Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 61, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SEMICLASSICAL GROUND STATES FOR NONLINEAR SCHRÖDINGER-POISSON SYSTEMS

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Communicated by Claudianor O. Alves

Abstract. In this article, we study the Schrödinger-Poisson system

$$-\epsilon^2\Delta u + V(x)u + \phi(x)u = Q(x)u^3, \quad x \in \mathbb{R}^3,$$
$$-\epsilon^2\Delta\phi = u^2, \quad x \in \mathbb{R}^3,$$

where $\epsilon > 0$ is a parameter, V and Q are positive bounded functions. We establish the existence of ground states for ϵ small, and describe the concentration phenomena of ground states as $\epsilon \to 0$.

1. Introduction and statement of main results

The Schrödinger-Poisson system

$$\begin{aligned} -\epsilon^2\Delta u + V(x)u + \phi u = l(x,u), \quad x \in \mathbb{R}^3,\\ -\epsilon^2\Delta\phi = u^2, \quad x \in \mathbb{R}^3, \end{aligned} \tag{1.1}$$

was first introduced in [6] as a physical model describing a charged wave interacting with its own electrostatic field in quantum mechanics. The unknowns u and ϕ represent the wave functions associated to the particle and electric potential, the function V is an external potential, and the nonlinearity $l(x,u)$ simulates the interaction between many particles or external nonlinear perturbations. For more information on the physical aspects, we refer the reader to [6].

There are many results on the existence and concentration of solutions for (1.1) and similar problems. Equation (1.1) is usually studied in two cases when ϵ is regarded as a small parameter, and when ϵ is fixed ($\epsilon = 1$). For fixed ϵ, see [1, 2, 3, 4, 5, 7, 16, 17, 18, 19, 21, 26, 27, 28] and references therein. In this article, we study onlythe case ϵ is small. So we shall recall some results for this case. In [15] the authors considered the system

$$\begin{aligned} -\epsilon^2\Delta u + V(x)u + \phi(x)u = f(u), \quad x \in \mathbb{R}^3,\\ -\Delta\phi = u^2, \quad x \in \mathbb{R}^3, \end{aligned} \tag{1.2}$$

and proved that (1.2) has a single bump solution, which concentrates on the critical points of $V(x)$. Later, D'Aprile and Wei [8] constructed positive radially symmetric

2010 *Mathematics Subject Classification.* 35J50, 35J60, 35A15.
Key words and phrases. Schrödinger-Poisson system; variational method; concentration; Nehari manifold.

Submitted September 29, 2017. Published March 5, 2018.

bound states of (1.2) with $f(u) = u^p$, $1 < p < \frac{11}{7}$. By applying a standard Lyapunov-Schmidt reduction methods, Ruiz and Vaira [22] proved the existence of multi-bump solutions of (1.2), whose bumps concentrate around a local minimum of the potential $V(x)$ when $f(u) = u^p$ and $3 < p < 5$. On the other hand, He [12] considered the system

$$\begin{gathered} -\epsilon^2\Delta u + V(x)u + \phi(x)u = f(u), \quad x \in \mathbb{R}^3, \\ -\epsilon^2\Delta\phi = u^2, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.3}$$

where f is of subcritical growth and:

$$\begin{gathered} f \in C^1(\mathbb{R}^3),\ f(s) = o(s^3) \text{ as } s \to 0,\ \frac{f(s)}{s^3} \text{ is increasing on } (0,\infty), \\ \text{there exists } \mu > 4 \text{ such that } 0 < \mu F(s) := \mu \int_0^s f(t)dt \le sf(s), \\ s > 0, \text{ and} \\ f'(s)s^2 - 3f(s)s \ge Cs^\sigma, \quad C > 0,\ \sigma \in (4,6). \end{gathered} \tag{1.4}$$

By using Ljusternik-Schnirelmann theory and minimax methods, he showed the multiplicity of positive solutions of (1.3) which concentrate on the minima of $V(x)$ as $\epsilon \to 0$. Later, Wang et al. [24] studied the system

$$\begin{gathered} -\epsilon^2\Delta u + V(x)u + \phi(x)u = b(x)f(u), \quad x \in \mathbb{R}^3, \\ -\epsilon^2\Delta\phi = u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{1.5}$$

Suppose that $V(x)$ has at least one minimum, $b(x)$ has at least one maximum, and f satisfies some weaker conditions than (1.4), namely

$$\begin{gathered} f \in C(\mathbb{R}^3), \quad f(s) = o(s^3) \text{ as } s \to 0, \quad \frac{f(s)}{s^3} \text{ is increasing on } (0,\infty), \\ \frac{F(s)}{s^4} \to \infty \quad \text{as } s \to \infty, \end{gathered}$$

Wang et al. obtained the existence and concentration of positive ground states for (1.5) using the method of Nehari manifold and minimax methods. He and Zou [13] considered the existence and concentration behavior of ground states of (1.1) with critical growth,

$$\begin{gathered} -\epsilon^2\Delta u + V(x)u + \phi(x)u = |u|^4u + f(u), \quad x \in \mathbb{R}^3, \\ -\epsilon^2\Delta\phi = u^2, \quad x \in \mathbb{R}^3, \end{gathered}$$

where f satisfies (1.4) and $f(t) \ge \lambda t^\sigma$ for all $t > 0$, where $\sigma \in (3,5)$. Recently, He et al. [14] studied the system

$$\begin{gathered} -\epsilon^2\Delta u + V(x)u + \phi(x)u = \lambda|u|^{p-2}u + |u|^4u, \quad x \in \mathbb{R}^3, \\ -\epsilon^2\Delta\phi = u^2, \quad x \in \mathbb{R}^3, \end{gathered}$$

where $3 < p \le 4$. Under certain assumptions on the potential V, they constructed a family of positive solutions which concentrates around a local minimum of V.

It seems that, the existence and concentration of ground states for (1.1) with three times growth have not been studied. So in the paper we shall fill this gap. In the sequel, we consider the system

$$\begin{gathered} -\epsilon^2\Delta u + V(x)u + \phi u = Q(x)u^3, \quad x \in \mathbb{R}^3, \\ -\epsilon^2\Delta\phi = u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{1.6}$$

To state the main results, we need some notation. Set:

$$\nu_{\min} = \min_{x\in\mathbb{R}^3} V(x), \quad \mathcal{V} := \{x \in \mathbb{R}^3 : V(x) = \nu_{\min}\}, \quad \nu_\infty := \liminf_{|x|\to\infty} V(x) < \infty,$$
$$q_{\max} = \max_{x\in\mathbb{R}^3} Q(x), \quad \mathcal{Q} := \{x \in \mathbb{R}^3 : Q(x) = q_{\max}\}, \quad q_\infty := \limsup_{|x|\to\infty} Q(x) < \infty.$$

We use the following assumptions

(A1) $V, Q \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ with $\nu_{\min} > 0$ and $\inf_{x\in\mathbb{R}^3} Q(x) > 0$;

(A2) $\nu_{\min} < \nu_\infty$, and there exist $R > 0$ and $x_{\min} \in \mathcal{V}$ such that $Q(x_{\min}) \geq Q(x)$ for all $|x| \geq R$.

(A3) $q_{\max} > q_\infty$, and there exist $R > 0$ and $x_{\max} \in \mathcal{Q}$ such that $V(x_{\max}) \leq V(x)$ for all $|x| \geq R$.

Observe that, for case (A2), we can assume that $Q(x_{\min}) = \max_{x\in\mathcal{V}} Q(x)$ and set

$$\mathcal{A}_V := \{x \in \mathcal{V} : Q(x) = Q(x_{\min})\} \cup \{x \notin \mathcal{V} : Q(x) > Q(x_{\min})\};$$

while for case (A3), we can assume that $V(x_{\max}) = \min_{x\in\mathcal{Q}} V(x)$ and set

$$\mathcal{A}_Q := \{x \in \mathcal{Q} : V(x) = V(x_{\max})\} \cup \{x \notin \mathcal{Q} : V(x) < V(x_{\max})\}.$$

This kind of structure was recently introduced by Ding and Liu [9] which generalized the case by Rabinowitz in [20].

The system (1.6) can be easily transformed into a Schrödinger equation with a nonlocal term. Actually, for all $u \in H^1(\mathbb{R}^3)$ and fixed $\epsilon > 0$, considering the linear functional L_u defined in $D^{1,2}(\mathbb{R}^3)$ by

$$L_u(v) = \int_{\mathbb{R}^3} u^2 v dx.$$

By the Hölder inequality and the Sobolev inequality, we have

$$|L_u(v)| \leq |u|^2_{\frac{12}{5}} |v|_6 \leq C|u|^2_{\frac{12}{5}} \|v\|_{D^{1,2}}. \tag{1.7}$$

Hence the Lax-Milgram theorem implies that there exists a unique $\phi_u^\epsilon \in D^{1,2}(\mathbb{R}^3)$ such that

$$\epsilon^2 \int_{\mathbb{R}^3} \nabla\phi_u^\epsilon \nabla v dx = L_u(v) = \int_{\mathbb{R}^3} u^2 v dx, \quad \forall v \in D^{1,2}(\mathbb{R}^3). \tag{1.8}$$

Namely, ϕ_u^ϵ is the unique solution of $-\epsilon^2\Delta\phi_u^\epsilon = u^2$. Moreover, ϕ_u^ϵ can be expressed as

$$\phi_u^\epsilon(x) = \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy.$$

Substituting ϕ_u^ϵ into the first equation of (1.6), we obtain

$$-\epsilon^2\Delta u + V(x)u + \phi_u^\epsilon(x)u = Q(x)u^3. \tag{1.9}$$

Let $\mathscr{L}_\epsilon$ denote the set of all positive ground states of (1.9). Now we state our main results.

Theorem 1.1. *Let* (A1) *and* (A2) *hold. Then for any* $\epsilon > 0$ *small we have:*
(1) Equation (1.6) *has a positive ground state* $\psi_\epsilon = (w_\epsilon, \phi^\epsilon_{w_\epsilon})$ *in* $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$*;*
(2) $\mathscr{L}_\epsilon$ *is compact in* $H^1(\mathbb{R}^3)$*;*
(3) If additionally V *and* Q *are uniformly continuous functions, then* w_ϵ *satisfies:*

(i) *there exists a maximum point* $x_\epsilon \in \mathbb{R}^3$ *of* w_ϵ, *such that* $\lim_{\epsilon\to 0}\operatorname{dist}(x_\epsilon, \mathcal{A}_V) = 0$. *Setting* $v_\epsilon(x) := w_\epsilon(\epsilon x + x_\epsilon)$, *for any sequence* $x_\epsilon \to x_0$, $\epsilon \to 0$, v_ϵ *converges in* $H^1(\mathbb{R}^3)$ *to a ground state* v *of*
$$-\Delta u + V(x_0)u + \phi_u(x)u = Q(x_0)u^3,$$
where $\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}dy$.
In particular, if $\mathcal{V}\cap\mathcal{Q} \neq \emptyset$, *then* $\lim_{\epsilon\to 0}\operatorname{dist}(x_\epsilon, \mathcal{V}\cap\mathcal{Q}) = 0$ *and up to a subsequence,* v_ϵ *converges in* $H^1(\mathbb{R}^3)$ *to a ground state* v *of*
$$-\Delta u + \nu_{\min}u + \phi_u(x)u = q_{\max}u^3,$$
(ii) $|w_\epsilon(x)| \le Cexp\big(-\frac{c}{\epsilon}|x - x_\epsilon|\big)$, *where* $C, c > 0$.

Theorem 1.2. *Suppose that* (A1), (A3) *hold. Then, all the conclusions of Theorem 1.1 remain true with* $\mathcal{A}_V$ *replaced by* $\mathcal{A}_Q$.

Outline for the proof. Compared with the previous results [12, 13, 14, 24], the main difficulty is the lack of the higher-order term and the competing effect of the nonlocal term with three times growth term, which causes that the standard method of Nehari manifold is invalid. Inspired by [10], by restricting the functional in a set, the functional has a unique maximum point along the nontrivial direction u in $H^1(\mathbb{R}^3)$. Then we use the one-to-one correspondence of the functionals on the manifold and an open set of the unit sphere to establish the new method of Nehari manifold. We also would like to point out that, using the similar ideas, we [25] showed the existence of classical ground states of system (1.1) with $\epsilon = 1$ when the potentials are asymptotically periodic. However, in this paper, we prove the existence and concentration of semiclassical ground states for system (1.1) with small enough ϵ. In addition, in the period of investigating the concentration behavior of ground states, the competing effect of the nonlocal term ϕu and three times growth term $Q(x)u^3$ makes that some estimations and verifications become complex. □

In this paper we use the following notation. For $1 \le p \le \infty$, the norm in $L^p(\mathbb{R}^3)$ is denoted by $|\cdot|_p$. $\int_{\mathbb{R}^3} f(x)dx$ will be represented by $\int_{\mathbb{R}^3} f(x)$. For any $r > 0$ and $x \in \mathbb{R}^3$, $B_r(x)$ denotes the ball centered at x with the radius r.

This article is organized as follows. In Section 2 we introduce the variational framework. In Section 3 we study the autonomous problem. In Section 4 we are devoted to investigating an auxiliary problem. In Section 5, we give the proof of Theorems 1.1 and 1.2.

2. The new method of Nehari manifold

For the proof of our theorems, we shall consider an equivalent equation to (1.9). By making the change of variable $x \to \epsilon x$, the problem (1.9) turns out to be
$$-\Delta u + V(\epsilon x)u + \phi_u(x)u = Q(\epsilon x)u^3, \quad u \in H^1(\mathbb{R}^3), \tag{2.1}$$
where $H^1(\mathbb{R}^3)$ is the Sobolev space with standard norm
$$\|u\|^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2).$$
Let $S_1 = \{u \in H^1(\mathbb{R}^3) : \|u\|^2 = 1\}$. From assumption (A1), it follows that
$$\|u\|_\epsilon^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + V(\epsilon x)u^2)$$

is an equivalent norm on $H^1(\mathbb{R}^3)$. The functional associated with the equation (2.1) is

$$I_\epsilon(u) = \frac{1}{2}\|u\|_\epsilon^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{4}\int_{\mathbb{R}^3} Q(\epsilon x)u^4.$$

Now we recall some standard properties of ϕ_u, see [25].

Lemma 2.1. *Let* (A1) *hold. For any $\epsilon > 0$, we have:*

(i) *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.*
(ii) *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\int_{\mathbb{R}^3} \phi_{u_n} u_n v \to \int_{\mathbb{R}^3} \phi_u uv$, for any $v \in C_0^\infty(\mathbb{R}^3)$.*

Below we describe the variational framework for our problem. Firstly we give the Nehari manifold N_ϵ corresponding to I_ϵ:

$$N_\epsilon = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I'_\epsilon(u), u\rangle = 0\},$$

where

$$\langle I'_\epsilon(u), u\rangle = |\nabla u|_2^2 + \int_{\mathbb{R}^3} V(\epsilon x)u^2 + \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} Q(\epsilon x)u^4,$$

and the least energy on N_ϵ is defined by $c_\epsilon := \inf_{N_\epsilon} I_\epsilon$.

Lemma 2.2. *Let* (A1) *hold. Then I_ϵ is coercive on N_ϵ.*

Proof. For all $u \in N_\epsilon$, we have

$$I_\epsilon(u) = I_\epsilon(u) - \frac{1}{4}\langle I'_\epsilon(u), u\rangle = \frac{1}{4}\|u\|_\epsilon^2. \tag{2.2}$$

Then $I_\epsilon|_{N_\epsilon}$ is coercive. □

Next we introduce a set to construct the new method of Nehari manifold as in [25]. Define

$$\Theta_\epsilon := \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \phi_u u^2 < \int_{\mathbb{R}^3} Q(\epsilon x)u^4\}.$$

As in [25], we can show that $\Theta_\epsilon \neq \emptyset$ since $\inf_{\mathbb{R}^3} Q > 0$ by (A1). Set

$$h_\epsilon(t) := I_\epsilon(tu) = \frac{t^2}{2}\|u\|_\epsilon^2 + \frac{t^4}{4}\Big[\int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} Q(\epsilon x)u^4\Big],$$

using the similar argument in [25], we obtain the following two lemmas.

Lemma 2.3. *Let* (A1) *hold. Then for any $\epsilon > 0$, we have:*

(i) *For all $u \in \Theta_\epsilon$, there exists a unique $t_\epsilon := t_\epsilon(u) > 0$ such that $h'_\epsilon(t) > 0$ for $0 < t < t_\epsilon$, and $h'_\epsilon(t) < 0$ for $t > t_\epsilon$. Moreover, $t_\epsilon u \in N_\epsilon$ and $I_\epsilon(t_\epsilon u) = \max_{t>0} I_\epsilon(tu)$.*
(ii) *If $u \notin \Theta_\epsilon$, then $tu \notin N_\epsilon$ for any $t > 0$.*
(iii) *For each compact subset W of $\Theta_\epsilon \cap S_1$, there exists $C_W > 0$ such that $t_w \leq C_W$ for all $w \in W$.*

Lemma 2.4. *Under assumption* (A1), *for $\epsilon > 0$ we have:*

(1) *$c_\epsilon > 0$;*
(2) *$\|u\|_\epsilon^2 \geq 4c_\epsilon$ for all $u \in N_\epsilon$.*

From Lemma 2.3 (i), we define the mapping $\hat{m}_\epsilon : \Theta_\epsilon \to N_\epsilon$ by $\hat{m}_\epsilon(u) = t_\epsilon u$. In addition, $\forall v \in \mathbb{R}^+ u$ we have $\hat{m}_\epsilon(v) = \hat{m}_\epsilon(u)$. Let $U_\epsilon := \Theta_\epsilon \cap S_1$, we easily infer that U_ϵ is an open subset of S_1. Define $m_\epsilon := \hat{m}_\epsilon|_{U_\epsilon}$. Then m_ϵ is a bijection from U_ϵ to N_ϵ. Moreover, by Lemmas 2.3 and 2.4, as in the proof of [23, Proposition 3.1], we have the following result.

Lemma 2.5. *If* (A1) *is satisfied, then the mapping m_ϵ is a homeomorphism between U_ϵ and N_ϵ, and the inverse of m_ϵ is given by $m_\epsilon^{-1}(u) = \frac{u}{\|u\|}$.*

By Lemma 2.5, the least energy c_ϵ has the following minimax characterization:

$$c_\epsilon := \inf_{u\in N_\epsilon} I_\epsilon(u) = \inf_{u\in U_\epsilon} \max_{t\geq 0} I_\epsilon(tu). \tag{2.3}$$

Considering the functional $\Psi_\epsilon : U_\epsilon \to \mathbb{R}$ given by $\Psi_\epsilon(w) := I_\epsilon(m_\epsilon(w))$, as in [23, Corollary 3.3] we easily deduce the following statement.

Lemma 2.6. *Let* (A1) *hold. Then the following results hold:*

1. *If $\{w_n\}$ is a PS sequence for Ψ_ϵ, then $\{m_\epsilon(w_n)\}$ is a PS sequence for I_ϵ. If $\{u_n\} \subset N_\epsilon$ is a bounded PS sequence for I_ϵ, then $\{m_\epsilon^{-1}(u_n)\}$ is a PS sequence for Ψ_ϵ.*
2. *w is a critical point of Ψ_ϵ if and only if $m_\epsilon(w)$ is a nontrivial critical point of I_ϵ. Moreover, $\inf_{N_\epsilon} I_\epsilon = \inf_{U_\epsilon} \Psi_\epsilon$.*

3. Autonomous problem

This section concerns the autonomous equation. Precisely, for any positive constants ν and q, we consider

$$-\Delta u + \nu u + \phi_u(x)u = qu^3,\ u \in H^1(\mathbb{R}^3). \tag{3.1}$$

The functional of (3.1) is denoted by

$$J_{\nu,q}(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \nu|u|^2) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{q}{4}\int_{\mathbb{R}^3} |u|^4.$$

The Nehari manifold corresponding to (3.1) is defined by

$$M_{\nu,q} = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle J'_{\nu,q}(u), u\rangle = 0\},$$

and the least energy on $M_{\nu,q}$ is defined by $m_{\nu,q} := \inf_{M_{\nu,q}} J_{\nu,q}$.

Denote

$$\Theta^q = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \phi_u u^2 < q\int_{\mathbb{R}^3} u^4\}.$$

Then as (2.3), $m_{\nu,q}$ has the following characterization:

$$m_{\nu,q} := \inf_{M_{\nu,q}} J_{\nu,q} = \inf_{w\in\Theta^q\cap S_1} \max_{t>0} J_{\nu,q}(tw). \tag{3.2}$$

From [25] we have the following result.

Lemma 3.1. *For any $\nu, q > 0$, problem* (3.1) *has a positive ground state $u_{\nu,q}$ with $J_{\nu,q}(u_{\nu,q}) = m_{\nu,q}$.*

The following lemma describes the behavior of the least energy for different parameters $\nu, q > 0$, which will play an important role in proving the existence results for (2.1).

Lemma 3.2. *Let $\nu_i, q_i > 0$, $i = 1, 2$, with $\min\{\nu_2 - \nu_1, q_1 - q_2\} \geq 0$. Then $m_{\nu_1,q_1} \leq m_{\nu_2,q_2}$. If additionally $\max\{\nu_2 - \nu_1, q_1 - q_2\} > 0$, then $m_{\nu_1,q_1} < m_{\nu_2,q_2}$. In particular, $m_{\nu_1,q_i} < m_{\nu_2,q_i}$ if $\nu_1 < \nu_2$ and $m_{\nu_i,q_2} < m_{\nu_i,q_1}$ if $q_2 > q_1$.*

Proof. We prove that $m_{\nu_1,q_1} \le m_{\nu_2,q_2}$ for example. From Lemma 3.1, choose u to be a positive ground state of problem (3.1) with $\nu = \nu_2$ and $q = q_2$. Then $u \in \Theta^{q_2}$ and

$$m_{\nu_2,q_2} = J_{\nu_2,q_2}(u) = \max_{t\ge 0} J_{\nu_2,q_2}(tu).$$

With the use of $q_1 \ge q_2$, $u \in \Theta^{q_1}$. Then there exists $t_0 > 0$ such that $J_{\nu_1,q_1}(t_0u) = \max_{t\ge 0} J_{\nu_1,q_1}(tu)$. By $\min\{\nu_2-\nu_1, q_1-q_2\} \ge 0$, we obtain $J_{\nu_1,q_1}(t_0u) \le J_{\nu_2,q_2}(t_0u)$. Then by (3.2) we have

$$\begin{aligned} m_{\nu_1,q_1} &= \inf_{w\in\Theta^{q_1}\cap S_1} \max_{t\ge 0} J_{\nu_1,q_1}(tw) \le \max_{t\ge 0} J_{\nu_1,q_1}(tu) = J_{\nu_1,q_1}(t_0u) \\ &\le J_{\nu_2,q_2}(t_0u) \le J_{\nu_2,q_2}(u) = m_{\nu_2,q_2}. \end{aligned}$$

□

4. An auxiliary problem

Now, we are ready to introduce the auxiliary problem for (2.1). For any $\nu_{\min} \le \tilde a \le \nu_\infty$, $q_\infty \le \tilde b \le q_{\max}$. Set

$$V_\epsilon^{\tilde a}(x) := \max\{\tilde a, V(\epsilon x)\}, \quad Q_\epsilon^{\tilde b}(x) := \min\{\tilde b, Q(\epsilon x)\},$$

and consider the auxiliary equation

$$-\Delta u + V_\epsilon^{\tilde a}(x)u + \phi_u(x)u = Q_\epsilon^{\tilde b}(x)u^3. \tag{4.1}$$

The functional is

$$I_\epsilon^{\tilde a,\tilde b}(u) = \frac12\int_{\mathbb{R}^3}(|\nabla u|^2 + V_\epsilon^{\tilde a}(x)u^2) + \frac14\int_{\mathbb{R}^3}\phi_u u^2 - \frac14\int_{\mathbb{R}^3} Q_\epsilon^{\tilde b}(x)u^4,$$

and the Nehari manifold is

$$N_\epsilon^{\tilde a,\tilde b} = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle (I_\epsilon^{\tilde a,\tilde b})'(u), u\rangle = 0\},$$

and the least energy on $N_\epsilon^{\tilde a,\tilde b}$ is denoted by $c_\epsilon^{\tilde a,\tilde b}$. Moreover, as in Section 2, denote

$$\Theta_\epsilon^{\tilde b} := \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3}\phi_u u^2 < \int_{\mathbb{R}^3} Q_\epsilon^{\tilde b}(x)u^4\}.$$

and $c_\epsilon^{\tilde a,\tilde b}$ can be characterized by

$$c_\epsilon^{\tilde a,\tilde b} = \inf_{u\in N_\epsilon^{\tilde a,\tilde b}} I_\epsilon^{\tilde a,\tilde b}(u) = \inf_{u\in\Theta_\epsilon^{\tilde b}\cap S_1} \max_{t>0} I_\epsilon^{\tilde a,\tilde b}(tu). \tag{4.2}$$

Lemma 4.1. $m_{\tilde a,\tilde b} \le c_\epsilon^{\tilde a,\tilde b}$.

Proof. For any $u \in \Theta_\epsilon^{\tilde b} \cap S_1$, we have $u \in \Theta^{\tilde b} \cap S_1$ since $\tilde b \ge Q_\epsilon^{\tilde b}(x)$. Then

$$m_{\tilde a,\tilde b} \le \max_{t>0} J_{\tilde a,\tilde b}(tu) \le \max_{t>0} I_\epsilon^{\tilde a,\tilde b}(tu).$$

By the arbitrary of u, from (4.2) we obtain that $m_{\tilde a,\tilde b} \le c_\epsilon^{\tilde a,\tilde b}$. □

5. Proof of the main results

In this part, we only prove Theorem 1.1, since the arguments for Theorem 1.2 are quite similar. Without loss of generality, we may assume that $x_{\min} := 0 \in \mathcal{V}$ in (A2) or $x_{\min} := 0 \in \mathcal{V} \cap \mathcal{Q}$ if $\mathcal{V} \cap \mathcal{Q} \neq \emptyset$. Denote

$$d := Q(0) = \max_{x \in \mathcal{V}} Q(x) \geq Q(x), \quad \forall |x| \geq R$$

and consider the functional I_ϵ.

Lemma 5.1. $\limsup_{\epsilon\to 0} c_\epsilon \leq m_{V(y_1),Q(y_1)}$*, where* $y_1 \in \mathbb{R}^3$*. In particular,* $\limsup_{\epsilon\to 0} c_\epsilon \leq m_{V(0),Q(0)} := m_{\nu_{\min},d}$.

Proof. Since (3.1) has a positive ground state for each $\nu, q > 0$, we can take $u \in M_{V(y_1),Q(y_1)}$ such that $J_{V(y_1),Q(y_1)}(u) = m_{V(y_1),Q(y_1)}$. Then

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + V(y_1)|u|^2) + \int_{\mathbb{R}^3} \phi_u u^2 = \int_{\mathbb{R}^3} Q(y_1)|u|^4. \tag{5.1}$$

Then for small $\epsilon > 0$, it holds

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\epsilon x + y_1)|u|^2) + \int_{\mathbb{R}^3} \phi_u u^2 = \int_{\mathbb{R}^3} Q(\epsilon x + y_1) u^4 + o_\epsilon(1).$$

Set $w_\epsilon(x) = u(x - \frac{y_1}{\epsilon})$. Then

$$\int_{\mathbb{R}^3} (|\nabla w_\epsilon|^2 + V(\epsilon x)|w_\epsilon|^2) + \int_{\mathbb{R}^3} \phi_{w_\epsilon} w_\epsilon^2 = \int_{\mathbb{R}^3} Q(\epsilon x) w_\epsilon^4 + o_\epsilon(1). \tag{5.2}$$

Since for small $\epsilon > 0$,

$$\int_{\mathbb{R}^3} Q(\epsilon x) w_\epsilon^4 - \int_{\mathbb{R}^3} \phi_{w_\epsilon} w_\epsilon^2 \geq C > 0, \tag{5.3}$$

we have that $w_\epsilon \in \Theta_\epsilon$. So there exists $t_\epsilon > 0$ such that $t_\epsilon w_\epsilon \in N_\epsilon$. Then

$$t_\epsilon^2 \int_{\mathbb{R}^3} (|\nabla w_\epsilon|^2 + V(\epsilon x)|w_\epsilon|^2) + t_\epsilon^4 \int_{\mathbb{R}^3} \phi_{w_\epsilon} w_\epsilon^2 = t_\epsilon^4 \int_{\mathbb{R}^3} Q(\epsilon x) w_\epsilon^4. \tag{5.4}$$

By (5.2) and (5.4) we obtain

$$(t_\epsilon^2 - 1)\Big[\int_{\mathbb{R}^3} Q(\epsilon x) w_\epsilon^4 - \int_{\mathbb{R}^3} \phi_{w_\epsilon} w_\epsilon^2\Big] = o_\epsilon(1).$$

Using (5.3) we have that $t_\epsilon \to 1$ as $\epsilon \to 0$. Since $t_\epsilon w_\epsilon \in N_\epsilon$, one has

$$\begin{aligned} c_\epsilon \leq I_\epsilon(t_\epsilon w_\epsilon) =& J_{V(y_1),Q(y_1)}(t_\epsilon w_\epsilon) + \frac{t_\epsilon^2}{2} \int_{\mathbb{R}^3} (V(\epsilon x) - V(y_1)) w_\epsilon^2 \\ &+ \frac{t_\epsilon^4}{4} \int_{\mathbb{R}^3} (Q(y_1) - Q(\epsilon x))|w_\epsilon|^4 \\ =& J_{V(y_1),Q(y_1)}(t_\epsilon w_\epsilon) + \frac{t_\epsilon^2}{2} \int_{\mathbb{R}^3} (V(\epsilon x + y_1) - V(y_1)) u^2 \\ &+ \frac{t_\epsilon^4}{4} \int_{\mathbb{R}^3} (Q(y_1) - Q(\epsilon x + y_1))|u|^4. \end{aligned}$$

Therefore,

$$c_\epsilon \leq J_{V(y_1),Q(y_1)}(t_\epsilon w_\epsilon) + o_\epsilon(1) = J_{V(y_1),Q(y_1)}(t_\epsilon u) + o_\epsilon(1) = J_{V(y_1),Q(y_1)}(u) + o_\epsilon(1).$$

Then

$$\limsup_{\epsilon\to 0} c_\epsilon \leq J_{V(y_1),Q(y_1)}(u) = m_{V(y_1),Q(y_1)}.$$

In particular, we take $y_1 = 0$, it follows that

$$\limsup_{\epsilon\to 0} c_\epsilon \le m_{V(0),Q(0)} = m_{\nu_{\min},d}.$$

$\square$

Lemma 5.2. *The minimax value c_ϵ is achieved if ϵ is small enough. Hence, problem* (2.1) *has a positive ground state if ϵ is small enough.*

Proof. Assume that $w_n \in U_\epsilon$ satisfies that $\Psi_\epsilon(w_n) \to \inf_{U_\epsilon} \Psi_\epsilon$. By the Ekeland variational principle, we may suppose that $\Psi'_\epsilon(w_n) \to 0$. Then from Lemma 2.6 (1) it follows that $I'_\epsilon(u_n) \to 0$, where $u_n = m_\epsilon(w_n) \in N_\epsilon$. By Lemma 2.6 (2), we have $I_\epsilon(u_n) = \Psi_\epsilon(w_n) \to c_\epsilon$. By Lemma 2.2, we obtain that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Up to a subsequence, we assume that $u_n \rightharpoonup \tilde{u}_\epsilon$ in $H^1(\mathbb{R}^3)$, $u_n \to \tilde{u}_\epsilon$ in $L^2_{loc}(\mathbb{R}^3)$ and $u_n \to \tilde{u}_\epsilon$ a.e. on $\mathbb{R}^3$. Then $I'_\epsilon(\tilde{u}_\epsilon) = 0$. Below we discuss for two cases.

(i) $\tilde{u}_\epsilon \ne 0$ if ϵ is small enough. Then $\tilde{u}_\epsilon \in N_\epsilon$. So $I_\epsilon(\tilde{u}_\epsilon) \ge c_\epsilon$. On the other hand,

$$\begin{aligned} I_\epsilon(\tilde{u}_\epsilon) - \frac{1}{4}\langle I'_\epsilon(\tilde{u}_\epsilon), \tilde{u}_\epsilon\rangle &= \frac{1}{4}\|\tilde{u}_\epsilon\|_\epsilon^2 \le \frac{1}{4}\|u_n\|_\epsilon^2 + o_n(1) \\ &= I_\epsilon(u_n) - \frac{1}{4}\langle I'_\epsilon(u_n), u_n\rangle + o_n(1) = c_\epsilon + o_n(1). \end{aligned} \tag{5.5}$$

Then $I_\epsilon(\tilde{u}_\epsilon) \le c_\epsilon$. Therefore, $I_\epsilon(\tilde{u}_\epsilon) = c_\epsilon$, and then $u_n \to \tilde{u}_\epsilon$ in $H^1(\mathbb{R}^3)$ by (5.5).

(ii) There exists a sequence ϵ_j with $\tilde{u}_{\epsilon_j} = 0$. For each fixed j, there exists a sequence $u_n \in N_{\epsilon_j}$ such that $u_n \rightharpoonup \tilde{u}_{\epsilon_j} = 0$ in $H^1(\mathbb{R}^3)$. By $c_{\epsilon_j} > 0$ in Lemma 2.4, it is easy to see that $\{u_n\}$ is non-vanishing. Then there exists $x_n \in \mathbb{R}^3$ and $\delta_0 > 0$ such that

$$\int_{B_1(x_n)} u_n^2(x)dx > \delta_0. \tag{5.6}$$

Select $\tilde{a} \in (\nu_{\min}, \nu_\infty)$. Since $u_n \in N_{\epsilon_j}$, we know that

$$|\nabla u_n|_2^2 + \int_{\mathbb{R}^3} V(\epsilon_j x)u_n^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = \int_{\mathbb{R}^3} Q(\epsilon_j x)u_n^4.$$

It is easy to see that

$$\int_{\mathbb{R}^3} (V^{\tilde{a}}_{\epsilon_j}(x) - V(\epsilon_j x))u_n^2 = \int_{\{x|V(\epsilon_j x)\le \tilde{a}\}} (\tilde{a} - V(\epsilon_j x))u_n^2 = o_n(1). \tag{5.7}$$

Similarly,

$$\int_{\mathbb{R}^3} (Q^d_{\epsilon_j}(x) - Q(\epsilon_j x))u_n^4 = \int_{\{x|Q(\epsilon_j x)\ge d\}} (Q(\epsilon_j x) - d)u_n^4 = o_n(1). \tag{5.8}$$

Then

$$|\nabla u_n|_2^2 + \int_{\mathbb{R}^3} V^{\tilde{a}}_{\epsilon_j}(x)u_n^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = \int_{\mathbb{R}^3} Q^d_{\epsilon_j}(x)u_n^4 + o_n(1).$$

By (5.6), we obtain that

$$\int_{\mathbb{R}^3} Q^d_{\epsilon_j}(x)u_n^4 - \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \ge \int_{\mathbb{R}^3} V^{\tilde{a}}_{\epsilon_j}(x)u_n^2 + o_n(1) \ge C\int_{\mathbb{R}^3} u_n^2 + o_n(1) > \delta > 0. \tag{5.9}$$

Hence, $u_n \in \Theta^d_{\epsilon_j}$. Then there exists $t_n > 0$ such that $t_n u_n \in N^{\tilde{a},d}_{\epsilon_j}$. We claim that t_n is bounded. Otherwise, assume that $t_n \to \infty$. By $t_n u_n \in N^{\tilde{a},d}_{\epsilon_j}$, we know that

$$|\nabla u_n|_2^2 + \int_{\mathbb{R}^3} V^{\tilde{a}}_{\epsilon_j}(x) u_n^2 + t_n^2 \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = t_n^2 \int_{\mathbb{R}^3} Q^d_{\epsilon_j}(x) u_n^4. \tag{5.10}$$

By (5.9), there is a contradiction for (5.10) if $t_n \to \infty$. Therefore, t_n is bounded. Then

$$\begin{aligned} c^{\tilde{a},d}_{\epsilon_j} &\le I^{\tilde{a},d}_{\epsilon_j}(t_n u_n) \\ &= I_{\epsilon_j}(t_n u_n) + \frac{1}{2}\int_{\mathbb{R}^3} [V^{\tilde{a}}_{\epsilon_j}(x) - V(\epsilon_j x)] t_n^2 u_n^2 - \frac{1}{4}\int_{\mathbb{R}^3} [Q^d_{\epsilon_j}(x) - Q(\epsilon_j x)] t_n^4 u_n^4. \end{aligned}$$

By the boundedness of t_n, (5.7) and (5.8), we obtain

$$c^{\tilde{a},d}_{\epsilon_j} \le I_{\epsilon_j}(t_n u_n) + o_n(1) \le I_{\epsilon_j}(u_n) + o_n(1).$$

Hence $c^{\tilde{a},d}_{\epsilon_j} \le c_{\epsilon_j}$ as $n \to \infty$. However, from Lemma 4.1, it holds

$$m_{\tilde{a},d} \le c^{\tilde{a},d}_{\epsilon_j},$$

leading to $m_{\tilde{a},d} \le c_{\epsilon_j}$. Taking the limit $j \to \infty$ and using Lemma 5.1, we obtain

$$m_{\tilde{a},d} \le m_{\nu_{\min},d},$$

which is a contradiction to Lemma 3.2 since $\tilde{a} > \nu_{\min}$.

Now we find the ground state $\tilde{u}_\epsilon$ for (2.1). Using the standard argument, we can further find a positive ground state for (2.1). This completes the proof. □

Denote $\mathscr{J}_\epsilon$ as the set of all positive ground states of (2.1).

Lemma 5.3. *Let* (A1), (A2) *hold. Then* $\mathscr{J}_\epsilon$ *is compact in* $H^1(\mathbb{R}^3)$ *for all small* $\epsilon > 0$.

Proof. Let the bounded sequence $\{u_n\} \subset \mathscr{J}_\epsilon \cap N_\epsilon$ be such that $I_\epsilon(u_n) = c_\epsilon$ and $I'_\epsilon(u_n) = 0$. Without loss of generality we assume that $u_n \rightharpoonup u \in H^1(\mathbb{R}^3)$. As done in the proof of Lemma 5.2, we obtain that $u_n \to u > 0$, $u \in N_\epsilon$ and $I_\epsilon(u) = c_\epsilon$. Then $u \in \mathscr{J}_\epsilon$. □

Lemma 5.4. *Suppose that* (A1), (A2) *are satisfied, and* V, Q *are uniformly continuous. Let* u_ϵ *be the positive ground state obtained in Lemma 5.2. Then there is* $y_\epsilon \in \mathbb{R}^3$ *such that* $\lim_{\epsilon\to 0} \operatorname{dist}(\epsilon y_\epsilon, \mathcal{A}_V) = 0$, *and for each sequence* $\epsilon y_\epsilon \to y_0$, $v_\epsilon(x) := u_\epsilon(x + y_\epsilon)$ *converges in* $H^1(\mathbb{R}^3)$ *to a ground state* v *of*

$$-\Delta u + V(y_0)u + \phi_u(x)u = Q(y_0)u^3, \quad u > 0.$$

In particular, if $\mathcal{V} \cap \mathcal{Q} \neq \emptyset$, *it follows that* $\lim_{\epsilon\to 0} \operatorname{dist}(\epsilon y_\epsilon, \mathcal{V} \cap \mathcal{Q}) = 0$, *and up to subsequences,* v_ϵ *converges in* $H^1(\mathbb{R}^3)$ *to a ground state* v *of*

$$-\Delta u + \nu_{\min} u + \phi_u(x)u = q_{\max} u^3, \quad u > 0.$$

Proof. Let u_n be the positive ground states of problem (2.1) with parameter $\epsilon_n \to 0$. It is easy to see that u_n is bounded and non-vanishing. Then there exists $\delta > 0$ such that

$$\int_{B_1(y_n)} |u_n|^2 dx \ge \delta. \tag{5.11}$$

Setting $v_n(x) = u_n(x + y_n)$, $\tilde{V}_{\epsilon_n}(x) = V(\epsilon_n(x + y_n))$ and $\tilde{Q}_{\epsilon_n}(x) = Q(\epsilon_n(x + y_n))$, we see that v_n solves the below problem

$$-\Delta u + \tilde{V}_{\epsilon_n}(x)u + \phi_u(x)u = \tilde{Q}_{\epsilon_n}(x)u^3,$$

with energy functional

$$\tilde{I}_{\epsilon_n}(v_n) = \frac{1}{2}\Big(\int_{\mathbb{R}^3} |\nabla v_n|^2 + \tilde{V}_{\epsilon_n}(x)v_n^2\Big) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 - \frac{1}{4}\int_{\mathbb{R}^3} \tilde{Q}_{\epsilon_n}(x)v_n^4.$$

Since v_n is also bounded in $H^1(\mathbb{R}^3)$, from (5.11), we may assume that $v_n \rightharpoonup v \neq 0$ in $H^1(\mathbb{R}^3)$.

Claim 1: The sequence $\epsilon_n y_n$ must be bounded. Otherwise if $\epsilon_n y_n \to \infty$, then we may suppose that $V(\epsilon_n y_n) \to V_0 \geq \nu_\infty > \nu_{\min}$ and $Q(\epsilon_n y_n) \to Q_0 \leq d := Q(0)$. Since V and Q are uniformly continuous functions, it follows that for $R > 0$ and $|x| \leq R$,

$$|\tilde{V}_{\epsilon_n}(x) - V_0| \leq |V(\epsilon_n(x + y_n)) - V(\epsilon_n y_n)| + |V(\epsilon_n y_n) - V_0| \to 0.$$

Similarly,

$$|\tilde{Q}_{\epsilon_n}(x) - Q_0| \to 0, \quad \forall |x| \leq R.$$

Then for each $\eta \in C_0^\infty(\mathbb{R}^3)$, we infer that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \tilde{V}_{\epsilon_n}(x)v_n\eta = \int_{\mathbb{R}^3} V_0 v\eta, \quad \lim_{n\to\infty}\int_{\mathbb{R}^3} \tilde{Q}_{\epsilon_n}(x)v_n^3\eta = \int_{\mathbb{R}^3} Q_0 v^3\eta. \tag{5.12}$$

Moreover, by Lemma 2.1 (ii), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \phi_{v_n} v_n\eta = \int_{\mathbb{R}^3} \phi_v v\eta.$$

Thus, v solves

$$-\Delta v + V_0 v + \phi_v(x)v = Q_0 v^3 \quad \text{in } \mathbb{R}^3. \tag{5.13}$$

From Fatou's lemma and Lemma 5.1 it follows that

$$\begin{aligned}
m_{\nu_{\min},d} &\geq \liminf_{n\to\infty} c_{\epsilon_n} = \liminf_{n\to\infty}[I_{\epsilon_n}(u_n) - \frac{1}{4}\langle I'_{\epsilon_n}(u_n), (u_n)\rangle] \\
&= \liminf_{n\to\infty}[\tilde{I}_{\epsilon_n}(v_n) - \frac{1}{4}\langle \tilde{I}'_{\epsilon_n}(v_n), (v_n)\rangle] \\
&= \liminf_{n\to\infty}\frac{1}{4}\int_{\mathbb{R}^3}(|\nabla v_n|^2 + \tilde{V}_{\epsilon_n}(x)|v_n|^2) \\
&\geq \frac{1}{4}\int_{\mathbb{R}^3}(|\nabla v|^2 + V_0|v|^2) \\
&= J_{V_0,Q_0}(v) - \frac{1}{4}\langle J'_{V_0,Q_0}(v), v\rangle \geq m_{V_0,Q_0}.
\end{aligned} \tag{5.14}$$

However, from the fact that $\nu_{\min} < V_0$ and $d \geq Q_0$, Lemma 3.2 implies that $m_{\nu_{\min},d} < m_{V_0,Q_0}$. This is a contradiction. Hence $\{\epsilon_n y_n\}$ is bounded and we suppose that $\epsilon_n y_n \to y_0$.

Claim 2:

$$y_0 \in \mathcal{A}_V := \{x \in \mathcal{V} : Q(x) = Q(x_{\min})\} \cup \{x \notin \mathcal{V} : Q(x) > Q(x_{\min})\}.$$

If $y_0 \notin \mathcal{A}_V$, then it is easy to see that

$$m_{\nu_{\min},d} < m_{V(y_0),Q(y_0)}. \tag{5.15}$$

Repeating the arguments of Claim 1 again, we have $m_{\nu_{\min},d} \geq m_{V(y_0),Q(y_0)}$. This contradicts with (5.15). Therefore, $y_0 \in \mathcal{A}_V$. Namely, $\lim_{\epsilon_n \to 0} \operatorname{dist}(\epsilon_n y_n, \mathcal{A}_V) = 0$. In particular, if $\mathcal{V} \cap \mathcal{Q} \neq \emptyset$, it follows that $\lim_{\epsilon \to 0} \operatorname{dist}(\epsilon y_\epsilon, \mathcal{V} \cap \mathcal{Q}) = 0$.

Claim 3: v_n converges strongly to v in $H^1(\mathbb{R}^3)$. Following the arguments in the proof of Claim 1, we know that v is a solution of the equation

$$-\Delta v + V(y_0)v + \phi_v(x)v = Q(y_0)v^3.$$

Moreover, as (5.14) we obtain

$$\begin{aligned} m_{V(y_0),Q(y_0)} &\leq \frac{1}{4}\int_{\mathbb{R}^3}(|\nabla v|^2 + V(y_0)v^2) \\ &\leq \liminf_{n\to\infty} \frac{1}{4}\int_{\mathbb{R}^3}(|\nabla v_n|^2 + \tilde{V}_{\epsilon_n}(x)|v_n|^2) = \liminf_{n\to\infty} c_{\epsilon_n}. \end{aligned}$$

By Lemma 5.1, we know that $\liminf_{n\to\infty} c_{\epsilon_n} \leq m_{V(y_0),Q(y_0)}$. Then

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla v_n|^2 = \int_{\mathbb{R}^3}|\nabla v|^2, \tag{5.16}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\tilde{V}_{\epsilon_n}(x)|v_n|^2 = \int_{\mathbb{R}^3}V(y_0)v^2. \tag{5.17}$$

In addition, since V is uniformly continuous, we know that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\tilde{V}_{\epsilon_n}(x)|v|^2 = \int_{\mathbb{R}^3}V(y_0)v^2.$$

This combining with (5.17) we obtain $v_n \to v$ in $L^2(\mathbb{R}^3)$. By (5.16), we know that $v_n \to v$ in $H^1(\mathbb{R}^3)$. Hence, if $\mathcal{V} \cap \mathcal{Q} \neq \emptyset$, up to subsequences, v_n converges in $H^1(\mathbb{R}^3)$ to a ground state v of

$$-\Delta u + \nu_{\min}u + \phi_u(x)u = q_{\max}u^3, \quad u > 0.$$

□

Lemma 5.5. *Suppose that* (A1), (A2) *are satisfied, and* V, Q *are uniformly continuous. Set* $v_n := u_n(x + y_n)$*, where* u_n *is the positive ground state obtained in Lemma 5.2 and* y_n *is given in* (5.11)*. Then:*

(i) *there exist* δ' *and* $M > 0$ *such that* $\delta' \leq |v_n|_\infty \leq M$ *for all* $n \in \mathbb{N}$.

(ii)
$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$
Moreover, there exist $C, c > 0$ *such that*
$$v_n(x) \leq Ce^{-c|x|}, \quad \forall x \in \mathbb{R}^3.$$

Proof. As in the proof of Lemma 5.4, we have that v_n is the solution of

$$-\Delta v_n + \tilde{V}_{\epsilon_n}(x)v_n + \phi_{v_n}(x)v_n = \tilde{Q}_{\epsilon_n}(x)v_n^3,$$

and $v_n \to v \neq 0$ in $H^1(\mathbb{R}^3)$. Then

$$\lim_{R\to\infty}\Big(\int_{|x|\geq R}(v_n^2 + v_n^6)\Big) = 0, \quad \text{uniformly for } n \in \mathbb{N}. \tag{5.18}$$

Using [13, Proposition 3.3], we obtain that $v_n \in L^t(\mathbb{R}^3)$ for all $t \geq 2$ and

$$|v_n|_t \leq C_t\|v_n\|_{H^1(\mathbb{R}^3)},$$

where C_t does not depend on n. Then for $t > 3$, $|v_n^3|_{\frac{t}{2}} \le C$ for all n. Thus by [11, Theorem 8.17], we infer that for all $y \in \mathbb{R}^3$,

$$\sup_{B_1(y)} v_n(x) \le C\big(|v_n|_{L^2(B_2(y))} + |v_n^3|_{L^{\frac{t}{2}}(B_2(y))}\big). \tag{5.19}$$

This implies that $|v_n|_\infty$ is uniformly bounded. Recall that by (5.11),

$$\delta \le \int_{B_1(y_n)} |u_n|^2 dx \le |B_1||v_n|_\infty^2.$$

Then $|v_n|_\infty \ge \delta'$, for all n. Combining (5.19) with (5.18), we obtain

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly for all } n \in \mathbb{N}.$$

Then we can take $\rho_0 > 0$ such that

$$\tilde{Q}_{\epsilon_n}(x) v_n^3 \le \frac{\nu_{\min}}{2} v_n,$$

for all $|x| > \rho_0$. Consequently,

$$-\Delta v_n + \frac{\tilde{V}_{\epsilon_n}(x)}{2} v_n \ leq \tilde{Q}_{\epsilon_n}(x) v_n^3 - \frac{\tilde{V}_{\epsilon_n}(x)}{2} v_n \le 0,$$

for all $|x| \ge \rho_0$. Let s and T be positive constants such that

$$s^2 < \frac{\nu_{\min}}{2}, \quad v_n(x) \le T e^{-s\rho_0},$$

for all $|x| = \rho_0$. Hence, the function $\psi(x) = Texp(-s|x|)$ satisfies

$$-\Delta\psi + \frac{\tilde{V}_{\epsilon_n}(x)}{2}\psi \ge (\frac{\nu_{\min}}{2} - s^2)\psi > 0,$$

for all $x \ne 0$. Thereby, taking $\eta = \max\{v_n - \psi, 0\} \in H_0^1(|x| > \rho_0)$ as a test function, we obtain

$$\begin{aligned} 0 &\ge \int_{\mathbb{R}^3} \big(\nabla v_n \nabla \eta + \frac{\tilde{V}_{\epsilon_n}(x)}{2} v_n \eta\big) \\ &\ge \int_{\mathbb{R}^3} \big((\nabla v_n - \nabla\psi)\nabla\eta + \frac{\tilde{V}_{\epsilon_n}(x)}{2}(v_n - \psi)\eta\big) \\ &\ge \frac{\nu_{\min}}{2} \int_{\{x\in\mathbb{R}^3 : v_n > \psi\}} (v_n - \psi)^2 \ge 0, \end{aligned}$$

for all $|x| > \rho_0$. Therefore, the set $\Omega_n := \{x \in \mathbb{R}^3 : |x| > \rho_0 \text{ and } v_n > \psi(x)\}$ is empty. Then we know that there exists $C, c > 0$ such that

$$v_n(x) \le Ce^{-c|x|}, \forall x \in \mathbb{R}^3.$$

This completes the proof. □

Proof of Theorem 1.1. Going back to (1.6) with the substitution of variables $x \mapsto \frac{x}{\epsilon}$, Lemma 5.2 implies that (1.6) has a positive ground state $w_\epsilon = u_\epsilon(\frac{x}{\epsilon})$ for $\epsilon > 0$ small. Lemma 5.3 implies that $\mathscr{L}_\epsilon$ is compact in $H^1(\mathbb{R}^3)$. Set $\epsilon_n \to 0$ as $n \to \infty$. If b_n denotes a maximum point of v_n, then from Lemma 5.5 (i), it follows that it is bounded. Then we assume that $b_n \in B_R(0)$. Thereby, the global maximum point of u_n is $z_n := b_n + y_n$ and then $x_n := \epsilon_n z_n$ is the maximum point of w_n. From the boundedness of b_n, by Lemma 5.4 we obtain that $\lim_{n\to\infty} x_n = y_0$, which together with the continuity of V gives

$$\lim_{n\to\infty} V(x_n) = V(y_0), \quad \lim_{n\to\infty} Q(x_n) = Q(y_0).$$

Then from Lemma 5.4, the proof of the conclusion (3)(i) in Theorem 1.1 is complete. Moreover, from Lemma 5.5, by the boundedness of b_n we obtain

$$w_n(x) = u_n(\frac{x}{\epsilon_n}) = v_n(\frac{x}{\epsilon_n} - y_n) = v_n(\frac{x}{\epsilon_n} - \frac{x_n}{\epsilon_n} - b_n) \le Ce^{-c|\frac{x}{\epsilon_n} - \frac{x_n}{\epsilon_n} - b_n|} \le Ce^{-\frac{c}{\epsilon_n}|x - x_n|}.$$

Thus, for small $\epsilon > 0$, we have that

$$w_\epsilon(x) \le Ce^{-\frac{c}{\epsilon}|x - x_\epsilon|}.$$

□

Proof of Theorem 1.2. If the potential functions V and Q satisfy condition (A3), we can assume that $x_{\max} := 0 \in \mathcal{Q}$ in (A3) or $x_{\max} := 0 \in \mathcal{V} \cap \mathcal{Q}$ if $\mathcal{V} \cap \mathcal{Q} \neq \emptyset$. Denote

$$e := V(0) = \min_{x\in\mathcal{Q}} V(x) \le V(x), \quad \forall |x| \ge R.$$

The rest is similar to the proof of Theorem 1.1. □

Acknowledgements. This work was supported by the National Natural Science Foundation of China (Nos. 11601204,11671077,11571140) and by the Natural Science Foundation of Outstanding Young Scholars of Jiangsu Province (No: BK20160063).

The authors would like to express their sincere gratitude to the anonymous referees for their helpful and insightful comments.

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