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OSCILLATORY BEHAVIOR FOR SECOND-ORDER DAMPED DIFFERENTIAL EQUATION WITH NONLINEARITIES INCLUDING RIEMANN-STIELTJES INTEGRALS

ERCAN TUNÇ, HAIDONG LIU

Abstract. In this article, we establish new oscillation criteria for forced second-order damped differential equations with nonlinearities that include Riemann-Stieltjes integrals. The results obtained here extend related results reported in the literature, and can easily be extended to more general equations of the type considered here. Two examples illustrate the results obtained here.

1. Introduction

This article concerns the oscillatory behavior of the forced second order differential equation with a nonlinear damping term,

$$\big(r(t)\phi_\alpha(x'(t))\big)' + p(t)\phi_\alpha(x'(t)) + f(t,x) = e(t), \quad t \ge t_0 \ge 0, \tag{1.1}$$

with

$$f(t,x) = q(t)\phi_\alpha(x(t)) + \int_a^b g(t,s)\phi_{\gamma(t,s)+\alpha-\alpha\beta(t)}(x(t))d\xi(s), \tag{1.2}$$

where $a, b \in \mathbb{R}$ with $b \in (a,\infty)$, $\alpha > 0$, and $\phi_*(u) := |u|^* \operatorname{sgn} u$.

In the remainder of this article we assume that:

- (i) r, p, q and $e : [t_0,\infty) \to \mathbb{R}$ are real valued continuous functions with $r(t) > 0$;
- (ii) $g : [t_0,\infty) \times [a,b] \to \mathbb{R}$ is a real valued continuous function;
- (iii) $\beta : [t_0,\infty) \to (0,\infty)$ and $\gamma : [t_0,\infty) \times [a,b] \to \mathbb{R}$ are real valued continuous function such that $\gamma(t,\cdot)$ is strictly increasing on $[a,b]$, and

$$0 < \gamma(t,a) < \alpha\beta(t) < \gamma(t,b) \quad \text{and} \quad \alpha\beta(t) \le \gamma(t,a) + \alpha, \quad \text{for } t \ge t_0; \tag{1.3}$$

- (iv) $\xi : [a,b] \to \mathbb{R}$ is a real valued strictly increasing function.

Here $\int_a^b f(s)d\xi(s)$ denotes the Riemann-Stieltjes integral of the function f on $[a,b]$ with respect to ξ.

As usual, a nontrivial solution $x(t)$ of equation (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

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We note that as special cases, when $\alpha = 1$ and $p(t) \equiv 0$, equation (1.1) reduces to the equation

$$(r(t)x'(t))' + q(t)x(t) + \int_a^b g(t,s)\phi_{\gamma(t,s)+1-\beta(t)}(x(t))d\xi(s) = e(t) : \tag{1.4}$$

when $p(t) \equiv 0$, $\beta(t) \equiv 1$, $\gamma(t,s) = \gamma(s)$ and $a = 0$, equation (1.1) reduces to

$$\big(r(t)\phi_\alpha(x'(t))\big)' + q(t)\phi_\alpha(x(t)) + \int_0^b g(t,s)\phi_{\gamma(s)}(x(t))d\xi(s) = e(t); \tag{1.5}$$

and when $\xi(s)$ is a step function, the integral term in the equation (1.5) reduces to a finite sum and hence equation (1.5) becomes

$$(r(t)\phi_\alpha(x'(t)))' + q(t)\phi_\alpha(x(t)) + \sum_{i=1}^n q_i(t)\phi_{\alpha_i}(x(t)) = e(t). \tag{1.6}$$

In recent years, differential equations and variational problems with variable exponent growth conditions have been investigated extensively. We refer the reader to [1, 2, 7, 8, 10, 13, 14, 16, 17, 18]. The study of such problems arise from nonlinear elasticity theory and electrorheological fluids, see [10, 18]. At the same time, some results on the oscillatory behavior of solutions of equations with variable exponent growth conditions were established in [9, 19] and the references therein. On the other hand, many authors have been interested in differential equations with nonlinearity given by a Riemann-Stieltjes integral $\int_a^b f(s)d\xi(s)$. Because the integral term becomes a finite sum when $\xi(s)$ is a step function and a Riemann integral when $\xi(s) = s$. We refer to [5, 9, 12] for more information. In particularly, Liu and Meng [9] discussed equation (1.4), Hassan and Kong [5] studied equation (1.5).

Motivated by the above, we will establish interval oscillation criteria for the general equation (1.1) which involves variable exponent growth conditions. Our work is of significance because equation (1.1) not only contains a α-Laplacian term but also contains a damping term and allows nonlinear terms given by variable exponents. It is our belief that the present paper will contribute significantly to the study of oscillatory behavior of solutions of second order damped differential equations with nonlinearities given by Riemann-Stieltjes integrals.

The paper is organized as follows. In Section 2 we establish interval oscillation criteria of both the El-Sayed type and the Kong type for equation (1.1). In Section 3 we apply our theory to two examples.

2. Main results

In the following, we denote by $L_\xi[a,b]$ the set of Riemann-Stieltjes integrable functions on $[a,b]$ with respect to ξ. We further assume that for any $t \in [t_0,\infty)$, $\gamma(t,\cdot)$, $1/\gamma(t,\cdot) \in L_\xi[a,b]$. To obtain our main results in this paper, we need the following lemmas.

Lemma 2.1 ([4]). *If X and Y are nonnegative and $\lambda > 1$, then*

$$\lambda XY^{\lambda-1} - X^\lambda \le (\lambda-1)Y^\lambda,$$

where equality holds if and only if $X = Y$.

The proofs of the following lemmas are similar to those of [9, Lemmas 2.1 and 2.2] and so the proofs will be omitted.

Lemma 2.2. *Assume that* (iii) *and* (1.3) *hold. Let* $h = \sup\{s \in (a,b) : \gamma(t,s) \le \alpha\beta(t),\ t \in [t_0,\infty)\}$*, and set*

$$m_1(t) := \int_h^b \frac{\alpha\beta^2(t)}{\gamma(t,s)}\Big(\int_h^b \mathrm{d}\xi(s)\Big)^{-1}\mathrm{d}\xi(s), \quad t \in [t_0,\infty),$$
$$m_2(t) := \int_a^h \frac{\alpha\beta^2(t)}{\gamma(t,s)}\Big(\int_a^h \mathrm{d}\xi(s)\Big)^{-1}\mathrm{d}\xi(s), \quad t \in [t_0,\infty).$$

Then for any function θ *satisfying* $\theta(t) \in (m_1(t), m_2(t))$ *for* $t \in [t_0,\infty)$*, there exists a function* $\eta : [t_0,\infty)\times[a,b] \to (0,\infty)$ *satisfying, for any* $t \in [t_0,\infty)$, $\eta(t,\cdot) \in L_\xi[a,b]$*, such that*

$$\int_a^b \gamma(t,s)\eta(t,s)d\xi(s) = \alpha\beta^2(t), \quad (t,s) \in [t_0,\infty)\times[a,b], \tag{2.1}$$
$$\int_a^b \eta(t,s)d\xi(s) = \theta(t), \quad (t,s) \in [t_0,\infty)\times[a,b]. \tag{2.2}$$

Lemma 2.3. *Let* $\theta : [t_0,\infty) \to (0,\infty)$ *and* $\eta : [t_0,\infty)\times[a,b] \to (0,\infty)$ *be functions such that* $\eta(t,\cdot) \in L_\xi[a,b]$ *for any* $t \in [t_0,\infty)$ *and* (2.2) *holds. Then, for any function* $w : [t_0,\infty)\times[a,b] \to [0,\infty)$ *satisfying, for any* $t \in [t_0,\infty)$, $w(t,\cdot) \in L_\xi[a,b]$*, we have*

$$\int_a^b \eta(t,s)w(t,s)d\xi(s) \ge \exp\Big(\frac{1}{\theta(t)}\int_a^b \eta(t,s)\ln[\theta(t)w(t,s)]d\xi(s)\Big), \tag{2.3}$$

where we use the convention that $\ln 0 = -\infty$ *and* $e^{-\infty} = 0$.

Following El-Sayed [3], for $c, d \in [t_0,\infty)$ with $c < d$, we define the function class $\mathcal{E}(c,d) := \{u \in C^1[c,d] : u(c) = 0 = u(d), u \not\equiv 0\}$. Our first main result provides an oscillation criterion for equation (1.1) of the El-Sayed type.

Theorem 2.4. *Suppose that for any* $T \ge t_0$*, there exist* $T \le a_1 < b_1 \le a_2 < b_2$ *such that for* $i = 1, 2$,

$$g(t,s) \ge 0 \quad \text{for } (t,s) \in [a_i,b_i]\times[a,b], \tag{2.4}$$
$$(-1)^i e(t) \ge 0 \quad \text{for } t \in [a_i,b_i]. \tag{2.5}$$

Let θ *be a function satisfying* $\theta(t) \in (m_1(t), \beta(t)]$ *for* $t \in [t_0,\infty)$*, and* $\eta : [t_0,\infty)\times[a,b] \to (0,\infty)$ *be a function such that* $1/\eta(t,\cdot) \in L_\xi[a,b]$ *and* (2.1)-(2.2) *hold. Suppose also that for* $i = 1, 2$*, there exists a function* $u_i \in \mathcal{E}(a_i,b_i)$ *such that*

$$\int_{a_i}^{bi} [\delta(t)Q(t)|u_i(t)|^{\alpha+1} - \delta(t)r(t)|u_i'(t)|^{\alpha+1}]dt > 0, \tag{2.6}$$

where

$$\delta(t) := \exp\Big(\int_{t_0}^t \frac{p(s)}{r(s)}ds\Big),$$

and

$$
\begin{aligned}
Q(t) = q(t) &+ \Big(\frac{\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big)|e(t)|}{\beta^2(t)-\theta(t)\beta(t)}\Big)^{\frac{\beta^2(t)-\theta(t)\beta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}} \\
&\times \exp\Big(\frac{\theta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}\Big[\ln\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big) \\
&+ \frac{\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\mathrm{d}\xi(s)}{\theta(t)}\Big]\Big).
\end{aligned}
\tag{2.7}
$$

Here we use the convention that $\ln 0 = -\infty$*,* $e^{-\infty} = 0$*, and* $0^0 = 1$ *due to the fact that* $\lim_{t\to 0^+} t^t = 1$*. Then equation* (1.1) *is oscillatory.*

Proof. Assume that (1.1) has an extendible solution $x(t)$ which is eventually positive or negative. Then, without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)$ such that $x(t) > 0$ for all $t \ge t_1$. When $x(t)$ is an eventually negative, the proof follows the same way except that the interval $[a_2, b_2]$ instead of $[a_1, b_1]$ is used. Define the function $w(t)$ by

$$
w(t) = \delta(t)\frac{r(t)\phi_\alpha(x'(t))}{\phi_\alpha(x(t))}, \quad t \ge t_1. \tag{2.8}
$$

Then, in view of (1.1) and (2.8), we obtain

$$
\begin{aligned}
&w'(t) \\
&= \delta'(t)\frac{r(t)\phi_\alpha(x'(t))}{\phi_\alpha(x(t))} + \delta(t)\Big[\frac{\big(r(t)\phi_\alpha(x'(t))\big)'}{\phi_\alpha(x(t))} - \frac{r(t)\phi_\alpha(x'(t))\,\big(\phi_\alpha(x(t))\big)'}{\big(\phi_\alpha(x(t))\big)^2}\Big] \\
&= \delta'(t)\frac{r(t)\phi_\alpha(x'(t))}{x^\alpha(t)} - \delta(t)\frac{p(t)\phi_\alpha(x'(t))}{x^\alpha(t)} - \delta(t)q(t) \\
&\quad - \delta(t)\int_a^b g(t,s)\big(x(t)\big)^{\gamma(t,s)-\alpha\beta(t)}d\xi(s) + \delta(t)\frac{e(t)}{x^\alpha(t)} \\
&\quad - \alpha\delta(t)r(t)\frac{\phi_\alpha(x'(t))x'(t)}{x^{\alpha+1}(t)} \\
&= -\delta(t)q(t) - \delta(t)\int_a^b g(t,s)\,(x(t))^{\gamma(t,s)-\alpha\beta(t)}\,d\xi(s) + \delta(t)\frac{e(t)}{x^\alpha(t)} \\
&\quad - \alpha\delta(t)r(t)\frac{\phi_\alpha(x'(t))x'(t)}{x^{\alpha+1}(t)} \\
&= -\delta(t)q(t) - \delta(t)\int_a^b g(t,s)\,(x(t))^{\gamma(t,s)-\alpha\beta(t)}\,d\xi(s) + \delta(t)\frac{e(t)}{x^\alpha(t)} \\
&\quad - \alpha\delta(t)r(t)\frac{|x'(t)|^{\alpha+1}}{x^{\alpha+1}(t)} \\
&= -\delta(t)q(t) - \delta(t)\int_a^b g(t,s)\,(x(t))^{\gamma(t,s)-\alpha\beta(t)}\,d\xi(s) + \delta(t)\frac{e(t)}{x^\alpha(t)} \\
&\quad - \alpha\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}},
\end{aligned}
\tag{2.9}
$$

for $t \ge t_1$.

From the assumption, there exists a nontrivial interval $[a_1, b_1] \subset [t_1, \infty)$ such that (2.4) and (2.5) hold with $i = 1$. Next, we consider two cases: case (I) $\theta(t) \equiv \beta(t)$, and case (II) $\theta(t) \in (m_1(t), \beta(t))$.

Assume that case (I) holds. Then, in view of (2.4), (2.5) and (2.9), we see that, for $t \in [a_1, b_1]$,

$$w'(t) \le -\delta(t)q(t) - \delta(t)\int_a^b g(t,s)\big(x(t)\big)^{\gamma(t,s)-\alpha\beta(t)} d\xi(s) - \alpha\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{\big(\delta(t)r(t)\big)^{1/\alpha}}. \tag{2.10}$$

Clearly, from the assumption on η, we have that

$$\int_a^b \eta(t,s)\left(\gamma(t,s) - \alpha\beta(t)\right) d\xi(s) = 0. \tag{2.11}$$

From (2.11) and Lemma 2.3, we obtain, for $t \in [a_1, b_1]$,

$$\begin{aligned}
&\int_a^b g(t,s)\big(x(t)\big)^{\gamma(t,s)-\alpha\beta(t)} d\xi(s)\\
&= \int_a^b \eta(t,s)\eta^{-1}(t,s)g(t,s)\left(x(t)\right)^{\gamma(t,s)-\alpha\beta(t)} d\xi(s)\\
&\ge \exp\Big(\frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln[\beta(t)\eta^{-1}(t,s)g(t,s)\big(x(t)\big)^{\gamma(t,s)-\alpha\beta(t)}]d\xi(s)\Big)\\
&= \exp\Big(\frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln[\beta(t)\eta^{-1}(t,s)g(t,s)]d\xi(s)\\
&\quad + \frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln[\big(x(t)\big)^{\gamma(t,s)-\alpha\beta(t)}]d\xi(s)\Big)\\
&= \exp\Big(\frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln[\beta(t)\eta^{-1}(t,s)g(t,s)]d\xi(s)\\
&\quad + \frac{\ln x(t)}{\beta(t)}\int_a^b \eta(t,s)\left(\gamma(t,s) - \alpha\beta(t)\right) d\xi(s)\Big)\\
&= \exp\Big(\frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln\big[\beta(t)\eta^{-1}(t,s)g(t,s)\big]\, d\xi(s)\Big)\\
&= \exp\Big(\ln[\beta(t)] + \frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln\big[\eta^{-1}(t,s)g(t,s)\big]\, d\xi(s)\Big).
\end{aligned}$$

Using this in (2.10), we see that, for $t \in [a_1, b_1]$,

$$\begin{aligned}
w'(t) &\le -\delta(t)q(t) - \delta(t)\exp\Big(\ln[\beta(t)]\\
&\qquad + \frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln[\eta^{-1}(t,s)g(t,s)]d\xi(s)\Big) - \alpha\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}}\\
&= -\delta(t)Q(t) - \alpha\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}},
\end{aligned} \tag{2.12}$$

where $Q(t)$ is defined by (2.7) with $\theta(t) \equiv \beta(t)$.

Multiplying both sides of (2.12) by $|u_1(t)|^{\alpha+1}$, integrating from a_1 to b_1, and using integration by parts, we obtain

$$\begin{aligned}
&\int_{a_1}^{b_1} \delta(t)Q(t)|u_1(t)|^{\alpha+1}dt \\
&\le -\int_{a_1}^{b_1} |u_1(t)|^{\alpha+1}w'(t)dt - \alpha\int_{a_1}^{b_1} |u_1(t)|^{\alpha+1}\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}}dt \\
&= (\alpha+1)\int_{a_1}^{b_1} \phi_\alpha(u_1(t))u_1'(t)w(t)dt - \alpha\int_{a_1}^{b_1} |u_1(t)|^{\alpha+1}\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}}dt \\
&\le \int_{a_1}^{b_1} \big[(\alpha+1)|u_1(t)|^{\alpha}|u_1'(t)||w(t)| - \alpha|u_1(t)|^{\alpha+1}\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}}\big]dt.
\end{aligned} \tag{2.13}$$

Applying Lemma 2.1 with

$$X = \Big(\alpha\frac{|u_1(t)|^{\alpha+1}}{(\delta(t)r(t))^{1/\alpha}}|w(t)|^{\frac{\alpha+1}{\alpha}}\Big)^{1/\lambda}, \quad \lambda = \frac{\alpha+1}{\alpha}, \quad Y = \Big(\frac{\alpha(\delta(t)r(t))^{\frac{1}{\alpha+1}}}{\alpha^{\frac{\alpha}{\alpha+1}}}|u_1'(t)|\Big)^{\alpha},$$

we see that

$$(\alpha+1)|u_1(t)|^{\alpha}|u_1'(t)||w(t)| - \alpha|u_1(t)|^{\alpha+1}\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}} \le \delta(t)r(t)|u_1'(t)|^{\alpha+1},$$

substituting this into (2.13) gives

$$\int_{a_1}^{b_1}[\delta(t)Q(t)|u_1(t)|^{\alpha+1} - \delta(t)r(t)|u_1'(t)|^{\alpha+1}]dt \le 0,$$

which contradicts (2.6) for $i = 1$.

Next, assume that case (II) holds. From (2.2) and (2.5), we have

$$\begin{aligned}
&\delta(t)\int_a^b g(t,s)[x(t)]^{\gamma(t,s)-\alpha\beta(t)}\mathrm{d}\xi(s) - \delta(t)\frac{e(t)}{x^\alpha(t)} \\
&= \delta(t)\int_a^b \big[g(t,s)[x(t)]^{\gamma(t,s)-\alpha\beta(t)} - \frac{e(t)}{x^\alpha(t)}\frac{\eta(t,s)}{\theta(t)}\big]\mathrm{d}\xi(s) \\
&= \delta(t)\int_a^b \big[g(t,s)[x(t)]^{\gamma(t,s)-\alpha\beta(t)} + \frac{|e(t)|}{x^\alpha(t)}\frac{\eta(t,s)}{\theta(t)}\big]\mathrm{d}\xi(s) \\
&= \delta(t)\int_a^b \frac{\eta(t,s)}{\theta(t)}\big[\frac{\theta(t)}{\eta(t,s)}g(t,s)[x(t)]^{\gamma(t,s)-\alpha\beta(t)} + \frac{|e(t)|}{x^\alpha(t)}\big]\mathrm{d}\xi(s).
\end{aligned} \tag{2.14}$$

If we let

$$p = \frac{\theta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}, \quad q = \frac{\beta^2(t)-\theta(t)\beta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}, \tag{2.15}$$

$$A = \frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)[x(t)]^{\gamma(t,s)-\alpha\beta(t)}, \quad B = \frac{1}{q}\frac{|e(t)|}{x^\alpha(t)}, \tag{2.16}$$

then from the Young inequality ($pA+qB\ge A^pB^q$, where $p+q=1$, $p,q>0$, $A\ge 0$, $B\ge 0$), we get

$$\begin{aligned}
&\frac{\theta(t)}{\eta(t,s)}g(t,s)[x(t)]^{\gamma(t,s)-\alpha\beta(t)}+\frac{|e(t)|}{x^\alpha(t)}\\
&\ge\Big(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)[x(t)]^{\gamma(t,s)-\alpha\beta(t)}\Big)^p\Big(\frac{1}{q}\frac{|e(t)|}{x^\alpha(t)}\Big)^q\\
&=\Big(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)\Big)^p\Big(\frac{|e(t)|}{q}\Big)^q[x(t)]^{(\gamma(t,s)-\alpha\beta(t))p-q\alpha}\\
&=\Big(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)\Big)^p\Big(\frac{|e(t)|}{q}\Big)^q[x(t)]^{\frac{\gamma(t,s)\theta(t)-\alpha\beta^2(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}}.
\end{aligned}\tag{2.17}$$

By (2.1) and (2.2), we get

$$\int_a^b\eta(t,s)[\gamma(t,s)\theta(t)-\alpha\beta^2(t)]\mathrm{d}\xi(s)\equiv 0,\quad\text{for any } t\in[t_0,\infty).\tag{2.18}$$

From (2.14)-(2.18) and Lemma 2.3, we see that, for $t\in[a_1,b_1]$,

$$\begin{aligned}
&\delta(t)\int_a^b g(t,s)[x(t)]^{\gamma(t,s)-\alpha\beta(t)}\mathrm{d}\xi(s)-\delta(t)\frac{e(t)}{x^\alpha(t)}\\
&\ge\delta(t)\int_a^b\frac{\eta(t,s)}{\theta(t)}\Big(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)\Big)^p\Big(\frac{|e(t)|}{q}\Big)^q\\
&\times[x(t)]^{\frac{\gamma(t,s)\theta(t)-\alpha\beta^2(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}}\mathrm{d}\xi(s)\\
&\ge\delta(t)\exp\Big(\frac{1}{\theta(t)}\int_a^b\eta(t,s)\ln\Big[\Big(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)\Big)^p\\
&\quad\times\Big(\frac{|e(t)|}{q}\Big)^q[x(t)]^{\frac{\gamma(t,s)\theta(t)-\alpha\beta^2(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}}\Big]\mathrm{d}\xi(s)\Big)\\
&=\delta(t)\exp\Big(\frac{1}{\theta(t)}\int_a^b\eta(t,s)\ln\Big[\Big(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)\Big)^p\Big(\frac{|e(t)|}{q}\Big)^q\Big]\mathrm{d}\xi(s)\Big)\\
&\times\exp\Big(\frac{1}{\theta(t)}\int_a^b\eta(t,s)\Big[\frac{\gamma(t,s)\theta(t)-\alpha\beta^2(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}\Big]\ln x(t)\mathrm{d}\xi(s)\Big)\\
&=\delta(t)\exp\Big(\frac{1}{\theta(t)}\int_a^b\eta(t,s)\ln\Big[\Big(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)\Big)^p\Big(\frac{|e(t)|}{q}\Big)^q\Big]\mathrm{d}\xi(s)\Big)\\
&\times\exp\Big(\frac{1}{\theta(t)}\frac{\ln x(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}\int_a^b\eta(t,s)\Big[\gamma(t,s)\theta(t)-\alpha\beta^2(t)\Big]\mathrm{d}\xi(s)\Big)\\
&=\delta(t)\exp\Big(\frac{p}{\theta(t)}\int_a^b\eta(t,s)\ln\Big[\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)\Big]\mathrm{d}\xi(s)\\
&\quad+\frac{1}{\theta(t)}\ln\big(\frac{|e(t)|}{q}\big)^q\int_a^b\eta(t,s)\mathrm{d}\xi(s)\Big)\\
&=\delta(t)\exp\Big(\frac{p}{\theta(t)}\int_a^b\eta(t,s)\Big[\ln\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big)+\ln\frac{g(t,s)}{\eta(t,s)}\Big]\mathrm{d}\xi(s)\\
&\quad+\ln\Big(\frac{|e(t)|}{q}\Big)^q\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \delta(t)\Big(\frac{|e(t)|}{q}\Big)^{q} \exp\Big(\frac{p}{\theta(t)} \ln\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big)\int_a^b \eta(t,s)\mathrm{d}\xi(s) \\
&\quad + \frac{p}{\theta(t)}\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\mathrm{d}\xi(s)\Big) \\
&= \delta(t)\Big(\frac{\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big)|e(t)|}{\beta^2(t)-\theta(t)\beta(t)}\Big)^{\frac{\beta^2(t)-\theta(t)\beta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}} \\
&\quad \times \exp\Big(\frac{\theta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}\Big[\ln\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big) \\
&\quad + \frac{1}{\theta(t)}\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\mathrm{d}\xi(s)\Big]\Big).
\end{aligned}
$$

Then from (2.9) and above inequality, we have

$$
\begin{aligned}
\omega'(t) &\le -\delta(t)q(t) - \delta(t)\Big(\frac{\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big)|e(t)|}{\beta^2(t)-\theta(t)\beta(t)}\Big)^{\frac{\beta^2(t)-\theta(t)\beta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}} \\
&\quad \times \exp\Big(\frac{\theta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}\Big[\ln\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big) \\
&\quad + \frac{1}{\theta(t)}\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\mathrm{d}\xi(s)\Big]\Big) - \alpha\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}} \\
&= -\delta(t)Q(t) - \alpha\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}}\,,
\end{aligned} \tag{2.19}
$$

where $Q(t)$ is defined by (2.7) with $\theta(t) \in (m_1(t), \beta(t))$. The rest of the proof is similar to that of case (I) and hence is omitted. This completes the proof of Theorem 2.4. □

Following Philos [11] and Kong [6], we say that for any $a, b \in \mathbb{R}$ with $a < b$, a function $H(t,s)$ belongs to a function class $\mathcal{H}(a,b)$, denoted by $H \in \mathcal{H}(a,b)$, if $H \in C(\mathbb{D}, [0,\infty))$, where $\mathbb{D} = \{(t,s) : b \ge t \ge s \ge a\}$, which satisfies

$$
H(t,t) = 0, \quad H(b,s) > 0, \quad H(s,a) > 0 \quad \text{for } b > s > a,
$$

and $H(t,s)$ has continuous partial derivative $\partial H(t,s)/\partial t$ and $\partial H(t,s)/\partial s$ on $[a,b] \times [a,b]$ such that

$$
\begin{aligned}
\frac{\partial H}{\partial t}(t,s) &= (\alpha+1)h_1(t,s)H^{\frac{\alpha}{\alpha+1}}(t,s), \\
\frac{\partial H}{\partial s}(t,s) &= (\alpha+1)h_2(t,s)H^{\frac{\alpha}{\alpha+1}}(t,s),
\end{aligned}
$$

where $h_1, h_2 \in L_{loc}(D, \mathbb{R})$.

Our next result uses the function class $\mathcal{H}(a,b)$ to establish an oscillation criterion for equation (1.1) of the Kong-type.

Theorem 2.5. *Suppose that for any $T \ge t_0$, there exist nontrivial subinterval $[a_1, b_1]$ and $[a_2, b_2]$ of $[T, \infty)$ such that (2.4) and (2.5) hold for $i = 1, 2$. Let θ and η be functions defined as in Theorem 2.4 such that $1/\eta(t,\cdot) \in L_\xi[a,b]$ and (2.1)-(2.2) hold. Suppose also that for $i = 1, 2$, there exists $c_i \in (a_i, b_i)$ and $H_i \in \mathcal{H}(a_i, b_i)$*

such that

$$\begin{aligned}&\frac{1}{H_i(c_i,a_i)}\int_{a_i}^{c_i}[\delta(s)Q(s)H_i(s,a_i)-\delta(s)r(s)|h_{i1}(s,a_i)|^{\alpha+1}]ds\\&+\frac{1}{H_i(b_i,c_i)}\int_{c_i}^{b_i}[\delta(s)Q(s)H_i(b_i,s)-\delta(s)r(s)|h_{i2}(b_i,s)|^{\alpha+1}]ds>0,\end{aligned}\tag{2.20}$$

where $\delta(t)$ *and* $Q(t)$ *are as in Theorem 2.4. Then equation* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.4, we again arrive at (2.12) and (2.19). In view of (2.12) and (2.19), we see that

$$w'(t)\le-\delta(t)Q(t)-\alpha\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{(\delta(t)r(t))^{1/\alpha}},\quad t\in[a_1,b_1].\tag{2.21}$$

Multiplying both sides of (2.21), with t replaced by s, by $H_1(s,a_1)$ and integrating from a_1 to c_1, we see that

$$\int_{a_1}^{c_1}\delta(s)Q(s)H_1(s,a_1)ds\le-\int_{a_1}^{c_1}H_1(s,a_1)w'(s)ds-\alpha\int_{a_1}^{c_1}H_1(s,a_1)\frac{|w(s)|^{\frac{\alpha+1}{\alpha}}}{(\delta(s)r(s))^{1/\alpha}}.$$

Integrating by parts, we obtain

$$\begin{aligned}&\int_{a_1}^{c_1}\delta(s)Q(s)H_1(s,a_1)ds\\&\le-H_1(c_1,a_1)w(c_1)+\int_{a_1}^{c_1}(\alpha+1)|h_{11}(s,a_1)|H_1^{\frac{\alpha}{\alpha+1}}(s,a_1)|w(s)|ds\\&\quad-\alpha\int_{a_1}^{c_1}H_1(s,a_1)\frac{|w(s)|^{\frac{\alpha+1}{\alpha}}}{(\delta(s)r(s))^{1/\alpha}}ds.\end{aligned}\tag{2.22}$$

Applying Lemma 2.1 with

$$X=\Big(\alpha\frac{H_1(s,a_1)|w(s)|^{\lambda}}{(\delta(s)r(s))^{1/\alpha}}\Big)^{1/\lambda},\quad\lambda=\frac{\alpha+1}{\alpha},\quad Y=\Big(\frac{\alpha(\delta(s)r(s))^{\frac{1}{\alpha+1}}}{\alpha^{\frac{\alpha}{\alpha+1}}}|h_{11}(s,a_1)|\Big)^{\alpha},$$

we see that

$$\begin{aligned}&(\alpha+1)|h_{11}(s,a_1)|H_1^{\frac{\alpha}{\alpha+1}}(s,a_1)|w(s)|-\alpha H_1(s,a_1)\frac{|w(s)|^{\frac{\alpha+1}{\alpha}}}{(\delta(s)r(s))^{1/\alpha}}\\&\le\delta(s)r(s)|h_{11}(s,a_1)|^{\alpha+1},\end{aligned}$$

substituting this into (2.22), we obtain

$$\int_{a_1}^{c_1}[\delta(s)Q(s)H_1(s,a_1)-\delta(s)r(s)|h_{11}(s,a_1)|^{\alpha+1}]ds\le-H_1(c_1,a_1)w(c_1)$$

or

$$\frac{1}{H_1(c_1,a_1)}\int_{a_1}^{c_1}[\delta(s)Q(s)H_1(s,a_1)-\delta(s)r(s)|h_{11}(s,a_1)|^{\alpha+1}]ds\le-w(c_1).\tag{2.23}$$

Similarly, multiplying both sides of (2.21), with t replaced by s, by $H_1(b_1,s)$ and integrating it from c_1 to b_1, and then applying Lemma 2.1, we see that

$$\frac{1}{H_1(b_1,c_1)}\int_{c_1}^{b_1}[\delta(s)Q(s)H_1(b_1,s)-\delta(s)r(s)|h_{12}(b_1,s)|^{\alpha+1}]ds\le w(c_1).\tag{2.24}$$

Combining (2.23) and (2.24), we arrive at

$$\begin{aligned} &\frac{1}{H_1(c_1,a_1)}\int_{a_1}^{c_1}[\delta(s)Q(s)H_1(s,a_1)-\delta(s)r(s)|h_{11}(s,a_1)|^{\alpha+1}]ds\\ &+\frac{1}{H_1(b_1,c_1)}\int_{c_1}^{b_1}[\delta(s)Q(s)H_1(b_1,s)-\delta(s)r(s)|h_{12}(b_1,s)|^{\alpha+1}]ds\le 0 \end{aligned}$$

which contradicts (2.20) for $i=1$, and completes the proof. □

Remark 2.6. When $p(t)\equiv 0$, $\beta(t)\equiv 1$, $\alpha=1$, $a=0$ and $\gamma(t,s)=\gamma(s)$, Theorems 2.4 and 2.5 reduce to [12, Theorems 2.1 and 2.2]. When $p(t)\equiv 0$, $\beta(t)\equiv 1$, $a=0$ and $\gamma(t,s)=\gamma(s)$, Theorems 2.4 and 2.5 reduce to [5, Theorems 2.1 and 2.2]. When $p(t)\equiv 0$ and $\alpha=1$, Theorems 2.4 and 2.5 reduce to [9, Theorems 2.1 and 2.2].

3. Examples

In this section, we will work out two numerical examples to illustrate our main results. Here we use the convention that $\ln 0=-\infty$ and $e^{-\infty}=0$.

Example 3.1. Consider equation (1.1) with $\alpha=2$, $r(t)=1$, $p(t)=0$, $q(t)=\lambda\sin 4t$ with $\lambda>0$ is a constant, $a=1$, $b=3$, $\gamma(t,s)=se^{-t}$, $g(t,s)\equiv 1$, $\beta(t)=e^{-t}$, $\xi(s)=s$, and $e(t)=-f(t)\cos 2t$ with $f\in C[0,\infty)$ is any nonnegative function. For any $T\ge 0$, we choose $k\in\mathbb{Z}$ large enough that $2k\pi\ge T$ and let $a_1=2k\pi$, $a_2=b_1=2k\pi+\frac{\pi}{4}$, and $b_2=2k\pi+\frac{\pi}{2}$. Then, (2.5) and (2.6) hold, and we have $m_1(t)=2\ln\frac{3}{2}e^{-t}$ and $m_2(t)=2\ln 2e^{-t}$. With

$$\begin{gathered} \theta(t)=\delta e^{-t},\quad \delta\in(2\ln(3/2),1],\quad p=\frac{\delta-2\ln(3/2)}{4\ln 2-2\ln 3},\\ \eta(t,s)=\begin{cases} 2pe^{-t}/s, & (t,s)\in[0,\infty)\times[1,2),\\ 2(1-p)e^{-t}/s, & (t,s)\in[0,\infty)\times[2,3], \end{cases} \end{gathered}$$

it is easy to verify that (2.1) and (2.2) hold. Letting $u_i(t)=\sin 4t$ for $t\in[a_i,b_i]$, $i=1,2$, and from the definition of $Q(t)$, we see that

$$\begin{aligned} Q(t)&=\lambda\sin 4t+\big[\big(1+\frac{\delta e^t}{1-\delta}\big)f(t)|\cos 2t|\big]^{\frac{1-\delta}{1-\delta+\delta e^t}}\\ &\quad\times\exp\Big(\frac{\delta e^t}{1-\delta+\delta e^t}\big[\ln\big(e^{-2t}-\delta e^{-2t}+\delta e^{-t}\big)-\frac{e^t}{\delta}\int_1^3\eta(t,s)\ln\eta(t,s)\mathrm{d}s\big]\Big)\\ &=:F(\lambda,\delta,t), \end{aligned}$$

from this and $\delta(t)=1$, we obtain

$$\begin{gathered} \int_{a_1}^{b_1}\delta(t)Q(t)|u_1(t)|^3\mathrm{d}t=\int_0^{\pi/4}\widetilde{F}(\lambda,\delta,t)\sin^3 4t\mathrm{d}t,\\ \int_{a_2}^{b_2}\delta(t)Q(t)|u_2(t)|^3\mathrm{d}t=-\int_{\frac{\pi}{4}}^{\pi/2}\widetilde{F}(\lambda,\delta,t)\sin^3 4t\mathrm{d}t, \end{gathered}$$

where

$$\begin{aligned} \widetilde{F}(\lambda,\delta,t)&=\lambda\sin 4t+\Big[\Big(1+\frac{\delta e^{t+2k\pi}}{1-\delta}\Big)f(t+2k\pi)|\cos 2t|\Big]^{\frac{1-\delta}{1-\delta+\delta e^{t+2k\pi}}}\\ &\quad\times\exp\Big(\frac{\delta e^{t+2k\pi}}{1-\delta+\delta e^{t+2k\pi}}\Big[\ln\big(e^{-2(t+2k\pi)}-\delta e^{-2(t+2k\pi)}+\delta e^{-(t+2k\pi)}\big) \end{aligned}$$

$$- \frac{e^{t+2k\pi}}{\delta} \int_1^3 \eta(t+2k\pi, s) \ln \eta(t+2k\pi, s) \mathrm{d}s \Big] \Big),$$

and

$$\int_{a_i}^{b_i} \delta(t) r(t) |u_i'(t)|^3 \mathrm{d}t = \int_{a_i}^{b_i} 64 |\cos^3 4t| \mathrm{d}t = \frac{64}{3}.$$

Thus, by Theorem 2.4 we see that (1.1) is oscillatory if $\int_0^{\pi/4} \widetilde{F}(\lambda, \delta, t) \sin^3 4t \mathrm{d}t > 64/3$ and $-\int_{\pi/4}^{\pi/2} \widetilde{F}(\lambda, \delta, t) \sin^3 4t \mathrm{d}t > 64/3$.

Example 3.2. Consider equation (1.1) with $\alpha = 3/2$, $r(t) = 1$, $p(t) = 1$, $q(t) = \lambda \sin t$ with $\lambda > 0$ is a constant, $a = 1$, $b = 3$, $\gamma(t,s) = s(\cos \frac{t}{2} + \frac{3}{2})$, $g(t,s) \equiv 1$, $\beta(t) = \cos \frac{t}{2} + \frac{3}{2}$, $\xi(s) = s$, and $e \in C[0,\infty)$ be any function satisfying $(-1)^i e(t) \geq 0$ on $[a_i, b_i]$ for $i = 1, 2$. For any $T \geq 0$, we choose $k \in \mathbb{Z}$ large enough that $2k\pi \geq T$ and let $a_1 = 2k\pi$, $a_2 = b_1 = 2k\pi + \frac{\pi}{4}$, $b_2 = 2k\pi + \frac{\pi}{2}$, $c_1 = 2k\pi + \frac{\pi}{8}$ and $c_2 = 2k\pi + \frac{3\pi}{8}$. Then, it is easy to see that (2.5) and (2.6) hold, and $m_1(t) = \ln 2(\cos \frac{t}{2} + \frac{3}{2})$ and $m_2(t) = 3 \ln \frac{3}{2}(\cos \frac{t}{2} + \frac{3}{2})$. With

$$\theta(t) = \delta(\cos \frac{t}{2} + \frac{3}{2}), \quad \delta \in (\ln 2, 1], \quad p = \frac{\delta - \ln 2}{3 \ln \frac{3}{2} - \ln 2},$$

$$\eta(t,s) = \begin{cases} 3p(\cos \frac{t}{2} + \frac{3}{2})/s, & (t,s) \in [0,\infty) \times [1, 3/2), \\ (1-p)(\cos \frac{t}{2} + \frac{3}{2})/s, & (t,s) \in [0,\infty) \times [/3/2, 3], \end{cases}$$

we see that (2.1) and (2.2) are valid, and from the definition of $Q(t)$, we obtain

$$\begin{aligned} Q(t) = \lambda \sin t + \Big[\Big(1 + \frac{\delta}{(1-\delta)(\cos \frac{t}{2} + \frac{3}{2})} \Big) |e(t)| \Big]^{\frac{(1-\delta)(\cos \frac{t}{2} + \frac{3}{2})}{(1-\delta)(\cos \frac{t}{2} + \frac{3}{2}) + \delta}} \\ \times \exp \Big(\frac{\delta}{(1-\delta)(\cos \frac{t}{2} + \frac{3}{2}) + \delta} \Big[\ln \big((\cos \frac{t}{2} + \frac{3}{2})^2 - \delta (\cos \frac{t}{2} + \frac{3}{2})^2 \\ + \delta (\cos \frac{t}{2} + \frac{3}{2}) \big) - \frac{1}{\delta (\cos \frac{t}{2} + \frac{3}{2})} \int_1^3 \eta(t,s) \ln \eta(t,s) \mathrm{d}s \Big] \Big). \end{aligned}$$

If we choose $H(t,s) = (t-s)^{5/2}$, then $h_1(t,s) = 1$, $h_2(t,s) = -1$. Since $\delta(t) = e^t$, by Theorem 2.5, we see that (1.1) is oscillatory if

$$\int_{2k\pi}^{2k\pi + \frac{\pi}{8}} Q(s) e^s (s - 2k\pi)^{5/2} \mathrm{d}s + \int_{2k\pi + \frac{\pi}{8}}^{2k\pi + \frac{\pi}{4}} Q(s) e^s (2k\pi + \pi/4 - s)^{5/2} \mathrm{d}s > e^{2k\pi} (e^{\pi/4} - 1),$$

and

$$\begin{aligned} \int_{2k\pi + \frac{\pi}{4}}^{2k\pi + \frac{3\pi}{8}} Q(s) e^s (s - 2k\pi - \pi/4)^{5/2} \mathrm{d}s + \int_{2k\pi + \frac{3\pi}{8}}^{2k\pi + \frac{\pi}{2}} Q(s) e^s (2k\pi + \pi/2 - s)^{5/2} \mathrm{d}s \\ > e^{2k\pi} (e^{\pi/2} - e^{\pi/4}). \end{aligned}$$

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Ercan Tunç

Gaziosmanpasa University, Department of Mathematics, Faculty of Arts and Sciences, 60240, Tokat, Turkey

E-mail address: `ercantunc72@yahoo.com`

Haidong Liu

Qufu Normal University, School of Mathematical Sciences, 273165, Qufu, China

E-mail address: `tomlhd983@163.com`