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BLOW UP OF SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS ARISING IN NONLINEAR DISPERSIVE PROBLEMS

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Abstract. We study a new class of ordinary differential equations with blow up solutions. Necessary and sufficient conditions for finite blow up time are proved. Based on the new differential equation, a revised version of the concavity method of Levine is proposed. As an application we investigate the non-existence of global solutions to the Cauchy problem of Klein-Gordon, and to the double dispersive equations. We obtain necessary and sufficient condition for finite time blow up with arbitrary positive energy. A very general sufficient condition for blow up is also given.

1. Introduction

The finite time blow up of the solutions to nonlinear dispersive equations has been intensively investigated in the previous decades. The blow up phenomena for semilinear wave equations, generalized Boussinesq equation, double dispersive equation and others have been studied basically by means of the concavity Levine's method. The main idea in Levine's method [8] is based on the fact that if a twice continuously differentiable function $z(t)$ is a concave function, i.e.

$$z''(t) \le 0,\ t > 0 \quad \text{and} \quad z(0) > 0,\ z'(0) < 0, \tag{1.1}$$

then there exists T_*, $0 < T_* < \infty$ such that

$$z(t) \to 0 \quad \text{as } t \to T_*, \quad t < T_*. \tag{1.2}$$

To apply Levine's method to global non-existence of solutions to a nonlinear evolution equation one has to find a positive, smooth function $\Psi(t)$, such that $z(t) = \Psi^{1-\gamma}(t)$ for some $\gamma > 1$ satisfies (1.1) or equivalently $\Psi(t)$ is a solution to the problem

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \ge 0, \quad t > 0, \quad \gamma > 1, \quad \Psi(0) > 0, \quad \Psi'(0) > 0. \tag{1.3}$$

Then $\Psi(t)$ tends to infinity for a finite time T_*.

In these applications $\Psi(t)$ is a nonnegative functional of the solution to the corresponding nonlinear dispersive equation. For example, for semilinear wave equations

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$\Psi(t)$ is defined as $\Psi(t) = \int_{\mathbb{R}} u^2(t,x)\,dx$; while for fourth and sixth order differential equations $\Psi(t)$ is a more complicated functional including the H^1 norm of the solution.

In a modification of the concavity method suggested in [4] the authors used that if the function $z(t)$, instead of (1.1), satisfies the second order differential inequality

$$z''(t) + \delta z'(t) + \mu z(t) \le 0, \quad t > 0, \quad \delta \ge 0, \quad \mu \ge 0 \tag{1.4}$$

with suitable initial data, then there exists T_* such that (1.2) holds. In this case if $\Psi(t)$ is a solution to the inequality

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \ge -2\delta\Psi(t)\Psi'(t) - \mu\Psi^2(t), \quad t > 0, \quad \gamma > 1, \quad \delta \ge 0, \quad \mu \ge 0$$

equipped with appropriate initial conditions, then the function $z(t) = \Psi^{1-\gamma}(t)$ satisfies (1.4).

Another application of the concavity method is done in [5, 13], where the inequality

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \ge -\beta\Psi(t), \quad t > 0, \quad \gamma > 1, \quad \beta > 0 \tag{1.5}$$

is proposed. Then the function $z(t) = \Psi^{1-\gamma}(t)$ satisfies (1.1) for a spacial choice of $\Psi(0)$ and $\Psi'(0)$.

In our previous paper [6] we suggest a new inequality

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \ge \alpha\Psi^2(t) - \beta\Psi(t), \quad t > 0, \quad \gamma > 1, \quad \alpha > 0, \quad \beta > 0. \tag{1.6}$$

Note, that for suitable chosen initial data $\Psi(0)$, $\Psi'(0)$, the function $z(t) = \Psi^{1-\gamma}(t)$ fulfills (1.1). In comparison with (1.5) the new inequality (1.6) includes an additional positive term $\alpha\Psi^2(t)$ on the right-hand side. This term naturally appears in the investigation of some nonlinear dispersive equations as Klein-Gordon equation, double dispersive equation with linear restoring force and others.

In [6] the finite time blow up of the solutions to inequality (1.6) is proved under very general conditions on the initial data. However, these conditions are only sufficient and not necessary ones.

Let us mention that in the concavity method there is no precise formulation of the blow up result of the solutions to (1.3) and its generalizations (see [4, 5, 8, 13]). Namely, the main assumption in this method is that $\Psi(t)$ is a twice continuously differentiable function for every $t \ge 0$. However, under some conditions on the initial data, it follows, that $\Psi(t)$ blows up for a finite time, i.e. $\Psi(t)$ is not defined for every $t \ge 0$.

To give a rigorous formulation of blow up for $\Psi(t)$, we replace inequality (1.6) by the corresponding differential equation

$$\begin{gathered} \Psi''(t)\Psi(t) - \gamma\Psi'^2(t) = \alpha\Psi^2(t) - \beta\Psi(t) + H(t), \quad t \in [0, T_m), \quad 0 < T_m \le \infty, \\ \gamma > 1, \quad \alpha > 0,\ \beta > 0. \end{gathered} \tag{1.7}$$

Here $\Psi(t) \in C^2([0, T_m))$ is a nonnegative solution to (1.7) defined in the maximal existence time interval $[0, T_m)$, $0 < T_m \le \infty$ and

$$H(t) \in C([0, T_m)), \quad H(t) \ge 0 \quad \text{for } t \in [0, T_m). \tag{1.8}$$

Note, that in the analysis of nonlinear dispersive equations $\Psi(t)$ is a solution to (1.7) with some function $H(t)$, see e.g. Lemma 4.4 below for Klein-Gordon equation. Usually $H(t)$ can not be expressed explicitly by $\Psi(t)$. That is why, up to now, the nonnegative term $H(t)$ has been neglected and the corresponding inequality (1.6) has been investigated.

By equation (1.7) we are able to formulate and to prove *necessary and sufficient condition* for blow up of $\Psi(t)$ at the right-end point T_m (see Theorem 2.4). Moreover, we prove that the blow up time T_m is finite (see Theorem 2.3). In Theorem 3.1 we give new easy checkable sufficient condition for finite time blow up of the solutions to (1.7). This condition generalizes the corresponding ones for blow up of the solutions to inequality (1.6), given in [1, 6]. The necessary and sufficient condition (2.7) sheds light on the reasons for blow up of the solutions to (1.7) and gives a better understanding of the different sufficient conditions and their interrelations.

We apply the results for ordinary differential equation (1.7) to Klein-Gordon equation

$$u_{tt} - u_{xx} + u = f(u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.9}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}, \tag{1.10}$$

$$u_0(x) \in \mathrm{H}^1(\mathbb{R}), \quad u_1(x) \in \mathrm{L}^2(\mathbb{R}) \tag{1.11}$$

and to double the dispersive equation with linear restoring force

$$u_{tt} - u_{xx} - u_{ttxx} + u_{xxxx} + u + f(u)_{xx} = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.12}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}, \tag{1.13}$$

$$\begin{gathered} u_0 \in \mathrm{H}^1(\mathbb{R}) \cap \dot{\mathrm{H}}^{-1}(\mathbb{R}), \quad \text{i.e. } u_0 \in \mathrm{H}^1(\mathbb{R}),\ (-\Delta)^{-1/2} u_0 \in \mathrm{L}^2(\mathbb{R}), \\ u_1 \in \mathrm{L}^2(\mathbb{R}) \cap \dot{\mathrm{H}}^{-1}(\mathbb{R}), \quad \text{i.e. } u_1 \in \mathrm{L}^2(\mathbb{R}), \quad (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R}). \end{gathered} \tag{1.14}$$

Here $(-\Delta)^{-s} u = \mathcal{F}^{-1}\big(|\xi|^{-2s}\mathcal{F}(u)\big)$ for $s > 0$, $\mathcal{F}(u)$, $\mathcal{F}^{-1}(u)$ are the Fourier transform and the inverse Fourier transform, respectively.

The nonlinearity $f(u)$ in (1.9) and (1.12) has one of the following two forms:

$$\begin{gathered} f(u) = \sum_{k=1}^{l} a_k |u|^{p_k - 1} u - \sum_{j=1}^{s} b_j |u|^{q_j - 1} u, \\ f(u) = a_1 |u|^{p_1} + \sum_{k=2}^{l} a_k |u|^{p_k - 1} u - \sum_{j=1}^{s} b_j |u|^{q_j - 1} u, \end{gathered} \tag{1.15}$$

where the constants a_k, p_k $(k = 1, 2, \dots, l)$ and b_j, q_j $(j = 1, 2, \dots, s)$ fulfill the conditions

$$\begin{gathered} a_1 > 0, \quad a_k \geq 0, \quad b_j \geq 0 \quad \text{for } k = 2, \dots, l,\ j = 1, \dots, s, \\ 1 < q_s < q_{s-1} < \dots < q_1 < p_1 < p_2 < \dots < p_l < 5. \end{gathered} \tag{1.16}$$

For example, the nonlinear term (1.15)-(1.16) includes the quadratic-cubic nonlinearity ($f(u) = u^2 + u^3$), which appears in a number of mathematical models of physical processes, e.g. dislocations of crystals [9], propagation of longitudinal strain waves in an isotropic cylindrical compressible elastic rod [11, 12] and others.

Let us recall that in the case of subcritical initial energy ($0 < E(0) < d$) the global behaviour of the solutions is fully investigated by means of the potential well method, suggested in [10] for semilinear wave equation. Further on, this method has been applied to Klein-Gordon equation [16], double dispersive equation [7, 17], more general classes of double dispersive nonlocal wave equations [2] and others. According to the potential well method, the global existence or finite time blow up of the solutions with subcritical initial energy is proved when the sign of the Nehari functional $I(0)$ is positive or negative, respectively.

For the supercritical initial energy ($E(0) > d$) there are a few results for finite time blow up of the solutions to (1.9)–(1.11) (see [5, 14, 15]) and (1.12)–(1.14) (see [7]).

In this article we give, for the first time in the literature, necessary and sufficient condition for finite time blow up of the solutions to (1.9)–(1.11) and (1.12)–(1.14) with arbitrary positive initial energy, see Theorem 4.6 and Theorem 4.13. Although the results in Theorem 4.6 and Theorem 4.13 are theoretical, they can be applied to the numerical study of nonlinear dispersive equations. More precisely, one can check whether the necessary and sufficient condition (4.13) is satisfied at some time $t = b > 0$ by means of some reliable numerical approach.

Moreover, we find out very general sufficient condition on u_0, u_1 for which the corresponding solution $u(t, x)$ blows up for a finite time. We demonstrate that all previous sufficient conditions in [1, 5, 6, 14, 15] for finite time blow up of $u(t, x)$ can be obtained as a consequence from this new sufficient condition.

The necessary and sufficient conditions (4.13) and (4.20) reveal the genesis of the finite time blow up of the solutions to (1.9)–(1.11) and (1.12)–(1.14), respectively. For example, when (4.13) is fulfilled at the initial time $t = 0$, we get the well known in the applications sufficient condition for finite time blow up of the solutions to Klein-Gordon equation [14, 15] (for the nonlinear wave equation see also [3]). Note, that condition (4.13) at $t = 0$ is only sufficient and not necessary one. Our research shows that any sufficient condition, prescribed at $t = 0$, ensures the satisfaction of condition (4.13) at some later time $t = b > 0$.

This article is organized in the following way. In Section 2 necessary and sufficient condition for finite time blow up of the solutions to (1.7) is proved. Easy checkable sufficient condition for finite time blow up of the solutions to (1.7) is given in Section 3. Section 4 deals with applications of the results from Section 2 and Section 3 to Klein-Gordon and double dispersive equations with linear restoring force.

2. Main results

We recall the definition for finite time blow up of a nonnegative smooth function.

Definition 2.1. The nonnegative function $\Psi(t) \in C^1([0, T_m))$, $0 < T_m \le \infty$, blows up at T_m if

$$\limsup_{t \to T_m, t < T_m} \Psi(t) = \infty. \tag{2.1}$$

Below we formulate a simple property of functions that blow up.

Lemma 2.2. *Suppose $\Psi(t) \in C^1([0, T_m))$, $0 < T_m \le \infty$, is a nonnegative function and M is an arbitrary constant. If $\Psi(t)$ blows up at T_m then there exists t_0, $t_0 \in [0, T_m)$ such that $\Psi(t_0) \ge M$ and $\Psi'(t_0) > 0$.*

Proof. If $\Psi(0) \ge M$ and $\Psi'(0) > 0$, then Lemma 2.2 holds for $t_0 = 0$. Otherwise, from Definition 2.1 it follows that there exist $t_1, t_3 \in (0, T_m)$, $t_3 > t_1$ such that $\Psi(t_3) > \Psi(t_1) > M$. We denote by $(t_2, t_3) \subset (t_1, t_3)$ the maximal interval where $\Psi(t) > \Psi(t_1)$ for every $t \in (t_2, t_3)$. From the mean value theorem there exists $t_0 \in (t_2, t_3)$ such that

$$\Psi(t_3) - \Psi(t_2) = (t_3 - t_2)\Psi'(t_0).$$

Since $\Psi(t_3) > \Psi(t_2)$ and $t_3 > t_2$ we obtain that $\Psi'(t_0) > 0$. From the choice of the interval (t_2, t_3) it follows that $\Psi(t_0) \ge M$. $\square$

The following theorem shows that blow up of the solutions to (1.7) under assumption (1.8) does not occur at infinity, i.e. only finite time blow up is possible.

Theorem 2.3. *Suppose $\Psi(t) \in C^2([0,T_m))$ is a nonnegative solution of the equation*

$$\begin{gathered}\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) = \alpha\Psi^2(t) - \beta\Psi(t) + H(t), \quad t \in [0,T_m),\\ \gamma > 1, \quad \alpha > 0, \quad \beta > 0,\end{gathered}$$

where $[0,T_m)$, $0 < T_m \le \infty$ is the maximal existence time interval for $\Psi(t)$ and (1.8) holds. If $\Psi(t)$ blows up at T_m then $T_m < \infty$.

Proof. Suppose $\Psi(t)$ blows at T_m. From Lemma 2.2 it follows that for $M = \beta/\alpha$ there exist $b \in [0,T_m)$ such that

$$\Psi(b) \ge \beta/\alpha \text{ and } \Psi'(b) > 0. \tag{2.2}$$

From equation (1.7) we have

$$\Psi''(b) = \frac{\gamma\Psi'^2(b)}{\Psi(b)} + \alpha\Psi(b) - \beta + \frac{H(b)}{\Psi(b)} \ge \frac{\gamma\Psi'^2(b)}{\Psi(b)} > 0,$$

thus $\Psi'(t) > \Psi'(b) > 0$ for $t \in [b, b+\varepsilon)$ for some sufficiently small $\varepsilon > 0$.

We will show that $\Psi'(t) > 0$ for every $t \in [b,T_m)$. If not, then there exists an interval (b,t_0), $t_0 \in (b,T_m)$, such that $\Psi'(t) > 0$ for $t \in [b,t_0)$ and $\Psi'(t_0) = 0$. Since $\Psi(t)$ is a strictly monotone increasing function for $t \in [b,t_0]$ it follows that $\Psi(t) > \Psi(b) \ge \beta/\alpha$ for every $t \in (b,t_0]$. Moreover, from (1.7) and (2.2) we have

$$\Psi''(t) = \frac{\gamma\Psi'^2(t)}{\Psi(t)} + \alpha\Psi(t) - \beta + \frac{H(t)}{\Psi(t)} > \alpha\Psi(b) - \beta \ge 0$$

for every $t \in (b,t_0]$. Hence $\Psi'(t)$ is a strictly increasing function for $t \in (b,t_0]$ and we get the following impossible chain of inequalities

$$0 = \Psi'(t_0) > \Psi'(b) > 0.$$

Thus $\Psi'(t) > 0$ for $t \in [b,T_m)$ and consequently

$$\Psi(t) > \Psi(b) \ge \beta/\alpha > 0 \quad \text{for } t \in (b,T_m). \tag{2.3}$$

We define a function

$$z(t) = \Psi^{1-\gamma}(t) \quad \text{for } t \in [b,T_m),$$

that satisfies

$$z'(t) = (1-\gamma)\Psi^{-\gamma}(t)\Psi'(t), \quad z''(t) = (1-\gamma)\Psi^{-1-\gamma}[\Psi''(t)\Psi(t) - \gamma\Psi'^2(t)]. \tag{2.4}$$

Function $z(t)$ is a solution to the initial value problem

$$\begin{gathered}z''(t) = -(\gamma-1)\left(\alpha z(t) - \beta z^{\frac{\gamma}{\gamma-1}}(t) + H(t) z^{\frac{\gamma+1}{\gamma-1}}(t)\right) \quad \text{for } t \in [b,T_m)\\ z(b) > 0, \quad z'(b) < 0.\end{gathered} \tag{2.5}$$

To prove that $T_m < \infty$ we assume by contradiction that $T_m = \infty$. From (1.8), (2.3) and (2.4) it follows that

$$z''(t) = -(\gamma-1)\Psi^{-\gamma}(t)\Big(\alpha\Psi(t) - \beta + \frac{H(t)}{\Psi(t)}\Big) \le 0$$

for $t \ge b$. Integrating $z''(t) \le 0$ twice from b to $t > b$, we get

$$z'(t) \le z'(b), \quad z(t) \le z'(b)(t-b) + z(b).$$

Consequently, there exists $T_* > b$ such that $z(T_*) = 0$, or equivalently $\Psi(T_*) = \infty$ for

$$T_* \le b - \frac{z(b)}{z'(b)} = b + \frac{\Psi(b)}{(\gamma-1)\Psi'(b)} < \infty, \tag{2.6}$$

which contradicts our assumption. Thus it follows that $T_m < \infty$ and Theorem 2.3 is proved. □

The following theorem is one of the the main results in this article. We formulate and prove necessary and sufficient condition for blow up of the solutions to (1.7) at the right-end point of the existence time interval.

Theorem 2.4. *Suppose $\Psi(t) \in C^2([0, T_m))$ is a nonnegative solution of the equation*

$$\begin{gathered}\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) = \alpha\Psi^2(t) - \beta\Psi(t) + H(t), \quad t \in [0, T_m),\\ \gamma > 1, \quad \alpha > 0, \ \beta > 0,\end{gathered}$$

where $[0, T_m)$, $0 < T_m \le \infty$ is the maximal existence time interval for $\Psi(t)$, $H(t) \in C([0,\infty))$, and $H(t) \ge 0$ for $t \in [0,\infty)$. Then $\Psi(t)$ blows up at T_m if and only if

$$\text{there exists } b \in [0, T_m) \text{ such that } \beta \le \alpha\Psi(b) \text{ and } \Psi'(b) > 0. \tag{2.7}$$

Moreover,

$$T_m \le b + \frac{\Psi(b)}{(\gamma-1)\Psi'(b)} < \infty\,. \tag{2.8}$$

Proof. (Necessity) Suppose $\Psi(t)$ blows up at T_m. Then condition (2.7) holds from Lemma 2.2 for $M = \beta/\alpha$ and $b = t_0$.

(Sufficiency) Suppose (2.7) holds. From the proof of Theorem 2.3 it follows that $T_m < \infty$. Moreover, $\Psi(t)$ is a strictly increasing function for $t \in [b, T_m)$.

If we assume that $\psi(t)$ does not blow up at T_m, i.e. (2.1) fails, then

$$\limsup_{t\to T_m, t<T_m} \Psi(t) < \infty.$$

From the monotonicity and boundedness of $\Psi(t)$ for $t \in [b, T_m)$ we get

$$\lim_{t\to T_m} \Psi(t) = \Psi(T_m) < \infty. \tag{2.9}$$

As in the proof of Theorem 2.3 after the substitution $z(t) = \Psi^{1-\gamma}(t)$, $t \in [b, T_m)$ we get that $z(t)$ satisfies problem (2.5). Integrating the equation in (2.5) from b to $t < T_m$ we get

$$z'(t) = z'(b) - (\gamma-1)\int_b^t \Big(\alpha z(s) - \beta z^{\frac{\gamma}{\gamma-1}}(s) + H(s) z^{\frac{\gamma+1}{\gamma-1}}(s)\Big)\, ds$$

or equivalently, from (2.4),

$$\Psi'(t) = \Psi^\gamma(t)\Big[\frac{\Psi'(b)}{\Psi^\gamma(b)} + \int_b^t \big(\alpha\Psi^{1-\gamma}(s) - \beta\Psi^{-\gamma}(s) + H(s)\Psi^{-\gamma-1}(s)\big)\, ds\Big]. \tag{2.10}$$

Thus from (2.3), (2.9) and (2.10) we have

$$\begin{aligned}\lim_{t\to T_m} \Psi'(t) =& \Psi^\gamma(T_m)\Big[\frac{\Psi'(b)}{\Psi^\gamma(b)} + \int_b^{T_m} \big(\alpha\Psi^{1-\gamma}(s) - \beta\Psi^{-\gamma}(s) + H(s)\Psi^{-\gamma-1}(s)\big)\, ds\Big]\\ =& \Psi'(T_m), \quad 0 < \Psi'(T_m) < \infty.\end{aligned}$$

According to the theory of the initial value problems for ordinary differential equations, the problem

$$\begin{gathered}\tilde{\Psi}''(t)\tilde{\Psi}(t)-\gamma\tilde{\Psi}'^2(t)=\alpha\tilde{\Psi}^2(t)-\beta\tilde{\Psi}(t)+H(t)\quad\text{for } t\geq T_m,\\ \tilde{\Psi}(T_m)=\Psi(T_m),\quad \tilde{\Psi}'(T_m)=\Psi'(T_m)\end{gathered}$$

has a classical solution $\tilde{\Psi}$ in the interval $[T_m, T_m+\delta)$, where $\delta>0$ is a sufficiently small number. Hence the function

$$\hat{\Psi}(t)=\begin{cases}\Psi(t) & \text{for } t\in[0,T_m),\\ \tilde{\Psi}(t) & \text{for } t\in[T_m,T_m+\delta),\end{cases}$$

$\hat{\Psi}(t)\in C^2([0,T_m+\delta))$, $\hat{\Psi}(t)\geq 0$ and is a classical nonnegative solution of (1.7) in the interval $[0,T_m+\delta)$ which contradicts the choice of T_m. Hence $\Psi(t)$ blows up at T_m and from (2.6) it follows that T_m satisfies (2.8). Thus Theorem 2.4 is proved. □

3. Sufficient conditions for finite time blow up

In this section we give some easy checkable sufficient condition on the initial data $\Psi(0)$ and $\Psi'(0)$ for finite time blow up of the solutions to (1.7). This result is important for the applications of Theorem 2.4 to nonlinear dispersive equations.

Theorem 3.1. *Suppose $\Psi(t)\in C^2([0,T_m))$ is a nonnegative solution of (1.7) in the maximal existence time interval $[0,T_m)$, $0<T_m\leq\infty$, $H(t)\in C([0,\infty))$ and $H(t)\geq 0$ for $t\in[0,\infty)$. If*

$$\beta<\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}+\frac{\alpha(2\gamma-1)}{2(\gamma-1)}\Psi(0)-\frac{\alpha^{2\gamma-1}\Psi^{2\gamma-1}(0)}{2(\gamma-1)\beta^{2\gamma-2}},\tag{3.1}$$

$$\Psi'(0)>0,\tag{3.2}$$

then $\Psi(t)$ blows up at $T_m<\infty$.

Proof. Firstly, we will show that $\Psi'(t)>0$ for every $t\in[0,T_m)$. If not, from (3.2) there exist an interval $[0,t_0)$ such that $\Psi'(t)>0$ for $t\in[0,t_0)$ and $\Psi'(t_0)=0$. From (1.7) and (3.2) it follows that $\Psi(0)>0$. Taking into account the monotonicity of $\Psi(t)$ in the interval $[0,t_0)$, we conclude that $\Psi(t)>0$ for $t\in[0,t_0]$. After the change $z(t)=\Psi^{1-\gamma}(t)$ for $t\in[0,t_0]$ and using identities (2.4), we obtain the equation

$$z''(t)=-(\gamma-1)\Big(\alpha z(t)-\beta z^{\frac{\gamma}{\gamma-1}}(t)+H(t)z^{\frac{\gamma+1}{\gamma-1}}(t)\Big)\quad\text{for } t\in[0,t_0].\tag{3.3}$$

Multiplying (3.3) by $z'(t)$ and integrating from 0 to $t\in(0,t_0]$ we get

$$\begin{aligned}z'^2(t)&=-\alpha(\gamma-1)z^2(t)+\frac{2\beta(\gamma-1)^2}{2\gamma-1}z^{\frac{2\gamma-1}{\gamma-1}}(t)\\ &\quad-2(\gamma-1)\int_0^t H(s)z'(s)z^{\frac{\gamma+1}{\gamma-1}}(s)\,ds+\tilde{C}.\end{aligned}\tag{3.4}$$

From (2.4) we obtain

$$\tilde{C}=z'^2(0)+\alpha(\gamma-1)z^2(0)-\frac{2\beta(\gamma-1)^2}{2\gamma-1}z^{\frac{2\gamma-1}{\gamma-1}}(0)=\frac{2(\gamma-1)^2}{2\gamma-1}\Psi^{1-2\gamma}(0)C,$$

where

$$C=\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}+\frac{\alpha(2\gamma-1)}{2(\gamma-1)}\Psi(0)-\beta.\tag{3.5}$$

From (3.1) we have $C > 0$.

By (2.4), equation (3.4) is equivalent to

$$\begin{aligned}\Psi'^2(t) = &- \frac{\alpha}{\gamma-1}\Psi^2(t) + \frac{2\beta}{2\gamma-1}\Psi(t) + \frac{2C}{2\gamma-1}\Psi^{1-2\gamma}(0)\Psi^{2\gamma}(t)\\ &+ 2\Psi^{2\gamma}(t)\int_0^t H(s)\Psi'(s)\Psi^{-2\gamma-1}(s)\,ds.\end{aligned}$$

Hence the inequality

$$\begin{aligned}\Psi'^2(t) &\ge \Big(-\frac{\alpha}{\gamma-1}\Psi(t) + \frac{2\beta}{2\gamma-1} + \frac{2C}{2\gamma-1}\Psi^{1-2\gamma}(0)\Psi^{2\gamma-1}(t)\Big)\Psi(t)\\ &= G(\Psi)\Psi(t)\end{aligned}$$

holds, where

$$G(y) = -\frac{\alpha}{\gamma-1}y + \frac{2\beta}{2\gamma-1} + \frac{2C}{2\gamma-1}\Psi^{1-2\gamma}(0)y^{2\gamma-1}.$$

From the identities

$$\begin{aligned}\frac{\partial G}{\partial y} &= -\frac{\alpha}{\gamma-1} + 2C\Psi^{1-2\gamma}(0)y^{2\gamma-2},\\ \frac{\partial^2 G}{\partial y^2} &= 4(\gamma-1)C\Psi^{1-2\gamma}(0)y^{2\gamma-3} > 0 \quad \text{for } y > 0\end{aligned}$$

it follows that $G(y)$ has a minimum in $[0,\infty)$ at the point

$$y_0 = \Big(\frac{\alpha\Psi^{2\gamma-1}(0)}{2(\gamma-1)C}\Big)^{\frac{1}{2\gamma-2}}.$$

Simple computations give us

$$G(y_0) = \frac{2}{2\gamma-1}\Big[\beta - \alpha\Big(\frac{\alpha\Psi^{2\gamma-1}(0)}{2(\gamma-1)C}\Big)^{\frac{1}{2\gamma-2}}\Big].$$

From (3.1) and (3.5) it follows that $G(y_0) > 0$, hence

$$\Psi'^2(t) \ge G(\Psi)\Psi(t) \ge G(y_0)\Psi(t) > 0 \quad \text{for } t \in [0, t_0]. \tag{3.6}$$

For $t = t_0$ we obtain the following impossible chain of inequalities

$$0 = \Psi'^2(t_0) \ge G(y_0)\Psi(t_0) > 0.$$

Thus $\Psi'(t) > 0$ and consequently $\Psi(t) > 0$ for every $t \in [0, T_m)$. Moreover, from (3.6) we get

$$\Psi'(t) \ge \sqrt{G(y_0)\Psi(t)}. \tag{3.7}$$

Integrating (3.7) we obtain

$$\Psi(t) \ge \Big(\frac{t}{2}\sqrt{G(y_0)} + \sqrt{\Psi(0)}\Big)^2.$$

For

$$t = b = \max\Big(2\Big(\sqrt{\beta/\alpha} - \sqrt{\Psi(0)}\Big)G^{-1/2}(y_0), 0\Big)$$

we get that $\Psi(b) \ge \beta/\alpha$. Since $\Psi'(b) > 0$ condition (2.7) of Theorem 2.4 is satisfied. Hence the function $\Psi(t)$ blows up at $T_m < \infty$. $\square$

Let us recall that sufficient conditions for finite time blow up of the solutions to (1.6) are obtained in [6] and [1]. Note, that every solution $\Psi(t)$ of equation (1.7) is also a solution to inequality (1.6). Hence, the sufficient conditions for blow up of the solutions to (1.7) are also sufficient ones for blow up of the solutions to (1.6). This allows us to compare the result in [1, 6] with the result in the present paper.

Below we will show that the proofs of the results in [1, 6] follow from Theorem 3.1 and Theorem 2.4. In this way we get a unified approach for proving blow up of solution to (1.6).

Theorem 3.2. *Suppose $\Psi(t) \in C^2([0,T_m))$ is a nonnegative solution of (1.7) in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$, $H(t) \in C([0,\infty))$ and $H(t) \ge 0$ for $t \in [0,\infty)$. If $\Psi'(0) > 0$ and one of the following conditions*

(i) $\beta < \alpha\Psi(0)$;

(ii) [6]
$$\beta < \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \alpha\Psi(0); \tag{3.8}$$

(iii) [1]
$$\beta < \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \alpha\Psi(0) + \frac{\alpha\Psi(0)}{2(\gamma-1)}(1-A^{2-2\gamma}), \quad A = \frac{\gamma-1}{\alpha}\frac{\Psi'^2(0)}{\Psi^2(0)} + 1$$

is satisfied, then $\Psi(t)$ blows up at $T_m < \infty$.

Proof. The proof of (i) follows directly from Theorem 2.4 for $b = 0$.

(ii) **Case 1:** If $\beta < \alpha\Psi(0)$, then (ii) follows from Theorem 3.2(i).

Case 2: Suppose that
$$\alpha\Psi(0) \le \beta < \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \alpha\Psi(0)\,. \tag{3.9}$$

Straightforward computations show that (3.1) also holds. Indeed, from (3.9) we get
$$\begin{aligned}
&\beta^{2\gamma-2}\Big(\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \frac{\alpha(2\gamma-1)}{2(\gamma-1)}\Psi(0) - \beta\Big)\\
&> \beta^{2\gamma-2}\frac{\alpha\Psi(0)}{2(\gamma-1)}\\
&\ge \alpha^{2\gamma-2}\Psi^{2\gamma-2}(0)\frac{\alpha\Psi(0)}{2(\gamma-1)} = \frac{\alpha^{2\gamma-1}\Psi^{2\gamma-1}(0)}{2(\gamma-1)},
\end{aligned}$$
which is equivalent to (3.1). Thus (ii) follows from Theorem 3.1

(iii) **Case 1:** If (3.8) holds, then (iii) follows from Theorem 3.2(ii).

Case 2: Suppose that
$$\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \alpha\Psi(0) \le \beta < \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \alpha\Psi(0) + \frac{\alpha\Psi(0)}{2(\gamma-1)}(1-A^{2-2\gamma})\,.$$

We will show that (3.1) is also fulfilled. Indeed, direct computations give us
$$\begin{aligned}
&\beta^{2\gamma-2}\Big(\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \frac{\alpha(2\gamma-1)}{2(\gamma-1)}\Psi(0) - \beta\Big)\\
&> \beta^{2\gamma-2}\frac{\alpha\Psi(0)}{2(\gamma-1)}A^{2-2\gamma}
\end{aligned}$$

$$
\begin{aligned}
&\geq \Big(\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}+\alpha\Psi(0)\Big)^{2\gamma-2}\frac{\alpha\Psi(0)}{2(\gamma-1)}A^{2-2\gamma} \\
&= \frac{\alpha^{2\gamma-1}\Psi^{2\gamma-1}(0)}{2(\gamma-1)}\Big[1+\frac{1}{2(\gamma-1)}\big(1-\frac{1}{A}\big)\Big]^{2\gamma-2} \\
&> \frac{\alpha^{2\gamma-1}\Psi^{2\gamma-1}(0)}{2(\gamma-1)},
\end{aligned}
$$

which is equivalent to (3.1). Thus (iii) follows from Theorem 3.1. □

4. Applications to nonlinear dispersive equation

In this section we consider the Cauchy problem for the Klein-Gordon equation (1.9)–(1.11) and double dispersive equation with linear restoring force (1.12)-(1.14). For functions depending on t and x we use the following short notation:

$$
\begin{gathered}
\|u\| = \|u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R})}, \quad \|u\|_1 = \|u(t,\cdot)\|_{\mathrm{H}^1(\mathbb{R})}, \\
(u,v) = (u(t,\cdot), v(t,\cdot)) = \int_{\mathbb{R}} u(t,x)v(t,x)\,dx.
\end{gathered}
$$

We recall the definition for blow up of the solutions to (1.9)-(1.11) and (1.12)-(1.14).

Definition 4.1. Suppose $u(t,x)$ is a solution to (1.9)-(1.11) or (1.12)-(1.14) in the maximal existence time interval $[0,T_m)$, $0<T_m\leq\infty$. Then $u(t,x)$ blows up at T_m if

$$
\limsup_{t\to T_m,\, t<T_m} \|u\|_1 = \infty. \tag{4.1}
$$

4.1. **Klein-Gordon equation.** We use the following well known local existence result for problem (1.9)-(1.11).

Theorem 4.2. *Problem* (1.9)-(1.11) *admits a unique local solution* $u(t,x)$ *that belongs to* $\mathrm{C}([0,T_m);\mathrm{H}^1(\mathbb{R}))\cap\mathrm{C}^1([0,T_m);\mathrm{L}^2(\mathbb{R}))\cap\mathrm{C}^2([0,T_m);\mathrm{H}^{-1}(\mathbb{R})$ *on a maximal existence time interval* $[0,T_m)$, $T_m\leq\infty$. *Moreover:*

- (i) *If* $\limsup_{t\to T_m, t<T_m}\|u\|_1<\infty$, *then* $T_m=\infty$;
- (ii) *For every* $t\in[0,T_m)$ *the solution* $u(t,x)$ *satisfies the conservation law* $E(t)=E(0)$, *where*

$$
E(t) := E(u(t,\cdot)) = \frac{1}{2}\left((u_t,u_t)+(u,u)+\|u_x\|^2\right) - \int_{\mathbb{R}}\int_0^u f(y)\,dy\,dx. \tag{4.2}
$$

The following lemma gives an equivalent form of Definition 4.1 for blow up of the solutions to (1.9)-(1.11) using the subquintic growth of the nonlinearity term (1.15), (1.16) i.e. $p_l<5$. Let us underline, that the restriction on the growth of the nonlinear term is essential for the result in Lemma 4.3 as well as in the further statements.

Lemma 4.3. *Suppose* $u(t,x)$ *is the solution to* (1.9)-(1.11) *in the maximal existence time interval* $[0,T_m)$, $0<T_m\leq\infty$. *Then the blow up of the* H^1 *norm of* $u(t,x)$ *is equivalent to the blow up of the* L^2 *norm of* $u(t,x)$ *at* T_m, *i.e.* $\limsup_{t\to T_m,\, t<T_m}\|u\|_1=\infty$ *if and only if*

$$
\limsup_{t\to T_m, t<T_m} \|u\| = \infty. \tag{4.3}
$$

Proof. (Necessity) Suppose (4.1) holds. From the definition of H^1 norm it follows that either $\limsup_{t\to T_m, t<T_m} \|u\| = \infty$, i.e. (4.3) is satisfied, or

$$\limsup_{t\to T_m, t<T_m} \|u_x\| = \infty \tag{4.4}$$

and $\limsup_{t\to T_m, t<T_m} \|u\| < \infty$. Hence

$$\|u(t,\cdot)\| \le C_1 \tag{4.5}$$

holds for every $t \in [0, T_m)$ and some constant C_1. From the Garliardo-Nirenberg inequality we get

$$\|u\|_{\mathrm{L}^{p+1}(\mathbb{R})} \le C_p \|u_x\|^{\frac{p-1}{2(p+1)}} \|u\|^{\frac{p+3}{2(p+1)}} \le C_p C_1^{\frac{p+3}{2(p+1)}} \|u_x\|^{\frac{p-1}{2(p+1)}} \tag{4.6}$$

for every $p > 1$ and some constant C_p depending only on p.

By Young's inequality for every $\varepsilon > 0$ we have

$$\int_{\mathbb{R}} |u(t,x)|^{p+1}\,dx \le \varepsilon\|u_x\|^2 + \frac{5-p}{4}\big(\frac{p-1}{4}\big)^{\frac{p-1}{5-p}} C_p^{\frac{4(p+1)}{5-p}} C_1^{\frac{2(p+3)}{5-p}} \varepsilon^{-\frac{p-1}{5-p}}. \tag{4.7}$$

Applying (4.7) for $p = p_k < 5$, $k = 1, 2, \dots, l$, and $\varepsilon = \big(4\sum_{k=1}^{l} \frac{a_k}{p_k+1}\big)^{-1}$ we obtain from the conservation law (4.2) the estimate

$$\begin{aligned} E(0) =& \frac{1}{2}\left((u_t,u_t) + (u,u) + \|u_x\|^2\right) - \sum_{k=1}^{l} \frac{a_k}{p_k+1}\int_{\mathbb{R}} |u|^{p_k+1}\,dx \\ &+ \sum_{j=1}^{s} \frac{b_j}{q_j+1}\int_{\mathbb{R}} |u|^{q_j+1}\,dx \ge \frac{1}{2}\|u_x\|^2 - \frac{1}{4}\|u_x\|^2 - C_2, \end{aligned}$$

where

$$C_2 = \sum_{k=1}^{l} \frac{a_k}{p_k+1}\frac{5-p_k}{4}\big(\frac{p_k-1}{4}\big)^{\frac{p_k-1}{5-p_k}} C_{p_k}^{\frac{4(p_k+1)}{5-p_k}} C_1^{\frac{2(p_k+3)}{5-p_k}} \Big(4\sum_{k=1}^{l}\frac{a_k}{p_k+1}\Big)^{\frac{p_k-1}{5-p_k}} < \infty.$$

Hence $\|u_x\|^2 \le 4(E(0) + C_2) < \infty$ for every $t \in [0, T_m)$ which contradicts (4.4). Thus $\limsup_{t\to T_m, t<T_m} \|u\| = \infty$.

(Sufficiency) Suppose (4.3) holds. Then from the inequality $\|u\| \le \|u\|_1$ it is obvious that (4.1) is satisfied. The proof is complete. $\square$

Later on we need the following auxiliary result.

Lemma 4.4. *Suppose $u(t,x)$ is the solution to* (1.9)-(1.11) *defined in the maximal existence time interval $[0, T_m)$, $0 < T_m \le \infty$. Then the function $\Psi(t) = (u,u)$ satisfies the equation*

$$\Psi''(t)\Psi(t) - \frac{p_1+3}{4}\Psi'^2(t) = (p_1-1)\Psi^2(t) - 2(p_1+1)E(0)\Psi(t) + H(t), \tag{4.8}$$

where

$$\begin{aligned} H(t) =& (p_1+3)\left[(u_t,u_t)(u,u) - (u,u_t)^2\right] \\ &+ \left[2(p_1+1)B(t) + (p_1-1)\|u_x\|^2)\right](u,u) \ge 0 \end{aligned} \tag{4.9}$$

and

$$B(t) = \sum_{k=2}^{l} \frac{a_k(p_k-p_1)}{(p_k+1)(p_1+1)}\int_{\mathbb{R}} |u|^{p_k+1}\,dx + \sum_{j=1}^{s} \frac{b_j(p_1-q_j)}{(q_j+1)(p_1+1)}\int_{\mathbb{R}} |u|^{q_j+1}\,dx. \tag{4.10}$$

Proof. By (1.9) and (4.2), we get the following identities for $\Psi(t)$:

$$
\begin{aligned}
&\Psi'(t) = 2(u, u_t),\\
&\Psi''(t) = 2(u_t, u_t) + 2(u, u_{tt}) = 2(u_t, u_t) - 2\|u\|_1^2 + 2\int_{\mathbb{R}} uf(u)\,dx\\
&\qquad = (p_1+3)(u_t,u_t) - 2(p_1+1)E(0) + (p_1-1)(u,u)\\
&\qquad\quad + (p_1-1)\|u_x\|^2 + 2(p_1+1)B(t),
\end{aligned}
$$

where $B(t)$ is given by (4.10). From (1.16) we have

$$B(t) \ge 0 \quad \text{for } t \in [0, T_m). \tag{4.11}$$

Substituting $\Psi'(t)$ and $\Psi''(t)$ in the left-hand side of (4.8), we get that $\Psi(t)$ is a solution to (4.8). Here $H(t)$ is given in (4.9) and $H(t) \ge 0$ from (4.11) and the Cauchy - Schwarz inequality. Lemma 4.4 is proved. $\square$

Theorem 4.5. *Suppose $u(t,x)$ is the solution to* (1.9)-(1.11) *defined in the maximal existence time interval* $[0,T_m)$, $0 < T_m \le \infty$ *and* $E(0) > 0$. *If $u(t,x)$ blows up at T_m, then $T_m < \infty$.*

Proof. From Lemma 4.4 we get that the function $\Psi(t) = (u,u)$ satisfies in $[0,T_m)$ equation (4.8). Hence, $\Psi(t)$ is a solution to (1.7) for

$$\alpha = p_1 - 1, \qquad \beta = 2(p_1+1)E(0) > 0, \qquad \gamma = \frac{p_1+3}{4} > 1 \tag{4.12}$$

and $H(t)$ defined in (4.9). From Lemma 4.3 it follows that $\Psi(t) = (u,u)$ blows at T_m. Applying Theorem 2.3 we get that $T_m < \infty$. Thus the solution $u(t,x)$ blows up for a finite time $T_m < \infty$. Theorem 4.5 is proved. $\square$

Theorem 4.6. *Suppose $u(t,x)$ is the solution to* (1.9)-(1.11) *defined in the maximal existence time interval* $[0,T_m)$, $0 < T_m \le \infty$ *and* $E(0) > 0$. *Then $u(t,x)$ blows up at T_m if and only if there exists $b \in [0,T_m)$ such that*

$$E(0) \le \frac{p_1-1}{2(p_1+1)}(u(b,\cdot), u(b,\cdot)) \quad \textit{and} \quad (u(b,\cdot), u_t(b,\cdot)) > 0. \tag{4.13}$$

Moreover,

$$T_m \le b + \frac{2}{(p_1-1)}\frac{(u(b,\cdot), u(b,\cdot))}{(u(b,\cdot), u_t(b,\cdot))} < \infty\,.$$

Proof. (Necessity) Suppose $u(t,x)$ blows up at T_m, i.e. (4.1) holds. By Lemma 4.3 it follows that $\limsup_{t\to T_m, t<T_m}\|u\| = \infty$, i.e. $\Psi(t) = (u,u)$ blows up at T_m. Then from Lemma 2.2 for $M = 2(p_1+1)E(0)/(p_1-1)$ and $b = t_0$ condition (4.13) is satisfied.

(Sufficiency) Suppose (4.13) holds. We assume by contradiction that $u(t,x)$ does not blow up at T_m, i.e according to Definition 4.1 we have

$$\limsup_{t\to T_m, t<T_m} \|u(t,\cdot)\|_1 < \infty. \tag{4.14}$$

From the local existence result in Theorem 4.2(i) it follows that $T_m = \infty$. Then $\Psi(t) = (u,u)$ satisfies (1.7) in $[0,\infty)$ for α, β, γ defined in (4.12). Note, that now $H(t)$, given in (4.9), is a nonnegative function for every $t \ge 0$. Moreover, condition (2.7) in Theorem 2.4 is fulfilled from (4.13). Applying Theorem 2.4 we get that $\Psi(t) = (u,u)$ blows up at T_m, which contradicts our assumption (4.14). The proof is complete. $\square$

In the following theorems we give general sufficient conditions on the initial data u_0 and u_1, which guarantee finite time blow up of the solutions to problem (1.9)-(1.11).

Theorem 4.7. *Suppose $u(t,x)$ is the solution to (1.9)-(1.11) with $E(0)>0$ defined in the maximal existence time interval $[0,T_m)$, $0<T_m\le\infty$. If $(u_0,u_1)>0$ and*

$$E(0)<\frac{1}{2}\frac{(u_0,u_1)^2}{(u_0,u_0)}+\frac{1}{2}(u_0,u_0)-\big(\frac{p_1-1}{2}\big)^{\frac{p_1-1}{2}}\big(\frac{(u_0,u_0)}{p_1+1}\big)^{\frac{p_1+1}{2}}E^{\frac{1-p_1}{2}}(0),$$

then $u(t,x)$ blows up at $T_m<\infty$.

Theorem 4.8. *Suppose $u(t,x)$ is the solution to (1.9)-(1.11) defined in the maximal existence time interval $[0,T_m)$, $0<T_m\le\infty$ and $E(0)>0$. If $(u_0,u_1)>0$ and one of the following conditions*

(i) [14, 15]

$$E(0)<\frac{p_1-1}{2(p_1+1)}(u_0,u_0)\,,$$

(ii) [6]

$$E(0)<\frac{1}{2}\frac{(u_0,u_1)^2}{(u_0,u_0)}+\frac{p_1-1}{2(p_1+1)}(u_0,u_0)\,,$$

(iii) [1]

$$E(0)<\frac{1}{2}\frac{(u_0,u_1)^2}{(u_0,u_0)}+\frac{p_1-1}{2(p_1+1)}(u_0,u_0)+\frac{(u_0,u_0)}{p_1+1}\Big[1-\Big(1+\frac{(u_0,u_1)^2}{(u_0,u_0)^2}\Big)^{\frac{1-p_1}{2}}\Big]$$

is satisfied, then $u(t,x)$ blows up at $T_m<\infty$.

Theorems 4.7 and 4.8 follow from Theorems 3.1 and 3.2, respectively, for α, β, γ defined in (4.12) and $\Psi(t)=(u,u)$, $\Psi(0)=(u_0,u_0)$, $\Psi'(0)=2(u_0,u_1)$.

Remark 4.9. From Lemma 4.4, neglecting the nonnegative terms $(p_1-1)\Psi^2(t)$ and $H(t)$ in (4.8), it follows that $\Psi(t)=(u,u)$ satisfies (1.5) for $\gamma=\frac{p_1+3}{4}>1$ and $\beta=2(p_1+1)E(0)>0$. By the idea of the proof of Theorem 3.1 one can get the result in [5, 13], i.e. under the conditions:

$$(u_0,u_1)>0,\quad 0<E(0)<\frac{1}{2}\frac{(u_0,u_1)^2}{(u_0,u_0)} \tag{4.15}$$

the solution to (1.9)-(1.11) blows up for a finite time.

Remark 4.10. For the first time condition (i) in Theorem 4.8 was proposed in [3] for proving blow up of the solution to semilinear wave equation with arbitrary high initial energy. Let us emphasize that this sufficient condition coincides with the necessary and sufficient one (4.13) formulated for $b=0$. The other sufficient conditions (ii), (iii) in Theorem 4.8 and (4.15) guarantee the validity of condition (4.13) for some later time $t=b$. We can conclude that for any other sufficient condition for finite time blow up, given at $t=0$, the corresponding solution must satisfy (4.13) for some $t=b>0$.

4.2. **Double dispersive equation with linear restoring force.** In the space $\mathrm{L}^2(\mathbb{R}) \cap \dot{\mathrm{H}}^{-1}(\mathbb{R})$ we define the scalar product

$$\langle u, v\rangle = \langle u(t,\cdot), v(t,\cdot)\rangle = (u,v) + \Big((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}v\Big). \tag{4.16}$$

Under the regularity conditions (1.14) problem (1.12)-(1.14) has a unique solution

$$u(t,x) \in \mathrm{C}([0,T_m);\mathrm{H}^1(\mathbb{R})) \cap \mathrm{C}^1([0,T_m);\mathrm{L}^2(\mathbb{R})) \cap \mathrm{C}^2([0,T_m);\mathrm{H}^{-1}(\mathbb{R}))$$

on a maximal existence time interval $[0,T_m)$, $T_m \le \infty$, and if

$$\limsup_{t\to T_m, t<T_m} \|u\|_1 < \infty \quad \text{then} \quad T_m = \infty.$$

Moreover, for every $t \in [0,T_m)$ the solution $u(t,x)$ to (1.12)-(1.14) satisfies the conservation law $E(t) = E(0)$, where

$$E(t) := E(u(\cdot,t)) = \frac{1}{2}\left(\langle u_t, u_t\rangle + \langle u, u\rangle + \|u_x\|^2\right) - \int_{\mathbb{R}}\int_0^u f(y)\,dy\,dx. \tag{4.17}$$

The results for double dispersive equation with linear restoring force (1.12)-(1.14) are identical with the results for Klein-Gordon equation (1.9)-(1.10), proved in Subsection 4.1. The main deference is that the standard scalar product $(\cdot,\cdot)$ has to be replaced with the scalar product $\langle\cdot,\cdot\rangle$ given in (4.16). In particular, Lemma 4.3 holds also for the solutions to (1.12)-(1.14). However, in addition to Lemma 4.3, we need the following equivalence of the blow up of H^1 norm of the solution $u(t,x)$ to (1.12)-(1.14) at T_m and the blow up of $\langle u, u\rangle$ at T_m.

Lemma 4.11. *Suppose $u(t,x)$ is the solution to (1.12)-(1.14) in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$. Then the blow up of H^1 norm of $u(t,x)$ at T_m is equivalent to the blow up of $\langle u(t,\cdot), u(t,\cdot)\rangle$ at T_m, i.e. $\limsup_{t\to T_m, t<T_m} \|u\|_1 = \infty$ if and only if*

$$\limsup_{t\to T_m, t<T_m} \langle u(t,\cdot), u(t,\cdot)\rangle = \infty. \tag{4.18}$$

Proof. If $\limsup_{t\to T_m} \|u\|_1 = \infty$ from Lemma 4.3 and definition (4.16) it follows that $\langle u(t,\cdot), u(t,\cdot)\rangle$ blows up at T_m. Conversely, suppose that (4.18) holds but

$$\limsup_{t\to T_m} \|u\|_1 < \infty. \tag{4.19}$$

From definition (4.16) we get $\limsup_{t\to T_m}\big((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}u\big) = \infty$. By the conservation law (4.17) it follows that at least one of the norms $\|u\|_{\mathrm{L}^{p_k}}$ tends to infinity for $t \to T_m$. Hence, from the embedding of H^1 into L^{p_k}, $p_k > 2$ we get that $\|u\|_1$ blows up at T_m, which contradict (4.19). The proof is complete. □

For a function $\Psi(t) = \langle u, u\rangle$ the statements in Lemma 4.4, Theorem 4.5, Theorem 4.6, and Theorem 4.7 are true for the solutions to problem (1.12)-(1.14) with the formal change of notation $(\cdot,\cdot)$ by $\langle\cdot,\cdot\rangle$. Below we only formulate the corresponding results without proofs.

Theorem 4.12. *Suppose $u(t,x)$ is the solution to (1.12)-(1.14) defined in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$ and $E(0) > 0$. If $u(t,x)$ blows up at T_m then $T_m < \infty$.*

Theorem 4.13. *Suppose $u(t,x)$ is the solution to* (1.12)-(1.14) *defined in the maximal existence time interval* $[0, T_m)$, $0 < T_m \le \infty$ *and* $E(0) > 0$. *Then $u(t,x)$ blows up at T_m if and only if there exists $b \in [0, T_m)$ such that*

$$E(0) \le \frac{p_1 - 1}{2(p_1 + 1)} \langle u(b,\cdot), u(b,\cdot)\rangle \quad \textit{and} \quad \langle u(b,\cdot), u_t(b,\cdot)\rangle > 0. \tag{4.20}$$

Moreover,

$$T_m \le b + \frac{2}{(p_1 - 1)} \frac{\langle u(b,\cdot), u(b,\cdot)\rangle}{\langle u(b,\cdot), u_t(b,\cdot)\rangle} < \infty\,.$$

Theorem 4.14. *Suppose $u(t,x)$ is the solution to* (1.12)-(1.14) *defined in the maximal existence time interval* $[0, T_m)$, $0 < T_m \le \infty$ *and* $E(0) > 0$. *If* $\langle u_0, u_1\rangle > 0$ *and*

$$E(0) < \frac{1}{2} \frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} + \frac{1}{2}\langle u_0, u_0\rangle - \Big(\frac{p_1 - 1}{2}\Big)^{\frac{p_1-1}{2}} \Big(\frac{\langle u_0, u_0\rangle}{p_1 + 1}\Big)^{\frac{p_1+1}{2}} E^{\frac{1-p_1}{2}}(0),$$

then $u(t,x)$ blows up at $T_m < \infty$.

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