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p-KIRCHHOFF TYPE PROBLEM WITH A GENERAL CRITICAL NONLINEARITY

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ABSTRACT. In this article, we consider the p-Kirchhoff type problem

$$\Big(1+\lambda\int_{\mathbb{R}^N}|\nabla u|^p+\lambda b\int_{\mathbb{R}^N}|u|^p\Big)(-\Delta_p u+b|u|^{p-2}u)=f(u), x\in\mathbb{R}^N,$$

where $\lambda>0$, the nonlinearity f can reach critical growth. Without the Ambrosetti-Robinowitz condition or the monotonicity condition on f, we prove the existence of positive solutions for the p-Kirchhoff type problem. In addition, we also study the asymptotic behavior of the solutions with respect to the parameter $\lambda\to 0$.

1. Introduction and statement of results

In this article, we study the p-Kirchhoff type problem

$$\Big(1+\lambda\int_{\mathbb{R}^N}|\nabla u|^p+\lambda b\int_{\mathbb{R}^N}|u|^p\Big)(-\Delta_p u+b|u|^{p-2}u)=f(u)\quad\text{in }\mathbb{R}^N, \tag{1.1}$$

where $b,\lambda>0$, $\Delta_p u:=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1<p<N$ and the nonlinearity f may be critical. Problem (1.1) with $p=2$ reduces to the Kirchhoff type problem

$$(1+\lambda\int_{\mathbb{R}^N}|\nabla u|^2+\lambda b\int_{\mathbb{R}^N}|u|^2)(-\Delta u+bu)=f(u)\quad\text{in }\mathbb{R}^N. \tag{1.2}$$

In the previous decades, the Kirchhoff type problem (1.2) has been object of intensive research as its strong relevance in applications. From a physical point of view, problem (1.2) on bounded domain $\Omega\subset\mathbb{R}^N$ is related to the stationary analogue of the equation

$$\rho\frac{\partial^2u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\,\mathrm{d}x\Big)\frac{\partial^2u}{\partial x^2}=0, \tag{1.3}$$

which is proposed by Kirchhoff in [10] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. After Lions [12] introduced a functional analysis approach to equation (1.3), he gave the equation

$$u_{tt}-\Big(a+b\int_\Omega|\nabla u|^2\,\mathrm{d}x\Big)\Delta u=f(x,u),\quad x\in\Omega,\ u=0,\ x\in\partial\Omega. \tag{1.4}$$

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When Kirchhoff's model takes into account the change of the string length caused by oscillations, u represents the displacement, $f(x,u)$ denotes the external force, b is the initial tension while a is related to the intrinsic properties of the string, such as Young's modulus. Moreover, problem (1.2) on bounded domain appears in many mathematical biological contexts. In [1], Kirchhoff type problem models some biological systems, where u describes a process which depends on the average of itself, such as population density.

In recent years, Kirchhoff type problems on $\mathbb{R}^N$ have been studied widely by the variational methods and results can be seen in [9, 11, 13, 14]. Especially, in [11], Li et al. considered problem (1.2) under the following conditions

- (A1) $f \in C(\mathbb{R}_+,\mathbb{R}_+)$ and $|f(t)| \le C_1(|t| + |t|^{p-1})$ for $t \ge 0$ and some $p \in (2, 2^*)$;
- (A2) $\lim_{t\to 0} f(t)/t = 0$;
- (A3) $\lim_{t\to\infty} \sup f(t)/t = \infty$.

Theorem 1.1 (see[11])**.** *Assume that $N \ge 3$ and* (A1)–(A3) *hold. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$, problem* (1.2) *has at least one positive solution.*

Subsequently, Liu, Liao and Tang [14] studied problem (1.2) under some weaker conditions than the ones in [11]. In [11, 14], the authors only considered problem (1.2) with subcritical growth. For the p-Kirchhoff type problem (1.1), there are also many results, see for example [2, 5, 6, 8]. Autuori, Colasuonno and Pucci [2] obtained two nontrivial solutions of possibly degenerate nonlinear eigenvalue problems involving the p-poly-harmonic Kirchhoff operator in bounded domains. Using the Nehari manifold method, Chen and Zhu [6] obtained positive solutions to the problem

$$\Big[a+\lambda\Big(\int_{\mathbb{R}^N}(|\nabla u|^p+b|u|^p)dx\Big)^{\tau}\Big](-\Delta_p u+b|u|^{p-2}u) = |u|^{m-2}u+\mu|u|^{q-2}u,\ x\in\mathbb{R}^N.$$

For fractional p-Kirchhoff problems, we refer to [18] and the references therein.

The papers cited above were all focused on p-Kirchhoff type problem with subcritical growth. Many of them need usual compactness conditions. Compared to p-Kirchhoff problems with subcritical growth, there are few results in term of p-Kirchhoff type problem involving critical growth. In [19], the author only considered the p-Kirchhoff type problem with specified critical growth term, not involving general critical growth. To the best of our knowledge, without usual compactness condition, there are few results conducted on problem (1.1) with general nonlinearity f reaching critical growth.

Main results. In this article, we study p-Kirchhoff type problem (1.1) with critical growth. Throughout the paper, $f \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\mathbb{R}_+ = [0,+\infty)$ and satisfies

- (A4) $\lim_{s\to 0} f(s)/s^{p-1} = 0$;
- (A5) $\lim_{s\to\infty} \sup f(s)/s^{p^*-1} \le 1$, where $p^* = Np/(N-p)$;
- (A6) There are $\alpha > 0$ and $q \in (p, p^*)$ such that $f(s) \ge \alpha s^{q-1}$ for all $s \ge 0$.

Condition (A5) implies that f has a critical growth at infinity and the limit of $f(s)/s^{p^*-1}$ at $+\infty$ is not necessary to exist.

Let S and C_s denote the best constants of Sobolev embeddings $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$, namely,

$$S\Big(\int_{\mathbb{R}^N}|u|^{p^*}\Big)^{p/p^*} \le \int_{\mathbb{R}^N}|\nabla u|^p \quad \text{for all } u \in D^{1,p}(\mathbb{R}^N),$$

$$C_s\Big(\int_{\mathbb{R}^N}|u|^s\Big)^{p/s}\le\int_{\mathbb{R}^N}(|\nabla u|^p+b|u|^p)\quad\text{for all } u\in W^{1,p}(\mathbb{R}^N).$$

Our main results read as follows.

Theorem 1.2. *Assume* (A4)–(A6) *hold. Then there exists $\lambda^*>0$ such that, for any $\lambda\in(0,\lambda^*)$, problem* (1.1) *possesses a nontrivial positive radial solution u_λ, provided that*

$$\alpha>S^{\frac{N(p-q)}{p^2}}C_q^{p/q}m^{\frac{N(p-q)}{p^2}+\frac{q}{p}}\Big(\frac{N}{p}-\frac{N}{q}\Big)^{\frac{q-p}{p}}.$$

Theorem 1.3. *Assume (A4)–(A6) hold. As $\lambda\to0$, $\{u_\lambda\}$ converges to u in $W_r^{1,p}(\mathbb{R}^N)$ (necessarily along a subsequence), where u is a ground state solution of*

$$-\Delta_p u+b|u|^{p-2}u=f(u)\ \text{in } \mathbb{R}^N.$$

For $p=2$ in Theorem 1.2, Li et al. [11] and Liu et al. [14] considered problem (1.2), but they only studied Kirchhoff type problem (1.2) with the general nonlinearity involving subcritical growth. Theorem 1.2 is concerned with a nonlinearity f reaching critical growth, which makes the problem much more complicated.

Main difficulties and ideas. To prove our results by variational methods, the difficulties are two-fold. The first difficulty is due to the appearance of $\int_{\mathbb{R}^N}(|\nabla u|^p+b|u|^p)$, which implies that (1.1) is no longer a pointwise identity. Namely, such a phenomenon causes some mathematical difficulties. The second difficulty lies in obtaining the boundedness of the Palais-Smale sequence (in short (PS) sequence) to the energy functional without usual Ambrosetti-Rabinowtiz condition. To overcome these difficulties, we adopt a local deformation argument from Byeon and Jeanjean [4] to obtain a bounded (PS) sequence. Then we use similar ideas in [21] to make a crucial modification on the min-max value as the presence of nonlocal term.

The rest of this is organized as follows. Section 2 is devoted to the limit problem. In Section 3, we define a min-max level and construct a bounded (PS) sequence. Finally, we give the proof of Theorem 1.2.

Notation

- $\|u\|_s:=\big(\int_{\mathbb{R}^N}|u|^s\big)^{1/s}$ for $s\in[1,\infty)$ and $u\in L^s(\mathbb{R}^N)$.
- Let $W^{1,p}(\mathbb{R}^N)$ be the Sobolev space equipped with the norm

$$\|u\|:=\Big(\int_{\mathbb{R}^N}(|\nabla u|^p+b|u|^p)\Big)^{1/p}$$

and $W_r^{1,p}(\mathbb{R}^N)=\{u\in W^{1,p}(\mathbb{R}^N):u(x)=u(|x|)\}$.

2. Limit problem

When $\lambda=0$, problem (1.1) reduces to the problem

$$-\Delta_p u+b|u|^{p-2}u=f(u)\quad\text{in } \mathbb{R}^N, \tag{2.1}$$

which is called the limit problem of problem (1.1). For $\lambda>0$ small, we may view the problem (1.1) as a corresponding perturbation problem to (2.1). In general, if problem (2.1) is well-behaved, then we may expect that the perturbed problem (1.1) possesses a solution in some neighborhood of solutions to problem (2.1). Indeed, the idea plays a critical role in establishing the existence of positive solutions to problem (1.1).

We define

$$J(u) := \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + b|u|^p) - \int_{\mathbb{R}^N} F(u),\ u \in W_r^{1,p}(\mathbb{R}^N),$$

where $F(t) = \int_0^t f(s)\,\mathrm{d}s$ and the mountain pass value $m := \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} J(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1], W_r^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

J. Zhang et al. [20] show that m is the least energy of problem (2.1) and can be achieved by a radially symmetric function. Let W_r denotes the set of positive ground state solutions U of (2.1) satisfying $U(0) = \max_{x\in\mathbb{R}^N} U(x)$. Then, $W_r \subset W_r^{1,p}(\mathbb{R}^N)$ and $W_r \neq \emptyset$.

Lemma 2.1 ([20])**.** *Under the assumptions in Theorem 1.2, W_r is compact in $W_r^{1,p}(\mathbb{R}^N)$.*

3. Proof of main results

Since we only seek positive solutions of problem (1.1), we may assume $f(s) = 0$ for all $s < 0$. In addition, we work in $W_r^{1,p}(\mathbb{R}^N)$ because problem (1.1) is autonomous. Define

$$\Phi_\lambda(u) = \frac{1}{p}\|u\|^p + \frac{\lambda}{2p}\|u\|^{2p} - \int_{\mathbb{R}^N} F(u),\ u \in W^{1,p}(\mathbb{R}^N).$$

By (A4)–(A6), $\Phi_\lambda \in C^1(W^{1,p}(\mathbb{R}^N), \mathbb{R})$ and for all $u, v \in W^{1,p}(\mathbb{R}^N)$,

$$\langle \Phi_\lambda'(u), v\rangle = (1 + \lambda\|u\|^p)\int_{\mathbb{R}^N}(|\nabla u|^{p-2}\nabla u\cdot\nabla v + b|u|^{p-2}uv) - \int_{\mathbb{R}^N} f(u)v.$$

It is standard to verify that the critical points of Φ_λ are weak solutions of (1.1).

Minimax level. Set $W_t(x) = W(x/t)$, $W \in W_r$, by the Pohozăev identity, we have

$$\begin{aligned} J(W_t) &= \frac{1}{p}t^{N-p}\int_{\mathbb{R}^N}|\nabla W|^p - t^N\int_{\mathbb{R}^N}\Big(F(W) - \frac{b|W|^p}{p}\Big) \\ &= \Big(\frac{1}{p}t^{N-p} - \frac{N-p}{Np}t^N\Big)\int_{\mathbb{R}^N}|\nabla W|^p. \end{aligned}$$

It is clear that $J(W_t) \to -\infty$ as $t \to \infty$ and $J(W_{t^*}) < -3$ for some $t^* > 1$. Let $A_\lambda = \max_{t\in[0,t^*]}\Phi_\lambda(W_t)$. By

$$\int_{\mathbb{R}^N}|\nabla W|^p = Nm,$$

we know

$$\lim_{\lambda\to 0} A_\lambda = \lim_{\lambda\to 0}\max_{t\in[0,t^*]} J(W_t) = m.$$

To get a uniformly bounded set of the mountain pathes, we have the following result.

Lemma 3.1. *There exist $\lambda^* > 0$ and $C_2 > 0$, such that for any $\lambda \in (0,\lambda^*)$, $\Phi_\lambda(W_{t^*}) < -3$, $\|W_t\| \le C_2$, $t \in (0,t^*]$ and $\|W\| \le C_2$ for any $W \in W_r$.*

Proof. For any $W \in W_r$, by Lemma 2.1, there exists $C_3 > 0$ such that $\|W\| \le C_3$ and

$$\begin{aligned}\|W_t\|^p &= t^{N-p}\|\nabla W\|_p^p + bt^N\|W\|_p^p \\ &\le (t^{N-p} + bt^N)\|W\|^p \\ &\le ((t^*)^{N-p} + b(t^*)^N)C_3^p.\end{aligned}$$

Let $C_2 = \max\{C_3, ((t^*)^{N-p} + b(t^*)^N)^{1/p}C_3\}$, then $\|W_t\| \le C_2$, $t \in (0, t^*]$. In addition,

$$\Phi_\lambda(W_{t^*}) = J(W_{t^*}) + \frac{\lambda}{2p}\|W_{t^*}\|^{2p} \le J(W_{t^*}) + \frac{\lambda}{2p}C_2^{2p}.$$

By $J(W_{t^*}) < -3$, there exists $\lambda^* > 0$ such that $\Phi_\lambda(W_{t^*}) < -3$ for $\lambda \in (0, \lambda^*)$. □

Next, we define a minmax value B_λ given by $B_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{s\in[0,t^*]} \Phi_\lambda(\gamma(s))$ where

$$\Gamma_\lambda = \{\gamma \in C([0, t^*], W_r^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(t^*) = W_{t^*}, \|\gamma(t)\| \le C_2 + 2\}.$$

It is clear that $\Gamma_\lambda \neq \emptyset$ and $B_\lambda \le A_\lambda$ for $\lambda \in (0, \lambda^*)$.

Lemma 3.2. *$B_\lambda \to m$ as $\lambda \to 0$.*

Proof. Obviously, $B_\lambda \le A_\lambda \to m$ as $\lambda \to 0$. Notice that $\Phi_\lambda(u) \ge J(u)$ for $u \in W_r^{1,p}(\mathbb{R}^N)$ and for any $\gamma \in \Gamma_\lambda, \tilde{\gamma}(\cdot) = \gamma(t^*) \in \Gamma$. Thus, $B_\lambda \ge m$. So, $\lim_{\lambda\to 0} B_\lambda = m$. □

For $c, d > 0$, set

$$\begin{gathered}\Phi_\lambda^c = \{u \in W_r^{1,p}(\mathbb{R}^N) : \Phi_\lambda(u) \le c\}, \\ W^d = \{u \in W_r^{1,p}(\mathbb{R}^N) : \inf_{v\in W_r} \|u - v\| \le d\}.\end{gathered}$$

Clearly, $W^d \neq \emptyset$ for all $d > 0$. In the following, we look for a solution $u \in W^d$ of problem (1.1) for $\lambda > 0$ small enough.

Lemma 3.3. *There exist $C' > 0$ and $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ and $W \in \Phi_\lambda^{A_\lambda} \cap (W^d \backslash W^{d/2})$, we have $\|\Phi'_\lambda(W)\| \ge C'$, provided that*

$$0 < d < \min\{1, (Nm)^{1/p}, \frac{1}{4}(\frac{p^*}{2p}s^{p^*/p})^{\frac{1}{p^*-p}}\}. \tag{3.1}$$

Proof. It suffices to prove that for d small with (3.1) and any $\{W_{\lambda_i}\} \subset W^d$ with

$$\begin{gathered}\lim_{i\to\infty} \Phi_{\lambda_i}(W_{\lambda_i}) \le m, \\ \lim_{i\to\infty} \|\Phi'_{\lambda_i}(W_{\lambda_i})\| \to 0,\end{gathered}$$

there exists $W_0 \in W_r$ such that $W_{\lambda_i} \to W_0$ in $W_r^{1,p}(\mathbb{R}^N)$, where $\lim_{i\to 0} \lambda_i = 0$. For convenience, we replace λ_i by λ. Because $W_\lambda \in W^d$, $W_\lambda = u_\lambda + v_\lambda$, where $u_\lambda \in W_r$ and $v_\lambda \in W^{1,p}(\mathbb{R}^N)$, such that $u_\lambda \to u_0$ strongly in $W_r^{1,p}(\mathbb{R}^N)$, $v_\lambda \to v_0$ weakly in $W^{1,p}(\mathbb{R}^N)$ and $v_\lambda \to v_0$ a.e. in $\mathbb{R}^N$. Set $W_0 = u_0 + v_0$, then $W_0 \in W^d$ and $W_\lambda \to W_0$ weakly in $W^{1,p}(\mathbb{R}^N)$. From $\lim_{\lambda\to\infty} \|\Phi'_\lambda(W_\lambda)\| = 0$, we get that $J'(W_\lambda) \to 0$ as $\lambda \to 0$. So $J'(W_0) = 0$. We claim that $W_0 \not\equiv 0$. On the contrary, if $W_0 \equiv 0$, then $\|u_0\| = \|v_0\| \le d$. By the Pohozǎev identity and $u_0 \in W_r$, we obtain that $\|\nabla u_0\|_p = (Nm)^{1/p}$. However, by (3.1), $\|\nabla u_0\|_p \le \|u_0\| \le d < (Nm)^{1/p}$. This is a contradiction. So $W_0 \not\equiv 0$ and $J(W_0) \ge m$. Moreover, by [3, Theorem

2.1](also[15, Lemma 2.8]) and its remark, since W_λ satisfies $J'(W_\lambda) = o(1)$, we know $\nabla W_\lambda \to \nabla W_0$ a. e. in $\mathbb{R}^N$. Then, we have as $\lambda \to 0$,

$$\Phi_\lambda(W_\lambda) = J(W_\lambda) + o(1) = J(W_0) + J(W_\lambda - W_0) + o(1).$$

Thus $J(W_\lambda - W_0) \le o(1)$. Together with the Sobolev's embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, by (A4)–(A6), we have

$$\|W_\lambda - W_0\|^p/2p \le \|W_\lambda - W_0\|_{p^*}^{p^*}/p^* \le S^{\frac{p}{p^*}}\|\nabla(W_\lambda - W_0)\|_p^{p^*}/p^*.$$

If $\|W_\lambda - W_0\| \not\to 0$ as $\lambda \to 0$, then $\|W_\lambda - W_0\| \ge (\frac{p^*}{2p}s^{p^*/p})^{\frac{1}{p^*-p}}$. On the other hand,

$$\|W_\lambda - W_0\| \le \|u_\lambda - u_0\| + \|v_\lambda\| + \|v_0\| \le o(1) + 2d \le \frac{1}{2}(\frac{p^*}{2p}s^{p^*/p})^{\frac{1}{p^*-p}}.$$

This is a contradiction. Thus, $W_\lambda \to W_0$ strongly in $W_r^{1,p}(\mathbb{R}^N)$. The proof is completed. □

Lemma 3.4. *Assume there exists $C_4 > 0$, for small $\lambda > 0$ such that $\Phi_\lambda(\gamma(s)) \ge B_\lambda - C_4$. Then $\gamma(s) \in W^{d/2}$, where $\gamma(s) = W(\cdot/s)$, $s \in (0, t^*]$.*

Proof. It follows from the Pohožaev's identity that for $s \in (0, t^*]$,

$$\Phi_\lambda(\gamma(s)) = \Big(\frac{1}{p}s^{N-p} - \frac{N-p}{Np}s^N\Big)\int_{\mathbb{R}^N}|\nabla W|^p + o(\lambda) = J(\gamma(s)) + o(\lambda).$$

Noting that $m = \max_{s\in(0,t^*]} J(\gamma(s)) = J(\gamma(1))$, for $C_5 > 0$ small, $\gamma(s) = W(\cdot/s) \in W^{d/2}$ for $|s-1| \le C_5$. Since $B_\lambda \to m$ as $\lambda \to 0$, there exists $C_4 > 0$, for $\lambda > 0$ small enough, such that $\Phi_\lambda(\gamma(s)) \ge B_\lambda - C_4$. Furthermore, $|s-1| \le C_5$ and $\gamma(s) \in W^{d/2}$. □

Next, we use the local deformation argument to get a bounded (PS) sequence.

Lemma 3.5. *For $\lambda > 0$ small, there exists a sequence $\{u_n\} \subset \Phi_\lambda^{A_\lambda} \cap W^d$ with $\lim_{n\to\infty} \Phi'_\lambda(u_n) \to 0$.*

Proof. Assume by contradiction, there is $\beta(\lambda) > 0$ such that $|\Phi'_\lambda(u)| \ge \beta(\lambda)$, $u \in \Phi_\lambda^{A_\lambda} \cap W^d$ for some small $\lambda > 0$. Similar arguments in [17] show that there exists a pseudo-gradient vector field Ψ_λ in $W_r^{1,p}(\mathbb{R}^N)$ on a neighborhood D_λ of $\Phi_\lambda^{A_\lambda} \cap W^d$ such that $\|\Phi_\lambda(u)\| \le 2\min\{1, |\Phi'_\lambda(u)|\}$ and $\langle \Phi'_\lambda(u), \Psi_\lambda(u)\rangle \ge \min\{1, |\Phi'_\lambda(u)|\}|\Phi'_\lambda(u)|$. Denote δ_λ be a Lipschitz continuous function on $W_r^{1,p}(\mathbb{R}^N)$ such that $\delta_\lambda \in [0,1]$ and

$$\delta_\lambda(u) = \begin{cases} 1, & u \in \Phi_\lambda^{A_\lambda} \cap W^d \\ 0, & u \in W_r^{1,p}(\mathbb{R}^N)\backslash D_\lambda. \end{cases}$$

Define ξ_λ be a Lipschitz continuous function on $\mathbb{R}$ such that $\xi_\lambda \in [0,1]$ and

$$\xi_\lambda(t) = \begin{cases} 1, & |t - B_\lambda| \le C_4/2, \\ 0, & |t - B_\lambda| \ge C_4, \end{cases}$$

where C_4 is given in Lemma 3.4. Set

$$E_\lambda(u) = \begin{cases} -\delta_\lambda(u)\xi_\lambda(\Phi_\lambda(u))\Psi_\lambda(u), & u \in D_\lambda, \\ 0, & u \in W_r^{1,p}(\mathbb{R}^N)\backslash D_\lambda. \end{cases}$$

Then, the initial-value problem

$$\frac{d}{dt}Y_\lambda(u,t) = E_\lambda(Y_\lambda(u,t)),$$
$$Y_\lambda(u,0) = u,$$

admits a unique global solution $Y_\lambda : W_r^{1,p}(\mathbb{R}^N) \times \mathbb{R}_+ \to W_r^{1,p}(\mathbb{R}^N)$ which satisfies

(i) $Y_\lambda(u,t) = u$, if $t = 0$ or $u \in D_\lambda$ or $|\Phi_\lambda(u) - B_\lambda| \geq C_4$;
(ii) $\|\frac{d}{dt}Y_\lambda(u,t)\| \leq 2$, for $(u,t) \in W_r^{1,p}(\mathbb{R}^N) \times \mathbb{R}_+$;
(iii) $\frac{d}{dt}\Phi_\lambda(Y_\lambda(u,t)) \leq 0$.

As in [7], we get that for any $s \in (0, t^*]$, there is $t_s > 0$ such that $Y_\lambda(\gamma(s), t_s) \in \Phi_\lambda^{B_\lambda - C_4/2}$, where $\gamma(s) = W(\cdot/s)$, $s \in (0, t^*]$. Let $\gamma_0(s) = Y_\lambda(\gamma(s), t_*(s))$, where $t_*(s) = \inf\{t \geq 0, Y_\lambda(\gamma(s), t) \in \Phi_\lambda^{B_\lambda - C_4/2}\}$. Then we can prove that $\gamma_0(s)$ is continuous in $[0, t^*]$ and $\|\gamma_0(s)\| \leq C_2+2$. Therefore, $\gamma_0 \in \Gamma_\lambda$ with $\max_{t\in[0,t^*]} \Phi_\lambda(\gamma_0(t)) \leq B_\lambda - C_4/2$, which contradicts the definition of B_λ. □

Proof of Theorem 1.2. For fixed $d > 0$ small which satisfies $d < S^{N/p^2}/3$, by Lemma 3.5, there exist $\lambda^* > 0$ with $\lambda \in (0, \lambda^*)$ and $\{u_n\} \subset \Phi_\lambda^{A_\lambda} \cap W^d$ such that $\Phi_\lambda(u_n) \leq A_\lambda, \Phi'_\lambda(u_n) \to 0$ as $n \to \infty$. We may assume $\lim_{n\to\infty} \|u_n\|^p := \kappa \leq \big(d + \sup_{u\in W_r} \|u\|\big)^p$ and $u_n \to u_\lambda$ weakly in $W_r^{1,p}(\mathbb{R}^N)$, then by [17, Corollary 1.26], up to a subsequence, $u_n \to u_\lambda$ strongly in $L^t(\mathbb{R}^N), t \in (p, p^*)$ and a. e. in $\mathbb{R}^N$. Since $u_n \in W^d$, there exist $U_n \in W_r$ and $w_n \in W_r^{1,p}(\mathbb{R}^N)$ such that $u_n = U_n + w_n$ and $\|w_n\| \leq d$. By Lemma 2.1, for some $U \in W_r$, $U_n \to U$ strongly in $W_r^{1,p}(\mathbb{R}^N)$. Let $v_n = u_n - u_\lambda$, then $\|v_n\| \leq 3d$ for n large.

Step 1. For any $\delta > 1$, up to a subsequence, it holds

$$\int_{\mathbb{R}^N} f(u_n)u_n \leq \int_{\mathbb{R}^N} f(u_\lambda)u_\lambda + \delta\int_{\mathbb{R}^N} |v_n|^{p^*} + o_n(1).$$

Obviously, there exists $s_0 > 1$ such that $f(s) \leq \delta s^{p^*}$ for all $s \geq s_0$. Choose $\chi(s) \in C(\mathbb{R})$ such that $\chi(s) = 0$ if $s \leq 1$, $\chi(s) = f(s)/s^{p^*}$ if $s \geq s_0$ and $\chi(s) \in [0, \delta]$ for any $s \in \mathbb{R}$. Let $g(s) = f(s) - \chi(s)s^{p^*}, s \geq 0$, then $\lim_{s\to 0^+} g(s)/s^{p-1} \to 0$ and $\lim_{s\to+\infty} g(s)/s^{p^*} \to 0$. It follows from the compactness lemma of Strauss[16] that

$$\int_{\mathbb{R}^N} g(u_n)u_n = \int_{\mathbb{R}^N} g(u_\lambda)u_\lambda + o_n(1).$$

Meanwhile, similar to Brezis-Lieb Lemma [17, Lemma 1.32], we have

$$\int_{\mathbb{R}^N} \chi(u_n)|u_n|^{p^*} = \int_{\mathbb{R}^N} \chi(u_n)|v_n|^{p^*} + \int_{\mathbb{R}^N} \chi(u_\lambda)|u_\lambda|^{p^*} + o_n(1).$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^N} f(u_n)u_n &= \int_{\mathbb{R}^N} g(u_n)u_n + \int_{\mathbb{R}^N} \chi(u_n)|u_n|^{p^*} \\
&= \int_{\mathbb{R}^N} f(u_\lambda)u_\lambda + \int_{\mathbb{R}^N} \chi(u_n)|v_n|^{p^*} + o_n(1) \\
&\leq \int_{\mathbb{R}^N} f(u_\lambda)u_\lambda + \delta\int_{\mathbb{R}^N} |v_n|^{p^*} + o_n(1).
\end{aligned}$$

Step 2. We show that $u_n \to u_\lambda$ strongly in $D^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. In fact, u_λ satisfies

$$(1+\lambda\kappa)(-\Delta_p u + b|u|^{p-2}u) = f(u), \quad u \in W^{1,p}(\mathbb{R}^N).$$

Similar as that in Lemma 3.3, $\nabla u_n \to \nabla u_\lambda$ a. e. in $\mathbb{R}^N$ as $n \to \infty$. It follows from the Brezis-Lieb Lemma that

$$\|u_n\|^p = \|v_n\|^p + \|u_\lambda\|^p + o(1).$$

By Step 1 and $\langle \Phi'_\lambda(u_n), u_n\rangle \to 0$, we have

$$(1+\lambda\kappa)(\|v_n\|^p + \|u_\lambda\|^p) \le \int_{\mathbb{R}^N} f(u_\lambda)u_\lambda + \delta \int_{\mathbb{R}^N} |v_n|^{p^*} + o_n(1).$$

Since

$$(1+\lambda\kappa)\|u_\lambda\|^p = \int_{\mathbb{R}^N} f(u_\lambda)u_\lambda,$$

we have

$$(1+\lambda\kappa)\|v_n\|^p \le \delta \int_{\mathbb{R}^N} |v_n|^{p^*} + o_n(1).$$

If $\|\nabla v_n\|_p \not\to 0$ as $n \to \infty$, then it follows from Sobolev's embedding that

$$\|\nabla v_n\|_p^p \le \delta \int_{\mathbb{R}^N} |v_n|^{p^*} + o_n(1) \le \delta S^{-p^*/p} \|\nabla v_n\|_p^{p^*} + o_n(1),$$

which implies

$$\liminf_{n\to\infty} \|\nabla v_n\|_p \ge \left(\delta^{-1} S^{p^*/p}\right)^{1/(p^*-p)}.$$

Then

$$\liminf_{n\to\infty} \|\nabla v_n\|_p \ge S^{N/p^2},$$

which is impossible since $d < S^{N/p^2}/3$. Thus, $\|\nabla v_n\|_p \to 0$ as $n \to \infty$.

Step 3. $u_n \to u_\lambda$ strongly in $W^{1,p}(\mathbb{R}^N)$. In fact, by Step 2, we have

$$(1+\lambda\kappa)(-\Delta_p u_\lambda + b|u_\lambda|^{p-2}u_\lambda) = f(u_\lambda), \quad u_\lambda \in W^{1,p}(\mathbb{R}^N).$$

By Step 1, $f(u_n)u_n \to f(u_\lambda)u_\lambda$ strongly in $L^1(\mathbb{R}^N)$. Thus, by $\langle \Phi'_\lambda(u_n), u_n\rangle \to 0$, we get

$$\begin{aligned}(1+\lambda\kappa)\|u_n\|^p &= \int_{\mathbb{R}^N} f(u_n)u_n + o_n(1) \\ &= \int_{\mathbb{R}^N} f(u_\lambda)u_\lambda + o_n(1) \\ &= (1+\lambda\kappa)\|u_\lambda\|^p + o_n(1).\end{aligned}$$

So, $\|u_n\| \to \|u_\lambda\|$ as $n \to \infty$. Therefore, $u_n \to u_\lambda$ strongly in $W^{1,p}(\mathbb{R}^N)$, which implies that $\Phi'_\lambda(u_\lambda) = 0$ and $u_\lambda \in \Phi_\lambda^{A_\lambda} \cap W^d$. For d small enough, $u_\lambda \ne 0$. □

Proof of Theorem 1.3. For $\lambda > 0$ small enough, problem (1.1) admits a positive solution u_λ with $u_\lambda \in \Phi_\lambda^{A_\lambda} \cap W^d$. That is, $u_\lambda \in W^d$ and $\Phi'_\lambda(u_\lambda) = 0$, $\Phi_\lambda(u_\lambda) \le A_\lambda$. Obviously, $\{u_\lambda\}$ is bounded in $W_r^{1,p}(\mathbb{R}^N)$. Up to a subsequence, we assume that for some $u \in W_r^{1,p}(\mathbb{R}^N)$, $u_\lambda \to u$ weakly in $W_r^{1,p}(\mathbb{R}^N)$, strongly in $L^t(\mathbb{R}^N)$ for $t \in (p, p^*)$ and a. e. in $\mathbb{R}^N$ as $\lambda \to 0$. Similar to Theorem 1.2, for $d < S^{N/p^2}/3$ given and small, $u \not\equiv 0$ and $u_\lambda \to u$ strongly in $W_r^{1,p}(\mathbb{R}^N)$ as $\lambda \to 0$. It implies that $u \in W^d$ and $J'(u) = 0$, $J(u) \le \lim_{\lambda\to 0} A_\lambda = m$. Since m is the least energy of problem (2.1), $J(u) \geq m$. It follows that $J(u) = m$, i. e. u is a ground state solution of problem (2.1). The proof is complete. $\square$

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