaining (1.4) forces us to introduce the perhaps unnecessary hypothesis that S lies on a set of graphs.

(3) The hypotheses on the singular set a priori do not exclude S to be a countable union of Cantor sets arranged on a family of parallel lines in the $x - t$ plane.

Proof of Theorem 2.1. For the convenience of the reader we begin by giving the proof in the simple case where S lies on a single differentiable graph contained inside a strip, $S \subset \Gamma := \{(x,t) \mid t = p(x),\ p \text{ differentiable},\ 0 < p(x) < 1,\ x \in \mathbb{R}\}$. Set

$$\Omega^+ = \{(x,t) \in \mathbb{R}^2 \mid t \le p(x)\}, \quad \Omega^- = \{(x,t) \in \mathbb{R}^2 \mid t \ge p(x)\}.$$

For $(x,t) \in \Omega^+$, we define

$$u^+(x,t) = \int_0^t \frac{ds}{\sqrt{h^2(x,s)+1}} + g^+(x), \tag{2.3}$$

where

$$g^+(x) = \int_0^x \frac{h(u,0)du}{\sqrt{h^2(u,0)+1}}$$

and for $(x,t) \in \Omega^-$, we define

$$u^-(x,t) = \int_1^t \frac{ds}{\sqrt{h^2(x,s)+1}} + g^-(x), \tag{2.4}$$

where

$$g^-(x) = \int_0^x \frac{h(u,1)du}{\sqrt{h^2(u,1)+1}}\,.$$

We begin with $u^+(x,t)$ for $t \le p(x)$, $(x,t) \in U := \mathbb{R}^2 \setminus S$, open. By our hypothesis

$$\begin{aligned} u_x^+(x,t) &= \int_0^t \frac{-h(x,s)h_x(x,s)}{\left(\sqrt{h^2(x,s)+1}\right)^3} ds + \frac{h(x,0)}{\sqrt{h^2(x,0)+1}} \\ &= \int_0^t \frac{h_s(x,s)}{\left(\sqrt{h^2(x,s)+1}\right)^3} ds + \frac{h(x,0)}{\sqrt{h^2(x,0)+1}} \\ &= \frac{h(x,t)}{\sqrt{h^2(x,t)+1}}. \end{aligned} \tag{2.5}$$

On the graph we have

$$u_x^+(x,p(x)) = \frac{h(x,p(x))}{\sqrt{h^2(x,p(x))+1}}, \quad (x,p(x)) \notin S. \tag{2.6}$$

Differentiating in t is straightforward, and holds quite generally that

$$u_t^+(x,t) = \frac{1}{\sqrt{h^2(x,t)+1}}, \quad u_t^+(x,p(x)) = \frac{1}{\sqrt{h^2(x,p(x))+1}}. \tag{2.7}$$

Thus from (2.5) and (2.7) we have

$$(u_x^+(x,t))^2 + (u_t^+(x,t))^2 = 1 \quad \text{in } \Omega^+ \setminus S. \tag{2.8}$$

Analogously we argue for $u^-(x,t)$ and we obtain

$$u_x^-(x,t) = \frac{h(x,t)}{\sqrt{h^2(x,t)+1}} \quad \text{in } \Omega^- \setminus S, \tag{2.9}$$

$$u_x^-(x,p(x)) = \frac{h(x,p(x))}{\sqrt{h^2(x,p(x))+1}}, \quad (x,p(x)) \notin S, \tag{2.10}$$

$$u_t^-(x,t) = \frac{1}{\sqrt{h^2(x,t)+1}}, \quad u_t^-(x,p(x)) = \frac{1}{\sqrt{h^2(x,p(x))+1}} \tag{2.11}$$

and so once more

$$(u_x^-(x,t))^2 + (u_t^-(x,t))^2 = 1 \quad \text{in } \Omega^- \setminus S. \tag{2.12}$$

Also from (2.6) and (2.10) we obtain

$$u_x^+(x,p(x)) = u_x^-(x,p(x)), \quad u_t^+(x,p(x)) = u_t^-(x,p(x)), \quad (x,p(x)) \notin S. \tag{2.13}$$

We now set

$$u(x,t) = \begin{cases} u^+(x,t), & (x,t) \in \Omega^+ \\ u^-(x,t) + \Delta(x), & (x,t) \in \Omega^- \end{cases} \tag{2.14}$$

where

$$\Delta(x) := u^+(x,p(x)) - u^-(x,p(x)), \quad x \in \mathbb{R}. \tag{2.15}$$

Note that $\Gamma \setminus S$ is open in Γ and so is its projection $\pi_x(\Gamma \setminus S) = \cup_{i=1}^{\infty}(a_i,b_i) =: O$, and for $x \in O$

$$\begin{aligned} \frac{d\Delta(x)}{dx} &= u_x^+(x,p(x)) + u_t^+(x,p(x))p'(x) - (u_x^-(x,p(x)) \\ &\quad + u_t^-(x,p(x))p'(x)) = 0 \end{aligned} \tag{2.16}$$

(by (2.13)). Therefore, by the continuity of h and p, $u(x,t)$ is differentiable on $\mathbb{R}^2 \setminus S$, and by (2.8), (2.12), (2.14) and (2.16),

$$(u_x(x,t))^2 + (u_t(x,t))^2 = 1 \quad \text{on } \mathbb{R}^2 \setminus S. \tag{2.17}$$

Hence, by the result in [1], u is of the form

$$u(x,t) = ax + bt + \gamma \quad (a^2 + b^2 = 1), \tag{2.18}$$

or

$$u(x,t) = c \pm \sqrt{(x-x_0)^2 + (t-t_0)^2}. \tag{2.19}$$

In the first case $u_t = b$ and so $h(x,t) \equiv$ constant.

On the other hand (2.19) gives

$$\begin{aligned} u_t(x,t) &= \pm \frac{t-t_0}{\sqrt{(x-x_0)^2 + (t-t_0)^2}} \\ \Rightarrow h(x,t) &= \frac{x-x_0}{t-t_0} \end{aligned} \tag{2.20}$$

which is singular on $\{t = t_0\}$, and thus is excluded by the hypothesis $H^1(S) = 0$. Therefore $h(x,t) \equiv$ constant is the only option.

Note that $\Delta(x)$ is continuous for $x \in \mathbb{R}; \mathcal{L}(\pi_x(S)) = 0$.

For the proof of the general case, we indicate the necessary modifications. Suppose $p_\ell(x) < p_{\ell+1}(x)$, $a_\ell(x) \in C^1$, $p_\ell(x) < a_\ell(x) < p_{\ell+1}(x)$, $\ell = 1,2,\dots$, $\ell = -2,-3,\dots$ (and $p_{-1}(x) < a_0(x) < p_1(x)$) where we have inserted the C^1 graphs $a_\ell(x)$ that will play the role of the horizontal lines $t = 0$ and $t = 1$ in the simple case treated above. Let

$$\Omega_1^+ = \{p_{-1}(x) \le t \le p_1(x)\}, \quad \Omega_1^- = \{p_1(x) \le t \le a_1(x)\}, \tag{2.21}$$

$$\left.\begin{aligned} u_1^+(x,t) &:= \int_{a_0(x)}^{t} \frac{ds}{\sqrt{h^2(x,s)+1}} + g_1^+(x), \\ g_1^+(x) &= \int_0^x \frac{h(s,a_0(s)) + a_0'(s)}{\sqrt{h^2(s,a_0(s))+1}} ds, \end{aligned}\right\} \text{on } \Omega_1^+; \tag{2.22}$$

$$
\left.\begin{aligned}
u_1^-(x,t) &:= \int_{a_1(x)}^t \frac{ds}{\sqrt{h^2(x,s)+1}} + g_1^-(x),\\
g_1^-(x) &= \int_0^x \frac{h(s,a_1(s))+a_1'(s)}{\sqrt{h^2(s,a_1(s))+1}}ds,
\end{aligned}\right\}\quad \text{on } \Omega_1^-; \tag{2.23}
$$

$$
\Delta_1(x) := u_1^+(x,p_1(x)) - u_1^-(x,p_1(x));
$$

$$
u_1(x,t) = \begin{cases} u_1^+(x,t), & \text{on } \Omega_1^+\\ u_1^-(x,t)+\Delta_1(x), & \text{on } \Omega_1^-. \end{cases} \tag{2.24}
$$

For $i = 2,3,\dots$, set

$$
\Omega_i^+ = \{a_{i-1}(x) \le t \le p_i(x)\}, \quad \Omega_i^- = \{p_i(x) \le t \le a_i(x)\}, \tag{2.25}
$$

$$
u_i^+(x,t) := u_{i-1}^-(x,t) + \Delta_{i-1}(x), \quad \text{on } \Omega_i^+, \tag{2.26}
$$

$$
\Delta_j(x) := (u_j^+ - u_j^-)(x,p_j(x)), \quad j = 1,2,\dots. \tag{2.27}
$$

Set

$$
\left.\begin{aligned}
u_i^-(x,t) &:= \int_{a_i(x)}^t \frac{ds}{\sqrt{h^2(x,s)+1}} + g_i^-(x),\\
g_i^-(x) &= \int_0^x \frac{h(s,a_i(s))+a_i'(s)}{\sqrt{h^2(s,a_i(s))+1}}ds,
\end{aligned}\right\}\quad \text{in } \Omega_i^-, \tag{2.28}
$$

$$
u_k(x,t) = \begin{cases} u_k^+(x,t), & \text{on } \Omega_k^+\\ u_k^-(x,t)+\Delta_k(x), & \text{on } \Omega_k^- \end{cases} \quad k = 1,2,\dots. \tag{2.29}
$$

Next we define u below $a_0(x)$.

$$
u_{-1}^+(x,t) = u_1^+(x,t) \quad \text{on } \Omega_{-1}^+ = \{p_{-1}(x) \le t \le a_0(x)\}, \tag{2.30}
$$

with

$$
u_{-1}^-(x,t) := \int_{a_{-1}(x)}^t \frac{ds}{\sqrt{h^2(x,s)+1}} + g_{-1}^-(x), \tag{2.31}
$$

on $\Omega_{-1}^- = \{a_{-1}(x) \le t \le p_{-1}(x)\}$, where

$$
g_{-1}^-(x) = \int_0^x \frac{h(s,a_{-1}(s))+a_{-1}'(s)}{\sqrt{h^2(s,a_{-1}(s))+1}}ds, \tag{2.32}
$$

$$
\Delta_{-1}(x) := u_{-1}^+(x,p_{-1}(x)) - u_{-1}^-(x,p_{-1}(x)), \tag{2.33}
$$

$$
u_{-1}(x,t) = \begin{cases} u_{-1}^+(x,t), & \text{in } \Omega_{-1}^+,\\ u_{-1}^-(x,t)+\Delta_{-1}(x), & \text{in } \Omega_{-1}^-. \end{cases} \tag{2.34}
$$

And further down $i = 2,3,\dots$, we set

$$
\Omega_{-i}^+ = \{p_{-i}(x) \le t \le a_{-i+1}(x)\}, \quad \Omega_{-i}^- = \{a_{-i}(x) \le t \le p_{-i}(x)\}, \tag{2.35}
$$

$$
u_{-i}^+(x,t) := u_{-i+1}^-(x,t) + \Delta_{-i+1}(x), \quad \text{on } \Omega_{-i}^+, \tag{2.36}
$$

$$
\Delta_{-i}(x) := (u_{-i}^+ - u_{-i}^-)(x,p_{-i}(x)), \tag{2.37}
$$

with

$$
u_{-i}^-(x,t) := \int_{a_{-i}(x)}^t \frac{ds}{\sqrt{h^2(x,s)+1}} + g_{-i}^-(x), \quad \text{on } \Omega_{-i}^- \tag{2.38}
$$

where

$$g^-_{-i}(x) = \int_0^x \frac{h(s, a_{-i}(s)) + a'_{-i}(s)}{\sqrt{h^2(s, a_{-i}(s)) + 1}} ds, \tag{2.39}$$

$$u_{-k}(x,t) = \begin{cases} u^+_{-k}(x,t), & \text{in } \Omega^+_{-k}, \\ u^-_{-k}(x,t) + \Delta_{-k}(x), & \text{in } \Omega^-_{-k}, \end{cases} \quad k = 2, 3, \dots . \tag{2.40}$$

Finally we set

$$u(x,t) = u_k(x,t) \quad \text{on } \Omega^+_k \cup \Omega^-_k, \ k \in Z \setminus \{0\}. \tag{2.41}$$

With this definition we note that $u(x,t)$ is differentiable on $\mathbb{R}^2 \setminus S$, and

$$\Big(\frac{\partial u}{\partial t}\Big)^2 + \Big(\frac{\partial u}{\partial x}\Big)^2 = 1 \quad \text{on } \mathbb{R}^2 \setminus S. \tag{2.42}$$

and thus we conclude as before that $h(x,t) \equiv$constant and $S = \emptyset$. The proof is complete. $\square$

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