Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 57, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

STABILITY OF TRAVELING WAVEFRONTS FOR A THREE-COMPONENT LOTKA-VOLTERRA COMPETITION SYSTEM ON A LATTICE

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Communicated by Zhaosheng Feng

Abstract. This article concerns the stability of traveling wavefronts for a three-component Lotka-Volterra competition system on a lattice. By means of the weighted energy method and the comparison principle, it is proved that the traveling wavefronts with large speed are exponentially asymptotically stable, when the initial perturbation around the traveling wavefronts decays exponentially as $j + ct \to -\infty$, where $j \in \mathbb{Z}$, $t > 0$ and $c > 0$, but the initial perturbation can be arbitrarily large on other locations.

1. Introduction

Consider the three-component Lotka-Volterra competition system on a lattice

$$
\begin{aligned}
\frac{du_j(t)}{dt} &= d_1\mathcal{D}[u_j](t) + r_1u_j(t)[1 - u_j(t) - b_{12}v_j(t)],\\
\frac{dv_j(t)}{dt} &= d_2\mathcal{D}[v_j](t) + r_2v_j(t)[1 - b_{21}u_j(t) - v_j(t) - b_{23}w_j(t)],\\
\frac{dw_j(t)}{dt} &= d_3\mathcal{D}[w_j](t) + r_3w_j(t)[1 - b_{32}v_j(t) - w_j(t)],
\end{aligned} \tag{1.1}
$$

with the initial data

$$u_j(0) = u_{j0}, \quad v_j(0) = v_{j0}, \quad w_j(0) = w_{j0},$$

where $j \in \mathbb{Z}$, $t > 0$, $d_n > 0$, $r_n > 0$, $b_{nm} > 0$, $m, n \in \{1, 2, 3\}$, $\mathcal{D}[z_j] = z_{j+1} + z_{j-1} - 2z_j$ for $z = u, v, w$. Here, u_j, v_j and w_j are the population densities of three different species (call them as species 1, 2, 3) at site j at time t, d_n is the migration coefficient of species n, r_n is the net birth rate of species n and b_{nm} is the competition coefficient of species m to species n. Also, we have taken the scales so that the carrying capacity of each species is normalized to be 1. Throughout this paper, we assume

(H1) b_{12}, $b_{32} > 1$, $b_{21} + b_{23} < 1$,

2010 *Mathematics Subject Classification.* 34A33, 34K20, 92D25.
Key words and phrases. Three-component competition system; lattice dynamical system; traveling wavefronts; stability.

Submitted September 8, 2017. Published March 1, 2018.

which means that the species 1, 3 are weak competitors to the species 2. Therefore, it is expected that the species 2 shall win the competition eventually. It is easy to see that the system (1.1) has constant equilibria $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,0,1)$.

To understand the invading phenomenon between the residents u, w and the invader v, the traveling wave solution connecting two equilibrium points $(1,0,1)$ and $(0,1,0)$ has been considered by many researchers [4, 14, 15]. We note that a traveling wave solution of (1.1) is a special translation invariant solution of the form

$$u_j(t)=\varphi(\xi),\quad v_j(t)=\psi(\xi),\quad w_j(t)=\theta(\xi),\quad \xi=j+ct,$$

where $c>0$ is the wave speed. If φ, ψ, θ are monotone, then (φ,ψ,θ) is called a traveling wavefront. Substituting $(\varphi(j+ct),\psi(j+ct),\theta(j+ct))$ into (1.1), we obtain the following wave profile system with the asymptotic boundary conditions

$$\begin{gathered}
c\varphi'(\xi)=d_1\mathcal{D}[\varphi](\xi)+r_1\varphi(\xi)[1-\varphi(\xi)-b_{12}\psi(\xi)],\\
c\psi'(\xi)=d_2\mathcal{D}[\psi](\xi)+r_2\psi(\xi)[1-b_{21}\varphi(\xi)-\psi(\xi)-b_{23}\theta(\xi)],\\
c\theta'(\xi)=d_3\mathcal{D}[\theta](\xi)+r_3\theta(\xi)[1-b_{32}\psi(\xi)-\theta(\xi)],\\
(\varphi,\psi,\theta)(-\infty)=(1,0,1),\ (\varphi,\psi,\theta)(+\infty)=(0,1,0),\\
0\le\varphi,\psi,\theta\le 1,
\end{gathered}\tag{1.2}$$

where $\mathcal{D}[u](\xi)=u(\xi+1)+u(\xi-1)-2u(\xi)$.

Clearly, when $\theta(\xi)=0$, system (1.2) reduces to the two-component Lotka-Volterra competition system, we refer to [3, 13]. For the three-component competition system (1.2), by considering a truncated problem with the help of a super-solution, Guo et al. [4] showed that there exists a positive constant c_{min} such that (1.2) has a strictly monotone solution if and only if $c\ge c_{min}$. At the same time, the linear determinacy for (1.2) was given in [4]. Later, Wu [15] established the asymptotic behavior of solutions of (1.2) at infinity, and constructed some entire solutions of (1.1). More recently, Wu [14] proved the monotonicity and uniqueness (up to translations) of solutions of (1.2) with speed $c\ge c_{min}$. A natural question is whether the traveling wavefronts of (1.1) (i.e., solutions of (1.2)) are stable for each admissible speed. In this paper, we give an answer to this question.

The stability of traveling wave solutions for various evolution equations with or without delay has been extensively studied, for example, see [2, 5, 7, 8, 9, 10, 11, 12, 13, 16, 18, 19, 21, 22, 23]. The main methods are the (technical) weighted energy method [5, 23], the sub- and supersolutions method and squeezing technique [1, 12], and the combination of the comparison principle and the weighted energy method [10, 13, 18]. To the best our knowledge, for evolution systems, little has been done to establish the stability of traveling wave solutions. In 2011, Yang, Li and Wu [16, 17] considered a diffusive epidemic system with delay and established the stability of traveling wavefronts. Lv and Wang [6] and Yu et al. [20] respectively investigated the stability of traveling wavefronts for two-component Lotka-Volterra cooperative and competitive systems with nonlocal dispersals. Encouraged by papers [6, 10, 13, 16, 20], in this paper, we take the weighted energy method together with the comparison principle to study the stability of traveling wavefronts for the three-component lattice competition system (1.1). We first give a comparison principle and then prove that the traveling wavefronts of (1.1) are stable, when the difference between initial data and traveling wavefront decays exponentially as $j+ct\to$

$-\infty$, but the initial data can be arbitrarily large on other locations. We should remark that although the main idea is same as that for two-component lattice competition system, some complexities and difficulties arise in the three component lattice competition system due to the coupling of the nonlinearities.

The rest of this paper is organized as follows. In section 2, we give the notations, the existence of traveling wavefronts, some necessary assumptions and the main theorem. Section 3 is devoted to the proof of the stability theorem.

2. Preliminaries and main result

In this section, we first recall some known results, then define a weight function and state our main result.

To study the stability of traveling wavefront of (1.1), it is convenient to work on (u_j^*, v_j, w_j^*), where $u_j^* = 1 - u_j, w_j^* = 1 - w_j$. For the sake of convenience, we drop the star. Then (1.1) is transformed into the system

$$
\begin{aligned}
\frac{du_j(t)}{dt} &= d_1\mathcal{D}[u_j](t) + r_1(1 - u_j(t))[-u_j(t) + b_{12}v_j(t)],\\
\frac{dv_j(t)}{dt} &= d_2\mathcal{D}[v_j](t) + r_2v_j(t)[1 - b_{21} - b_{23} - v_j(t) + b_{21}u_j(t) + b_{23}w_j(t)],\\
\frac{dw_j(t)}{dt} &= d_3\mathcal{D}[w_j](t) + r_3(1 - w_j(t))[-w_j(t) + b_{32}v_j(t)],
\end{aligned} \tag{2.1}
$$

with the initial data

$$
\begin{aligned}
u_j(0) &= 1 - u_{j0},\\
v_j(0) &= v_{j0},\\
w_j(0) &= 1 - w_{j0}.
\end{aligned} \tag{2.2}
$$

Let $u_j(t) = \varphi(\xi)$, $v_j(t) = \psi(\xi)$, $w_j(t) = \theta(\xi)$, $\xi = j + ct$. Then the wave profile system of (2.1) is

$$
\begin{aligned}
c\varphi'(\xi) &= d_1\mathcal{D}[\varphi](\xi) + r_1(1 - \varphi(\xi))[-\varphi(\xi) + b_{12}\psi(\xi)],\\
c\psi'(\xi) &= d_2\mathcal{D}[\psi](\xi) + r_2\psi(\xi)[1 - b_{21} - b_{23} - \psi(\xi) + b_{21}\varphi(\xi) + b_{23}\theta(\xi)],\\
c\varphi'(\xi) &= d_3\mathcal{D}[\theta](\xi) + r_3(1 - \theta(\xi))[-\theta(\xi) + b_{32}\psi(\xi)],
\end{aligned} \tag{2.3}
$$

with the boundary condition

$$
(\varphi, \psi, \theta)(-\infty) = (0, 0, 0) \quad \text{and} \quad (\varphi, \psi, \theta)(+\infty) = (1, 1, 1). \tag{2.4}
$$

The existence of traveling wavefront of (2.1) comes from Guo et al. [4].

Proposition 2.1 (Existence)**.** *Assume that* (H1) *holds. Then there exists $c_{min} > 0$ such that for any $c \geq c_{min}$,* (2.1) *admits a traveling wavefront $(\varphi(\xi), \psi(\xi), \theta(\xi))$ connecting $(0, 0, 0)$ and $(1, 1, 1)$, and satisfying $\varphi'(\cdot) > 0$, $\psi'(\cdot) > 0$ and $\theta'(\cdot) > 0$ on $\mathbb{R}$. For any $c < c_{min}$, there is no such traveling wave.*

Before stating our main result, let us make the following notation. Throughout the paper, l_w^2 denotes a weighted l^2-space with a weighted function $0 < w(\xi) \in C(\mathbb{R})$, that is

$$
l_w^2 := \Big\{\zeta = \{\zeta_i\}_{i\in\mathbb{Z}}, \zeta_i \in \mathbb{R} : \sum_i w(i + ct)\zeta_i^2 < \infty\Big\},
$$

and its norm is defined by

$$\|\zeta\|_{l^2_w} = \Big(\sum_i w(i+ct)\zeta_i^2\Big)^{1/2} \quad \text{for } \zeta \in l^2_w.$$

In particular, when $w \equiv 1$, we denote l^2_w by l^2.

To obtain our stability result, we need the following assumption.

(H2) $0 < b_{21} + b_{23} < \frac{2}{3}$, $b_{12} > 2 + \frac{r_2 b_{21}}{2r_1}$, $b_{32} > 2 + \frac{r_2 b_{23}}{2r_3}$.

Define three functions on λ as follows:

$$\begin{aligned}
\mathcal{M}_1(\lambda) &= 2d_1 - 4r_1 + 2r_1 b_{12} - r_2 b_{21} - d_1(e^{\lambda}+1),\\
\mathcal{M}_2(\lambda) &= 2d_2 + 2r_2 - 3r_2(b_{21}+b_{23}) - d_2(e^{\lambda}+1),\\
\mathcal{M}_3(\lambda) &= 2d_3 - 4r_3 + 2r_3 b_{32} - r_2 b_{23} - d_3(e^{\lambda}+1).
\end{aligned}$$

From assumption (H2), we obtain

$$\begin{gathered}
\mathcal{M}_1(0) = -4r_1 + 2r_1 b_{12} - r_2 b_{21} > 0,\\
\mathcal{M}_2(0) = 2r_2 - 3r_2(b_{21}+b_{23}) > 0,\\
\mathcal{M}_3(0) = -4r_3 + 2r_3 b_{32} - r_2 b_{23} > 0.
\end{gathered}$$

Then by the continuity of $\mathcal{M}_1(\lambda)$, $\mathcal{M}_2(\lambda)$ and $\mathcal{M}_3(\lambda)$ with respect to λ, there exists $\lambda_0 > 0$ such that

$$\mathcal{M}_1(\lambda_0) > 0,\ \mathcal{M}_2(\lambda_0) > 0,\ \mathcal{M}_2(\lambda_0) > 0. \tag{2.5}$$

Furthermore, we define

$$\begin{aligned}
\mathcal{N}_1(\xi) &= 2d_1 - 4r_1 + 2r_1 b_{12}\psi(\xi) + r_1 b_{12}\varphi(\xi) - r_1 b_{12} - r_2 b_{21} - d_1(e^{\lambda_0}+1),\\
\mathcal{N}_2(\xi) &= 2d_2 - 2r_2 + 4r_2\psi(\xi) - 3r_2(b_{21}+b_{23}) - r_1 b_{12} - r_3 b_{32} + r_1 b_{12}\varphi(\xi)\\
&\quad + r_3 b_{32}\theta(\xi) - d_2(e^{\lambda_0}+1),\\
\mathcal{N}_3(\xi) &= 2d_3 - 4r_3 + 2r_3 b_{32}\psi(\xi) + r_3 b_{32}\theta(\xi) - r_3 b_{32} - r_2 b_{23} - d_3(e^{\lambda_0}+1),
\end{aligned}$$

where $(\varphi(\xi), \psi(\xi), \theta(\xi))$ is a traveling wavefront given in Proposition 2.1.

By (2.4), we have

$$\begin{gathered}
\lim_{\xi\to+\infty} \mathcal{N}_1(\xi) = \mathcal{M}_1(\lambda_0) > 0,\\
\lim_{\xi\to+\infty} \mathcal{N}_2(\xi) = \mathcal{M}_2(\lambda_0) > 0,\\
\lim_{\xi\to+\infty} \mathcal{N}_3(\xi) = \mathcal{M}_3(\lambda_0) > 0,
\end{gathered}$$

which imply that there exists a number $\xi_0 > 0$ large enough such that

$$\begin{aligned}
\mathcal{N}_1(\xi_0) &= 2d_1 - 4r_1 + 2r_1 b_{12}\psi(\xi_0) + r_1 b_{12}\varphi(\xi_0) - r_1 b_{12} - r_2 b_{21} - d_1(e^{\lambda_0}+1) > 0,\\
\mathcal{N}_2(\xi_0) &= 2d_2 - 2r_2 + 4r_2\psi(\xi_0) - 3r_2(b_{21}+b_{23}) - r_1 b_{12} - r_3 b_{32} + r_1 b_{12}\varphi(\xi_0)\\
&\quad + r_3 b_{32}\theta(\xi_0) - d_2(e^{\lambda_0}+1) > 0,\\
\mathcal{N}_3(\xi_0) &= 2d_3 - 4r_3 + 2r_3 b_{32}\psi(\xi_0) + r_3 b_{32}\theta(\xi_0) - r_3 b_{32} - r_2 b_{23} - d_3(e^{\lambda_0}+1) > 0.
\end{aligned}$$

Define the weighted function

$$w(\xi) = \begin{cases} e^{-\lambda_0(\xi-\xi_0)}, & \xi \le \xi_0,\\ 1, & \xi > \xi_0, \end{cases} \tag{2.6}$$

where λ_0 is defined by (2.5). Let

$$c_1 = 4r_1 + r_1 b_{12} + r_2 b_{21} + d_1(e^{\lambda_0} + e^{-\lambda_0} + 1), \tag{2.7}$$

$$c_2 = 2r_2 + 3r_2(b_{21} + b_{23}) + r_1 b_{12} + r_3 b_{32} + d_2(e^{\lambda_0} + e^{-\lambda_0} + 1), \tag{2.8}$$

$$c_3 = 4r_3 + r_3 b_{32} + r_2 b_{23} + d_3(e^{\lambda_0} + e^{-\lambda_0} + 1). \tag{2.9}$$

Theorem 2.2 (Stability). *Assume that* (H2) *holds. For any given traveling wavefront* $(\varphi(\xi(t,j)), \psi(\xi(t,j)), \theta(\xi(t,j)))$ *with the wave speed* $c > \max\{c_{min}, \tilde{c}\}$, *where*

$$\tilde{c} = \frac{\max\{c_1, c_2, c_3\}}{\lambda_0}.$$

If the initial data satisfies

$$(0,0,0) \le (u_j(0), v_j(0), w_j(0)) \le (1,1,1), \quad j \in \mathbb{Z},$$

and the initial perturbations satisfy

$$\begin{aligned} u_j(0) - \varphi(j) &\in l_w^2, \\ v_j(0) - \psi(j) &\in l_w^2, \\ w_j(0) - \theta(j) &\in l_w^2, \end{aligned}$$

then the nonnegative solution of the Cauchy problem (2.1) *and* (2.2) *uniquely exists and satisfies*

$$(0,0,0) \le (u_j(t), v_j(t), w_j(t)) \le (1,1,1), \quad j \in \mathbb{Z}, \ t > 0,$$

and

$$\begin{aligned} u_j(t) - \varphi(j+ct) &\in C\left((0,+\infty); l_w^2\right), \\ v_j(t) - \psi(j+ct) &\in C\left((0,+\infty); l_w^2\right), \\ w_j(t) - \theta(j+ct) &\in C\left((0,+\infty); l_w^2\right), \end{aligned}$$

where $w(\xi)$ *is defined by* (2.6). *Moreover,* $(u_j(t), v_j(t), w_j(t))$ *converges to the traveling wavefront* $(\varphi(j+ct), \psi(j+ct), \theta(j+ct))$ *exponentially in time* t*, i.e.,*

$$\begin{aligned} \sup_{j\in\mathbb{Z}} |u_j(t) - \varphi(j+ct)| &\le Ce^{-\mu t}, \\ \sup_{j\in\mathbb{Z}} |v_j(t) - \psi(j+ct)| &\le Ce^{-\mu t}, \\ \sup_{j\in\mathbb{Z}} |w_j(t) - \theta(j+ct)| &\le Ce^{-\mu t}, \end{aligned}$$

for all $t > 0$*, where* C *and* μ *are some positive constants.*

3. Stability of traveling wavefronts

We first state the boundedness and the comparison principle for the Cauchy problem (2.1) and (2.2) and then prove the main theorem by using the weighted energy method combined with the comparison principle.

Lemma 3.1 (Boundedness). *Assume that* (H1) *holds and that the initial data* $(u_j(0), v_j(0), w_j(0))$ *satisfy*

$$(0,0,0) \le (u_j(0), v_j(0), w_j(0)) \le (1,1,1)$$

for $j \in \mathbb{Z}$. Then the solution $(u_j(t), v_j(t), w_j(t))$ of the Cauchy problem (2.1) and (2.2) exists and satisfies

$$(0,0,0) \le (u_j(t), v_j(t), w_j(t)) \le (1,1,1)$$

for $t \in (0,+\infty)$, $j \in \mathbb{Z}$.

Lemma 3.2 (Comparison principle)**.** *Assume that* (H1) *holds. Let $(u_j^-(t), v_j^-(t), w_j^-(t))$ and $(u_j^+(t), v_j^+(t), w_j^+(t))$ be the solution of* (2.1) *with the initial data $(u_j^-(0), v_j^-(0), w_j^-(0))$ and $(u_j^+(0), v_j^+(0), w_j^+(0))$, respectively. If*

$$(0,0,0) \le (u_j^-(0), v_j^-(0), w_j^-(0)) \le (u_j^+(0), v_j^+(0), w_j^+(0)) \le (1,1,1)$$

for $j \in \mathbb{Z}$, then

$$(0,0,0) \le (u_j^-(t), v_j^-(t), w_j^-(t)) \le (u_j^+(t), v_j^+(t), w_j^+(t)) \le (1,1,1)$$

for $t \in (0,+\infty)$, $j \in \mathbb{Z}$.

Let the initial data $(u_j(0), v_j(0), w_j(0))$ be such that

$$(0,0,0) \le (u_j(0), v_j(0), w_j(0)) \le (1,1,1)$$

for $j \in \mathbb{Z}$, and let

$$\begin{gathered}
u_j^-(0) = \min\{u_j(0), \varphi(j)\}, \quad j \in \mathbb{Z}, \\
u_j^+(0) = \max\{u_j(0), \varphi(j)\}, \quad j \in \mathbb{Z}, \\
v_j^-(0) = \min\{v_j(0), \psi(j)\}, \quad j \in \mathbb{Z}, \\
v_j^+(0) = \max\{v_j(0), \psi(j)\}, \quad j \in \mathbb{Z}, \\
w_j^-(0) = \min\{w_j(0), \theta(j)\}, \quad j \in \mathbb{Z}, \\
w_j^+(0) = \max\{w_j(0), \theta(j)\}, \quad j \in \mathbb{Z}.
\end{gathered}$$

Then we can easily get

$$\begin{gathered}
0 \le u_j^-(0) \le u_j(0) \le u_j^+(0) \le 1, \quad j \in \mathbb{Z}, \\
0 \le u_j^-(0) \le \varphi(j) \le u_j^+(0) \le 1, \quad j \in \mathbb{Z}, \\
0 \le v_j^-(0) \le v_j(0) \le v_j^+(0) \le 1, \quad j \in \mathbb{Z}, \\
0 \le v_j^-(0) \le \psi(j) \le v_j^+(0) \le 1, \quad j \in \mathbb{Z}, \\
0 \le w_j^-(0) \le w_j(0) \le w_j^+(0) \le 1, \quad j \in \mathbb{Z}, \\
0 \le w_j^-(0) \le \theta(j) \le w_j^+(0) \le 1, \quad j \in \mathbb{Z}.
\end{gathered} \tag{3.1}$$

Define $u_j^+(t)$, $u_j^-(t)$, $v_j^+(t)$, $v_j^-(t)$, $w_j^+(t)$, $w_j^-(t)$ as the corresponding solutions of (2.1) with the initial data $u_j^+(0)$, $u_j^-(0)$, $v_j^+(0)$, $v_j^-(0)$, $w_j^+(0)$, $w_j^-(0)$ respectively. Then by the comparison principle in Lemma 3.2, we obtain

$$\begin{gathered}
0 \le u_j^-(t) \le u_j(t) \le u_j^+(t) \le 1, \quad t \in (0,+\infty),\ j \in \mathbb{Z}, \\
0 \le u_j^-(t) \le \varphi(j+ct) \le u_j^+(t) \le 1, \quad t \in (0,+\infty),\ j \in \mathbb{Z}, \\
0 \le v_j^-(t) \le v_j(t) \le v_j^+(t) \le 1, \quad t \in (0,+\infty),\ j \in \mathbb{Z}, \\
0 \le v_j^-(t) \le \psi(j+ct) \le v_j^+(t) \le 1, \quad t \in (0,+\infty),\ j \in \mathbb{Z}, \\
0 \le w_j^-(t) \le w_j(t) \le w_j^+(t) \le 1, \quad t \in (0,+\infty),\ j \in \mathbb{Z}, \\
0 \le w_j^-(t) \le \theta(j+ct) \le w_j^+(t) \le 1, \quad t \in (0,+\infty),\ j \in \mathbb{Z}.
\end{gathered} \tag{3.2}$$

Let

$$\begin{aligned}
U_j(t) &= u_j^+(t) - \varphi(j+ct), \quad U_{j0}(0) = u_j^+(0) - \varphi(j),\\
V_j(t) &= v_j^+(t) - \psi(j+ct), \quad V_{j0}(0) = v_j^+(0) - \psi(j),\\
W_j(t) &= w_j^+(t) - \theta(j+ct), \quad W_{j0}(0) = w_j^+(0) - \theta(j),
\end{aligned}$$

where $t \in (0,+\infty)$, $j \in \mathbb{Z}$. Then by (2.1) and (2.3), $(U_j(t), V_j(t), W_j(t))$ satisfies

$$\begin{aligned}
\frac{dU_j(t)}{dt} &= d_1[U_{j+1}(t) + U_{j-1}(t) - 2U_j(t)] + U_j(t)[2r_1\varphi(\xi(t,j)) - r_1 - r_1b_{12}V_j(t)\\
&\quad - r_1b_{12}\psi(\xi(t,j))] + r_1U_j^2(t) + r_1b_{12}(1-\varphi(\xi(t,j)))V_j(t),\\
\frac{dV_j(t)}{dt} &= d_2[V_{j+1}(t) + V_{j-1}(t) - 2V_j(t)] + V_j(t)[r_2 - r_2b_{21} - r_2b_{23} - 2r_2\psi(\xi(t,j))\\
&\quad + r_2b_{21}U_j(t) + r_2b_{23}W_j(t) + r_2b_{21}\varphi(\xi(t,j)) + r_2b_{23}\theta(\xi(t,j))]\\
&\quad - r_2V_j^2(t) + [r_2b_{21}U_j(t) + r_2b_{23}W_j(t)]\psi(\xi(t,j)),\\
\frac{dW_j(t)}{dt} &= d_3[W_{j+1}(t) + W_{j-1}(t) - 2W_j(t)] + W_j(t)[2r_3\theta(\xi(t,j)) - r_3 - r_3b_{32}V_j(t)\\
&\quad - r_3b_{32}\psi(\xi(t,j))] + r_3W_j^2(t) + r_3b_{32}(1-\theta(\xi(t,j)))V_j(t),
\end{aligned} \tag{3.3}$$

with the initial data $U_j(0) = U_{j0}(0)$, $V_j(0) = V_{j0}(0)$, $W_j(0) = W_{j0}(0)$, $j \in \mathbb{Z}$. It follows from (3.1) and (3.2) that

$$\begin{gathered}
(0,0,0) \le (U_j(t), V_j(t), W_j(t)) \le (1,1,1),\\
(0,0,0) \le (U_{j0}(0), V_{j0}(0), V_{j0}(0)) \le (1,1,1).
\end{gathered}$$

We define

$$B_{\mu,w}^i(t,j) = A_w^i(t,j) - 2\mu, \quad i = 1,2,3, \tag{3.4}$$

where

$$\begin{aligned}
A_w^1(t,j) &= 2\Big(2d_1 - \frac{c}{2}\frac{w_\xi'(\xi(t,j))}{w(\xi(t,j))} - 2r_1\varphi(\xi(t,j)) + r_1 + r_1b_{12}V_j(t)\\
&\quad + r_1b_{12}\psi(\xi(t,j))\Big) - 2r_1U_j(t) - r_1b_{12}(1-\varphi(\xi(t,j)) - r_2b_{21}\psi(\xi(t,j))\\
&\quad - d_1\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big),
\end{aligned}$$

$$\begin{aligned}
A_w^2(t,j) &= 2\Big(2d_2 - \frac{c}{2}\frac{w_\xi'(\xi(t,j))}{w(\xi(t,j))} - r_2 + r_2b_{21} + r_2b_{23} + 2r_2\psi(\xi(t,j))\\
&\quad - r_2b_{21}U_j(t) - r_2b_{21}\varphi(\xi(t,j)) - r_2b_{23}W_j(t) - r_2b_{23}\theta(\xi(t,j))\Big) + 2r_2V_j(t)\\
&\quad - r_2b_{21}\psi(\xi(t,j)) - r_2b_{23}\psi(\xi(t,j)) - r_1b_{12}(1-\varphi(\xi(t,j))\\
&\quad - r_3b_{32}(1-\theta(\xi(t,j))) - d_2\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big),
\end{aligned}$$

and

$$\begin{aligned}
A_w^3(t,j) &= 2\Big(2d_3 - \frac{c}{2}\frac{w_\xi'(\xi(t,j))}{w(\xi(t,j))} - 2r_3\varphi(\xi(t,j)) + r_3 + r_3b_{32}V_j(t)\\
&\quad + r_3b_{32}\psi(\xi(t,j))\Big) - 2r_3W_j(t) - r_3b_{32}(1-\theta(\xi(t,j)) - r_2b_{23}\psi(\xi(t,j))
\end{aligned}$$

$$- d_3\Big(2+\frac{w(\xi(t,j-1))}{w(\xi(t,j))}+\frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big).$$

Clearly, $\xi(t,j+1)=\xi(t,j)+1$ and $\xi(t,j-1)=\xi(t,j)-1$.

Now we establish some key inequalities.

Lemma 3.3. *Assume that* (H2) *holds. For any $c>\max\{c_{min},\tilde{c}\}$, there exist some positive constants C_i such that*

$$A_w^i(t,j)\geq C_i,\quad i=1,2,3,$$

for all $t>0$ and $j\in\mathbb{Z}$.

Proof. Since $c>\max\{c_{min},\tilde{c}\}$, we obtain $c\lambda_0>c_1$, $c\lambda_0>c_2$, and $c\lambda_0>c_3$, where c_1, c_2 and c_3 can be seen in (2.7), (2.8) and (2.9). That is,

$$\begin{gathered}
c\lambda_0-4r_1-r_1b_{12}-r_2b_{21}-d_1(e^{\lambda_0}+e^{-\lambda_0}+1)>0,\\
c\lambda_0-2r_2-3r_2(b_{21}+b_{23})-r_1b_{12}-r_3b_{32}-d_2(e^{\lambda_0}+e^{-\lambda_0}+1)>0,\\
c\lambda_0-4r_3-r_3b_{32}-r_2b_{23}-d_3(e^{\lambda_0}+e^{-\lambda_0}+1)>0.
\end{gathered}$$

Firstly, we prove that $A_w^1(t,j)\geq C_1$ for some positive constant C_1.

Case 1: $\xi(t,j)<\xi_0-1$. It is clear that $\xi(t,j)<\xi_0$, $\xi(t,j+1)<\xi_0$ and $\xi(t,j-1)<\xi_0$. Then $w(\xi(t,j))=e^{-\lambda_0(\xi(t,j)-\xi_0)}$, $w(\xi(t,j-1))=e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j+1))=e^{-\lambda_0(\xi(t,j)+1-\xi_0)}$. Thus, we have

$$\begin{aligned}
A_w^1(t,j)=&4d_1-c\frac{w_\xi'(\xi(t,j))}{w(\xi(t,j))}-4r_1\varphi(\xi(t,j))+2r_1+2r_1b_{12}V_j(t)+2r_1b_{12}\psi(\xi(t,j))\\
&-2r_1U_j(t)-r_1b_{12}(1-\varphi(\xi(t,j)))-r_2b_{21}\psi(\xi(t,j))\\
&-d_1\Big(2+\frac{w(\xi(t,j-1))}{w(\xi(t,j))}+\frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big)\\
>&2d_1+c\lambda_0-4r_1-r_1b_{12}-r_2b_{21}-d_1(e^{\lambda_0}+e^{-\lambda_0})\\
=&c\lambda_0-4r_1-r_1b_{12}-r_2b_{21}-d_1(e^{\lambda_0}+e^{-\lambda_0}+1)+3d_1\\
>&3d_1>0.
\end{aligned}$$

Case 2: $\xi_0-1\leq\xi(t,j)\leq\xi_0$. In this case, $\xi(t,j-1)<\xi_0$ and $\xi(t,j+1)\geq\xi_0$. Then $w(\xi(t,j))=e^{-\lambda_0(\xi(t,j)-\xi_0)}$, $w(\xi(t,j-1))=e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j+1))=1$. Hence, we obtain

$$\begin{aligned}
A_w^1(t,j)=&4d_1-c\frac{w_\xi'(\xi(t,j))}{w(\xi(t,j))}-4r_1\varphi(\xi(t,j))+2r_1+2r_1b_{12}V_j(t)+2r_1b_{12}\psi(\xi(t,j))\\
&-2r_1U_j(t)-r_1b_{12}(1-\varphi(\xi(t,j))-r_2b_{21}\psi(\xi(t,j))\\
&-d_1\Big(2+\frac{w(\xi(t,j-1))}{w(\xi(t,j))}+\frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big)\\
>&2d_1+c\lambda_0-4r_1-r_1b_{12}-r_2b_{21}-d_1\big(e^{\lambda_0}+e^{\lambda_0(\xi(t,j)-\xi_0)}\big)\\
\geq&c\lambda_0-4r_1-r_1b_{12}-r_2b_{21}-d_1(e^{\lambda_0}+1+e^{-\lambda_0})+d_1e^{-\lambda_0}+2d_1\\
>&d_1e^{-\lambda_0}+2d_1>0.
\end{aligned}$$

Case 3: $\xi_0 < \xi(t,j) \le \xi_0 + 1$**.** In this case, $\xi(t,j-1) \le \xi_0$ and $\xi(t,j+1) > \xi_0$. Then $w(\xi(t,j-1)) = e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j)) = w(\xi(t,j+1)) = 1$. Thus, one has

$$\begin{aligned}
A_w^1(t,j) =& 4d_1 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 4r_1\varphi(\xi(t,j)) + 2r_1 + 2r_1b_{12}V_j(t) + 2r_1b_{12}\psi(\xi(t,j)) \\
&- 2r_1U_j(t) - r_1b_{12}(1-\varphi(\xi(t,j)) - r_2b_{21}\psi(\xi(t,j)) \\
&- d_1\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big) \\
>& 2d_1 - 4r_1 + 2r_1b_{12}\psi(\xi_0) + r_1b_{12}\varphi(\xi_0) - r_1b_{12} - r_2b_{21} \\
&- d_1(e^{-\lambda_0(\xi(t,j)-1-\xi_0)} + 1) \\
>& 2d_1 - 4r_1 + 2r_1b_{12}\psi(\xi_0) + r_1b_{12}\varphi(\xi_0) - r_1b_{12} - r_2b_{21} - d_1(e^{\lambda_0} + 1) \\
=& \mathcal{N}_1(\xi_0) > 0.
\end{aligned}$$

Case 4: $\xi(t,j) > \xi_0 + 1$**.** In this case, $\xi(t,j) > \xi_0$, $\xi(t,j+1) > \xi_0$ and $\xi(t,j-1) > \xi_0$. Then $w(\xi(t,j)) = w(\xi(t,j-1)) = w(\xi(t,j+1)) = 1$. Hence, we obtain

$$\begin{aligned}
A_w^1(t,j) =& 4d_1 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 4r_1\varphi(\xi(t,j)) + 2r_1 + 2r_1b_{12}V_j(t) + 2r_1b_{12}\psi(\xi(t,j)) \\
&- 2r_1U_j(t) - r_1b_{12}(1-\varphi(\xi(t,j)) - r_2b_{21}\psi(\xi(t,j)) \\
&- d_1\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big) \\
>& -4r_1 + 2r_1b_{12}\psi(\xi_0) + r_1b_{12}\varphi(\xi_0) - r_1b_{12} - r_2b_{21} \\
=& \mathcal{N}_1(\xi_0) + d_1(e^{\lambda_0} + 1) - 2d_1 \\
>& d_1(e^{\lambda_0} - 1) > 0.
\end{aligned}$$

Therefore, we can obtain $A_w^1(t,j) \ge C_1 > 0$ by choosing a suitable C_1 small enough.

Secondly, we show $A_w^2(t,j) \ge C_2$ for some positive constant C_2.

Case 1: $\xi(t,j) < \xi_0 - 1$**.** It is clear that $\xi(t,j) < \xi_0$, $\xi(t,j+1) < \xi_0$ and $\xi(t,j-1) < \xi_0$. Hence, $w(\xi(t,j)) = e^{-\lambda_0(\xi(t,j)-\xi_0)}$, $w(\xi(t,j-1)) = e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j+1)) = e^{-\lambda_0(\xi(t,j)+1-\xi_0)}$. Then one has

$$\begin{aligned}
A_w^2(t,j) =& 4d_2 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 2r_2 + 2r_2b_{21} + 2r_2b_{23} + 4r_2\psi(\xi(t,j)) + 2r_2V_j(t) \\
&- 2r_2b_{21}U_j(t) - 2r_2b_{21}\varphi(\xi(t,j)) - 2r_2b_{23}W_j(t) - 2r_2b_{23}\theta(\xi(t,j)) \\
&- r_2b_{21}\psi(\xi(t,j)) - r_2b_{23}\psi(\xi(t,j)) - r_1b_{12}(1-\varphi(\xi(t,j))) \\
&- r_3b_{32}(1-\theta(\xi(t,j))) - d_2\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big) \\
>& 2d_2 + c\lambda_0 - 2r_2 - 3r_2(b_{21}+b_{23}) - r_1b_{12} - r_3b_{32} - d_2(e^{\lambda_0} + e^{-\lambda_0}) \\
=& c\lambda_0 - 2r_2 - 3r_2(b_{21}+b_{23}) - r_1b_{12} - r_3b_{32} - d_2(e^{\lambda_0} + e^{-\lambda_0} + 1) + 3d_2 \\
>& 3d_2 > 0.
\end{aligned}$$

Case 2: $\xi_0 - 1 \le \xi(t,j) \le \xi_0$**.** In this case, $\xi(t,j-1) < \xi_0$ and $\xi(t,j+1) \ge \xi_0$. Then $w(\xi(t,j)) = e^{-\lambda_0(\xi(t,j)-\xi_0)}$, $w(\xi(t,j-1)) = e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j+1)) = 1$.

Hence, we obtain

$$\begin{aligned}
A_w^2(t,j) =& 4d_2 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 2r_2 + 2r_2b_{21} + 2r_2b_{23} + 4r_2\psi(\xi(t,j)) + 2r_2V_j(t)\\
&- 2r_2b_{21}U_j(t) - 2r_2b_{21}\varphi(\xi(t,j)) - 2r_2b_{23}W_j(t) - 2r_2b_{23}\theta(\xi(t,j))\\
&- r_2b_{21}\psi(\xi(t,j)) - r_2b_{23}\psi(\xi(t,j)) - r_1b_{12}(1-\varphi(\xi(t,j))\\
&- r_3b_{32}(1-\theta(\xi(t,j))) - d_2\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big)\\
>& 2d_2 + c\lambda_0 - 2r_2 - 3r_2(b_{21}+b_{23}) - r_1b_{12} - r_3b_{32}\\
&- d_2(e^{\lambda_0} + e^{\lambda_0(\xi(t,j)-\xi_0)})\\
\geq& c\lambda_0 - 2r_2 - 3r_2(b_{21}+b_{23}) - r_1b_{12} - r_3b_{32} - d_2(e^{\lambda_0} + 1 + e^{-\lambda_0})\\
&+ d_2e^{-\lambda_0} + 2d_2\\
>& d_2e^{-\lambda_0} + 2d_2 > 0.
\end{aligned}$$

Case 3: $\xi_0 < \xi(t,j) \leq \xi_0 + 1$**.** In this case, $\xi(t,j-1) \leq \xi_0$ and $\xi(t,j+1) > \xi_0$. Then $w(\xi(t,j-1)) = e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j)) = w(\xi(t,j+1)) = 1$. Thus, we obtain

$$\begin{aligned}
A_w^2(t,j) =& 4d_2 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 2r_2 + 2r_2b_{21} + 2r_2b_{23} + 4r_2\psi(\xi(t,j)) + 2r_2V_j(t)\\
&- 2r_2b_{21}U_j(t) - 2r_2b_{21}\varphi(\xi(t,j)) - 2r_2b_{23}W_j(t) - 2r_2b_{23}\theta(\xi(t,j))\\
&- r_2b_{21}\psi(\xi(t,j)) - r_2b_{23}\psi(\xi(t,j)) - r_1b_{12}(1-\varphi(\xi(t,j))\\
&- r_3b_{32}(1-\theta(\xi(t,j))) - d_2\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big)\\
>& 2d_2 - 2r_2 + 4r_2\psi(\xi_0) - 3r_2(b_{21}+b_{23}) - r_1b_{12} - r_3b_{32} + r_1b_{12}\varphi(\xi_0)\\
&+ r_3b_{32}\theta(\xi_0) - d_2(e^{-\lambda_0(\xi(t,j)-1-\xi_0)} + 1)\\
>& 2d_2 - 2r_2 + 4r_2\psi(\xi_0) - 3r_2(b_{21}+b_{23}) - r_1b_{12} - r_3b_{32} + r_1b_{12}\varphi(\xi_0)\\
&+ r_3b_{32}\theta(\xi_0) - d_2(e^{\lambda_0} + 1)\\
=& \mathcal{N}_2(\xi_0) > 0.
\end{aligned}$$

Case 4: $\xi(t,j) > \xi_0 + 1$**.** In this case, $\xi(t,j) > \xi_0$, $\xi(t,j+1) > \xi_0$ and $\xi(t,j-1) > \xi_0$. Then $w(\xi(t,j)) = w(\xi(t,j-1)) = w(\xi(t,j+1)) = 1$. Hence, we have

$$\begin{aligned}
A_w^2(t,j) =& 4d_2 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 2r_2 + 2r_2b_{21} + 2r_2b_{23} + 4r_2\psi(\xi(t,j)) + 2r_2V_j(t)\\
&- 2r_2b_{21}U_j(t) - 2r_2b_{21}\varphi(\xi(t,j)) - 2r_2b_{23}W_j(t) - 2r_2b_{23}\theta(\xi(t,j))\\
&- r_2b_{21}\psi(\xi(t,j)) - r_2b_{23}\psi(\xi(t,j)) - r_1b_{12}(1-\varphi(\xi(t,j))\\
&- r_3b_{32}(1-\theta(\xi(t,j))) - d_2\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big)\\
>& -2r_2 + 4r_2\psi(\xi_0) - 3r_2(b_{21}+b_{23}) - r_1b_{12} - r_3b_{32} + r_1b_{12}\varphi(\xi_0)\\
&+ r_3b_{32}\theta(\xi_0)\\
=& \mathcal{N}_2(\xi_0) + d_2(e^{\lambda_0} + 1) - 2d_2
\end{aligned}$$

$$>d_2(e^{\lambda_0}-1)>0.$$

Therefore, we obtain $A_w^2(t,j)\geq C_2>0$ by choosing a suitable C_2 small enough.

Thirdly, we prove $A_w^3(t,j)\geq C_3$ for some positive constant C_3.

Case 1: $\xi(t,j)<\xi_0-1$**.** It is clear that $\xi(t,j)<\xi_0$, $\xi(t,j+1)<\xi_0$ and $\xi(t,j-1)<\xi_0$. Hence, $w(\xi(t,j))=e^{-\lambda_0(\xi(t,j)-\xi_0)}$, $w(\xi(t,j-1))=e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j+1))=e^{-\lambda_0(\xi(t,j)+1-\xi_0)}$. Then one has

$$\begin{aligned}
A_w^3(t,j)=&4d_3-c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))}-4r_3\theta(\xi(t,j))+2r_3+2r_3b_{32}V_j(t)+2r_3b_{32}\psi(\xi(t,j))\\
&-2r_3W_j(t)-r_3b_{32}(1-\theta(\xi(t,j))-r_2b_{23}\psi(\xi(t,j))\\
&-d_3\Big(2+\frac{w(\xi(t,j-1))}{w(\xi(t,j))}+\frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big)\\
>&2d_3+c\lambda_0-4r_3-r_3b_{32}-r_2b_{23}-d_3(e^{\lambda_0}+e^{-\lambda_0})\\
=&c\lambda_0-4r_3-r_3b_{32}-r_2b_{23}-d_3(e^{\lambda_0}+e^{-\lambda_0}+1)+3d_3\\
>&3d_3>0.
\end{aligned}$$

Case 2: $\xi_0-1\leq\xi(t,j)\leq\xi_0$**.** In this case, $\xi(t,j-1)<\xi_0$ and $\xi(t,j+1)\geq\xi_0$. Then $w(\xi(t,j))=e^{-\lambda_0(\xi(t,j)-\xi_0)}$, $w(\xi(t,j-1))=e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j+1))=1$. Hence, we obtain

$$\begin{aligned}
A_w^3(t,j)=&4d_3-c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))}-4r_3\theta(\xi(t,j))+2r_3+2r_3b_{32}V_j(t)+2r_3b_{32}\psi(\xi(t,j))\\
&-2r_3W_j(t)-r_3b_{32}(1-\theta(\xi(t,j))-r_2b_{23}\psi(\xi(t,j))\\
&-d_3\Big(2+\frac{w(\xi(t,j-1))}{w(\xi(t,j))}+\frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big)\\
>&2d_3+c\lambda_0-4r_3-r_3b_{32}-r_2b_{23}-d_3\big(e^{\lambda_0}+e^{\lambda_0(\xi(t,j)-\xi_0)}\big)\\
\geq&c\lambda_0-4r_3-r_3b_{32}-r_2b_{23}-d_3(e^{\lambda_0}+1+e^{-\lambda_0})+d_3e^{-\lambda_0}+2d_3\\
>&d_3e^{-\lambda_0}+2d_3>0.
\end{aligned}$$

Case 3: $\xi_0<\xi(t,j)\leq\xi_0+1$**.** In this case, $\xi(t,j-1)\leq\xi_0$ and $\xi(t,j+1)>\xi_0$. Then $w(\xi(t,j-1))=e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j))=w(\xi(t,j+1))=1$. Thus, we have

$$\begin{aligned}
A_w^3(t,j)=&4d_3-c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))}-4r_3\theta(\xi(t,j))+2r_3+2r_3b_{32}V_j(t)+2r_3b_{32}\psi(\xi(t,j))\\
&-2r_3W_j(t)-r_3b_{32}(1-\theta(\xi(t,j))-r_2b_{23}\psi(\xi(t,j))\\
&-d_3\Big(2+\frac{w(\xi(t,j-1))}{w(\xi(t,j))}+\frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big)\\
>&2d_3-4r_3+2r_3b_{32}\psi(\xi_0)+r_3b_{32}\theta(\xi_0)-r_3b_{32}-r_2b_{23}\\
&-d_3(e^{-\lambda_0(\xi(t,j)-1-\xi_0)}+1)\\
>&2d_3-4r_3+2r_3b_{32}\psi(\xi_0)+r_3b_{32}\theta(\xi_0)-r_3b_{32}-r_2b_{23}-d_3(e^{\lambda_0}+1)\\
=&\mathcal{N}_3(\xi_0)>0.
\end{aligned}$$

Case 4: $\xi(t,j) > \xi_0+1$. In this case, $\xi(t,j) > \xi_0$, $\xi(t,j+1) > \xi_0$ and $\xi(t,j-1) > \xi_0$. Then $w(\xi(t,j)) = w(\xi(t,j-1)) = w(\xi(t,j+1)) = 1$. Hence, we have

$$
\begin{aligned}
A_w^3(t,j) =&4d_3 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 4r_3\theta(\xi(t,j)) + 2r_3 + 2r_3b_{32}V_j(t) + 2r_3b_{32}\psi(\xi(t,j)) \\
&- 2r_3W_j(t) - r_3b_{32}(1-\theta(\xi(t,j)) - r_2b_{23}\psi(\xi(t,j)) \\
&- d_3\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big) \\
>& -4r_3 + 2r_3b_{32}\psi(\xi_0) + r_3b_{32}\theta(\xi_0) - r_3b_{32} - r_2b_{23} \\
=&\mathcal{N}_3(\xi_0) + d_3(e^{\lambda_0}+1) - 2d_3 \\
>&d_3(e^{\lambda_0}-1) > 0.
\end{aligned}
$$

Therefore, we show $A_w^3(t,j) \geq C_3 > 0$ by choosing a suitable C_3 small enough. The proof is complete. □

Lemma 3.4. *Assume that* (H2) *holds. For any $c > \max\{c_{min}, \tilde{c}\}$, there exist some positive constants C_i such that*

$$
B^i_{\mu,w}(t,j) \geq C_i, \quad i=1,2,3,
$$

for all $t > 0$, $j \in \mathbb{Z}$ and $0 < \mu < \frac{\min_{i=1,2,3}\{C_i\}}{2}$.

The proof of the above lemma can be easily obtained by Lemma 3.3, so we omit here. Next, we will give the energy estimates.

Lemma 3.5. *Assume that* (H2) *hold. For any $c > \max\{c_{min}, \tilde{c}\}$, it holds*

$$
\begin{aligned}
&\|U_j(t)\|^2_{l^2_w} + \|V_j(t)\|^2_{l^2_w} + \|W_j(t)\|^2_{l^2_w} \\
&+ \int_0^t e^{-2\mu(t-s)}\Big(\|U_j(s)\|^2_{l^2_w} + \|V_j(s)\|^2_{l^2_w} + \|W_j(t)\|^2_{l^2_w}\Big)ds \\
&\leq Ce^{-2\mu t}\Big(\|U_{j0}(0)\|^2_{l^2_w} + \|V_{j0}(0)\|^2_{l^2_w} + \|W_{j0}(0)\|^2_{l^2_w}\Big)
\end{aligned}
\tag{3.5}
$$

for some positive constant C.

Proof. Multiplying the equations in (3.3) by $e^{2\mu t}w(\xi(t,j))U_j(t)$, $e^{2\mu t}w(\xi(t,j))V_j(t)$ and $e^{2\mu t}w(\xi(t,j))W_j(t)$ respectively, where $\mu > 0$ is defined in Lemma 3.4, we obtain

$$
\begin{aligned}
&\Big(\frac{1}{2}e^{2\mu t}w(\xi(t,j))U_j^2(t)\Big)_t - d_1e^{2\mu t}w(\xi(t,j))U_j(t)(U_{j+1}(t) + U_{j-1}(t)) \\
&+ \Big(2d_1 - \frac{c}{2}\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - \mu - 2r_1\varphi(\xi(t,j)) + r_1 + r_1b_{12}V_j(t) \\
&+ r_1b_{12}\psi(\xi(t,j))\Big)e^{2\mu t}w(\xi(t,j))U_j^2(t) \\
&= r_1e^{2\mu t}w(\xi(t,j))U_j^3(t) + r_1b_{12}(1-\varphi(\xi(t,j)))e^{2\mu t}w(\xi(t,j))U_j(t)V_j(t),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&\Big(\frac{1}{2}e^{2\mu t}w(\xi(t,j))V_j^2(t)\Big)_t - d_2 e^{2\mu t}w(\xi(t,j))V_j(t)(V_{j+1}(t)+V_{j-1}(t))\\
&+\Big(2d_2 - \frac{c}{2}\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - \mu - r_2 + r_2b_{21} + r_2b_{23} + 2r_2\psi(\xi(t,j)) - r_2b_{21}U_j(t)\\
&- r_2b_{23}W_j(t) - r_2b_{21}\varphi(\xi(t,j)) - r_2b_{23}\theta(\xi(t,j))\Big)e^{2\mu t}w(\xi(t,j))V_j^2(t)\\
&= -r_2e^{2\mu t}w(\xi(t,j))V_j^3(t) + r_2b_{21}\psi(\xi(t,j))e^{2\mu t}w(\xi(t,j))U_j(t)V_j(t)\\
&\quad + r_2b_{23}\psi(\xi(t,j))e^{2\mu t}w(\xi(t,j))W_j(t)V_j(t),
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&\Big(\frac{1}{2}e^{2\mu t}w(\xi(t,j))W_j^2(t)\Big)_t - d_3 e^{2\mu t}w(\xi(t,j))W_j(t)(W_{j+1}(t)+W_{j-1}(t))\\
&+\Big(2d_3 - \frac{c}{2}\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - \mu - 2r_3\theta(\xi(t,j)) + r_3 + r_3b_{32}V_j(t)\\
&+ r_3b_{32}\psi(\xi(t,j))\Big)e^{2\mu t}w(\xi(t,j))W_j^2(t)\\
&= r_3e^{2\mu t}w(\xi(t,j))W_j^3(t) + r_3b_{32}(1-\theta(\xi(t,j)))e^{2\mu t}w(\xi(t,j))W_j(t)V_j(t).
\end{aligned}
\tag{3.8}
$$

By the Cauchy-Schwarz inequality $2ab \le a^2 + b^2$, we obtain

$$
\begin{gathered}
2U_{j+1}(t)U_j(t) \le U_{j+1}^2(t) + U_j^2(t),\\
2V_{j+1}(t)V_j(t) \le V_{j+1}^2(t) + V_j^2(t),\\
2W_{j+1}(t)W_j(t) \le W_{j+1}^2(t) + W_j^2(t).
\end{gathered}
$$

By summing over all $j \in \mathbb{Z}$ for (3.6), (3.7) and (3.8), then integrating over $[0,t]$, one has

$$
\begin{aligned}
&e^{2\mu t}\|U_j(t)\|_{l_w^2}^2 + \int_0^t \sum_j \Big[2\Big(2d_1 - \frac{c}{2}\frac{w'_\xi(\xi(s,j))}{w(\xi(s,j))} - \mu - 2r_1\varphi(\xi(s,j))\\
&+ r_1 + r_1b_{12}V_j(s) + r_1b_{12}\psi(\xi(s,j))\Big) - d_1\frac{w(\xi(s,j+1))}{w(\xi(s,j))}\\
&- d_1\frac{w(\xi(s,j-1))}{w(\xi(s,j))} - 2d_1\Big]e^{2\mu s}w(\xi(s,j))U_j^2(s)ds\\
&\le \|U_{j0}(0)\|_{l_w^2}^2 + 2r_1\int_0^t\sum_j e^{2\mu s}w(\xi(s,j))U_j(s)U_j^2(s)ds\\
&\quad + \int_0^t\sum_j r_1b_{12}(1-\varphi(\xi(s,j)))e^{2\mu s}w(\xi(s,j))(U_j^2(s)+V_j^2(s))ds,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&e^{2\mu t}\|V_j(t)\|_{l_w^2}^2 + \int_0^t \sum_j \Big[2\Big(2d_2 - \frac{c}{2}\frac{w'_\xi(\xi(s,j))}{w(\xi(s,j))} - \mu - r_2 + r_2b_{21} + r_2b_{23}\\
&+ 2r_2\psi(\xi(s,j)) - r_2b_{21}U_j(s) - r_2b_{23}W_j(s) - r_2b_{21}\varphi(\xi(s,j)) - r_2b_{23}\theta(\xi(s,j))\Big)\\
&- d_2\frac{w(\xi(s,j+1))}{w(\xi(s,j))} - d_2\frac{w(\xi(s,j-1))}{w(\xi(s,j))} - 2d_2\Big]e^{2\mu s}w(\xi(s,j))V_j^2(s)ds\\
&\le \|V_{j0}(0)\|_{l_w^2}^2 - 2r_2\int_0^t\sum_j e^{2\mu s}w(\xi(s,j))V_j(s)V_j^2(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \sum_j r_2 b_{21}\psi(\xi(s,j)))e^{2\mu s}w(\xi(s,j))(U_j^2(s)+V_j^2(s))ds\\
&+\int_0^t \sum_j r_2 b_{23}\psi(\xi(s,j)))e^{2\mu s}w(\xi(s,j))(W_j^2(s)+V_j^2(s))ds,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&e^{2\mu t}\|W_j(t)\|_{l_w^2}^2+\int_0^t\sum_j\Big[2\Big(2d_3-\frac{c}{2}\frac{w'_\xi(\xi(s,j))}{w(\xi(s,j))}-\mu-2r_3\theta(\xi(s,j))\\
&+r_3+r_3b_{32}V_j(s)\Big)+r_3b_{32}\psi(\xi(s,j))-d_3\frac{w(\xi(s,j+1))}{w(\xi(s,j))}\\
&-d_3\frac{w(\xi(s,j-1))}{w(\xi(s,j))}-2d_3\Big]e^{2\mu s}w(\xi(s,j))W_j^2(s)ds\\
&\le \|W_{j0}(0)\|_{l_w^2}^2+2r_3\int_0^t\sum_j e^{2\mu s}w(\xi(s,j))W_j(s)W_j^2(s)ds\\
&\quad+\int_0^t\sum_j r_3b_{32}(1-\theta(\xi(s,j)))e^{2\mu s}w(\xi(s,j))(W_j^2(s)+V_j^2(s))ds.
\end{aligned}
\tag{3.11}
$$

Adding the inequalities (3.9), (3.10), (3.11), we have

$$
\begin{aligned}
&e^{2\mu t}\Big(\|U_j(t)\|_{l_w^2}^2+\|V_j(t)\|_{l_w^2}^2+\|W_j(t)\|_{l_w^2}^2\Big)+\int_0^t\sum_j e^{2\mu s}\Big(B_{\mu,w}^1(s,j)U_j^2(s)\\
&+B_{\mu,w}^2(s,j)V_j^2(s)+B_{\mu,w}^3(s,j)W_j^2(s)\Big)w(\xi(s,j))ds\\
&\le \|U_{j0}(0)\|_{l_w^2}^2+\|V_{j0}(0)\|_{l_w^2}^2+|W_{j0}(0)\|_{l_w^2}^2,
\end{aligned}
$$

where $B_{\mu,w}^1(t,j)$, $B_{\mu,w}^2(t,j)$ and $B_{\mu,w}^2(t,j)$ are defined in (3.4). By Lemma 3.4, we obtain (3.5), i.e.,

$$
\begin{aligned}
&\|U_j(t)\|_{l_w^2}^2+\|V_j(t)\|_{l_w^2}^2+\|W_j(t)\|_{l_w^2}^2\\
&+\int_0^t e^{-2\mu(t-s)}\Big(\|U_j(s)\|_{l_w^2}^2+\|V_j(s)\|_{l_w^2}^2+\|W_j(s)\|_{l_w^2}^2\Big)ds\\
&\le Ce^{-2\mu t}\Big(\|U_{j0}(0)\|_{l_w^2}^2+\|V_{j0}(0)\|_{l_w^2}^2+\|W_{j0}(0)\|_{l_w^2}^2\Big)
\end{aligned}
\tag{3.12}
$$

for some positive constant C. The proof is complete. $\square$

Proof of Theorem 2.2. Since $w(\xi)\ge 1$ defined by (2.6), we obtain $\|\cdot\|_{l^2}\le\|\cdot\|_{l_w^2}$. Furthermore, by the Sobolev's embedding inequality $l^2\hookrightarrow l^\infty$, one has

$$
\begin{aligned}
\sup_{j\in\mathbb{Z}}|U_j(t)|&\le C\|U_j(t)\|_{l^2}\le C\|U_j(t)\|_{l_w^2},\\
\sup_{j\in\mathbb{Z}}|V_j(t)|&\le C\|V_j(t)\|_{l^2}\le C\|V_j(t)\|_{l_w^2},\\
\sup_{j\in\mathbb{Z}}|W_j(t)|&\le C\|W_j(t)\|_{l^2}\le C\|W_j(t)\|_{l_w^2}.
\end{aligned}
$$

Then by (3.12), we obtain

$$
\sup_{j\in\mathbb{Z}}|u_j^+(t)-\varphi(j+ct)|=\sup_{j\in\mathbb{Z}}|U_j(t)|\le Ce^{-\mu t},
$$

$$
\begin{aligned}
\sup_{j\in\mathbb{Z}}|v_j^+(t)-\psi(j+ct)| &= \sup_{j\in\mathbb{Z}}|V_j(t)| \le Ce^{-\mu t},\\
\sup_{j\in\mathbb{Z}}|w_j^+(t)-\theta(j+ct)| &= \sup_{j\in\mathbb{Z}}|W_j(t)| \le Ce^{-\mu t},
\end{aligned}
$$

where $t>0$. Similarly, we can obtain

$$
\begin{aligned}
\sup_{j\in\mathbb{Z}}|u_j^-(t)-\varphi(j+ct)| &= \sup_{j\in\mathbb{Z}}|U_j(t)| \le Ce^{-\mu t},\\
\sup_{j\in\mathbb{Z}}|v_j^-(t)-\psi(j+ct)| &= \sup_{j\in\mathbb{Z}}|V_j(t)| \le Ce^{-\mu t},\\
\sup_{j\in\mathbb{Z}}|w_j^-(t)-\theta(j+ct)| &= \sup_{j\in\mathbb{Z}}|W_j(t)| \le Ce^{-\mu t}.
\end{aligned}
$$

Then the squeezing technique yields

$$
\begin{aligned}
&\sup_{j\in\mathbb{Z}}|u_j(t)-\varphi(j+ct)| \le Ce^{-\mu t}, \quad t>0,\\
&\sup_{j\in\mathbb{Z}}|v_j(t)-\psi(j+ct)| \le Ce^{-\mu t}, \quad t>0,\\
&\sup_{j\in\mathbb{Z}}|w_j(t)-\theta(j+ct)| \le Ce^{-\mu t}, \quad t>0.
\end{aligned}
$$

This completes the proof. □

Acknowledgments. G.-B. Zhang was supported by NSF of China (11401478).

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