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OBLIQUE DERIVATIVE PROBLEM FOR ELLIPTIC SECOND-ORDER SEMI-LINEAR EQUATIONS IN A DOMAIN WITH A CONICAL BOUNDARY POINT

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Abstract. This article concerns the oblique boundary value problem for elliptic semi-linear equations in a domain with a conical point on the boundary. We investigate the asymptotic behavior of strong solutions near a boundary conical point. New regularity theorems are established under the least possible assumptions on the equation coefficients. The investigation of asymptotic properties of solutions can be used to obtain new solvability theorems. The results obtained in this paper are extensions of our previous results to a wider class of elliptic equations.

1. Introduction

Problems in which the boundary value condition has the form $\mathcal{B}(x, u, Du) = 0$, where $\mathcal{B}$ depends on the gradient Du of the unknown function u in a suitable way, are called *oblique derivative problems*. The two-dimensional basic theory of linear oblique derivative problems is quite old. For two-dimensional domains Talenti [31] established $W^{2,2}$-solvability under the assumption that a^{ij} are measurable functions only. For higher-dimensional case the $W^{2,2}$-regularity and invertibility properties for some linear oblique derivative problems are obtained if $a^{ij} \in W^{1,n}(\Omega)$ (see Miranda [27], Chicco [10]) or if a^{ij} are measurable functions satisfying the Cordes condition (Chicco [10]). Agmon, Douglis and Nirenberg in [1], using explicit representations for solutions to derive suitable L^p-estimates, established that the condition $a^{ij} \in C^0(\overline{\Omega})$ is sufficient to $W^{2,p}$-regularity of solutions for all values of $p \in (0, \infty)$.

It should be noted that investigations in the aforementioned works refer to linear boundary value problems in sufficiently smooth domains. However, many problems of physics and technology lead to boundary-value problems in domains with a non-smooth boundary, in particular, in domains which have a finite number of angular ($n = 2$) or conical ($n \geq 3$) points on the boundary. The theory of linear boundary value problems in non-smooth domains was described in well-known survey of Kondrat'ev and Oleinik [19] and in the work of Kufner and Sändig [22], as well as in the monograph of Kozlov et al. [20].

In non-smooth domains, linear oblique derivative problems were studied earlier by Faierman [13], Garroni, Solonnikov and Vivaldi [15], Grisvard [16], Lieberman

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[24, 23, 26], Reisman [29], and others. Lieberman considered the oblique derivative problem in Lipschitz domains. His results concern elliptic equations with Hölder-continuous coefficients. The local and global maximum principle for general second-order linear and quasi-linear elliptic oblique derivative problems were established by him. Grisvard in his work investigated the properties of the second weak derivatives of the oblique problem for the Laplace operator in a plane domain with a polygonal boundary. Solonnikov et al. proved the uniqueness of solutions and obtained *a priori* estimates for weak solutions of the Laplace operator in the Sobolev-Kondrat'ev weighted spaces.

Some properties of solutions of the semi-linear problem in a smooth domain and in a neighbourhood of an isolated singular point were studied by Kondrat'ev et al., see e.g. [17, 18]. Other problems for elliptic semi-linear equations were considered by Veron et al., see e.g. [2, 12, 28].

The oblique derivative problem plays a major role in the study of reflected shocks in transonic flow [9]. Another important application of this theory is the capillary problem (see e.g. [14]). In geodesy, the most fundamental problems of the gravity field determination from boundary observations are translated into exterior boundary value problems for the Laplace or Poisson equations, see e.g. [11, 30].

The aim of this paper is to describe the asymptotic behavior of strong solutions to the oblique problem for general semi-linear second-order elliptic equations near the boundary conical point, i.e. we obtain the estimation of the type $|u(x)| = O(|x|^\alpha)$ with the sharp exponent α. In our previous papers ([3, 4, 5, 6, 7, 8]) we obtained similar results for linear and quasi-linear oblique problems. The result presented in this paper extends our previous results to a wider class of elliptic equations. New regularity theorems were established. Our results refer to general equations of second-order. It should be pointed out that assumptions concerning of the equation coefficients are the least restrictive possible, i.e. the leading coefficients of the equation must be Dini-continuous at the conical point and the lower coefficients can grow in a particular way.

This paper is organized as follows. At first, we introduce notations for a domain with a conical boundary point and introduce function spaces that are used in the following sections. Next, we formulate the boundary-value problem with oblique derivative for semi-linear elliptic equations in a domain with a conical boundary point. The problem assumptions are also formulated. In section 4 we describe the main results, i.e. Theorems 4.1–4.3. In Theorem 4.1 we assume that the leading coefficients are Dini-continuous at zero. Then, we generalize the result assuming that the condition of Dini-continuity is not satisfied. It gives less accurate regularity of solutions. In Theorem 4.3, we obtain estimates for a particular function describing the growth of the leading coefficients. In the next two sections, we derive global and local weighted estimates that are used in the last section to prove the main theorems.

2. Preliminaries

Let $\mathcal{K}$ be an open cone $\{(r,\omega) : 0 < r < \infty, \omega \in \Omega\}$ with the vertex at $\mathcal{O}$ with boundary $\partial\mathcal{K} = \{(r,\omega) : 0 < r < \infty, \omega_1 = \frac{\omega_0}{2} \in (0,\pi), \omega_i \in \Omega, i \geq 2\}$. Let $G \subset \mathbb{R}^n$ be a bounded domain.

Let us introduce the following notations for a domain G which has a conical point at $\mathcal{O} \in \partial G$.

- Ω: is a subdomain of the unit sphere S^{n-1};
- $\partial\Omega$: the boundary of Ω;
- $G_a^b := G \cap \{(r,\omega) : 0 \le a < r < b, \omega \in \Omega\}$: a layer in $\mathbb{R}^n$;
- $\Gamma_a^b := \partial G \cap \{(r,\omega) : 0 \le a < r < b, \omega \in \partial\Omega\}$: the lateral surface of the layer G_a^b;
- $G_d := G \backslash G_0^d$, $\Gamma_d := \partial G \backslash \Gamma_0^d$, $\Omega_\varrho := \overline{G_0^d} \cap \partial B_\varrho(0)$, $\varrho \le d$;
- $G^{(k)} := G_{2^{-(k+1)}d}^{2^{-k}d}$, $k = 0, 1, 2, \dots$.

Without loss of generality we assume that there exists $0 < b \ll 1$ such that G_0^b is a *rotational cone* with the vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (0,\pi)$ (see Figure 1), thus

$$\Gamma_0^b = \Big\{(r,\omega)x_1^2 = \cot^2\frac{\omega_0}{2}\sum_{i=2}^n x_1^2,\ r \in (0,b),\ \omega_1 = \frac{\omega_0}{2},\ \omega_0 \in (0,\pi)\Big\}.$$

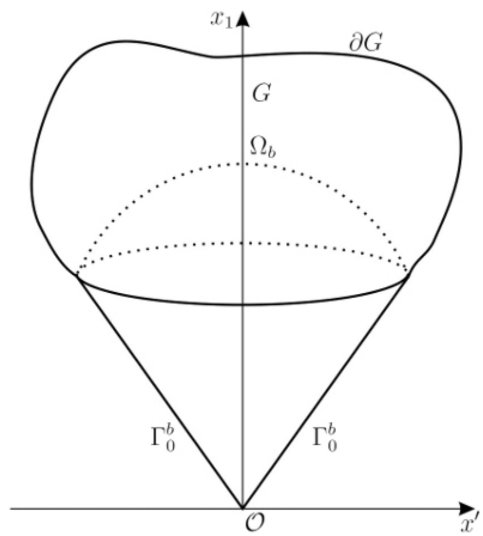

FIGURE 1. n-dimensional bounded domain with a boundary conical point

We use standard function spaces: $C^k(\overline{G})$, $C_0^k(G)$ with the norm $|u|_{k,G}$; the Lebesgue space $L^p(G)$, $p \ge 1$, with the norm $\|u\|_{L^p(G)} = \big(\int_G |u|^p dx\big)^{1/p}$; the Sobolev space $W^{k,p}(G)$ for integer $k \ge 0$, $1 \le p < \infty$, which is a set of all functions $u \in L^p(G)$ such that for every multi-index β with $|\beta| \le k$ weak partial derivatives $D^\beta u$ belong to $L^p(G)$, equipped with the finite norm

$$\|u\|_{W^{k,p}(G)} = \Big(\int_G \sum_{|\beta|\le k} |D^\beta u|^p dx\Big)^{1/p};$$

the weighted Sobolev space $V_{p,\alpha}^k(G)$ for integer $k \ge 0$, $1 < p < \infty$ and $\alpha \in \mathbb{R}$ is the space of distributions $u \in \mathcal{D}'(G)$ with the finite norm

$$\|u\|_{V_{p,\alpha}^k(G)} = \Big(\int_G \sum_{|\beta|\le k} r^{\alpha+p(|\beta|-k)}|D^\beta u|^p dx\Big)^{1/p}$$

and $V_{p,\alpha}^{k-\frac{1}{p}}(\Gamma)$ that is the space of functions g, given on ∂G, with the norm

$$\|g\|_{V_{p,\alpha}^{k-\frac{1}{p}}(\partial G)} = \inf \|\mathcal{G}\|_{V_{p,\alpha}^{k}(G)},$$

where the infimum is taken over all functions g such that $\mathcal{G}\big|_{\partial G} = g$ in the sense of traces.

For $p = 2$ we use the notation

$$W^k(G) = W^{k,2}(G), \quad \mathring{W}_\alpha^k(G) = V_{2,\alpha}^k(G), \quad \mathring{W}_\alpha^{k-\frac{1}{2}}(\Gamma) = V_{2,\alpha}^{k-\frac{1}{2}}(\Gamma).$$

3. Setting of the problem

Let $G \subset \mathbb{R}^n$ be a bounded domain with the boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near $\mathcal{O}$ it is a conical surface. We consider the semi-linear problem

$$\begin{gathered} a^{ij}(x)u_{x_ix_j} + a^i(x)u_{x_i} + a(x)u(x) = h(u) + f(x), \quad x \in G, \\ h(u) = a_0(x)u(x)|u(x)|^{q-1}, \quad q \in (0,1), \\ \frac{\partial u}{\partial \vec{n}} + \chi(\omega)\frac{\partial u}{\partial r} + \frac{1}{|x|}\gamma(\omega)u(x) = g(x), \quad x \in \partial G\backslash\mathcal{O}. \end{gathered} \tag{3.1}$$

where $\vec{n}$ denotes the unite exterior normal vector to $\partial G\backslash\mathcal{O}$; (r,ω) are spherical coordinates in $\mathbb{R}^n$ with the pole $\mathcal{O}$.

Remark 3.1. For $q = 1$, problem (3.1) takes the form of linear problem with $a(x) \longmapsto a(x) - a_0(x)$, which was considered in [4].

Definition 3.2. A function u is called a strong solution of problem (3.1) provided that for any $\epsilon > 0$ function $u \in W^{2,n}_{\rm loc}(G) \cap W^2(G_\epsilon) \cap C^0(\overline{G})$ and satisfies the equation of (3.1) for almost all $x \in G_\epsilon$ as well as the boundary condition in the sense of traces on Γ_ϵ.

Regarding the problem we assume that the following conditions are satisfied:

(A1) the uniform ellipticity condition

$$\nu|\xi|^2 \le \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \le \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \ x \in \overline{G} \tag{3.2}$$

with the ellipticity constants $\nu, \mu > 0$; $a^{ij} = a^{ji}$; $a^{ij}(0) = \delta_i^j$;

(A2) $a^{ij} \in C^0(\overline{G})$, $a^i \in L^p(G)$, $p > n$; $a \in L^n(G) \cap \mathring{W}^0_{4-n}(G)$, $a(x) \le 0$, $a_0 \in L^{\frac{n}{1-q}}(G) \cap V^0_{\frac{2}{1-q}, \frac{4}{1-q}-n}(G)$; there exists a monotonically increasing nonnegative continuous function $\mathcal{A}$, $\mathcal{A}(0) = 0$, such that for $x, y \in \overline{G}$

$$\Big(\sum_{i,j=1}^n |a^{ij}(x) - \delta_i^j|^2\Big)^{1/2} + |x|\Big(\sum_{i=1}^n |a^i(x)|^2\Big)^{1/2} + |x|^2|a(x)| \le \mathcal{A}(|x|); \tag{3.3}$$

(A3) $\gamma(\omega), \chi(\omega) \in C^1(\overline{\Omega})$ and there exist numbers $\chi_0 \ge 0$, $\gamma_0 > 0$, such that $\gamma(\omega) \ge \gamma_0 > 0$, $\chi_0 \ge \chi(\omega) > 0$;

(A4) $f \in L^n(G) \cap \overset{\circ}{W}{}^0_{4-n}(G)$, $g \in V^{1-\frac{1}{n}}_{n,0}(\partial G) \cap \overset{\circ}{W}{}^{1/2}_{4-n}(\partial G)$, there exist nonnegative numbers $f_1,\ g_0,\ g_1,\ a_1$ and $s > 0$, such that

$$\begin{gathered} |f(x)| \leq f_1 |x|^{s-2}, \quad |g(x)| \leq g_0 |x|^{s-1}, \quad |\nabla g| \leq g_1 |x|^{s-2}, \\ \|a_0\|^{\frac{1}{1-q}}_{\frac{2}{1-q}, \frac{4}{1-q}-n(G_0^\varrho)} \leq a_1 \varrho^s, \ \varrho \in (0,1); \end{gathered} \tag{3.4}$$

(A5) $a_0 \in V^0_{\frac{n}{1-q}, \frac{2qn}{1-q}}(G)$.

4. Main result

Let us consider the following eigenvalue problem for the Laplace-Beltrami operator Δ_ω on the unit sphere

$$\begin{gathered} \Delta_\omega \psi + \lambda(\lambda + n - 2)\psi(\omega) = 0, \quad \omega \in \Omega, \\ \frac{\partial \psi}{\partial \vec{\nu}} + [\lambda \chi(\omega) + \gamma(\omega)]\psi(\omega) = 0, \quad \omega \in \partial\Omega, \end{gathered} \tag{4.1}$$

which consists of the determination of all values λ (eigenvalues), for which (4.1) has non-zero weak solutions $\psi(\omega)$ (eigenfunctions). Here $\vec{\nu}$ denotes the unite exterior normal vector to $\partial\mathcal{K}$ at the points of $\partial\Omega$ and functions $\chi(\omega) \geq 0$, $\gamma(\omega) > 0$ are $C^0(\partial\Omega)$-functions (see [3, 4]). Let us define the number

$$k_s = \sqrt{g_0^2 + a_1^2 + \frac{1}{2s}(f_1^2 + g_1^2)}. \tag{4.2}$$

Our main results are the following statements, whose proofs are Section 7.

Theorem 4.1. *Let u be a strong solution of problem* (3.1) *and $\lambda > 1$ be the smallest positive eigenvalue of problem* (4.1)*. Suppose that assumptions* (A1)–(A5) *with $\mathcal{A}(r)$ being Dini-continuous at zero are satisfied. Suppose in addition that there exists a nonnegative constant k_0, such that*

$$\|a_0\|_{L^{\frac{n}{1-q}}(G^{2\varrho}_{\varrho/4})} \leq k_0 \varrho^{1-2q} \psi^q(\varrho), \tag{4.3}$$

where

$$\psi(\varrho) = \begin{cases} \varrho^\lambda & \text{if } s > \lambda, \\ \varrho^\lambda \ln \frac{1}{\varrho} & \text{if } s = \lambda, \\ \varrho^s & \text{if } s < \lambda, \end{cases} \tag{4.4}$$

with $0 < \varrho < b$. Then there are positive constants $d \in (0, b)$ and c_1, c_2, which depend only on ν, μ, s, b, λ, γ_0, χ_0, k_0, f_1, g_0, g_1, a_1, $\|\gamma\|_{C^1(\partial\Omega)}$, $\|\chi\|_{C^1(\partial\Omega)}$, $\operatorname{diam} G$, $\operatorname{meas} G$, *on the modulus of continuity of the leading coefficients and on the quantity $\int_0^1 \frac{\mathcal{A}(r)}{r} dr$, and do not depend on u, such that for all $x \in G_0^d$:*

- *if $0 < q < 1 - \frac{2}{\lambda}$ and $\lambda > s$, then*

$$|u(x)| \leq c_1 |x|^{\frac{2}{1-q}}; \tag{4.5}$$

- *if $1 - \frac{2}{\lambda} \leq q \leq 1$, then*

$$|u(x)| \leq c_2 \begin{cases} |x|^\lambda & \text{if } \lambda < s, \\ |x|^\lambda \ln \frac{1}{|x|} & \text{if } \lambda = s, \\ |x|^s & \text{if } \lambda > s. \end{cases} \tag{4.6}$$

Theorem 4.2. *Let u be a strong solution of problem* (3.1) *and $\lambda > 1$ be the smallest positive eigenvalue of problem* (4.1). *Suppose that assumptions* (A1)–(A5) *are satisfied with $\mathcal{A}(r)$, which is a continuous at zero function, but* not Dini *continuous. Suppose in addition that there exists a nonnegative constant k_0, such that* (4.3) *holds with*

$$\psi(\varrho) = \begin{cases} \varrho^{\lambda-\epsilon} & \text{if } s > \lambda, \\ \varrho^{s-\epsilon} & \text{if } s \le \lambda, \end{cases} \tag{4.7}$$

and $0 < \varrho < b$. Then, for any $\epsilon > 0$, there are positive constants $d \in (0, b)$ and c_1, c_2, which depend only on ν, μ, s, b, λ, γ_0, χ_0, k_0, f_1, g_0, g_1, a_1, $\|\gamma\|_{C^1(\partial\Omega)}$, $\|\chi\|_{C^1(\partial\Omega)}$, $\operatorname{diam} G$, $\operatorname{meas} G$, *on the modulus of continuity of the leading coefficients and do not depend on u, such that for all $x \in G_0^d$*

- *if $0 < q < 1 - \frac{2}{\lambda}$ and $\lambda > s$, then*
$$|u(x)| \le c_1 |x|^{\frac{2}{1-q}+\epsilon};$$
- *if $1 - \frac{2}{\lambda} \le q \le 1$, then*
$$|u(x)| \le c_2 \begin{cases} |x|^{\lambda-\epsilon} & \text{if } \lambda \le s, \\ |x|^{s-\epsilon} & \text{if } \lambda > s. \end{cases}$$

Theorem 4.3. *Let u be a strong solution of problem* (3.1) *and $\lambda > 1$ be the smallest positive eigenvalue of problem* (4.1). *Suppose that assumptions* (A1)–(A5) *are satisfied with $\mathcal{A}(r) \sim \frac{1}{\ln \frac{1}{r}}$. Suppose in addition that there exists a nonnegative constant k_0, such that* (4.3) *holds with*

$$\psi(\varrho) = \ln^{c_s(\lambda)}\big(\frac{1}{\varrho}\big) \begin{cases} \varrho^{\lambda} & \text{if } s > \lambda, \\ \varrho^{s} & \text{if } s \le \lambda, \end{cases} \tag{4.8}$$

and $0 < \varrho < b$. Then there are positive constants $d \in (0, b)$ and c_1, c_2, c_s, which depend only on ν, μ, s, b, λ, γ_0, χ_0, k_0, f_1, g_0, g_1, a_1, $\|\gamma\|_{C^1(\partial\Omega)}$, $\|\chi\|_{C^1(\partial\Omega)}$, $\operatorname{diam} G$, $\operatorname{meas} G$, *on the modulus of continuity of the leading coefficients and do not depend on u, such that for all $x \in G_0^d$*

- *if $0 < q < 1 - \frac{2}{\lambda}$ and $\lambda > s$, then*
$$|u(x)| \le c_1 \ln^{c_s(\lambda)}\big(\frac{1}{|x|}\big) |x|^{\frac{2}{1-q}};$$
- *if $1 - \frac{2}{\lambda} \le q \le 1$, then*
$$|u(x)| \le c_2 \ln^{c_s(\lambda)}\big(\frac{1}{|x|}\big) \begin{cases} |x|^{\lambda} & \text{if } \lambda \le s, \\ |x|^{s} & \text{if } \lambda > s. \end{cases}$$

5. Global integral weighted estimate

Let us introduce the function

$$M(\epsilon) = \max_{x \in \overline{\Omega}_\epsilon} |u(x)| \tag{5.1}$$

then because of $u \in C^0(\overline{G})$,

$$\lim_{\epsilon \to 0^+} M(\epsilon) = |u(0)|. \tag{5.2}$$

Lemma 5.1. *Let u be a strong solution of problem* (3.1) *and assumptions* (A1)–(A3) *be satisfied. Then there exists a positive constant c_0 depending only on ν, μ, G, $\max_{x,y\in G}\mathcal{A}(|x-y|)$, $\|\chi\|_{C^1(\partial\Omega)}$, $\|\gamma\|_{C^1(\partial\Omega)}$, such that*

$$\lim_{\epsilon\to 0^+}\epsilon^{2-n}\Big|\int_{\Omega_\epsilon} u\frac{\partial u}{\partial r}d\Omega_\epsilon\Big| \le c_0|u(0)|^2. \tag{5.3}$$

Proof. Let us consider the set $G_\epsilon^{2\epsilon}$, $\Omega_\epsilon\subset\partial G_\epsilon^{2\epsilon}$. By [25, Lemma 6.36] we have

$$\int_{\Omega_\epsilon}|w|d\Omega_\epsilon \le c\int_{G_\epsilon^{2\epsilon}}(|w|+|\nabla w|)\,dx,$$

where c depends only on the domain G. Setting $w=u\frac{\partial u}{\partial r}$ and using the Cauchy inequality, we obtain

$$\int_{\Omega_\epsilon}\Big|u\frac{\partial u}{\partial r}\Big|d\Omega_\epsilon \le c\int_{G_\epsilon^{2\epsilon}}\big(r^2u_{xx}^2+|\nabla u|^2+r^{-2}u^2\big)\,dx. \tag{5.4}$$

Let us consider two sets $G_{\epsilon/2}^{5\epsilon/2}$ and $G_\epsilon^{2\epsilon}\subset G_{\epsilon/2}^{5\epsilon/2}$, and a new variable x' defined by $x=\epsilon x'$. Thus, the function $w(x')=u(\epsilon x')$ satisfies in $G_{1/2}^{5/2}$ the problem

$$\begin{gathered}
a^{ij}(\epsilon x')w_{x_i'x_j'}+\epsilon a^i(\epsilon x')w_{x_i'}+\epsilon^2a(\epsilon x')w(x')\\
=\epsilon^2h(w(x'))+\epsilon^2f(\epsilon x'),\quad x'\in G_{1/2}^{5/2},\\
h(w(x'))=a_0(\epsilon x')w(x')|w(x')|^{q-1},\quad q\in(0,1),\\
\frac{\partial w}{\partial\vec n'}+\chi(\omega)\frac{\partial w}{\partial r'}+\frac{1}{|x'|}\gamma(\omega)w(x')=\epsilon g(\epsilon x'),\quad x'\in\Gamma_{1/2}^{5/2}.
\end{gathered} \tag{5.5}$$

We apply L^2-estimate for the solution of problem (5.5) (see [1, Theorem 15.3]). As a result we obtain the following estimation

$$\begin{aligned}
&\int_{G_1^2}\big(w_{x'x'}^2+|\nabla' w|^2+w^2\big)\,dx'\\
&\le c_1\int_{G_{1/2}^{5/2}}[\epsilon^4f^2+\epsilon^4h^2(w)+w^2]dx'+c_2\epsilon^2\inf\int_{G_{1/2}^{5/2}}\big(|\nabla'\mathcal{G}|^2+|\mathcal{G}|^2\big)\,dx',
\end{aligned}$$

here infimum is taken over all $\mathcal{G}\in W^1(G_{1/2}^{5/2})$, such that $\mathcal{G}\big|_{\Gamma_{1/2}^{5/2}}=g$ and constants c_1 and c_2 are positive and depend only on ν, μ, G, $\max_{x,y\in G_{1/2}^{5/2}}\mathcal{A}(|x-y|)$, $\|\chi\|_{C^1(\Gamma_{1/2}^{5/2})}$, $\|\gamma\|_{C^1(\Gamma_{1/2}^{5/2})}$. Returning to the variable x we obtain

$$\begin{aligned}
&\int_{G_\epsilon^{2\epsilon}}\big(r^{4-n}u_{xx}^2+r^{2-n}|\nabla u|^2+r^{-n}u^2\big)\,dx\\
&\le c_3\int_{G_{\epsilon/2}^{5\epsilon/2}}\big(r^{4-n}f^2+r^{4-n}h^2+r^{-n}u^2\big)dx\\
&\quad+c_4\inf\int_{G_{\epsilon/2}^{5\epsilon/2}}\big(r^{4-n}|\nabla\mathcal{G}|^2+r^{2-n}|\mathcal{G}|^2\big)\,dx.
\end{aligned} \tag{5.6}$$

Now, by the mean value theorem with regard to $u \in C^0(\overline{G})$ and (5.1), we obtain

$$\begin{aligned}\int_{G_{\epsilon/2}^{5\epsilon/2}} r^{-n} u^2 dx &= \int_{\epsilon/2}^{5\epsilon/2} r^{-1} \int_\Omega u^2(r,\omega) d\Omega dr \le 2\epsilon(\theta_1\epsilon)^{-1} \int_\Omega u^2(\theta_1\epsilon,\omega) d\Omega \\ &\le 2{\theta_1}^{-1} M^2(\theta_1\epsilon) \operatorname{meas}\Omega\end{aligned} \tag{5.7}$$

for some $\frac{1}{2} < \theta_1 < \frac{5}{2}$. By (5.4), (5.6) and (5.7), it follows that

$$\begin{aligned}&\epsilon^{2-n} \Big| \int_{\Omega_\epsilon} u \frac{\partial u}{\partial r} d\Omega_\epsilon \Big| \\ &\le c_5 M^2(\theta_1\epsilon) + c_6 \int_{G_{\epsilon/2}^{5\epsilon/2}} (r^{4-n}h^2 + r^{4-n}f^2) dx + c_7 \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{4-n}(\Gamma^{5\epsilon/2}_{\epsilon/2})}.\end{aligned} \tag{5.8}$$

Taking into account that $q < 1$, by the Young inequality with $\frac{1}{q}$ and $\frac{1}{1-q}$, we deduce

$$\begin{aligned}r^{4-n}h^2 &= r^{4-n} a_0^2 |u|^{2q} = \big(r^{-nq}|u|^{2q}\big)\big(r^{4-n+nq}a_0^2\big) \\ &\le \delta r^{-n}u^2 + c(\delta,q) r^{\frac{4}{1-q}-n} a_0^{\frac{2}{1-q}}, \quad \forall \delta > 0.\end{aligned} \tag{5.9}$$

By assumptions about functions f, g and a_0, from (5.7) and (5.9), we obtain that

$$\lim_{\epsilon\to 0^+} \Big\{ \int_{G_{\epsilon/2}^{5\epsilon/2}} \big[r^{4-n}h^2 + r^{4-n}f^2\big] dx + \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{4-n}(\Gamma^{5\epsilon/2}_{\epsilon/2})} \Big\} = 0 \tag{5.10}$$

and therefore, by (5.2), (5.8) and (5.10), we finally obtain (5.3). $\square$

Theorem 5.2. *Let u be a strong solution of problem* (3.1) *and assumptions* (A1)–(A3) *be satisfied. Then $u \in \overset{\circ}{W}{}^2_{4-n}(G)$ and*

$$\begin{aligned}&\|u\|_{\overset{\circ}{W}{}^2_{4-n}(G)} + \Big(\int_G a_0 r^{2-n}|u|^{1+q} dx\Big)^{1/2} + \Big(\int_{\partial G} r^{1-n}\gamma(\omega) u^2 ds\Big)^{1/2} \\ &\le C\Big(|u|_{2,G} + \|a_0\|^{\frac{1}{1-q}}_{V^0_{\frac{2}{1-q},\frac{4}{1-q}-n}(G)} + \|f\|_{\overset{\circ}{W}{}^0_{4-n}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-n}(\partial G)}\Big),\end{aligned} \tag{5.11}$$

where $C > 0$ depends on ν, μ, b, q, n, $\operatorname{meas} G$, $\|\chi\|_{C^1(\partial\Omega)}$, $\|\gamma\|_{C^1(\partial\Omega)}$ and on modulus of continuity of the leading coefficients.

Proof. Let us rewrite the equation of (3.1) in the form

$$\Delta u = h(u) + f(x) - \Big[\big(a^{ij}(x) - \delta_i^j\big)u_{x_ix_j} + a^i(x)u_{x_i} + a(x)u(x)\Big], \tag{5.12}$$

multiply both sides by $r^{2-n}u(x)$ and integrate over G_ϵ. As a result we obtain

$$\begin{aligned}\int_{G_\epsilon} r^{2-n} u \Delta u dx &= \int_{G_\epsilon} r^{2-n} u\,(h+f)\,dx \\ &\quad - \int_{G_\epsilon} r^{2-n} u\big[\big(a^{ij} - \delta_i^j\big)u_{x_ix_j} + a^i u_{x_i} + au\big] dx.\end{aligned} \tag{5.13}$$

Calculating the integral from the left side by parts, using the boundary condition, the representation $\partial G_\epsilon = \Gamma_\epsilon \cup \Omega_\epsilon$, $d\Omega_\epsilon = \epsilon^{n-1}d\Omega$, and the fact that

$$x_i \cos(\vec{n}, x_i)\Big|_{\Omega_\epsilon} = \epsilon, \quad x_i \cos(\vec{n}, x_i)\Big|_{\Gamma_0^d} = 0,$$

(see [8, Lemma 1.10]), we obtain

$$\begin{aligned}
&\int_{G_\epsilon} r^{2-n}|\nabla u|^2dx + \int_{\Gamma_\epsilon} \gamma(\omega)r^{1-n}u^2ds + \frac{n-2}{2}\int_{\Omega} u^2 d\Omega \\
&+ \int_{G_\epsilon} r^{2-n}a_0(x)|u|^{q+1}dx \\
&= \int_{\Gamma_\epsilon} r^{2-n}ug\,ds - \int_{\Gamma_\epsilon} \chi(\omega)r^{2-n}u\frac{\partial u}{\partial r}ds - \epsilon^{2-n}\int_{\Omega_\epsilon} u\frac{\partial u}{\partial r}d\Omega_\epsilon \\
&\quad + \frac{n-2}{2}\int_{\Gamma_d} r^{-n}u^2x_i\cos(\vec{n},x_i)ds - \int_{G_\epsilon} r^{2-n}uf\,dx \\
&\quad + \int_{G_\epsilon} r^{2-n}u\big[\big(a^{ij}-\delta_i^j\big)u_{x_ix_j} + a^iu_{x_i} + au\big]dx
\end{aligned} \tag{5.14}$$

(see [4, formulas (13)-(17)]). Repeating verbatim estimations of formulas (26)-(33) of the proof of [4, Theorem 3.1], taking $\delta_1 = 1$, identity (5.14) takes the form

$$\begin{aligned}
&\int_{G_\epsilon} r^{2-n}|\nabla u|^2dx + \int_{\Gamma_\epsilon} \gamma(\omega)r^{1-n}u^2ds + \int_{G_\epsilon} r^{2-n}a_0(x)|u|^{q+1}dx \\
&\le \epsilon^{2-n}\Big|\int_{\Omega_\epsilon} u\frac{\partial u}{\partial r}d\Omega_\epsilon\Big| + \frac{1}{2}\int_{\partial\Omega} \chi\left(\frac{\omega_0}{2},\omega'\right)u^2\left(\epsilon,\frac{\omega_0}{2},\omega'\right)d\omega' \\
&\quad + 2\int_{G_\epsilon} \mathcal{A}(r)\left(r^{4-n}u_{xx}^2 + r^{2-n}|\nabla u|^2 + r^{-n}u^2\right)dx + \frac{\delta}{2}\int_{G_\epsilon} r^{-n}u^2dx \\
&\quad + C(\chi_0,d)\int_{G_d}\left(u_{xx}^2 + |\nabla u|^2 + u^2\right)dx + \frac{1}{2\delta}\|f\|^2_{\mathring{W}^0_{4-n}(G)} \\
&\quad + \frac{c}{\gamma_0}\|g\|^2_{\mathring{W}^{1/2}_{4-n}(\partial G)},
\end{aligned} \tag{5.15}$$

for all $\delta > 0$. By assumption (3.3), function $\mathcal{A}(r)$ is continuous at zero and $\mathcal{A}(0) = 0$. Therefore for all $\delta > 0$ there exists $d > 0$ such that $\mathcal{A}(r) < \delta$ for all $0 < r < d < b$. Hence, by (5.6), (5.7) and (5.9), assuming that $2\epsilon < d$, we obtain for all $\delta > 0$

$$\begin{aligned}
&\int_{G_\epsilon} \mathcal{A}(r)\left(r^{4-n}u_{xx}^2 + r^{2-n}|\nabla u|^2 + r^{-n}u^2\right)dx \\
&= \int_{G_\epsilon^{2\epsilon}} \mathcal{A}(r)\left(r^{4-n}u_{xx}^2 + r^{2-n}|\nabla u|^2 + r^{-n}u^2\right)dx \\
&\quad + \int_{G_{2\epsilon}^d} \mathcal{A}(r)\left(r^{4-n}u_{xx}^2 + r^{2-n}|\nabla u|^2 + r^{-n}u^2\right)dx \\
&\quad + \int_{G_d} \mathcal{A}(r)\Big(r^{4-n}u_{xx}^2 + r^{2-n}|\nabla u|^2 + r^{-n}u^2\Big)dx \\
&= C\mathcal{A}(2\epsilon)\Big(M^2(\epsilon) + \int_{G_{\epsilon/2}^{5\epsilon/2}} r^{\frac{4}{1-q}-n}a_0^{\frac{2}{1-q}}(x)dx + \|f\|^2_{\mathring{W}^0_{4-n}(G_{\epsilon/2}^{5\epsilon/2})} \\
&\quad + \|g\|^2_{\mathring{W}^{1/2}_{4-n}\left(\Gamma_{\epsilon/2}^{5\epsilon/2}\right)}\Big) + \delta\int_{G_{2\epsilon}^d}\left(r^{4-n}u_{xx}^2 + r^{2-n}|\nabla u|^2 + r^{-n}u^2\right)dx \\
&\quad + C_1(d,\operatorname{diam} G)\int_{G_d}\left(u_{xx}^2 + |\nabla u|^2 + u^2\right)dx.
\end{aligned}$$

Substituting $\epsilon = 2^{-k-1}d$ to (5.6), we obtain

$$
\begin{aligned}
&\int_{G^{(k)}} \left(r^{4-n}u_{xx}^2 + r^{2-n}|\nabla u|^2 + r^{-n}u^2\right) dx \\
&\le C_2 \int_{G^{(k-1)}\cup G^{(k)}\cup G^{(k+1)}} \left(r^{-n}u^2 + r^{4-n}f^2 + r^{4-n}h^2\right) dx \\
&\quad + C_3 \inf \int_{G^{(k-1)}\cup G^{(k)}\cup G^{(k+1)}} \left(r^{4-n}|\nabla \mathcal{G}|^2 + r^{2-n}|\mathcal{G}|^2\right) dx,
\end{aligned}
$$

here infimum is taken over all $\mathcal{G} \in \mathring{W}^1_{4-n}(G)$ such that $\mathcal{G}\big|_{\partial G} = g$. Summing these inequalities over $k = 0, 1, \dots, \lfloor \log_2\left(\frac{d}{4\epsilon}\right)\rfloor$, for any $\epsilon \in \left(0, \frac{d}{2}\right)$, we obtain

$$
\begin{aligned}
&\int_{G^d_{2\epsilon}} \left(r^{4-n}u_{xx}^2 + r^{2-n}|\nabla u|^2 + r^{-n}u^2\right) dx \\
&\le C_2 \int_{G^{2d}_{\epsilon}} \left(r^{-n}u^2 + r^{4-n}f^2 + r^{4-n}h^2\right) dx + C_3\|g\|^2_{\mathring{W}^{1/2}_{4-n}(\Gamma^{2d}_{\epsilon})}.
\end{aligned} \tag{5.16}
$$

By (5.15) and (5.16), we have

$$
\begin{aligned}
&\int_{G_\epsilon} r^{2-n}|\nabla u|^2 dx + \int_{\Gamma_\epsilon} r^{1-n}\gamma(\omega)u^2 ds + \int_{G_\epsilon} r^{2-n}a_0(x)|u|^{q+1}dx \\
&\le \epsilon^{2-n}\Big|\int_{\Omega_\epsilon} u\frac{\partial u}{\partial r} d\Omega_\epsilon\Big| + \int_{\partial\Omega} \chi\left(\frac{\omega_0}{2}, \omega'\right) u^2\left(\epsilon, \frac{\omega_0}{2}, \omega'\right) d\omega' \\
&\quad + \delta \int_{G_\epsilon} r^{\frac{4}{1-q}-n} a_0^{\frac{2}{1-q}}(x)dx + \mathcal{A}(2\epsilon)\Big(M^2(\epsilon) + \int_{G^{5\epsilon/2}_{\epsilon/2}} r^{\frac{4}{1-q}-n} a_0^{\frac{2}{1-q}}(x)dx \\
&\quad + \|f\|^2_{\mathring{W}^0_{4-n}\left(G^{5\epsilon/2}_{\epsilon/2}\right)} + \|g\|^2_{\mathring{W}^{1/2}_{4-n}\left(\Gamma^{5\epsilon/2}_{\epsilon/2}\right)}\Big) + \delta\int_{G_\epsilon} r^{-n}u^2 dx \\
&\quad + C(\chi_0, d, \operatorname{diam} G)\int_{G_d} \left(u_{xx}^2 + |\nabla u|^2 + u^2\right) dx \\
&\quad + C_4\Big(\|f\|^2_{\mathring{W}^0_{4-n}(G)} + \|g\|^2_{\mathring{W}^{1/2}_{4-n}(\partial G)}\Big), \quad \forall\ \delta > 0,
\end{aligned} \tag{5.17}
$$

where $C_4 > 0$ does not depend on ϵ. By [4, Lemma 2.5], taking into account that $\chi(\omega) \le \chi_0$ and $0 < \gamma_0 \le \gamma(\omega)$, we find

$$
\begin{aligned}
\int_{G_\epsilon} r^{-n}u^2 dx &\le \frac{1}{\lambda(\lambda+n-2)} \int_{G_\epsilon} r^{2-n}|\nabla u|^2 dx \\
&\quad + \frac{1}{\lambda(\lambda+n-2)}\Big(1 + \frac{\lambda\chi_0}{\gamma_0}\Big) \int_{\Gamma_\epsilon} r^{1-n}\gamma(\omega)u^2 ds \\
&\quad + d^{-n}\int_G u^2 dx.
\end{aligned} \tag{5.18}
$$

We apply L^2-estimate for the solution of problem (3.1) (see [1, Theorem 15.3]). As a result we obtain the estimate

$$\begin{aligned}
&\int_{G_d}\big(u_{xx}^2+|\nabla u|^2+u^2\big)\,dx\\
&\le C_5\int_{G_{d/2}}\big(f^2+h^2+u^2\big)\,dx+C_6\|g\|^2_{W^{1/2}(\Gamma_{d/2})}\\
&\le C_7\Big(\int_{G_{d/2}} r^{\frac{4}{1-q}-n}a_0^{\frac{2}{1-q}}\,dx+|u|^2_{2,G}+\|f\|^2_{\mathring{W}^0_{4-n}(G)}+\|g\|^2_{\mathring{W}^{1/2}_{4-n}(\partial G)}\Big),
\end{aligned}\tag{5.19}$$

where the constant $C_7>0$ depends only on ν, μ, d, G, $\max_{x,y\in G}\mathcal{A}(|x-y|)$, $\|\chi\|_{C^1(\partial G)}$, $\|\gamma\|_{C^1(\partial G)}$. Now, let us choose

$$\delta\le\min\big\{\frac{\gamma_0(\lambda+n-2)}{4(\gamma_0+\lambda\chi_0)},\frac{\lambda(\lambda+n-2)}{2}\big\}.$$

Thus, by (5.17)–(5.19), we obtain

$$\begin{aligned}
&\int_{G_\epsilon} r^{2-n}|\nabla u|^2dx+\int_{\Gamma_\epsilon} r^{1-n}\gamma(\omega)u^2ds+\int_{G_\epsilon} r^{2-n}a_0|u|^{q+1}dx\\
&\le \epsilon^{2-n}\Big|\int_{\Omega_\epsilon} u\frac{\partial u}{\partial r}d\Omega_\epsilon\Big|+\int_{\partial\Omega}\chi\Big(\frac{\omega_0}{2},\omega'\Big)u^2\Big(\epsilon,\frac{\omega_0}{2},\omega'\Big)\,d\omega'\\
&\quad+C_8\int_G r^{\frac{4}{1-q}-n}a_0^{\frac{2}{1-q}}(x)dx+\mathcal{A}(2\epsilon)\Big(M^2(\epsilon)\\
&\quad+\int_{G^{5\epsilon/2}_{\epsilon/2}} r^{\frac{4}{1-q}-n}a_0^{\frac{2}{1-q}}\,dx+\|f\|^2_{\mathring{W}^0_{4-n}(G^{5\epsilon/2}_{\epsilon/2})}+\|g\|^2_{\mathring{W}^{1/2}_{4-n}(\Gamma^{5\epsilon/2}_{\epsilon/2})}\Big)\\
&\quad+C_9\Big(|u|^2_{2,G}+\|f\|^2_{\mathring{W}^0_{4-n}(G)}+\|g\|^2_{\mathring{W}^{1/2}_{4-n}(\partial G)}\Big).
\end{aligned}\tag{5.20}$$

By Lemma 5.1, (5.18), as well as $u\in C^0(\overline{G})$, using the Fatou's lemma, we can pass in (5.20) to the limit $\epsilon\to 0^+$. As a result we obtain

$$\begin{aligned}
&\int_G\big(r^{2-n}|\nabla u|^2+r^{-n}u^2\big)\,dx+\int_{\partial G} r^{1-n}\gamma(\omega)u^2ds+\int_G r^{2-n}a_0|u|^{q+1}dx\\
&\le C_8\int_G r^{\frac{4}{1-q}-n}a_0^{\frac{2}{1-q}}\,dx+C_{10}\Big(|u|^2_{2,G}+\|f\|^2_{\mathring{W}^0_{4-n}(G)}+\|g\|^2_{\mathring{W}^{1/2}_{4-n}(\partial G)}\Big).
\end{aligned}\tag{5.21}$$

Passing to the limit $\epsilon\to 0^+$ in (5.16) and taking into account (5.21), we obtain

$$\begin{aligned}
\int_{G_0^d} r^{4-n}u_{xx}^2dx&\le C_{11}\int_G r^{-n}u^2dx+C_2\int_G r^{\frac{4}{1-q}-n}a_0^{\frac{2}{1-q}}\,dx\\
&\quad+C_2\|f\|^2_{\mathring{W}^0_{4-n}(G)}C_3\|g\|^2_{\mathring{W}^{1/2}_{4-n}(\partial G)}.
\end{aligned}\tag{5.22}$$

Finally, by (5.21) and (5.22), we obtain the required estimation (5.11). $\square$

Theorem 5.3. *Let u be a strong solution of problem* (3.1) *and assumptions* (A1)–(A3) *be satisfied. Then* $(u - u(0)) \in \mathring{W}^2_{4-n}(G)$ *and*

$$\begin{aligned} &\|u-u(0)\|_{\mathring{W}^2_{4-n}(G)} + \Big(\int_G a_0 r^{2-n}|u-u(0)|^{1+q}dx\Big)^{1/2} \\ &+ \Big(\int_{\partial G} r^{1-n}\gamma(\omega)|u-u(0)|^2 ds\Big)^{1/2} \\ &\le C\Big(|u(0)|\cdot\|a\|_{\mathring{W}^0_{4-n}(G)} + |u|_{2,G} + \|a_0\|^{\frac{1}{1-p}}_{V^0_{\frac{2}{1-q},\frac{4}{1-q}-n}(G)} \\ &\quad + \|f\|_{\mathring{W}^0_{4-n}(G)} + \|g\|_{\mathring{W}^{1/2}_{4-n}(\partial G)}\Big) \end{aligned} \tag{5.23}$$

where $C > 0$ depends on ν, μ, b, q, n, $\operatorname{diam} G$, $\|\chi\|_{C^1(\partial\Omega)}$, $\|\gamma\|_{C^1(\partial\Omega)}$ and on modulus of continuity of the leading coefficients.

Proof. Setting $v(x) = u(x) - u(0)$, we have $v(x) \in C^0(\overline{G})$, $v(0) = 0$ and v is a strong solution of the problem

$$\begin{gathered} a^{ij}(x)v_{x_ix_j} + a^i(x)v_{x_i} + a(x)v(x) \\ = h(u) + f(x) - a(x)u(0) \equiv h(u) + f_0(x), \quad x \in G, \\ h(v) = a_0(x)\,(v+u(0))\,|v+u(0)|^{q-1}, \quad q \in (0,1), \\ \frac{\partial v}{\partial \vec{n}} + \chi(\omega)\frac{\partial v}{\partial r} + \frac{1}{|x|}\gamma(\omega)v(x) = g(x) - \frac{1}{|x|}\gamma(\omega)u(0) \equiv g_0(x), \quad x \in \partial G\setminus\mathcal{O}. \end{gathered}$$

Without loss of generality we can suppose that $u(0) \ge 0$. Then $g_0(x) \le g(x)$, since $\gamma(\omega) > 0$. We have that $f_0(x) \in \mathring{W}^0_{4-n}(G)$. Proceeding step by step the arguments of the proof of Theorem 5.2 for the function v we obtain the required estimation (5.23). $\square$

Corollary 5.4. *Let u be a strong solution of problem* (3.1) *and assumptions* (A1)–(A3) *be satisfied. Then $u(0) = 0$.*

Proof. By the Cauchy inequality we have

$$\frac{1}{2}|u(0)|^2 \le |u(x)|^2 + |u(x) - u(0)|^2.$$

Thus

$$\frac{1}{2}|u(0)|^2 \int_{G_0^d} r^{-n}dx \le \int_{G_0^d} r^{-n}|u(x)|^2dx + \int_{G_0^d} r^{-n}|u(x)-u(0)|^2dx. \tag{5.24}$$

By Theorems 5.2 and 5.3 both integrals from the right side are finite, i.e. the right side of (5.24) is finite. On the other hand, because of $\int_{G_0^d} r^{-n}dx \sim \int_0^d \frac{dr}{r} = \infty$, the left side of this inequality is infinite if $u(0) \ne 0$. It leads to a contradiction. Thus, it must be $u(0) = 0$. $\square$

6. Local integral weighted estimates

Theorem 6.1. *Let u be a strong solution of problem* (3.1) *and assumptions* (A1)–(A4) *be satisfied with $\mathcal{A}(r)$ that is Dini-continuous at zero. Then there are $d \in (0, b)$*

and a constant $c > 0$ depending only on ν, μ, b, s, λ, q, γ_0, χ_0, $\operatorname{meas} G$, $\operatorname{diam} G$, $\|\chi\|_{C^1(\partial\Omega)}$, $\|\gamma\|_{C^1(\partial\Omega)}$ *and on the quantity $\int_0^d \frac{\mathcal{A}(\tau)}{\tau} d\tau$, such that for all $\varrho \in (0, d)$,*

$$\begin{aligned} \|u\|_{\mathring{W}^2_{4-n}(G_0^\varrho)} &\le c\Big(\|u\|_{L^2(G)} + \|a_0\|_{V^0_{\frac{2}{1-q},\frac{4}{1-q}-n}(G)} + \|f\|_{\mathring{W}^0_{4-n}(G)} \\ &\quad + \|g\|_{\mathring{W}^{1/2}_{4-n}(\partial G)} + k_s\Big) \begin{cases} \varrho^\lambda & \text{if } s > \lambda, \\ \varrho^\lambda \ln \frac{1}{\varrho} & \text{if } s = \lambda, \\ \varrho^s & \text{if } s < \lambda, \end{cases} \end{aligned} \tag{6.1}$$

where k_s is defined by (4.2).

Proof. By Theorem 5.2 we have that $u \in \mathring{W}^2_{4-n}(G)$. Let us now introduce the function

$$\widetilde{U}(\varrho) = \int_{G_0^\varrho} r^{2-n} |\nabla u|^2 dx + \int_{\Gamma_0^\varrho} r^{1-n} \gamma(\omega) u^2 ds, \quad 0 < \varrho < d < b.$$

Multiplying both sides of (5.12) by $r^{2-n}u(x)$ and integrating it over the domain G_0^ϱ, $0 < \varrho < d$, we obtain

$$\begin{aligned} \int_{G_0^\varrho} r^{2-n} u \Delta u dx &= \int_{G_0^\varrho} r^{2-n} u(h + f) dx \\ &\quad - \int_{G_0^\varrho} r^{2-n} u[(a^{ij} - \delta_i^j) u_{x_i x_j} + a^i u_{x_i} + au] dx. \end{aligned} \tag{6.2}$$

Calculating the integral from the left side by parts, similarly to (5.14), we obtain

$$\begin{aligned} &\int_{G_0^\varrho} r^{2-n} |\nabla u|^2 dx + \int_{\Gamma_0^\varrho} \chi(\omega) r^{2-n} u \frac{\partial u}{\partial r} ds + \int_{\Gamma_0^\varrho} \gamma(\omega) r^{1-n} u^2 ds \\ &+ \int_{G_0^\varrho} r^{2-n} u h \, dx \\ &= \int_\Omega \Big(\varrho u \frac{\partial u}{\partial r} + \frac{n-2}{2} u^2\Big) d\Omega + \int_{\Gamma_0^\varrho} r^{2-n} u g \, ds - \int_{G_0^\varrho} r^{2-n} u f dx \\ &\quad + \int_{G_0^\varrho} r^{2-n} u[(a^{ij} - \delta_i^j) u_{x_i x_j} + a^i u_{x_i} + au] dx. \end{aligned} \tag{6.3}$$

Similarly to (5.16), we have

$$\int_{G_0^\varrho} r^{4-n} u_{xx}^2 dx \le C_1 \int_{G_0^{2\varrho}} \big(r^{-n} u^2 + r^{4-n} h^2 + r^{4-n} f^2\big) dx + C_2 \|g\|^2_{\mathring{W}^{1/2}_{4-n}(\Gamma_0^{2\varrho})}.$$

Repeating verbatim estimations of formulas (52)–(55) of the proof of [4, Theorem 4.1], identity (6.3) together with the above inequality, takes the form

$$\begin{aligned} &[1 - (\mathcal{A}(\varrho) + \delta)]\widetilde{U}(\varrho) + \int_{G_0^\varrho} r^{2-n} a_0 |u|^{1+q} dx \\ &\le \frac{\varrho}{2\lambda} \widetilde{U}'(\varrho) + \mathcal{A}(\varrho) \widetilde{U}(2\varrho) + c_1 \delta^{-1} \Big(\|f\|^2_{\mathring{W}^0_{4-n}(G_0^{2\varrho})} \\ &\quad + \|a_0\|^{\frac{2}{1-q}}_{V^0_{\frac{2}{1-q},\frac{4}{1-q}-n}(G_0^{2\varrho})} + \|g\|^2_{\mathring{W}^{1/2}_{4-n}(\Gamma_0^{2\varrho})}\Big), \quad \forall \delta > 0, \end{aligned}$$

where constant $c_1 > 0$ depends on γ_0, χ_0, λ. We used inequalities (5.9) for $\delta = 1$ and (5.16) for $\epsilon = 0$. Now, using assumption (A4), we finally obtain

$$[1 - (\mathcal{A}(\varrho) + \delta)]\widetilde{U}(\varrho) \leq \frac{\varrho}{2\lambda}\widetilde{U}'(\varrho) + \mathcal{A}(\varrho)\widetilde{U}(2\varrho) + c_2 k_s^2 \delta^{-1} \varrho^{2s}, \quad \forall \delta > 0. \tag{6.4}$$

Moreover, by Theorem 5.2, we have the initial condition

$$\begin{aligned} \widetilde{U}(d) \leq C\Big(&\|u\|^2_{L^2(G)} + \|f\|^2_{\mathring{W}^0_{4-n}(G)} + \|a_0\|^{\frac{2}{1-q}}_{V^0_{\frac{2}{1-q},\frac{4}{1-q}-n}(G)} \\ &+ \|g\|^2_{\mathring{W}^{1/2}_{4-n}(\partial G)}\Big) \equiv U_0. \end{aligned} \tag{6.5}$$

The differential inequality (6.4) with the initial condition (6.5) is the Cauchy problem of [8, Theorem 1.57] and it is the same type as (57) and (58) in [4]. Repeating verbatim investigations for $s > \lambda$, $s = \lambda$ and $s < \lambda$ in the proof of [4, Theorem 4.1], we obtain

$$\widetilde{U}(\varrho) \leq c\left(U_0 + k_s^2\right) \begin{cases} \varrho^{2\lambda} & \text{if } s > \lambda, \\ \varrho^{2\lambda} \ln \frac{1}{\varrho} & \text{if } s = \lambda, \\ \varrho^{2s} & \text{if } s < \lambda, \end{cases} \tag{6.6}$$

here constant $c > 0$ depends only on λ, d, s and on $\int_0^d \frac{\mathcal{A}(\varsigma)}{\varsigma} d\varsigma$. Finally, taking into account (5.9), (5.16), (5.18), (6.5) and (6.6), we obtain the required estimate (6.1). ☐

Theorem 6.2. *Let u be a strong solution of problem* (3.1) *and assumptions* (A1)–(A4) *be satisfied with function $\mathcal{A}(r)$ continuous at zero, but* not Dini *continuous. Then for all $\epsilon > 0$ there are $d \in (0, b)$ and a constant $c_\epsilon > 0$ depending only on ν, μ, b, s, λ, q, ϵ, γ_0, χ_0,* $\operatorname{meas} G$, $\operatorname{diam} G$, *$\|\chi\|_{C^1(\partial\Omega)}$, $\|\gamma\|_{C^1(\partial\Omega)}$, such that for all $\varrho \in (0, d)$*

$$\begin{aligned} \|u\|_{\mathring{W}^2_{4-n}(G_0^\varrho)} \leq c\Big(&\|u\|_{L^2(G)} + \|a_0\|_{V^0_{\frac{2}{1-q},\frac{4}{1-q}-n}(G)} + \|f\|_{\mathring{W}^0_{4-n}(G)} \\ &+ \|g\|_{\mathring{W}^{1/2}_{4-n}(\partial G)} + k_s\Big) \begin{cases} \varrho^{\lambda-\epsilon} & \text{if } s > \lambda, \\ \varrho^{s-\epsilon} & \text{if } s \leq \lambda, \end{cases} \end{aligned} \tag{6.7}$$

where k_s is defined by (4.2).

Proof. Similarly to the proof of Theorem 6.1, we obtain the Cauchy problem (6.4)-(6.5). Repeating verbatim the proof of [4, Theorem 4.2] and taking into account Theorem 5.2 we obtain estimation (6.7). ☐

Theorem 6.3. *Let u be a strong solution of problem* (3.1) *and assumptions* (A1)–(A4) *be satisfied with function $\mathcal{A}(r) \sim \frac{1}{\ln \frac{1}{r}}$. Then there are $d \in (0, b)$ and constants $c > 0$ and $c_s > 0$ depending only on ν, μ, b, s, λ, q, ϵ, γ_0, χ_0,* $\operatorname{meas} G$, $\operatorname{diam} G$, *$\|\chi\|_{C^1(\partial\Omega)}$, $\|\gamma\|_{C^1(\partial\Omega)}$, such that for all $\varrho \in (0, d)$,*

$$\begin{aligned} \|u\|_{\mathring{W}^2_{4-n}(G_0^\varrho)} \leq c\Big(&\|u\|_{L^2(G)} + \|a_0\|_{V^0_{\frac{2}{1-q},\frac{4}{1-q}-n}(G)} + \|f\|_{\mathring{W}^0_{4-n}(G)} \\ &+ \|g\|_{\mathring{W}^{1/2}_{4-n}(\partial G)} + k_s\Big) \ln^{c_s(\lambda)}\left(\frac{1}{\varrho}\right) \begin{cases} \varrho^{\lambda} & \text{if } s > \lambda, \\ \varrho^{s} & \text{if } s \leq \lambda, \end{cases} \end{aligned} \tag{6.8}$$

where k_s is defined by (4.2).

Proof. Similarly to the proof of Theorem 6.1, we obtain the Cauchy problem (6.4)–(6.5). Repeating verbatim the proof of [4, Theorem 4.3] and taking into account Theorem 5.2 we obtain estimation (6.8). □

7. Power modulus of continuity

Theorem 7.1. *Let $u \in W^{2,n}(G)$ be a strong solution of* (3.1)*. Suppose that assumptions* (A1)–(A5) *are satisfied. Then there is a positive constant c, such that*

$$\|u\|_{V^2_{n,0}(G)} \le c\Big(\|f\|_{L^n(G)} + \|a_0\|^{\frac{1}{1-q}}_{V^0_{\frac{n}{1-q},\frac{2qn}{1-q}}(G)} + \|g\|_{V^{1-\frac{1}{n}}_{n,0}(\partial G)} + \|u\|_{L^n(G')}\Big),$$

for an arbitrary nonempty open set $G' \subset\subset G$.

Proof. By [21, Theorem 1.4.1], for a solution $u \in V^2_{n,0}(G)$ of the problem

$$\begin{gathered}
a^{ij}(x)u_{x_ix_j} + a^i(x)u_{x_i} + a(x)u(x) = F(x), \quad x \in G,\\
\frac{\partial u}{\partial \vec{n}} + \chi(\omega)\frac{\partial u}{\partial r} + \frac{1}{|x|}\gamma(\omega)u(x) = g(x), \quad x \in \partial G\backslash\mathcal{O},
\end{gathered}$$

the estimate

$$\|u\|_{V^2_{n,0}(G)} \le c_0\Big(\|F\|_{L^n(G)} + \|g\|_{V^{1-\frac{1}{n}}_{n,0}(\partial G)} + \|u\|_{L^n(G')}\Big)$$

holds for any nonempty open set $G' \subset\subset G$, provided that $\lambda > 1$ and $F \in L^n(G)$, where constant c_0 depends only on ν, μ, n, χ_0, γ_0, $\max_{x\in\overline{G}} \mathcal{A}(|x|)$, $\|a^i\|_{p,G}$, $\|a\|_{\frac{p}{2},G}$, $p > n$, and the domain G. Thus, using the Jensen inequality with $F(x) = f(x) + a_0(x)u|u|^{q-1}$, we obtain

$$\begin{aligned}
&\int_G \big(r^{-2n}|u|^n + r^{-n}|\nabla u|^n + |u_{xx}|^n\big)\,dx\\
&\le C\Big\{\int_G \big(|a_0|^n|u|^{qn} + |f|^n\big)\,dx + \int_{G'}|u|^n dx + \|g\|^n_{V^{1-\frac{1}{n}}_{n,0}(\partial G)}\Big\}.
\end{aligned}\tag{7.1}$$

Using the Young inequality and taking into account $q \in (0,1)$, we deduce

$$|a_0|^n|u|^{qn} = \big(r^{-2qn}|u|^{qn}\big)\big(r^{2qn}|a_0(x)|^n\big) \le \epsilon^{\frac{qn}{q-1}} r^{\frac{2qn}{1-q}}|a_0|^{\frac{n}{1-q}} + \epsilon^n r^{-2n}|u|^n, \tag{7.2}$$

for all $\epsilon > 0$. Choosing $\epsilon = 2^{-n}$, from (7.1) and (7.2), we obtain the required estimation. □

Remark 7.2. Choosing in Theorem 7.1 the domain G' such that $(\operatorname{diam} G')^{2n} < 1/2$, we have

$$\int_{G'}|u|^n dx \le \int_{G'} r^{2n}r^{-2n}|u|^n dx \le (\operatorname{diam} G')^{2n}\int_{G'} r^{-2n}|u|^n dx \le \frac{1}{2}\int_{G'} r^{-2n}|u|^n dx.$$

Thus, formula (7.1) takes the form

$$\begin{aligned}
&\int_G \big(r^{-2n}|u|^n + r^{-n}|\nabla u|^n + |u_{xx}|^n\big)\,dx\\
&\le C_1\Big\{\int_G \big(|a_0|^n|u|^{qn} + |f|^n\big)\,dx + \|g\|^n_{V^{1-\frac{1}{n}}_{n,0}(\partial G)}\Big\}.
\end{aligned}$$

and the statement of Theorem 7.1 takes the form

$$\|u\|_{V^2_{n,0}(G)} \le c_1\Big(\|f\|_{L^n(G)} + \|a_0\|^{\frac{1}{1-q}}_{V^0_{\frac{n}{1-q},\frac{2qn}{1-q}}(G)} + \|g\|_{V^{1-\frac{1}{n}}_{n,0}(\partial G)}\Big).$$

Now we will prove the main results.

Proof of Theorem 4.1. We consider two sets $G^{2\varrho}_{\varrho/4}$ and $G^{\varrho}_{\varrho/2} \subset G^{2\varrho}_{\varrho/4}$, $\varrho > 0$. We make the transformation: $x = \varrho x'$, $u(\varrho x') = \psi(\varrho)v(x')$, where function $\psi(\varrho)$ is defined by (4.4). The function $v(x')$ satisfies the problem

$$
\begin{aligned}
&a^{ij}(\varrho x')v_{x'_ix'_j} + \varrho a^i(\varrho x')v_{x'_i} + \varrho^2 a(\varrho x')v \\
&= \frac{\varrho^2}{\psi(\varrho)} f(\varrho x') + a_0(\varrho x')\varrho^2\psi^{q-1}(\varrho)v|v|^{q-1}, \quad x' \in G^2_{1/4}, \\
&\frac{\partial v}{\partial \vec{n'}} + \chi(\omega)\frac{\partial v}{\partial r'} + \frac{1}{|x'|}\gamma(\omega)v = \frac{\varrho}{\psi(\varrho)} g(\varrho x'), \quad x' \in \Gamma^2_{1/4}.
\end{aligned}
$$

Based on the local maximum principle (see [24, Theorem 3.3], [26, Theorem 4.3] and [25, Corollary 7.34]) we conclude that

$$
\begin{aligned}
&\sup_{x' \in G^1_{1/2}} |v(x')| \\
&\le C\Big[\Big(\int_{G^2_{1/4}} v^2 dx'\Big)^{1/2} + \frac{\varrho^2}{\psi(\varrho)}\|f\|_{L^n(G^2_{1/4})} \\
&\quad + \varrho^2\psi^{q-1}(\varrho)\Big(\int_{G^2_{1/4}} |a_0(\varrho x')|^n|v|^{qn}dx'\Big)^{1/n} + \frac{\varrho}{\psi(\varrho)}\sup_{x' \in G^2_{1/4}} |g(\varrho x')|\Big],
\end{aligned}
\tag{7.3}
$$

where constant $C > 0$ depends only on ν, μ, b, γ_0, χ_0, n, G, ω_0, $\max_{\omega\in\partial\Omega}\gamma(\omega)$, g_1, $\|a\|_{L^n(G^4_{1/4})}$, $\|(\sum_{i=1}^n |a^i|^2)^{1/2}\|_{L^p(G^2_{1/2})}$, $p > n$. Now, we use the Young inequality. Taking into account that $q \in (0,1)$, we deduce

$$
\begin{aligned}
|a_0(\varrho x')|^n|v|^{qn} &= \big(|x'|^{-2qn}|v|^{qn}\big)\big(|x'|^{2qn}|a_0(\varrho x')|^n\big) \\
&\le \epsilon^{\frac{qn}{q-1}}|x'|^{\frac{2qn}{1-q}}|a_0(\varrho x')|^{\frac{n}{1-q}} + \epsilon^n|x'|^{-2n}|v|^n, \quad \forall \epsilon > 0.
\end{aligned}
\tag{7.4}
$$

By (7.3) and (7.4), we obtain

$$
\begin{aligned}
&\sup_{x' \in G^1_{1/2}} |v(x')| \\
&\le C\Big(\int_{G^2_{1/4}} v^2 dx'\Big)^{1/2} + C\frac{\varrho^2}{\psi(\varrho)}\|f\|_{L^n\left(G^2_{1/4}\right)} + C\frac{\varrho}{\psi(\varrho)}\sup_{x' \in G^2_{1/4}} |g(\varrho x')| \\
&\quad + C\varrho^2\psi^{q-1}(\varrho)\Big[\epsilon\Big(\int_{G^2_{1/4}} |x'|^{-2n}|v|^n dx'\Big)^{1/n} \\
&\quad + \epsilon^{\frac{q}{q-1}}\Big(\int_{G^2_{1/4}} |x'|^{\frac{2qn}{1-q}}|a_0(\varrho x')|^{\frac{n}{1-q}}dx'\Big)^{1/n}\Big],
\end{aligned}
\tag{7.5}
$$

for all $\epsilon > 0$. Returning to the variable x and to the function u, we have

$$\begin{aligned} &\Big(\int_{G^2_{1/4}} v^2 dx'\Big)^{1/2} \\ &= \Big(\frac{1}{\psi^2(\varrho)}\int_{G^2_{1/4}} u^2(\varrho x')dx'\Big)^{1/2} \le \frac{2^{\frac{n}{2}}}{\psi(\varrho)}\Big(\int_{G^{2\varrho}_{\varrho/4}} r^{-n}u^2(x)dx\Big)^{1/2} \\ &\le c\Big(\|u\|_{L^2(G)} + \|a_0\|_{V^0_{\frac{2}{1-q},\frac{4}{1-q}-n}(G)} + \|f\|_{\mathring{W}^0_{4-n}(G)} \\ &\quad + \|g\|_{\mathring{W}^{1/2}_{4-n}(\partial G)} + k_s\Big) = \text{const.}, \end{aligned} \tag{7.6}$$

by the definition of the function $\psi(\varrho)$ and Theorem 6.1. By assumption (A4),

$$\begin{aligned} \frac{\varrho^2}{\psi(\varrho)}\|f\|_{L^n\left(G^2_{1/4}\right)} &= \frac{\varrho^2}{\psi(\varrho)}\Big(\int_{G^2_{1/4}} |f(\varrho x')|^n dx'\Big)^{1/n} \le \frac{\varrho}{\psi(\varrho)}\Big(\int_{G^{2\varrho}_{\varrho/4}} |f(x)|^n dx\Big)^{1/n} \\ &\le f_1\frac{\varrho}{\psi(\varrho)}\Big(\int_{\varrho/4}^{2\varrho} r^{n(s-2)}r^{n-1}dr\,\text{meas}\,\Omega\Big)^{1/n} \\ &\le c_1 f_1\frac{\varrho^s}{\psi(\varrho)} = c_1 f_1\begin{cases} \varrho^{s-\lambda} < 1 & \text{if } s > \lambda, \\ \frac{1}{\ln\frac{1}{\varrho}} < 1 & \text{if } s = \lambda, \\ 1 & \text{if } s < \lambda. \end{cases} \end{aligned}$$

Hence

$$\frac{\varrho^2}{\psi(\varrho)}\|f\|_{L^n\left(G^2_{1/4}\right)} \le c_1 f_1 = \text{const.} \tag{7.7}$$

Similarly

$$\frac{\varrho}{\psi(\varrho)}\sup_{x'\in G^2_{1/4}} |g(\varrho x')| \le \frac{\varrho}{\psi(\varrho)} g_1\varrho^{s-1} = g_1\begin{cases} \varrho^{s-\lambda} < 1 & \text{if } s > \lambda, \\ \frac{1}{\ln\frac{1}{\varrho}} < 1 & \text{if } s = \lambda, \\ 1 & \text{if } s < \lambda. \end{cases}$$

Thus

$$\frac{\varrho}{\psi(\varrho)}\sup_{x'\in G^2_{1/4}} |g(\varrho x')| \le g_1 = \text{const.} \tag{7.8}$$

We calculate

$$\begin{aligned} \Big(\int_{G^2_{1/4}} |x'|^{-2n}|v|^n dx'\Big)^{1/n} &= \varrho\Big(\int_{G^{2\varrho}_{\varrho/4}} r^{-2n}|v|^n dx\Big)^{1/n} \\ &= \frac{\varrho}{\psi(\varrho)}\Big(\int_{G^{2\varrho}_{\varrho/4}} r^{-2n}|u|^n dx\Big)^{1/n} \end{aligned} \tag{7.9}$$

and

$$\Big(\int_{G^2_{1/4}} |x'|^{\frac{2qn}{1-q}}|a_0(\varrho x')|^{\frac{n}{1-q}}dx'\Big)^{1/n} \le c(q,n)\varrho^{-1}\Big(\int_{G^{2\varrho}_{\varrho/4}} |a_0|^{\frac{n}{1-q}}dx\Big)^{1/n}. \tag{7.10}$$

Choosing $\epsilon = \psi(\varrho)/\varrho$ in (7.5), because of (7.1) and (7.2), by (7.9) and (7.10), we obtain

$$\begin{aligned} &\epsilon\Big(\int_{G^2_{1/4}} |x'|^{-2n}|v|^n dx'\Big)^{1/n} \\ &\le c(n,q,b)\Big[\int_G \Big(|a_0(x)|^{\frac{n}{1-q}} + |f(x)|^n\Big)\,dx\Big]^{1/n} + c(n,q,b)\|g\|^n_{V^{1-\frac{1}{n}}_{n,0}(\Gamma)} \\ &\le c(n,q,b,s,f_1,k_0) = \text{const.} \end{aligned} \tag{7.11}$$

and

$$\begin{aligned} &\epsilon^{\frac{q}{q-1}}\Big(\int_{G^2_{1/4}} |x'|^{\frac{2qn}{1-q}}|a_0(\varrho x')|^{\frac{n}{1-q}}\,dx'\Big)^{1/n} \\ &\le c_2\Big(\frac{\varrho}{\psi(\varrho)}\Big)^{\frac{q}{1-q}}\frac{1}{\varrho}\Big(\int_G |a_0|^{\frac{n}{1-q}}\,dx\Big)^{1/n} \le c_2k_0 = \text{const.}, \end{aligned} \tag{7.12}$$

by (4.3) and assumption (A4). From (7.6)–(7.8), (7.11) and (7.12), with regard to (7.5), we have

$$\sup_{x'\in G^1_{1/2}} |v(x')| \le c_3(1+\varrho^2\psi^{q-1}(\varrho)). \tag{7.13}$$

We need to show that for all $\varrho > 0$,

$$\varrho^2\psi^{q-1}(\varrho) < \infty. \tag{7.14}$$

Let us assume $0 < q < 1 - \frac{2}{\lambda}$ and $\lambda > s$. In this case we have that if $s \le \frac{2}{1-q}$, then

$$\varrho^2\psi^{q-1}(\varrho) = \varrho^{s(q-1)+2} < \infty$$

holds for all $\varrho > 0$. Choosing the best exponent $s = \frac{2}{1-q} < \lambda$, we obtain the required estimation (4.5). In fact, by (7.13) and (7.14), it follows

$$|v(x')| \le M_0' = \text{const.}$$

for all $x' \in G^1_{1/2}$. Returning to the variable x, we obtain

$$|u(x)| \le M_0'\psi(\varrho) = M_0'\varrho^{\frac{2}{1-q}}$$

for all $x \in G^{\varrho}_{\varrho/2}$, $0 < \varrho < b$. Setting $|x| = 2\varrho/3$, we obtain (4.5).

Let us assume that $1 - \frac{2}{\lambda} \le q \le 1$. Thus for all $\varrho > 0$

$$\varrho^2\psi^{q-1}(\varrho) = \begin{cases} \varrho^{2+\lambda(q-1)} < \infty & \text{if } s > \lambda, \\ \varrho^{2+\lambda(q-1)}\ln^{\frac{3}{2}(q-1)}\frac{1}{\varrho} < \infty & \text{if } s = \lambda, \\ \varrho^{2+s(q-1)} \le \varrho^{2q}\varrho^{(\lambda-2)(q-1)} < \infty & \text{if } s < \lambda. \end{cases}$$

Repeating verbatim the proof of (4.5), we obtain the estimate (4.6). The proof is complete. □

Proof of Theorem 4.2. Repeating verbatim the proof of Theorem 4.1, taking into account (4.7) and applying Theorem 6.2, we obtain the desired result. □

Proof of Theorem 4.3. Repeating verbatim the proof of Theorem 4.1, taking into account (4.8) and applying Theorem 6.3, we obtain the desired result. □

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