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RADIAL SOLUTIONS FOR INHOMOGENEOUS BIHARMONIC ELLIPTIC SYSTEMS

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Abstract. In this article we obtain weak radial solutions for the inhomogeneous elliptic system

$$\begin{gathered} \Delta^2 u + V_1(|x|)|u|^{q-2}u = Q(|x|)F_u(u,v) \quad \text{in } \mathbb{R}^N, \\ \Delta^2 v + V_2(|x|)|v|^{q-2}v = Q(|x|)F_v(u,v) \quad \text{in } \mathbb{R}^N, \\ u, v \in D_0^{2,2}(\mathbb{R}^N), \quad N \geq 5, \end{gathered}$$

where Δ^2 is the biharmonic operator, V_i, $Q \in C^0((0,+\infty),[0,+\infty))$, $i = 1, 2$, are radially symmetric potentials, $1 < q < N$, $q \neq 2$, and F is a s-homogeneous function. Our approach relies on an application of the Symmetric Mountain Pass Theorem and a compact embedding result proved in [17].

1. Introduction

In this article concerns the existence of nontrivial solutions for the inhomogeneous biharmonic elliptic system

$$\begin{gathered} \Delta^2 u + V_1(|x|)|u|^{q-2}u = Q(|x|)F_u(u,v) \quad \text{in } \mathbb{R}^N, \\ \Delta^2 v + V_2(|x|)|v|^{q-2}v = Q(|x|)F_v(u,v) \quad \text{in } \mathbb{R}^N, \\ u, v \in D_0^{2,2}(\mathbb{R}^N), \quad N \geq 5, \end{gathered} \tag{1.1}$$

where Δ^2 is the biharmonic operator, V_i, $Q \in C^0((0,+\infty),[0,+\infty))$, $i = 1, 2$, are radially symmetric potentials, $1 < q < N$, $q \neq 2$, and F is a s-homogeneous function satisfying the following assumptions:

(A1) $V_i \in C^0((0,+\infty),[0,+\infty))$, such that
$$\liminf_{r\to+\infty} \frac{V_i(r)}{r^a} > 0, \quad \liminf_{r\to 0} \frac{V_i(r)}{r^{a_0}} > 0, \tag{1.2}$$
for some real numbers a and a_0.

(A2) $Q \in C^0((0,+\infty),[0,+\infty))$, is such that
$$\limsup_{r\to+\infty} \frac{Q(r)}{r^b} < \infty, \quad \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty, \tag{1.3}$$
for some real numbers b and b_0.

(A3) $F \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a homogeneous function of degree s, with $s > \max\{2, q\}$.

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(A4) There exists $C > 0$ such that
$$
\begin{aligned}
|F_u(u,v)| \le C(|u|^{s-1} + |v|^{s-1}),\ (u,v) \in \mathbb{R}^2,\\
|F_v(u,v)| \le C(|u|^{s-1} + |v|^{s-1}),\ (u,v) \in \mathbb{R}^2.
\end{aligned} \tag{1.4}
$$

(A5) $F(u,v) > 0$, $\forall\ u,\ v > 0$.

Nonlinear elliptic problems of fourth order without singularities in bounded domains have been extensively studied by several authors, see [9, 20, 38, 40], and references therein.

For application or motivation, we note that, when $\Omega \subset \mathbb{R}^N$ is a bounded domain, the problem
$$
\begin{gathered}
\Delta^2 u + c\Delta u = f(x,u) \quad \text{in } \Omega,\\
u = \Delta u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$
which arises in the study of traveling waves in suspension bridges (see [16, 24, 28]) and in the study of the static deflection of an elastic plate in a fluid.

For studies on the existence and multiplicity of solutions for nonlinear biharmonic problems in unbounded domains, the reader is referred to [19, 30, 35] in the radial case, and to [1] in the non-radial sub-(sup) linear case. Maximum principle results for biharmonic equation in unbounded domains are obtained in [32]. Also for unbounded domains, nontrivial solutions and multiplicity results are obtained in [4, 5, 6, 15, 31, 39] and in references therein. Additional results in the scalar case may be found in [34, 36, 37, 41, 42].

Elliptic systems may be used to describe the multiplicative chemical reactions catalyzed by catalyst. For the existence of nontrivial solutions to nonvariational systems, potential systems and Hamiltonian systems including critical exponents case see, for instance, [3, 21, 25, 26, 42]. See also [2, 7, 12, 13, 14, 27, 29].

For results for fourth-order equations with singular potential see [30] and [6]. Alves et. al. in [6] proved the existence of solutions to the problem
$$
\begin{gathered}
\Delta^2 u + V(x)|u|^{q-1}u = |u|^{2^*-2}u, \quad \text{in } \Omega \subset \mathbb{R}^N\\
u \in D_0^{2,2}(\Omega), \quad N \ge 5,
\end{gathered}
$$
where $1 \le q \le 2^* - 1$ and $V = V(x)$ is a potential that changes sign and has singularities in Ω. Wang and Shen in [39] proved existence of sign-changing solutions for the problem
$$
\begin{gathered}
\Delta^2 u = \lambda \frac{|u|^{2^{**}(s)-2}u}{|x|^s} + \beta a(x)|u|^{r-2}u, \quad \text{in } \Omega \subset \mathbb{R}^N\\
u \in D_0^{2,2}(\Omega), \quad N \ge 5,
\end{gathered}
$$
motivated by the Hardy-Rellich's inequality
$$
\bar{\lambda} \int_{\mathbb{R}^N} \frac{u^2}{|x|^4} dx \le \int_{\mathbb{R}^N} |\Delta u|^2 dx,
$$
as improved in their work. Radial solutions for the biharmonic equation
$$
\Delta^2 u + V(|x|)|u|^{q-2}u = Q(|x|)f(u),
$$
were obtained in [11], when $q = 2$, and in [17], when $q \ne 2$. Motivated by the work of Alves [2] for elliptic systems, a natural question is whether or not the results of

[17] can be extended to the elliptic system (1.1). Here we answer this question in the affirmative when (A3) and (A4) are satisfied.

Before stating our results, we to introduce some notation. Let $D_0^{2,2}(\mathbb{R}^N)$ be the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|\Delta u\|_2$ and $D_{0,r}^{2,2}(\mathbb{R}^N)$ the set of radially symmetric functions in $D_0^{2,2}(\mathbb{R}^N)$.

For $p \geq 1$ and a function $\nu : \mathbb{R}^N \to \mathbb{R}$ define

$$L^p(\mathbb{R}^N;\nu) = \big\{u : \mathbb{R}^N \to \mathbb{R}\ u \text{ is Lebesgue measurable and } \int_{\mathbb{R}^N} \nu(x)|u|^p dx < +\infty\big\}$$

endowed with the norm

$$\|u\|_{p,\nu} := \Big(\int_{\mathbb{R}^N} \nu(x)|u|^p dx\Big)^{1/p}.$$

Define the Banach space $X_{V_i} := D_0^{2,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N;V_i)$, with the norm

$$\|u\|_{V_i} = \|\Delta u\|_2 + \|u\|_{p,V_i}$$

and $X_{V_i,r}$ the set of radially symmetric functions in X_{V_i}, $i = 1, 2$.

We consider the product space $X := X_{V_1} \times X_{V_2}$ endowed with the norm

$$\|(u,v)\| := \Big(\int_{\mathbb{R}^N} (|\Delta u|^2 + |\Delta v|^2)dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^N} (V_1(|x|)|u|^q + V_2(|x|)|v|^q)dx\Big)^{1/q},$$

and $X_r := X_{V_1,r} \times X_{V_2,r}$. Also, we endow the space $L^s(\mathbb{R}^N;Q) \times L^s(\mathbb{R}^N;Q)$ endowed with the norm

$$\|(u,v)\|_{s,Q} = \Big(\int_{\mathbb{R}^N} Q(|x|)(|u|^s + |v|^s)dx\Big)^{1/s}.$$

Let $\alpha^* := \frac{N-4}{2} + \frac{q-1}{q}(a+N)$ and $\alpha_0^* := \frac{N-4}{2} + \frac{q-1}{q}(a_0+N)$. Now, as in [17], we define some indexes that will appear in our results.

The bottom indices are defined as

$$s_* := \begin{cases} q, & b \leq a,\ b \leq -N \text{ or } b \geq -N + \frac{q(N-4)}{2} - \varepsilon, \\ \frac{2(N+b+\varepsilon)}{N-4}, & b \leq a \text{ and } -N < b < -N + \frac{q(N-4)}{2} - \varepsilon, \\ q + \frac{q(b-a)}{\alpha^*}, & b > a \geq -N + \frac{q(N-4)}{2}, \\ q + \frac{2(b-a)}{N-4}, & b > a, b > -N \text{ and } -N + \frac{q(N-4)}{2} - \varepsilon < a < -N + \frac{q(N-4)}{2}, \\ \frac{2(N+b+\varepsilon)}{N-4}, & b > a, b > -N \text{ and } a \leq -N + \frac{q(N-4)}{2} - \varepsilon, \\ q + \frac{2(b-a)}{N-4}, & a < b \leq -N, \end{cases}$$

and the top indices are defined as

$$s^* := \begin{cases} \frac{2(N+b_0-\varepsilon)}{N-4}, & a_0 \geq b_0 > -N \text{ or } b_0 \geq a_0 \geq -N + \frac{q(N-4)}{2} + \varepsilon, \\ q + \frac{2(b_0-a_0)}{N-4}, & b_0 \geq a_0 \text{ and } -N + \frac{q(N-4)}{2} \leq a_0 < -N + \frac{q(N-4)}{2} + \varepsilon, \\ q + \frac{q(b_0-a_0)}{\alpha_0^*}, & b_0 \geq a_0 \text{ and } -N - \frac{q(N-4)}{2(q-1)} < a_0 < -N + \frac{q(N-4)}{2}, \\ +\infty, & b_0 \geq a_0 \text{ and } a \leq -N - \frac{q(N-4)}{2(q-1)}. \end{cases}$$

Consider also $s_{**} := q + \frac{q(b_0-a_0)}{\alpha_0^*}$, with $b_0 \leq a_0 < -N - \frac{q(N-4)}{2(q-1)}$.

Our main result is the following.

Theorem 1.1. *Let* (A1)–(A5) *be satisfied. If* $s_* < s < s^*$*, then system* (1.1) *has a nontrivial solution* $(u,v) \in X_r$ *by which we mean*

$$\begin{aligned} &\int_{\mathbb{R}^N} (\Delta u.\Delta\varphi + \Delta v.\Delta\psi)dx + \int_{\mathbb{R}^N} (V_1(|x|)|u|^{q-2}u\varphi + V_2(|x|)|v|^{q-2}v\psi)dx \\ &= \int_{\mathbb{R}^N} Q(|x|)(\varphi F_u(u,v) + \psi F_v(u,v))dx\,, \end{aligned} \tag{1.5}$$

for all $(\varphi,\psi) \in X$*. Moreover, if* $F(u,v) = F(-(u,v))$ *and there exists* $\eta > 0$ *such that* $F(u,v) \geq \eta(|u|^s + |v|^s)$*, for all* $(u,v) \in \mathbb{R}^2$*, then system* (1.1) *has infinitely many radial solutions* $(u,v) \in X_r$*,* $i = 1,2$*.*

The proof of Theorem 1.1 will be given using arguments similar to those developed in [17]. First we define an Euler functional $I : X_r \to \mathbb{R}$ associated with the equation (1.5). Then, we obtain a Principle of Symmetric Criticality result, which yields that the critical points of I are solutions of the system. Finally, we prove that this functional has the mountain pass geometry and apply the Symmetric Mountain Pass Theorem to obtain the result.

2. Existence results

In this section we will prove our main result. To do this, we will divide the proof in some lemmas. Firstly, let us present the following embedding theorem established in [17].

Theorem 2.1. *Let* V_i*,* $i = 1,2$*, and* Q *be functions satisfying* (1.2) *and* (1.3)*. If* $s_* < s^*$*, then the embedding*

$$X_{V_i,r} \hookrightarrow L^s(\mathbb{R}^N;Q),$$

is continuous for all $s_* \leq s \leq s^*$ *when* $s^* < \infty$*,* $s_* \leq s < \infty$ *when* $s^* = \infty$ *or* $\max\{s_*,\, s_{**}\} \leq s < \infty$*. Furthermore, the embedding is compact for all* $s_* < s < s^*$ *or* $\max\{s_*,\, s_{**}\} < s < \infty$*.*

Now let us define the Euler functional $I : X_r \to \mathbb{R}$ by

$$\begin{aligned} I(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u|^2 + |\Delta v|^2)dx + \frac{1}{q}\int_{\mathbb{R}^N} (V_1(|x|)|u|^q + V_2(|x|)|v|^q)dx \\ - \int_{\mathbb{R}^N} Q(|x|)F(u,v)\,dx. \end{aligned}$$

By conditions (A1)–(A4) and the continuous embeddings obtained in Theorem 2.1, we have that $I \in C^1(X_r;\mathbb{R})$ with Fréchet derivative in $(u,v) \in X_r$ given by

$$\begin{aligned} &\langle I'(u,v), (\varphi,\psi)\rangle \\ &= \int_{\mathbb{R}^N} (\Delta u.\Delta\varphi + \Delta v.\Delta\psi)dx + \int_{\mathbb{R}^N} (V_1(|x|)|u|^{q-2}u\varphi + V_2(|x|)|v|^{q-2}v\psi)dx \\ &\quad - \int_{\mathbb{R}^N} Q(|x|)(\varphi F_u(u,v) + \psi F_v(u,v))dx, \end{aligned}$$

for all $(\varphi,\psi) \in X_r$.

The proof of the next lemma follows the arguments presented in [17].

Lemma 2.2. *Every critical point of the functional* $I : X_r \to \mathbb{R}$ *satisfies* (1.5)*.*

Proof. Let $(u,v) \in X_r$ be a critical point of I. Given $(\varphi,\psi) \in X$, define

$$\bar{\varphi}(r) := \frac{1}{|\partial B_r|}\int_{\partial B_r} \varphi(\xi)dS(\xi), \quad \bar{\psi}(r) := \frac{1}{|\partial B_r|}\int_{\partial B_r} \psi(\xi)dS(\xi), \tag{2.1}$$

where ∂B_r denotes the sphere of center 0 and radius r and $|\partial B_r|$ denotes its Lebesgue measure.

Proceeding as in the proof of the mean-value formulas for Laplace's equation (see [18]), using polar coordinates in $\mathbb{R}^N$ and divergence theorem, we conclude that

$$\begin{aligned}
\frac{d}{dr}\bar{\varphi}(r) &= \frac{r}{N|B_r|}\int_{B_r} \Delta\varphi(\xi)d\xi, \\
\frac{d}{dr}\bar{\psi}(r) &= \frac{r}{N|B_r|}\int_{B_r} \Delta\psi(\xi)d\xi, \\
\frac{d^2}{dr^2}\bar{\varphi}(r) &= -\frac{N-1}{r}\frac{d}{dr}\bar{\varphi}(r) + \frac{1}{|\partial B_r|}\int_{\partial B_r} \Delta\varphi(\xi)dS(\xi) \\
\frac{d^2}{dr^2}\bar{\psi}(r) &= -\frac{N-1}{r}\frac{d}{dr}\bar{\psi}(r) + \frac{1}{|\partial B_r|}\int_{\partial B_r} \Delta\psi(\xi)dS(\xi).
\end{aligned}$$

Since $\Delta\bar{\varphi} = \frac{d^2}{dr^2}\bar{\varphi} + \frac{N-1}{r}\frac{d}{dr}\bar{\varphi}$ and $\Delta\bar{\psi} = \frac{d^2}{dr^2}\bar{\psi} + \frac{N-1}{r}\frac{d}{dr}\bar{\psi}$, we obtain

$$\Delta\bar{\varphi} = \frac{1}{|\partial B_r|}\int_{\partial B_r} \Delta\varphi(\xi)dS(\xi) \text{ and } \Delta\bar{\psi} = \frac{1}{|\partial B_r|}\int_{\partial B_r} \Delta\psi(\xi)dS(\xi). \tag{2.2}$$

From this we see that $(\bar{\varphi},\bar{\psi}) \in X_r$ and then

$$\begin{aligned}
&\langle I'(u,v),(\bar{\varphi},\bar{\psi})\rangle \\
&= \int_{\mathbb{R}^N} (\Delta u.\Delta\bar{\varphi} + \Delta v.\Delta\bar{\psi})dx + \int_{\mathbb{R}^N} (V_1(|x|)|u|^{q-2}u\bar{\varphi} + V_2(|x|)|v|^{q-2}v\bar{\psi})dx \\
&\quad - \int_{\mathbb{R}^N} Q(|x|)(\bar{\varphi}F_u(u,v) + \bar{\psi}F_v(u,v))dx = 0.
\end{aligned}$$

Therefore, using polar coordinates in $\mathbb{R}^N$ and Fubini's Theorem again and the identities (2.1) and (2.2) we obtain result. □

Before we prove the Palais-Smale condition for the functional I, we need to make some remarks about assumptions of the function F.

Remark 2.3. (a) Since F is a C^1 homogeneous function of degree s, then $sF(u,v) = uF_u(u,v) + vF_v(u,v)$ and ∇F is a homogeneous function of degree $s-1$.

(b) From (F_1), (a) and the Young inequality we have $|F(u,v)| \le C(|u|^s + |v|^s)$ for all $(u,v) \in \mathbb{R}^2$.

(c) Our prototype of F is $F(u,v) = (a|u| + b|v|)^s + c|u|^\alpha|v|^\beta$, u, $v \in \mathbb{R}$; a, b, $c > 0$ and $\alpha + \beta = s$, with α, $\beta > 1$.

Lemma 2.4. *The functional $I : X_r \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. Let $\{(u_n,v_n)\}$ be a sequence X_r such that $I'(u_n,v_n) \to 0$ and $I(u_n,v_n) \to c$, as $n \to +\infty$. We shall see that $\{(u_n,v_n)\}$ is bounded in X. Indeed, since $I'(u_n,v_n) \to 0$, we have $\|I'(u_n,v_n)\| < 1$ for all n sufficiently large, and so, $|\langle I'(u_n,v_n),(u_n,v_n)\rangle| \le \|(u_n,v_n)\|$. Since $\{I(u_n,v_n)\}$ is convergent sequence, there exists a positive constant C_0 such that $|I(u_n,v_n)| \le C_0$. In this case, from (F_0) and the Remark (a), we have

$$\begin{aligned} &C_0 + \frac{1}{s}\|(u_n,v_n)\| \\ &\ge I(u_n,v_n) - \frac{1}{s}\langle I'(u_n,v_n),(u_n,v_n)\rangle \\ &\ge C\Big[\int_{\mathbb{R}^N}(|\Delta u_n|^2+|\Delta v_n|^2)dx + \int_{\mathbb{R}^N}(V_1(|x|)|u_n|^q+V_2(|x|)|v_n|^q)dx\Big], \end{aligned} \tag{2.3}$$

where C_0 and C are positive constants. To conclude that $\{(u_n,v_n)\}$ is bounded, we will split our arguments in the cases: $1<q<2$ and $q>2$.

Case $q>2$. Suppose $\{(u_n,v_n)\}$ is unbounded. Then, up to a subsequence, $\|(u_n,v_n)\| \to +\infty$, as $n\to+\infty$. From (2.3) we see that

$$\begin{aligned} &\frac{C_0}{\|(u_n,v_n)\|^2} + \frac{1}{s}\frac{1}{\|(u_n,v_n)\|} \\ &\ge \frac{C}{\|(u_n,v_n)\|^2}\Big[\int_{\mathbb{R}^N}(|\Delta u_n|^2+|\Delta v_n|^2)dx \\ &\quad + \int_{\mathbb{R}^N}(V_1(|x|)|u_n|^q+V_2(|x|)|v_n|^q)dx\Big], \end{aligned} \tag{2.4}$$

for some positive constants C_0 and C.

If $\big\{\int_{\mathbb{R}^N}(V_1(|x|)|u_n|^q+V_2(|x|)|v_n|^q)dx\big\}$ is an unbounded sequence, then, up to a subsequence,

$$\int_{\mathbb{R}^N}(V_1(|x|)|u_n|^q+V_2(|x|)|v_n|^q)dx \to +\infty, \quad \text{as } n\to+\infty.$$

This implies that $\int_{\mathbb{R}^N}(V_1(|x|)|u_n|^q+V_2(|x|)|v_n|^q)dx > 1$, for n sufficiently large. Consequently, since $q>2$, we obtain

$$\|(u_n,v_n)\|^2 \le 2\Big[\int_{\mathbb{R}^N}(|\Delta u_n|^2+|\Delta v_n|^2)dx + \int_{\mathbb{R}^N}(V_1(|x|)|u_n|^q+V_2(|x|)|v_n|^q)dx\Big].$$

Combining this with (2.4) we deduce that

$$\frac{C_0}{\|(u_n,v_n)\|^2} + \frac{1}{s}\frac{1}{\|(u_n,v_n)\|} \ge \frac{C}{2},$$

for some constants C_0, $C>0$ and for n sufficiently large. So we obtain a contradiction.

On the other hand, if $\big\{\int_{\mathbb{R}^N}(V_1(|x|)|u_n|^q+V_2(|x|)|v_n|^q)dx\big\}$ is bounded, we conclude that

$$\int_{\mathbb{R}^N}(|\Delta u_n|^2+|\Delta v_n|^2)dx \to +\infty, \quad \text{as } n\to+\infty,$$

up to a subsequence. Using (2.4) we see that, for some positive constants C_0 and C,

$$\begin{aligned} \frac{C_0}{\|(u_n,v_n)\|^2} + \frac{1}{s}\frac{1}{\|(u_n,v_n)\|^2} &\ge C\frac{\|\Delta u_n\|_2^2+\|\Delta v_n\|_2^2+\|u_n\|_{q,V_1}^q+\|v_n\|_{q,V_2}^q}{\|(u_n,v_n)\|^2} \\ &= C\frac{1+\frac{\|u_n\|_{q,V_1}^q+\|v_n\|_{q,V_2}^q}{\|\Delta u_n\|_2^2+\|\Delta v_n\|_2^2}}{\Big[1+\frac{(\|u_n\|_{q,V_1}^q+\|v_n\|_{q,V_2}^q)^{1/q}}{(\|\Delta u_n\|_2^2+\|\Delta v_n\|_2^2)^{1/2}}\Big]^2} \to C, \end{aligned}$$

as $n \to +\infty$. Here we also have a contradiction.

Case $1 < q < 2$. If $\{ \int_{\mathbb{R}^N} (|\Delta u_n|^2 + |\Delta v_n|^2) dx \}$ is an unbounded sequence, then, up to a subsequence,

$$\int_{\mathbb{R}^N} (|\Delta u_n|^2 + |\Delta v_n|^2) dx \to +\infty, \quad \text{as } n \to +\infty.$$

As a consequence $\int_{\mathbb{R}^N} (|\Delta u_n|^2 + |\Delta v_n|^2) dx > 1$, for n sufficiently large. Since $1 < q < 2$, it follows that

$$\begin{aligned}
&\|(u_n, v_n)\|^q \\
&\le 2^q \Big[\Big(\int_{\mathbb{R}^N} (|\Delta u_n|^2 + |\Delta v_n|^2) dx \Big)^{\frac{q}{2}} + \int_{\mathbb{R}^N} (V_1(|x|)|u_n|^q + V_2(|x|)|v_n|^q) dx \Big] \\
&\le 2^q \Big[\int_{\mathbb{R}^N} (|\Delta u_n|^2 + |\Delta v_n|^2) dx + \int_{\mathbb{R}^N} (V_1(|x|)|u_n|^q + V_2(|x|)|v_n|^q) dx \Big].
\end{aligned}$$

Using this and (2.3) we conclude that, for some positive constants C_0 and C,

$$C_0 + \frac{1}{s} \|(u_n, v_n)\| \ge \frac{C}{2^q} \|(u_n, v_n)\|^q,$$

for n sufficiently large. But this is a contradiction.

Now, if $\{ \int_{\mathbb{R}^N} (|\Delta u_n|^2 + |\Delta v_n|^2) dx \}$ is bounded, then, up to a subsequence,

$$\int_{\mathbb{R}^N} (V_1(|x|)|u_n|^q + V_2(|x|)|v_n|^q) dx \to +\infty, \text{ as } n \to +\infty.$$

Hence, using (2.3) again, we obtain

$$\begin{aligned}
\frac{C_0}{\|(u_n, v_n)\|^q} + \frac{1}{s} \frac{1}{\|(u_n, v_n)\|^{q-1}} &\ge C \frac{\|\Delta u_n\|_2^2 + \|\Delta v_n\|_2^2 + \|u_n\|_{q,V_1}^q + \|v_n\|_{q,V_2}^q}{\|(u_n, v_n)\|^q} \\
&= C \frac{\frac{\|\Delta u_n\|_2^2 + \|\Delta v_n\|_2^2}{\|u_n\|_{q,V_1}^q + \|v_n\|_{q,V_2}^q} + 1}{\Big[\frac{(\|\Delta u_n\|_2^2 + \|\Delta v_n\|_2^2)^{1/2}}{(\|u_n\|_{q,V_1}^q + \|v_n\|_{q,V_2}^q)^{1/q}} + 1 \Big]^q} \to C,
\end{aligned}$$

as $n \to +\infty$. We have again a contradiction and, therefore, $\{(u_n, v_n)\}$ is bounded in X_r. Consequently, $\{u_n\}$ and $\{v_n\}$ are also bounded in X_{r,V_1} and X_{r,V_2}, respectively. Using the fact that X_{r,V_i}, $i \in \{1, 2\}$, is reflexive, we conclude that there exist $u \in X_{r,V_1}$ and $v \in X_{r,V_2}$ such that $u_n \rightharpoonup u$ weakly in X_{r,V_1} and $v_n \rightharpoonup v$ weakly in X_{r,V_2}, as $n \to +\infty$, up to subsequences. Hence $(u_n, v_n) \rightharpoonup (u, v)$ weakly in X_r, as $n \to +\infty$, up to a subsequence. Since X_{r,V_i} is compactly imbedded in $L^s(\mathbb{R}^N; Q)$, $i \in \{1, 2\}$ (see Theorem 2.1), we deduce that $u_n \to u$ and $v_n \to v$ strongly in $L^s(\mathbb{R}^N; Q)$, as $n \to +\infty$. As a consequence, $u_n \to u$ and $v_n \to v$ a.e. in $\mathbb{R}^N$, as $n \to +\infty$.

Now we shall prove that

$$\langle I'(u_n, v_n), (\varphi, \psi) \rangle \to \langle I'(u, v), (\varphi, \psi) \rangle,$$

for all $(\varphi, \psi) \in X_r$, as $n \to +\infty$.

For $(\varphi, \psi) \in X_r$, we define

$$F_{(\varphi,\psi)}(u, v) := \int_{\mathbb{R}^N} [\Delta u \Delta \varphi + \Delta v \Delta \psi] dx.$$

Note that $F_{(\varphi,\psi)} \in X_r'$ and

$$\langle F'_{(\varphi,\psi)}(u,v),(z,w)\rangle = \int_{\mathbb{R}^N} [\Delta z \Delta\varphi + \Delta w \Delta\psi] dx,$$

for all $(z,w) \in X_r$. Since $(u_n, v_n) \rightharpoonup (u,v)$ weakly in X_r, as $n \to +\infty$, we deduce that $F_{(\varphi,\psi)}(u_n,v_n) \to F_{(\varphi,\psi)}(u,v)$ strongly in X_r, as $n \to +\infty$, for all $(\varphi,\psi) \in X_r$, that is,

$$\int_{\mathbb{R}^N} [\Delta u_n \Delta\varphi + \Delta v_n \Delta\psi] dx \to \int_{\mathbb{R}^N} [\Delta u \Delta\varphi + \Delta v \Delta\psi] dx, \tag{2.5}$$

as $n \to +\infty$, for all $(\varphi,\psi) \in X_r$.

We consider $(\varphi,\psi) \in X_r$ and define

$$g_n := (V_1)^{\frac{q-1}{q}} |u_n|^{q-2} u_n, \quad h_n := (V_2)^{\frac{q-1}{q}} |v_n|^{q-2} v_n,$$
$$g := (V_1)^{\frac{q-1}{q}} |u|^{q-2} u, \quad h := (V_2)^{\frac{q-1}{q}} |v|^{q-2} v.$$

So, $g_n \to g$ and $h_n \to h$ a.e. in $\mathbb{R}^N$. Moreover, $\{g_n\}$ and $\{h_n\}$ are bounded in $L^{q/(q-1)}(\mathbb{R}^N)$. It follows from Brézis and Lieb lemma [10] (see also [22, Lemma 4.8]) that

$$\int_{\mathbb{R}^N} g_n \varphi dx \to \int_{\mathbb{R}^N} g\varphi dx \quad \text{and} \quad \int_{\mathbb{R}^N} h_n \psi dx \to \int_{\mathbb{R}^N} h\psi dx,$$

as $n \to +\infty$, for all $\varphi, \psi \in L^q(\mathbb{R}^N)$. In particular, given $(\varphi,\psi) \in X_r$, we have $(V_1)^{1/q}\varphi$, $(V_2)^{1/q}\psi \in L^q(\mathbb{R}^N)$, so that,

$$\int_{\mathbb{R}^N} g_n (V_1(|x|))^{1/q} \varphi dx \to \int_{\mathbb{R}^N} g (V_1(|x|))^{1/q} \varphi dx$$

and

$$\int_{\mathbb{R}^N} h_n (V_2(|x|))^{1/q} \psi dx \to \int_{\mathbb{R}^N} h (V_2(|x|))^{1/q} \psi dx,$$

as $n \to +\infty$. Hence,

$$\int_{\mathbb{R}^N} V_1(|x|) |u_n|^{q-2} u_n \varphi dx \to \int_{\mathbb{R}^N} V_1(|x|) |u|^{q-2} u \varphi dx$$

and

$$\int_{\mathbb{R}^N} V_2(|x|) |v_n|^{q-2} v_n \psi dx \to \int_{\mathbb{R}^N} V_2(|x|) |v|^{q-2} v \psi dx,$$

as $n \to +\infty$. Consequently,

$$\begin{aligned} &\int_{\mathbb{R}^N} (V_1(|x|)|u_n|^{q-2} u_n \varphi + V_2(|x|)|v_n|^{q-2} v_n \psi) dx \\ &\to \int_{\mathbb{R}^N} (V_1(|x|)|u|^{q-2} u \varphi + V_2(|x|)|v|^{q-2} v \psi) dx, \end{aligned} \tag{2.6}$$

as $n \to +\infty$, for all $(\varphi,\psi) \in X_r$.

We define $K : X_r \to X_r'$ by

$$\langle K(u,v),(\varphi,\psi)\rangle := \int_{\mathbb{R}^N} Q(|x|)[\varphi F_u(u,v) + \psi F_v(u,v)] dx.$$

First, we prove that

$$\|F_u(u_n,v_n) - F_u(u,v)\|_{\frac{s}{s-1},Q} \to 0 \quad \text{and} \quad \|F_v(u_n,v_n) - F_v(u,v)\|_{\frac{s}{s-1},Q} \to 0,$$

as $n \to +\infty$. By using (A4), we will deduce that

$$Q(|x|)|F_u(u_n, v_n) - F_u(u, v)|^{\frac{s}{s-1}} \leq h(x)$$

a.e. $x \in \mathbb{R}^N$, for some function $h \in L^1(\mathbb{R}^N)$. In fact, since $\|Q^{\frac{1}{s}} u_n - Q^{1/s} u\|_s \to 0$ and $\|Q^{\frac{1}{s}} v_n - Q^{1/s} v\|_s \to 0$, as $n \to +\infty$, we conclude that

(1) $Q^{1/s} u_n \to Q^{1/s} u$ and $Q^{1/s} v_n \to Q^{1/s} v$ a.e. in $\mathbb{R}^N$, as $n \to +\infty$;
(2) $|Q^{1/s} u_n| \leq h_1$ and $|Q^{\frac{1}{s}} v_n| \leq h_2$ a.e. in $\mathbb{R}^N$, where h_1, $h_2 \in L^1(\mathbb{R}^N)$.

Hence, for some positive constant C,

$$\begin{aligned}
&Q(|x|)|F_u(u_n, v_n) - F_u(u, v)|^{\frac{s}{s-1}} \\
&\leq 2^{\frac{s}{s-1}} Q(|x|)\Big(|F_u(u_n, v_n)|^{\frac{s}{s-1}} + |F_u(u, v)|^{\frac{s}{s-1}}\Big) \\
&\leq C\Big(Q(|x|)|u_n|^s + Q(|x|)|v_n|^s + Q(|x|)|u|^s + Q(|x|)|v|^s\Big) \leq h(x),
\end{aligned}$$

where $h(x) = C\Big((h_1(x))^s + (h_2(x))^s + Q(|x|)|u|^s + Q(|x|)|v|^s\Big)$.

Since $Q(|x|)|F_u(u_n, v_n) - F_u(u, v)|^{\frac{s}{s-1}} \to 0$ a.e. in $\mathbb{R}^N$, as $n \to +\infty$, we see, by the Dominated Convergence Theorem of Lebesgue, that $\|F_u(u_n, v_n) - F_u(u, v)\|_{\frac{s}{s-1}, Q} \to 0$, as $n \to +\infty$. Similarly, $\|F_v(u_n, v_n) - F_v(u, v)\|_{\frac{s}{s-1}, Q} \to 0$, as $n \to +\infty$.

On the other hand, using Hölder's inequality and the continuous embedding $X_{r,Vi} \hookrightarrow L^s(\mathbb{R}^N, Q)$, $i \in \{1, 2\}$, we have, for all $(\varphi, \psi) \in X_r$,

$$\begin{aligned}
&|\langle K(u_n, v_n) - K(u, v), (\varphi, \psi)\rangle| \\
&\leq \int_{\mathbb{R}^N} Q(|x|)|F_u(u_n, v_n) - F_u(u, v)||\varphi| dx \\
&\quad + \int_{\mathbb{R}^N} Q(|x|)|F_v(u_n, v_n) - F_v(u, v)||\psi| dx \\
&\leq \|F_u(u_n, v_n) - F_u(u, v)\|_{\frac{s}{s-1}, Q}\|\varphi\|_{s,Q} + \|F_v(u_n, v_n) - F_v(u, v)\|_{\frac{s}{s-1}, Q}\|\psi\|_{s,Q} \\
&\leq C\|F_u(u_n, v_n) - F_u(u, v)\|_{\frac{s}{s-1}, Q}\|(\varphi, \psi)\| \\
&\quad + C\|F_v(u_n, v_n) - F_v(u, v)\|_{\frac{s}{s-1}, Q}\|(\varphi, \psi)\|,
\end{aligned}$$

for some positive constant C. Using this we see that

$$\begin{aligned}
\|K(u_n, v_n) - K(u, v)\|_{X_r'} \leq C[&\|F_u(u_n, v_n) - F_u(u, v)\|_{\frac{s}{s-1}, Q} \\
&+ \|F_v(u_n, v_n) - F_v(u, v)\|_{\frac{s}{s-1}, Q}] \to 0,
\end{aligned}$$

as $n \to +\infty$, so that, we obtain $\langle K(u_n, v_n) - K(u, v), (\varphi, \psi)\rangle \to 0$, as $n \to +\infty$; that is,

$$\langle K(u_n, v_n), (\varphi, \psi)\rangle \to \langle K(u, v), (\varphi, \psi)\rangle,$$

as $n \to +\infty$, for all $(\varphi, \psi) \in X_r$. Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^N} Q(|x|)[\varphi F_u(u_n, v_n) + \psi F_v(u_n, v_n)] dx \\
&\to \int_{\mathbb{R}^N} Q(|x|)[\varphi F_u(u, v) + \psi F_v(u, v)] dx,
\end{aligned} \tag{2.7}$$

as $n \to +\infty$, for all $(\varphi, \psi) \in X_r$. Moreover, since $(u_n, v_n) \rightharpoonup (u, v)$ in X_r and $K(u_n, v_n) \to K(u, v)$ in X_r', as $n \to +\infty$, it follows that

$$\langle K(u_n, v_n), (u_n, v_n)\rangle \to \langle K(u, v), (u, v)\rangle, \tag{2.8}$$

as $n \to +\infty$; that is,

$$\int_{\mathbb{R}^N} Q(|x|)[F_u(u_n,v_n)u_n + F_v(u_n,v_n)v_n]dx \to \int_{\mathbb{R}^N} Q(|x|)[F_u(u,v)u + F_v(u,v)v]dx,$$

as $n \to +\infty$. Combining (2.5), (2.6) and (2.7) we obtain

$$\langle I'(u_n,v_n),(\varphi,\psi)\rangle \to \langle I'(u,v),(\varphi,\psi)\rangle,$$

as $n \to +\infty$, for all $(\varphi,\psi) \in X_r$. Hence, as $I'(u_n,v_n) \to 0$, as $n \to +\infty$, we deduce that $I'(u,v) = 0$. This implies

$$\begin{aligned} 0 &= \langle I'(u,v),(u,v)\rangle \\ &= \int_{\mathbb{R}^N} [|\Delta u|^2 + |\Delta v|^2]dx + \int_{\mathbb{R}^N} [V_1(|x|)|u|^q + V_2(|x|)|v|^q]dx - \langle K(u,v),(u,v)\rangle. \end{aligned}$$

Therefore,

$$\langle K(u,v),(u,v)\rangle = \int_{\mathbb{R}^N} [|\Delta u|^2 + |\Delta v|^2]dx + \int_{\mathbb{R}^N} [V_1(|x|)|u|^q + V_2(|x|)|v|^q]dx. \tag{2.9}$$

On the other hand,

$$\begin{aligned} &\int_{\mathbb{R}^N} [|\Delta u_n|^2 + |\Delta v_n|^2]dx + \int_{\mathbb{R}^N} [V_1(|x|)|u_n|^q + V_2(|x|)|v_n|^q]dx \\ &= \langle I'(u_n,v_n),(u_n,v_n)\rangle + \langle K(u_n,v_n),(u_n,v_n)\rangle. \end{aligned} \tag{2.10}$$

From (2.8), (2.9) and (2.10) we have

$$\begin{aligned} &\int_{\mathbb{R}^N} [|\Delta u_n|^2 + |\Delta v_n|^2]dx + \int_{\mathbb{R}^N} [V_1(|x|)|u_n|^q + V_2(|x|)|v_n|^q]dx \\ &\to \int_{\mathbb{R}^N} [|\Delta u|^2 + |\Delta v|^2]dx + \int_{\mathbb{R}^N} [V_1(|x|)|u|^q + V_2(|x|)|v|^q]dx, \end{aligned} \tag{2.11}$$

as $n \to +\infty$. As before, from the Brezis-Lieb Lemma, we can show that

$$\begin{gathered} \int_{\mathbb{R}^N} V_1(|x|)|u_n|^q dx - \int_{\mathbb{R}^N} V_1(|x|)|u_n - u|^q dx \to \int_{\mathbb{R}^N} V_1(|x|)|u|^q dx, \\ \int_{\mathbb{R}^N} V_2(|x|)|v_n|^q dx - \int_{\mathbb{R}^N} V_2(|x|)|v_n - v|^q dx \to \int_{\mathbb{R}^N} V_2(|x|)|v|^q dx, \\ \int_{\mathbb{R}^N} |\Delta u_n|^2 dx - \int_{\mathbb{R}^N} |\Delta(u_n - u)|^2 dx \to \int_{\mathbb{R}^N} |\Delta u|^2 dx, \\ \int_{\mathbb{R}^N} |\Delta v_n|^2 dx - \int_{\mathbb{R}^N} |\Delta(v_n - v)|^2 dx \to \int_{\mathbb{R}^N} |\Delta v|^2 dx, \end{gathered}$$

as $n \to +\infty$. This implies

$$\begin{aligned} &\int_{\mathbb{R}^N} [V_1(|x|)|u_n|^q + V_2(|x|)|v_n|^q]dx - \int_{\mathbb{R}^N} [V_1(|x|)|u_n - u|^q + V_2(|x|)|v_n - v|^q]dx \\ &\to \int_{\mathbb{R}^N} [V_1(|x|)|u|^q + V_2(|x|)|v|^q]dx \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} [|\Delta u_n|^2 + |\Delta v_n|^2]dx - \int_{\mathbb{R}^N} [|\Delta(u_n - u)|^2 + |\Delta(v_n - v)|^2]dx \\ &\to \int_{\mathbb{R}^N} [|\Delta u|^2 + |\Delta v|^2]dx, \end{aligned} \tag{2.13}$$

as $n \to +\infty$. By (2.11), (2.12) and (2.13), we obtain

$$\int_{\mathbb{R}^N} [|\Delta(u_n - u)|^2 + |\Delta(v_n - v)|^2]dx + \int_{\mathbb{R}^N} [V_1(|x|)|u_n - u|^q + V_2(|x|)|v_n - v|^q]dx \to 0,$$

as $n \to +\infty$. As a consequence, we deduce that

$$\begin{aligned} \|(u_n, v_n) - (u, v)\| &= \Big(\int_{\mathbb{R}^N} (|\Delta(u_n - u)|^2 + |\Delta(v_n - v)|^2)dx \Big)^{1/2} \\ &\quad + \Big(\int_{\mathbb{R}^N} (V_1(|x|)|u_n - u|^q + V_2(|x|)|v_n - v|^q)dx \Big)^{1/q} \to 0, \end{aligned}$$

as $n \to +\infty$. Therefore, $(u_n, v_n) \to (u, v)$ in X_r, as $n \to +\infty$. □

Lemma 2.5 (Geometry of the Mountain Pass Theorem). *The functional $I : X_r \to \mathbb{R}$ satisfies the following conditions:*

(a) *$I(0,0) = 0$ and there exist $c > 0$, $\rho > 0$ such that $I(u, v) \geq c$ for $\|(u, v)\| = \rho$;*

(b) *There exists $(u, v) \in X_r$, with $\|(u, v)\| > \rho$, such that $I(u, v) < 0$.*

Proof. First we note that $I(0,0) = 0$. Now, taking $q_0 := \max\{2, q\}$ and using the Remark 2.3 item (b), we conclude that

$$\begin{aligned} I(u, v) \geq \frac{1}{q_0} \Big[& \int_{\mathbb{R}^N} [|\Delta u|^2 + |\Delta v|^2]dx + \int_{\mathbb{R}^N} [V_1(|x|)|u|^q + V_2(|x|)|v|^q]dx \Big] \\ & - C \int_{\mathbb{R}^N} Q(|x|)[|u|^s + |v|^s]dx, \end{aligned} \tag{2.14}$$

for some constant $C > 0$. By the continuous embedding $X_{r,V_i} \hookrightarrow L^s(\mathbb{R}^N; Q)$, $i = 1$, 2, we deduce that

$$\int_{\mathbb{R}^N} Q(|x|)|u|^s dx \leq C\|(u, v)\|^s, \quad \int_{\mathbb{R}^N} Q(|x|)|v|^s dx \leq C\|(u, v)\|^s,$$

for some positive constant C. This and (2.14) implies that

$$\begin{aligned} I(u, v) \geq \frac{1}{q_0} \Big[& \int_{\mathbb{R}^N} [|\Delta u|^2 + |\Delta v|^2]dx + \int_{\mathbb{R}^N} [V_1(|x|)|u|^q + V_2(|x|)|v|^q]dx \Big] \\ & -C\|(u, v)\|^s, \end{aligned} \tag{2.15}$$

for some constant $C > 0$. For $0 < \|(u, v)\| < 1$, we have

$$\begin{aligned} &\|(u, v)\|^{q_0} \\ &\leq 2^{q_0} \Big[\Big(\int_{\mathbb{R}^N} (|\Delta u|^2 + |\Delta v|^2)dx \Big)^{q_0/2} + \Big(\int_{\mathbb{R}^N} (V_1(|x|)|u|^q + V_2(|x|)|v|^q]dx \Big)^{q_0/q} \Big] \\ &\leq 2^{q_0} \Big[\int_{\mathbb{R}^N} (|\Delta u|^2 + |\Delta v|^2)dx + \int_{\mathbb{R}^N} (V_1(|x|)|u|^q + V_2(|x|)|v|^q)dx \Big]. \end{aligned} \tag{2.16}$$

Combining (2.15) with (2.16) we obtain

$$I(u, v) \geq \frac{1}{q_0 2^{q_0}} \|(u, v)\|^{q_0} - C\|(u, v)\|^s,$$

for some positive constant C and for $0 < \|(u, v)\| < 1$. So, there exist $0 < \rho < 1$ sufficiently small and $c > 0$ such that $I(u, v) \geq c > 0$ for all $(u, v) \in X$, with $\|(u, v)\| = \rho$. This completes the proof of (a).

Fixing $(u_0, v_0) \in X_r$ such that $F(u_0, v_0) > 0$, we have, for all $t > 0$,

$$\begin{aligned} I(t(u_0, v_0)) = &\frac{t^2}{2}\int_{\mathbb{R}^N}[|\Delta u_0|^2 + |\Delta v_0|^2]dx + \frac{t^q}{q}\int_{\mathbb{R}^N}[V_1(|x|)(u_0)^q + V_2(|x|)(v_0)^q]dx \\ &- t^s\int_{\mathbb{R}^N} Q(|x|)F(u_0, v_0)dx. \end{aligned}$$

This implies

$$I(t(u_0, v_0)) \to -\infty \quad \text{as } t \to +\infty.$$

Thus, for $t > 0$, sufficiently large, $\|t(u_0, v_0)\| > \rho$ and $I(t(u_0, v_0)) < 0$. Therefore, (b) follows. This completes the proof. $\square$

Finally, we can prove our main result.

Proof of Theorem 1.1. As a consequence of Lemma 2.4 and Lemma 2.5, we conclude, by using the Mountain Pass Theorem, due to Ambrosetti-Rabinowitz [8], that there exists a sequence $\{(u_n, v_n)\}$ in X so that

$$I(u_n, v_n) \to c > 0 \text{ and } I'(u_n, v_n) \to 0,$$

as $n \to +\infty$. By Lemma 2.4, $(u_n, v_n) \to (u, v)$ in X_r, as $n \to +\infty$, up to a subsequence. In view of $I \in C^1(X_r, \mathbb{R})$, it follows that

$$I(u_n, v_n) \to I(u, v) \quad \text{and} \quad I'(u_n, v_n) \to I'(u, v),$$

as $n \to +\infty$. This implies that $I'(u, v) = 0$ and $I(u, v) = c \neq 0$, that is, $(u, v) \in X_r$ is a nontrivial critical point of I. By Lemma 2.2, we conclude that (u, v) is a radial solution for the system (1.1) in the sense of equation (1.5).

Our next goal is to apply the Symmetric Mountain Pass Theorem [33, Theorem 6.5] to complete the proof of Theorem 1.1. So, we need to show that I satisfies the following conditions:

(a) $I(-(u, v)) = I(u, v)$, for all $(u, v) \in X_r$;
(b) For any nontrivial finite dimensional subspace $U \subset X_r$, there exists $R > 0$ such that $I(u, v) \leq 0$ for all $(u, v) \in U$, with $\|(u, v)\| \geq R$.

Since $F(u, v) = F(-(u, v))$, (a) occurs.

Now, suppose that (b) is not true. Therefore, there exists a nontrivial finite dimensional subspace $U \subset X_r$ and a sequence $\{(u_n, v_n)\}$ in U such that $\|(u_n, v_n)\| \to +\infty$, as $n \to +\infty$, and $I(u_n, v_n) > 0$, for all $n \in \mathbb{N}$. Since U has finite dimension, all norms are equivalent on U. In this case, since $F(u, v) \geq \eta(|u|^s + |v|^s)$ for all $(u, v) \in \mathbb{R}^2$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} Q(|x|)F(u_n, v_n)dx &\geq \eta\int_{\mathbb{R}^N} Q(|x|)(|u_n|^s + |v_n|^s)dx \\ &= \eta\|(u_n, v_n)\|_{s,Q}^s \geq C\|(u_n, v_n)\|^s, \end{aligned}$$

for some positive constant C. Since $s > \max\{2, q\} = q_0$ and $\|(u_n, v_n)\| \to +\infty$, as $n \to +\infty$, we deduce that

$$\begin{aligned} I(u_n, v_n) &\leq \frac{1}{2}\int_{\mathbb{R}^N}[|\Delta u_n|^2 + |\Delta v_n|^2]dx + \frac{1}{q}\int_{\mathbb{R}^N}[V_1(|x|)|u_n|^q + V_2(|x|)|v_n|^q]dx \\ &\quad - C\|(u_n, v_n)\|^s \\ &\leq \frac{1}{2}\|(u_n, v_n)\|^2 + \frac{1}{q}\|(u_n, v_n)\|^q - C\|(u_n, v_n)\|^s \end{aligned}$$

$$\leq \Big(\frac{1}{2}+\frac{1}{q}\Big)\|(u_n,v_n)\|^{q_0} - C\|(u_n,v_n)\|^{s},$$

for n sufficiently large and for some positive constant C. As a consequence,

$$\lim_{n\to+\infty} I(u_n,v_n) = -\infty;$$

that is, there exists n, sufficiently large, such that $I(u_n,v_n)<0$, which is a contradiction. This completes the proof of (b).

So, by Symmetric Mountain Pass Theorem, there exists an unbounded sequence of critical values for I, which corresponds to the existence of a sequence of nontrivial critical points for I. Consequently, by Lemma 2.2, equation (1.5) holds, which completes the proof of Theorem 1.1. □

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