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STABLE SOLUTIONS TO WEIGHTED QUASILINEAR PROBLEMS OF LANE-EMDEN TYPE

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ABSTRACT. We prove that all entire stable $W^{1,p}_{\mathrm{loc}}$ solutions of weighted quasilinear problem

$$-\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) = f(x)|u|^{q-1}u$$

must be zero. The result holds true for $p \ge 2$ and $p-1 < q < q_c(p, N, a, b)$. Here $b > a-p$ and $q_c(p, N, a, b)$ is a new critical exponent, which is infinity in low dimension and is always larger than the classic critical one, while $w, f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative functions such that $w(x) \le C_1|x|^a$ and $f(x) \ge C_2|x|^b$ for large $|x|$. We also construct an example to show the sharpness of our result.

1. Introduction and statement of main results

In this article we assume that $q > p-1 \ge 1$ and $w, f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative functions. Let us consider the following weighted quasilinear equation

$$-\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) = f(x)|u|^{q-1}u \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

If $w \equiv 1$, the left hand side of (1.1) becomes the well-known p-Laplace operator. The terms $w(x)$ and $f(x)$ are usually regarded as weights while $|u|^{q-1}u$ is the so-called Lane-Emden nonlinearity. Because of the degenerate nature of the term $|\nabla u|^{p-2}$ when $p > 2$, solutions to (1.1) must be understood in the weak sense. Moreover, solutions to elliptic equations with Hardy potentials may possess singularities (see Proposition 1.8 for an example). Therefore, it is natural to study weak solutions of (1.1) in a suitable weighted Sobolev space. For this purpose, let us define

$$\|\varphi\|_w = \Big(\int_{\mathbb{R}^N} w(x)|\nabla\varphi|^p\,dx\Big)^{1/p}$$

for $\varphi \in C_c^\infty(\mathbb{R}^N)$ and denote by $W_0^{1,p}(\mathbb{R}^N, w)$ the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the $\|\cdot\|_w$-norm. Remark that for $w \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ we have $C_c^1(\mathbb{R}^N) \subset W_0^{1,p}(\mathbb{R}^N, w)$ and $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N, w)$ means that if for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, there holds $u\varphi \in W_0^{1,p}(\mathbb{R}^N, w)$. Let us make also the meaning of weak solution and stable solution more precisely.

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Definition 1.1. A function $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N, w)$ is said to be a *weak solution* of (1.1) if $f(x)|u|^q \in L^1_{\rm loc}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}(\nabla u, \nabla\varphi)\, dx = \int_{\mathbb{R}^N} f(x)|u|^{q-1}u\varphi\, dx \tag{1.2}$$

for all $\varphi \in C^1_c(\mathbb{R}^N)$.

Definition 1.2. A weak solution u of (1.1) is *stable* if

$$\int_{\mathbb{R}^N} w(x)\left[|\nabla u|^{p-2}|\nabla\varphi|^2 + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla\varphi)^2\right] dx \ge q\int_{\mathbb{R}^N} f(x)|u|^{q-1}\varphi^2\, dx \tag{1.3}$$

for all $\varphi \in C^1_c(\mathbb{R}^N)$.

We recall that the stability condition translates into the fact that the second variation at u of the energy functional

$$E(u) = \int_{\mathbb{R}^N}\left(\frac{w(x)|\nabla u|^p}{p} - \frac{f(x)|u|^{q+1}}{q+1}\right) dx$$

is nonnegative. Therefore all the local minima of the functional are stable weak solutions of (1.1).

Proposition 1.3. *If u is a stable solution of (1.1), then*

$$(p-1)\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}|\nabla\varphi|^2\, dx \ge q\int_{\mathbb{R}^N} f(x)|u|^{q-1}\varphi^2\, dx \tag{1.4}$$

for every $\varphi \in C^1_c(\mathbb{R}^N)$.

We remark that (1.3) and (1.4) hold for any $\varphi \in W^{1,p}_0(\mathbb{R}^N, w)$ by density arguments.

In this article we prove a Liouville type theorem for stable solutions of (1.1). We recall that Liouville type theorems concern about the nonexistence of nontrivial solution in the entire Euclidean space $\mathbb{R}^N$. This type of theorems for (1.1) has drawn much attention in the last four decades. Let us mention the pioneering article [19], where Gidas and Spruck established the optimal nonexistence result for positive solutions to the equation $-\Delta u = |u|^{q-1}u \quad \text{in } \mathbb{R}^N$. They proved that this equation has no positive solution if and only if q is less than the critical exponent $\frac{N+2}{N-2}$, which is ∞ if $N = 2$.

In recent years, not only weak and positive solutions but also other types of solutions to equation (1.1) such as stable solutions have been studied immensely by several authors. Readers can find physical motivation and recent development on the topic of stable solutions in monograph [10] by Dupaigne and references therein.

We should refer to the works [13, 14] by Farina for Lane-Emden equation

$$-\Delta u = |u|^{q-1}u \quad \text{in } \mathbb{R}^N,$$

where he proved that all stable C^2 solutions must be zero if $1 < q < q_c(N)$, where $q_c(N)$ is explicitly given and is always greater than the classic critical exponent $\frac{N+2}{N-2}$. Later, similar results were proved in [8] for stable C^1 solutions of quasilinear equation $-\Delta_p u = |u|^{q-1}u$.

The weighted semilinear elliptic equation

$$-\operatorname{div}(w(x)\nabla u) = f(x)|u|^{q-1}u \quad \text{in } \mathbb{R}^N$$

was also studied recently by some authors. In [7], several Liouville type theorems for classical stable solutions of this equation were established under different assumptions on w and f. Paper [26] deals with more specific equation $-\Delta u = |x|^b|u|^{q-1}u$ but for stable solutions of class $H^1_{\rm loc}$, which covers solutions having singularities. Related works on existence, nonexistence and bifurcation results for singular elliptic problems can be found in [1, 12, 15, 16, 17, 18, 20, 23, 24] and references therein.

For other types of nonlinearities, we refer to paper [11] for stable C^2 solutions of semilinear equation $-\Delta u = f(u)$ and papers [4, 22, 21, 25] for stable C^1 solutions of quasilinear equation $-\Delta_p u = f(u)$. In general, Liouville type theorems for stable solutions of nonlinear elliptic equations are usually guaranteed in low dimensional case.

The main purpose of this paper is to obtain a sharp Liouville type theorem for stable solutions of class $W^{1,p}_{\rm loc}$ to equation (1.1). Our result therefore directly extends the result in [6], which deals with equation

$$-\Delta_p u = f(x)|u|^{q-1}u \quad \text{in } \mathbb{R}^N.$$

It should be noted that in [6], the author only considered the case $p < N$ and $C^{1,\delta}_{\rm loc}(\mathbb{R}^N)$ solutions, which are locally bounded. This $C^{1,\delta}_{\rm loc}(\mathbb{R}^N)$ regularity assumption is natural when $w \equiv f \equiv 1$. However, if the weights w and f are Hardy potentials, then solutions of equation (1.1) may have singularities and do not belong to class $C^{1,\delta}_{\rm loc}(\mathbb{R}^N)$ anymore. Therefore, weak solutions of class $W^{1,p}_{\rm loc}$ are more suitable settings for (1.1) and we will work with this type of solutions in this paper. Furthermore, we also construct an example to show the sharpness of our result.

We begin with the following a priori estimate for stable solutions of (1.1).

Proposition 1.4. *Suppose that $q > p-1$ and u is a stable solution of (1.1). Then for any*

$$\alpha \in \Big(1, \frac{2q-p+1+2\sqrt{q(q-p+1)}}{p-1}\Big),$$

there exists a constant $C = C(p,q,\alpha) > 0$ such that for any function $\eta \in C^1_c(\mathbb{R}^N)$ with $0 \le \eta \le 1$ and $\nabla\eta = 0$ in a neighborhood of $\{x \in \mathbb{R}^N : f(x) = 0\}$ we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(w(x)|\nabla u|^p|u|^{\alpha-1} + f(x)|u|^{\alpha+q}\big)\,\eta^{\frac{p(\alpha+q)}{q-p+1}}\,dx \\ &\le C\int_{\mathbb{R}^N} w(x)^{\frac{\alpha+q}{q-p+1}} f(x)^{-\frac{\alpha+p-1}{q-p+1}} |\nabla\eta|^{\frac{p(\alpha+q)}{q-p+1}}\,dx. \end{aligned} \tag{1.5}$$

With the help of Proposition 1.4, it is not hard to obtain our main result.

Theorem 1.5. *Let $b > a-p$ and $C_1, C_2, R_0 > 0$. Suppose that $w(x) \le C_1|x|^a$ and $f(x) \ge C_2|x|^b$ for a.e. $x \in \mathbb{R}^N \setminus B(0,R_0)$, in addition, $w(x)+f(x) > 0$ for a.e. $x \in B(0,R_0)$. Let u be a stable solution of equation (1.1). Assume that*

$$p-1 < q < q_c(p,N,a,b), \quad \text{if } N > \frac{(p-a)(p+3)+4b}{p-1},$$
$$p-1 < q < \infty, \quad \text{if } N \le \frac{(p-a)(p+3)+4b}{p-1}$$

with the critical exponent

$$q_c(p,N,a,b) = \frac{2(p-a+b)\sqrt{(p-1)(p-a+b)(Np+bp+a-b-p)}}{(N+a-p)[(p-1)N-(p-a)(p+3)-4b]}$$

$$
\begin{aligned}
&+\Big(p-1)[N^2(p-1)-p(N(p+2)-p)+a(N(2a+p)-2p+a)\\
&+b(N(p-4)-p^2+pa-2b)]\Big)\\
&\div\Big((N+a-p)[(p-1)N-(p-a)(p+3)-4b]\Big).
\end{aligned}
$$

Then $u\equiv 0$.

Remark 1.6. Since $b>a-p$, if $N>\frac{(p-a)(p+3)+4b}{p-1}$, we deduce that $a-p>-N$ and $q_c(p,N,a,b)$ is well-defined. The assumption on q in Theorem 1.5 is equivalent to

$$
N<\frac{(p-a+b)(2q-p+1+2\sqrt{q(q-p+1)})+q(p-a)(p-1)+b(p-1)^2}{(p-1)(q-p+1)}.
$$

Indeed, the critical exponent $q_c(p,N,a,b)$ is explicitly computed by solving the above quadratic inequation in q.

Remark 1.7. If $a=0$, then

$$
\begin{aligned}
q_c(p,N,0,b)=&\frac{2(p+b)\sqrt{(p-1)(p+b)(Np+bp-b-p)}}{(N-p)[(p-1)N-p(p+3)-4b]}\\
&+\frac{(p-1)[N^2(p-1)-p(N(p+2)-p)+b(N(p-4)-p^2-2b)]}{(N-p)[(p-1)N-p(p+3)-4b]},
\end{aligned}
$$

which is the critical exponent q_c in [6]. Furthermore, if $a=b=0$, then we obtain

$$
q_c(p,N,0,0)=\frac{2p^2\sqrt{(p-1)(N-1)}+(p-1)[N^2(p-1)-p(N(p+2)-p)]}{(N-p)[(p-1)N-p(p+3)]},
$$

which equals the critical exponent p_c in [8]. We observe that the critical exponent $q_c(p,N,0,0)$ is always greater than the classic critical exponent $\frac{N(p-1)+p}{N-p}$. If $a=0$ and $p=2$, we find

$$
q_c(2,N,0,b)=\frac{2(b+2)\sqrt{(b+2)(b+2N-2)}+(N-2)^2-2(b+2)(b+N)}{(N-2)(N-4b-10)},
$$

which is the critical exponent $\overline{p}(b)$ in [9]. Finally, if $a=b=0$ and $p=2$, we have

$$
q_c(2,N,0,0)=\frac{8\sqrt{N-1}+N^2-8N+4}{(N-2)(N-10)}.
$$

It is the critical exponent p_c in [14]. Therefore, our conclusion in Theorem 1.5 extends results in [6, 8, 9, 14] to stable solutions of class $W^{1,p}_{\rm loc}$.

The assumption on q in Theorem 1.5 is optimal. Indeed, let us consider the limit problem

$$
-\operatorname{div}(|x|^a|\nabla u|^{p-2}\nabla u)=|x|^b|u|^{q-1}u\quad\text{in } \mathbb{R}^N. \tag{1.6}
$$

We have the following result.

Proposition 1.8. *Let* $b>a-p$. *Suppose that* $N>\frac{(p-a)(p+3)+4b}{p-1}$ *and* $q\geq q_c(p,N,a,b)$, *which is defined in Theorem 1.5, then* $U(x)=m/|x|^n$ *is a stable solution of equation* (1.6). *Here,*

$$
n=\frac{p-a+b}{q-p+1}\quad\text{and}\quad m=[n^{p-1}(N+a-1-(n+1)(p-1))]^{1/(q-p+1)}.
$$

2. Proofs

This section is devoted to the proofs of Proposition 1.4, Theorem 1.5 and Proposition 1.8. For convenience, we always denote by C a generic constant whose concrete values may change from line to line or even in the same line. If this constant depends on an arbitrary small number ε, then we may denote it by C_ε. We also use Young inequality in the form $ab \le \varepsilon a^p + C_\varepsilon b^q$ for $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Proof of Proposition 1.4. For each $k \in \mathbb{N}$ we define

$$a_k(t) = \begin{cases} |t|^{\frac{\alpha-1}{2}} t, & |t| < k, \\ k^{\frac{\alpha-1}{2}} t, & |t| \ge k, \end{cases} \quad \text{and} \quad b_k(t) = \begin{cases} |t|^{\alpha-1} t, & |t| < k, \\ k^{\alpha-1} t, & |t| \ge k. \end{cases}$$

It is easy to check that

$$\begin{gathered} a_k(t)^2 \ge t b_k(t), \quad a_k'(t)^2 \le \frac{(\alpha+1)^2}{4\alpha} b_k'(t), \\ |a_k(t)|^p a_k'(t)^{2-p} + |b_k(t)|^p b_k'(t)^{1-p} \le C|t|^{\alpha+p-1} \end{gathered} \tag{2.1}$$

for all $t \in \mathbb{R}$, where C depends only on p and α. Moreover, since $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N, w)$, clearly $a_k(u), b_k(u) \in W^{1,p}_{\rm loc}(\mathbb{R}^N, w)$ for any $k \in \mathbb{N}$. We split the proof into four steps.

Step 1. For any $\varepsilon \in (0,1)$, any $k \in \mathbb{N}$ and any nonnegative function $\psi \in C^1_c(\mathbb{R}^N)$, there exists a constant $C_\varepsilon = C(p, \varepsilon) > 0$ such that

$$\begin{aligned} &(1-\varepsilon) \int_{\mathbb{R}^N} w(x) |\nabla u|^p b_k'(u) \psi^p \, dx \\ &\le C_\varepsilon \int_{\mathbb{R}^N} w(x) |b_k(u)|^p b_k'(u)^{1-p} |\nabla \psi|^p \, dx + \int_{\mathbb{R}^N} f(x) |u|^{q-1} u b_k(u) \psi^p \, dx. \end{aligned} \tag{2.2}$$

To prove this, using $\varphi = b_k(u)\psi^p$ as a test function. Since

$$\nabla \varphi = b_k'(u) \psi^p \nabla u + p b_k(u) \psi^{p-1} \nabla \psi,$$

using (1.2) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} w(x) |\nabla u|^p b_k'(u) \psi^p \, dx + p \int_{\mathbb{R}^N} w(x) |\nabla u|^{p-2} b_k(u) \psi^{p-1} (\nabla u, \nabla \psi) \, dx \\ &= \int_{\mathbb{R}^N} f(x) |u|^{q-1} u b_k(u) \psi^p \, dx. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^N} w(x) |\nabla u|^p b_k'(u) \psi^p \, dx \\ &\le p \int_{\mathbb{R}^N} w(x) |\nabla u|^{p-1} |b_k(u)| \psi^{p-1} |\nabla \psi| \, dx + \int_{\mathbb{R}^N} f(x) |u|^{q-1} u b_k(u) \psi^p \, dx \\ &\le \int_{\mathbb{R}^N} \varepsilon \Big(w(x)^{\frac{p-1}{p}} |\nabla u|^{p-1} b_k'(u)^{\frac{p-1}{p}} \psi^{p-1} \Big)^{\frac{p}{p-1}} \\ &\quad + C_\varepsilon \Big(w(x)^{1/p} |b_k(u)| b_k'(u)^{\frac{1-p}{p}} |\nabla \psi| \Big)^p \, dx + \int_{\mathbb{R}^N} f(x) |u|^{q-1} u b_k(u) \psi^p \, dx \\ &= \varepsilon \int_{\mathbb{R}^N} w(x) |\nabla u|^p b_k'(u) \psi^p \, dx + C_\varepsilon \int_{\mathbb{R}^N} w(x) |b_k(u)|^p b_k'(u)^{1-p} |\nabla \psi|^p \, dx \end{aligned}$$

$$+\int_{\mathbb{R}^N} f(x)|u|^{q-1}ub_k(u)\psi^p\,dx,$$

which implies (2.2).

Step 2. For any $\varepsilon\in(0,1)$, any $k\in\mathbb{N}$ and any nonnegative function $\psi\in C_c^1(\mathbb{R}^N)$, there exists a constant $C_\varepsilon=C(p,\varepsilon)>0$ such that

$$\begin{aligned} q\int_{\mathbb{R}^N} f(x)|u|^{q-1}a_k(u)^2\psi^p\,dx &\le (p-1+\varepsilon)\int_{\mathbb{R}^N} w(x)|\nabla u|^p a_k'(u)^2\psi^p\,dx \\ &\quad + C_\varepsilon\int_{\mathbb{R}^N} w(x)|a_k(u)|^p a_k'(u)^{2-p}|\nabla\psi|^p\,dx. \end{aligned} \tag{2.3}$$

To prove this, we use the stability assumption with $\varphi=a_k(u)\psi^{p/2}$. Since

$$\nabla\varphi=a_k'(u)\psi^{p/2}\nabla u+\frac{p}{2}a_k(u)\psi^{\frac{p-2}{2}}\nabla\psi,$$

using (1.4) we obtain

$$\begin{aligned} &q\int_{\mathbb{R}^N} f(x)|u|^{q-1}a_k(u)^2\psi^p\,dx \\ &\le (p-1)\int_{\mathbb{R}^N} w(x)|\nabla u|^p a_k'(u)^2\psi^p\,dx \\ &\quad + (p-1)p\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-1}|a_k(u)|a_k'(u)\psi^{p-1}|\nabla\psi|\,dx \\ &\quad + \frac{(p-1)p^2}{4}\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}a_k(u)^2\psi^{p-2}|\nabla\psi|^2\,dx. \end{aligned} \tag{2.4}$$

Now we use Young inequality to estimate the last two terms

$$\begin{aligned} &(p-1)p\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-1}|a_k(u)|a_k'(u)\psi^{p-1}|\nabla\psi|\,dx \\ &\le \int_{\mathbb{R}^N}\frac{\varepsilon}{2}\Big(w(x)^{\frac{p-1}{p}}|\nabla u|^{p-1}a_k'(u)^{\frac{2(p-1)}{p}}\psi^{p-1}\Big)^{\frac{p}{p-1}} \\ &\quad + C_\varepsilon\Big(w(x)^{1/p}|a_k(u)|a_k'(u)^{\frac{2-p}{p}}|\nabla\psi|\Big)^p\,dx \\ &= \frac{\varepsilon}{2}\int_{\mathbb{R}^N} w(x)|\nabla u|^p a_k'(u)^2\psi^p\,dx + C_\varepsilon\int_{\mathbb{R}^N} w(x)|a_k(u)|^p a_k'(u)^{2-p}|\nabla\psi|^p\,dx \end{aligned}$$

and

$$\begin{aligned} &\frac{(p-1)p^2}{4}\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}a_k(u)^2\psi^{p-2}|\nabla\psi|^2\,dx \\ &\le \int_{\mathbb{R}^N}\frac{\varepsilon}{2}\Big(w(x)^{\frac{p-2}{p}}|\nabla u|^{p-2}a_k'(u)^{\frac{2(p-2)}{p}}\psi^{p-2}\Big)^{\frac{p}{p-2}} \\ &\quad + C_\varepsilon\Big(w(x)^{\frac{2}{p}}a_k(u)^2a_k'(u)^{\frac{2(2-p)}{p}}|\nabla\psi|^2\Big)^{p/2}\,dx \\ &= \frac{\varepsilon}{2}\int_{\mathbb{R}^N} w(x)|\nabla u|^p a_k'(u)^2\psi^p\,dx + C_\varepsilon\int_{\mathbb{R}^N} w(x)|a_k(u)|^p a_k'(u)^{2-p}|\nabla\psi|^p\,dx. \end{aligned}$$

Using these two estimates into (2.4), we obtain (2.3).

Step 3. We claim that there exists a constant $C = C(p, q, \alpha) > 0$ such that for any nonnegative function $\psi \in C_c^1(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \left(w(x)|\nabla u|^p|u|^{\alpha-1} + f(x)|u|^{\alpha+q}\right) \psi^p \, dx \le C \int_{\mathbb{R}^N} w(x)|u|^{\alpha+p-1}|\nabla\psi|^p \, dx. \quad (2.5)$$

To prove this, we set $\beta_\varepsilon = 1 - \frac{(p-1+\varepsilon)(\alpha+1)^2}{4(1-\varepsilon)\alpha q}$. Since $\lim_{\varepsilon\to 0^+} \beta_\varepsilon = 1 - \frac{(p-1)(\alpha+1)^2}{4\alpha q} > 0$, we can fix some $\varepsilon \in (0, 1)$ depending on p, q and α such that $\beta_\varepsilon > 0$.

Collecting (2.2), (2.3) and with the help of (2.1) we obtain

$$\begin{aligned}
& q \int_{\mathbb{R}^N} f(x)|u|^{q-1} a_k(u)^2 \psi^p \, dx \\
& \le (p-1+\varepsilon) \int_{\mathbb{R}^N} w(x)|\nabla u|^p a_k'(u)^2 \psi^p \, dx \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} w(x)|a_k(u)|^p a_k'(u)^{2-p} |\nabla\psi|^p \, dx \\
& \le \frac{(p-1+\varepsilon)(\alpha+1)^2}{4\alpha} \int_{\mathbb{R}^N} w(x)|\nabla u|^p b_k'(u)\psi^p \, dx \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} w(x)|a_k(u)|^p a_k'(u)^{2-p} |\nabla\psi|^p \, dx \\
& \le \frac{(p-1+\varepsilon)(\alpha+1)^2}{4(1-\varepsilon)\alpha} \int_{\mathbb{R}^N} f(x)|u|^{q-1} u b_k(u)\psi^p \, dx \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} w(x) \left[|a_k(u)|^p a_k'(u)^{2-p} + |b_k(u)|^p b_k'(u)^{1-p}\right] |\nabla\psi|^p \, dx \\
& \le \frac{(p-1+\varepsilon)(\alpha+1)^2}{4(1-\varepsilon)\alpha} \int_{\mathbb{R}^N} f(x)|u|^{q-1} a_k(u)^2 \psi^p \, dx \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} w(x)|u|^{\alpha+p-1}|\nabla\psi|^p \, dx.
\end{aligned}$$

Therefore,

$$q\beta_\varepsilon \int_{\mathbb{R}^N} f(x)|u|^{q-1} a_k(u)^2 \psi^p \, dx \le C_\varepsilon \int_{\mathbb{R}^N} w(x)|u|^{\alpha+p-1}|\nabla\psi|^p.$$

Letting $k \to \infty$, by the monotone convergence theorem we obtain

$$\int_{\mathbb{R}^N} f(x)|u|^{\alpha+q}\psi^p \, dx \le C \int_{\mathbb{R}^N} w(x)|u|^{\alpha+p-1}|\nabla\psi|^p, \quad (2.6)$$

where C depends only on p, q and α. On the other hand, applying (2.2) for $\varepsilon = 1/2$,

$$\begin{aligned}
& \int_{\mathbb{R}^N} w(x)|\nabla u|^p b_k'(u)\psi^p \, dx \\
& \le C \int_{\mathbb{R}^N} w(x)|b_k(u)|^p b_k'(u)^{1-p}|\nabla\psi|^p \, dx + 2\int_{\mathbb{R}^N} f(x)|u|^{q-1} u b_k(u)\psi^p \, dx \\
& \le C \int_{\mathbb{R}^N} w(x)|u|^{\alpha+p-1}|\nabla\psi|^p \, dx + 2\int_{\mathbb{R}^N} f(x)|u|^{q-1} a_k(u)^2 \psi^p \, dx.
\end{aligned}$$

Letting $k \to \infty$, by the monotone convergence theorem and (2.6) we obtain

$$\int_{\mathbb{R}^N} w(x)|\nabla u|^p|u|^{\alpha-1}\psi^p \, dx \le C \int_{\mathbb{R}^N} w(x)|u|^{\alpha+p-1}|\nabla\psi|^p \, dx. \quad (2.7)$$

Combining (2.6) and (2.7) we obtain (2.5).

Step 4. We are now in a position to prove a priori estimate (1.5). Applying (2.5) for $\psi = \eta^{\frac{\alpha+q}{q-p+1}}$ to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \big(w(x)|\nabla u|^p|u|^{\alpha-1} + f(x)|u|^{\alpha+q}\big)\,\eta^{\frac{p(\alpha+q)}{q-p+1}}\,dx \\
&\le C\int_{\mathbb{R}^N} w(x)|u|^{\alpha+p-1}|\nabla\eta|^p\eta^{\frac{p(\alpha+p-1)}{q-p+1}}\,dx \\
&\le \int_{\mathbb{R}^N} \frac{1}{2}\Big(f(x)^{\frac{\alpha+p-1}{\alpha+q}}|u|^{\alpha+p-1}\eta^{\frac{p(\alpha+p-1)}{q-p+1}}\Big)^{\frac{\alpha+q}{\alpha+p-1}} \\
&\quad + C\Big(w(x)f(x)^{-\frac{\alpha+p-1}{\alpha+q}}|\nabla\eta|^p\Big)^{\frac{\alpha+q}{q-p+1}}\,dx \\
&\le \frac{1}{2}\int_{\mathbb{R}^N} f(x)|u|^{\alpha+q}\eta^{\frac{p(\alpha+q)}{q-p+1}}\,dx + C\int_{\mathbb{R}^N} w(x)^{\frac{\alpha+q}{q-p+1}}f(x)^{-\frac{\alpha+p-1}{q-p+1}}|\nabla\eta|^{\frac{p(\alpha+q)}{q-p+1}}\,dx.
\end{aligned}
$$

Hence, (1.5) follows. □

Proof of Theorem 1.5. Applying (1.5) for a test function $\eta_R \in C_c^1(\mathbb{R}^N)$ satisfying $0 \le \eta_R \le 1$ in $\mathbb{R}^N$ and

$$
\begin{gathered}
\eta_R = 1 \quad \text{in } B(0,R),\\
\eta_R = 0 \quad \text{in } \mathbb{R}^N \setminus B(0,2R),\\
|\nabla\eta_R| \le \frac{C}{R} \quad \text{in } B(0,2R)\setminus B(0,R).
\end{gathered}
$$

Consequently, for all $R > R_0$ there exists a constant C independent of R such that

$$
\int_{B(0,R)} \big(w(x)|\nabla u|^p|u|^{\alpha-1} + f(x)|u|^{\alpha+q}\big)\,dx \le CR^\theta, \tag{2.8}
$$

where

$$
\theta = N - \frac{(p-a)(\alpha+q) + b(\alpha+p-1)}{q-p+1}.
$$

Note that $\alpha \in (1, \alpha_0(q))$ where

$$
\alpha_0(t) = \frac{2t - p + 1 + 2\sqrt{t(t-p+1)}}{p-1}.
$$

Let us define the function

$$
g(t) = \frac{(p-a)(\alpha_0(t)+t) + b(\alpha_0(t)+p-1)}{t-p+1}, \quad \text{for } t > p-1.
$$

Since

$$
g'(t) = \frac{p-a+b}{(t-p+1)^2}\Big(-p - \sqrt{\frac{t-p+1}{t}}\Big) < 0,
$$

the function $g(t)$ is decreasing in $t > p-1$. On the other hand,

$$
\lim_{t\to(p-1)^+} g(t) = +\infty, \quad \lim_{t\to+\infty} g(t) = \frac{(p-a)(p+3)+4b}{p-1}.
$$

Therefore, if $N \le \frac{(p-a)(p+3)+4b}{p-1}$, then $N < g(q)$ since $q > p-1$. Hence if we fix $\alpha \in [1, \alpha_0(q))$, suitably near $\alpha_0(q)$, we obtain

$$
N < \frac{(p-a)(\alpha+q) + b(\alpha+p-1)}{q-p+1},
$$

which means that $\theta < 0$. Then the desired result follows by letting $R \to \infty$ in (2.8).

Assume now $N > \frac{(p-a)(p+3)+4b}{p-1}$. Since g is decreasing, we obtain in this case a critical value $q_c(p,N,a,b)$ such that $N < g(q)$ for $1 < q < q_c(p,N,a,b)$. From this, the desired result follows again by letting $R \to \infty$ in (2.8). Clearly, $q_c(p,N,a,b)$ may be deduced from the equation $N = g(q)$, which is given the value in Theorem 1.5 (see also Remark 1.6). Then we complete the proof. □

Proof of Proposition 1.8. Direct calculation yields that U is a weak solution of (1.6). In order to show that U is stable, we need the following inequality (see [3, 5]).

Lemma 2.1 (Caffarelli-Kohn-Nirenberg inequality)**.** *Let $r < \frac{N-2}{2}$, then for all $\varphi \in C_c^1(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} \frac{|\nabla\varphi|^2}{|x|^{2r}}\,dx \ge \Big(\frac{N-2-2r}{2}\Big)^2 \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^{2r+2}}\,dx. \tag{2.9}$$

Applying (2.9) with $r = \frac{(n+1)(p-2)-a}{2}$ we obtain

$$\int_{\mathbb{R}^N} \frac{|\nabla\varphi|^2}{|x|^{(n+1)(p-2)-a}}\,dx \ge \Big(\frac{N-2-(n+1)(p-2)+a}{2}\Big)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^{n(q-1)-b}}\,dx. \tag{2.10}$$

Since U is radially symmetric and decreasing in $|x|$, by arguing as in [2, Remark 1.7] it is necessary to check stability of U for all radially symmetric test function $\varphi \in C_c^1(\mathbb{R}^N)$. For such φ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[|x|^a|\nabla U|^{p-2}|\nabla\varphi|^2 + (p-2)|x|^a|\nabla U|^{p-4}(\nabla U,\nabla\varphi)^2 - q|x|^b|U|^{q-1}\varphi^2\right]dx\\
&= \int_{\mathbb{R}^N} \left[(p-1)|x|^a|\nabla U|^{p-2}|\nabla\varphi|^2 - q|x|^b|U|^{q-1}\varphi^2\right]dx\\
&= \int_{\mathbb{R}^N} \left[(mn)^{p-2}(p-1)\frac{|\nabla\varphi|^2}{|x|^{(n+1)(p-2)-a}} - qm^{q-1}\frac{\varphi^2}{|x|^{n(q-1)-b}}\right]dx\\
&\ge \int_{\mathbb{R}^N} \left[(mn)^{p-2}(p-1)\left(\frac{N-2-(n+1)(p-2)+a}{2}\right)^2 - qm^{q-1}\right]\frac{\varphi^2}{|x|^{n(q-1)-b}}\,dx,
\end{aligned}$$

where we have used (2.10) in the last estimate. Direct computation yields

$$\begin{aligned}
&(mn)^{p-2}(p-1)\Big(\frac{N-2-(n+1)(p-2)+a}{2}\Big)^2 - qm^{q-1}\\
&= (mn)^{p-2}\Big[(p-1)\left(\frac{N-2-(n+1)(p-2)+a}{2}\right)^2\\
&- nq\left(N+a-1-(n+1)(p-1)\right)\Big].
\end{aligned}$$

We want to show that

$$(p-1)\Big(\frac{N-2-(n+1)(p-2)+a}{2}\Big)^2 - nq\left[N+a-1-(n+1)(p-1)\right] \ge 0.$$

After substituting $n = \frac{p-a+b}{q-p+1}$, this inequality is equivalent to

$$N \ge \frac{(p-a+b)(2q-p+1+2\sqrt{q(q-p+1)}) + q(p-a)(p-1) + b(p-1)^2}{(p-1)(q-p+1)}.$$

The last inequality is verified by Remark 1.6 and assumption $q \geq q_c(p, N, a, b)$. Thus, U is stable. □

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