Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 91, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

HARNACK INEQUALITY FOR QUASILINEAR ELLIPTIC EQUATIONS WITH (p,q) GROWTH CONDITIONS AND ABSORPTION LOWER ORDER TERM

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Communicated by Marco Squassina

Abstract. In this article we study the quasilinear elliptic equation with absorption lower term

$$-\operatorname{div}\Big(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big)+f(u)=0,\quad u\geq 0.$$

Despite of the lack of comparison principle, we prove a priori estimate of Keller-Osserman type. Particularly, under some natural assumptions on the functions g,f for nonnegative solutions we prove an estimate of the form

$$\int_0^{u(x)} f(s)\,ds\leq c\frac{u(x)}{r}g\big(\frac{u(x)}{r}\big),\quad x\in\Omega, B_{8r}(x)\subset\Omega,$$

with constant c, independent on $u(x)$. Using this estimate we give a simple proof of the Harnack inequality.

1. Introduction

In this article we consider nonnegative solutions of the quasilinear elliptic equation

$$-\operatorname{div}A(x,\nabla u)+a_0(u)=0, x\in\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n\geq 2$. We suppose that the functions $A=(a_1,a_2,\dots,a_n)$ and a_0 satisfy the Caratheodory conditions and the following structural conditions

$$\begin{gathered}A(x,\xi)\xi\geq\nu_1 g(|\xi|)|\xi|,\quad |A(x,\xi)|\leq\nu_2 g(|\xi|),\\ \nu_1 f(u)\leq a_0(u)\leq\nu_2 f(u),\end{gathered} \tag{1.2}$$

where ν_1,ν_2 are positive constants and g is positive function satisfying conditions

$$g\in C(\mathbb{R}^1_+),\quad \big(\frac{t}{\tau}\big)^{p-1}\leq\frac{g(t)}{g(\tau)}\leq\big(\frac{t}{\tau}\big)^{q-1},\quad t\geq\tau>0,\ 1<p\leq q<n. \tag{1.3}$$

Harnack's inequality for linear elliptic equations established by Moser [16] is one of the most important results in the theory of partial differential equations. Serrin[21], Trudinger [25, 26], Di Benedetto and Trudinger [2], generalized Moser's

2010 *Mathematics Subject Classification.* 35J15, 35J60, 35J62.
Key words and phrases. Harnack inequality; quasilinear elliptic equation;
Keller-Osserman type estimate; absorption lower term.

Submitted June 14, 2017. Published April 16, 2018.

result to the case of quasilinear elliptic equations with lower order terms from L_s spaces. We also would like to comment on the Harnack type estimates for elliptic equations with absorption term. The strong maximum principle for the equation $-\Delta u+f(u)=0$ was proved by Benilan, Brezis and Grandall [1] under the conditions

$$\int_0^1 \frac{du}{\sqrt{F(u)}}=\infty, \quad F(u)=\int_0^u f(s)\,ds.$$

Further it was extended by Vazquez [29] for the equation $-\Delta_p u+f(u)=0$ and for the equation (1.1) by Pucci and Serrin [18]-[20] and by Felmer, Montenegro and Quaas [27]. Finn and McQwen [28], Dindos [3]. Mohammed and Porru [15] proved the Harnack inequality for the non-divergence linear elliptic equations with absorption term. This was extended by Julin [5, 6] to the linear divergence and non-divergence elliptic equations with absorption term. Harnack's inequality for the equations of the type $-\Delta_p u+f(u)=0$ and $u_t-\Delta_p u+f(u)=0$ was proved in [24]. It is natural to conjecture that the Harnack inequality holds for the elliptic equations with non-standard growth conditions perturbed by absorption term. Our strategy of the proof of the Harnack inequality is similar to that in [24].

In the paper we prove estimates of Keller-Osserman type for solutions to elliptic equation with nonstandard growth conditions and absorption lower term; after that we give a simple proof of the Harnack inequality.

Before formulating the main result let us remind the reader the definition of the weak solution to the equation (1.1).

Definition 1.1. Let $G(t)=tg(t)$. Then note by $W^{1,G}(\Omega)$ the class of functions u that are weakly differentiable in Ω and satisfy the condition

$$\int_\Omega G(|\nabla u|)\,dx<\infty.$$

Definition 1.2. We say that u is a weak solution to (1.1), if $u\in W^{1,G}(\Omega)$ and satisfies the integral equation

$$\int_\Omega \{A(x,\nabla u)\nabla\varphi+a_0(u)\varphi\}\,dx=0, \tag{1.4}$$

for any $\varphi\in \mathring{W}^{1,G}(\Omega)$.

Let $x_0\in\Omega$. For any $\rho>0$ we set

$$F(u)=\int_0^u f(s)\,ds, \delta(u)=\frac{F(u)}{f(u)}, M(\rho)=\sup_{B_\rho(x_0)} u,$$
$$\delta(\rho)=\sup_{B_\rho(x_0)}\delta(u), \quad F(\rho)=\sup_{B_\rho(x_0)}F(u),$$

where $B_\rho(x_0)$ is ball $\{x:|x-x_0|<\rho\}$.

The next theorem is an a priori estimate of Keller-Osserman type, which is interesting in itself and which can be used in the theory of "large" solutions (see for example [11, 30], [8]–[10]).

Theorem 1.3. *Let conditions* (1.2), (1.3) *be fulfilled and u be a nonnegative weak solution to the equation* (1.1) *in Ω. Let $x_0\in\Omega$. Fix $\sigma\in(0,1)$. Then there exist a positive numbers c_1,c_2, depending only on n,p,q,ν_1,ν_2 such that*

$$F(\sigma\rho)\le c_1(1-\sigma)^{-c_2}\frac{\delta(\rho)}{\rho}\Big(g\big(\frac{M(\rho)}{\rho}\big)+g\big(\frac{\delta(\rho)}{\rho}\big)\Big), \tag{1.5}$$

for all $B_{8\rho}(x_0) \subset \Omega$.

Remark 1.4. Conditions (1.1), (1.3) imply the local boundedness and Hölder continuity of solutions (see, for example [14]).

Remark 1.5. For the case $p = q$ inequality (1.5) was proved in [24]. In the case $p = q$, using the comparison theorem an radial type solutions, inequality of the type (1.5) was proved in [8].

To prove the Harnack inequality for equations with absorption lower terms we need the following condition.

Definition 1.6. We say that a continuous function ψ satisfies condition (A) if there exists $\mu > 0$ such that

$$\frac{\psi(t)}{\psi(\tau)} \le \big(\frac{t}{\tau}\big)^{\mu}, \tag{1.6}$$

for all $t \ge \tau > 0$.

Condition (A) arises due to presence of absorption lower order terms in the equation (1.1): this condition was not presented in [14], but it is closely connected with analogous conditions in the works [18]–[20].

Theorem 1.7. *Let G^{-1} be the inverse function to the function $G(t) = tg(t)$, and let conditions* (1.2), (1.3) *be fulfilled. Let also u be a nonnegative weak solution to the equation* (1.1)*, function $f(u)$ be nondecreasing and $\psi(u) = u^{-1}G^{-1}(F(u))$ satisfies condition* (A)*.Then there exists positive number c_3, depending only on n, p, q, ν_1, ν_2, c such that*

$$F(u(x)) \le c_3 \frac{u(x)}{\rho} g\left(\frac{u(x)}{\rho}\right), \tag{1.7}$$

for almost all $x \in B_\rho(x_0)$ and for any $x_0 \in \Omega$, such that $B_{8\rho}(x_0) \subset \Omega$.

The following theorem is Harnack inequality for the nonnegative weak solutions to the equation (1.1), which is simple consequence of the Theorem 1.7.

Theorem 1.8. *Let u be a nonnegative weak solution to the equation* (1.1)*, let conditions* (1.2), (1.3) *be fulfilled. Assume that function $f(u)$ is nondecreasing and $\psi(u) = u^{-1}G^{-1}(F(u))$ satisfies condition (A).Then there exists positive number c_4, depending only on n, p, q, ν_1, ν_2, such that*

$$\sup_{B_\rho(x_0)} u(x) \le c_4 \inf_{B_\rho(x_0)} u(x), \tag{1.8}$$

for almost all $x \in B_\rho(x_0)$, and for any $x_0 \in \Omega$, such that $B_{8\rho}(x_0) \subset \Omega$.

Remark 1.9. The formulation of the Theorem 1.7 is the same as in [14], however due to presence of absorption lower order term, the results of [14] cannot be used. The main novelty of our result that the constant c_4 is independent on u.

Remark 1.10. If $f(u) = g(u)f_1(u)$, where function $f_1(u)$ satisfies condition (A) with $\mu_1 > q - p$, then the function $u^{-1}G^{-1}(F(u))$ satisfies condition (A) with $\mu = \frac{\mu_1 - q + p}{q} > 0$. A simple example of the function $f_1(u)$, which satisfies condition (A) for $\mu_1 = 1$ is a function $f_1 \in C^1(\mathbb{R}^1_+, f_1$ is nondecreasing and $f_1(0) = 0$.

Remark 1.11. If $f(u) = g^s(u)f_1(u)$, where f_1 is nondecreasing and $s > \frac{q-1}{p-1}$, then the function $u^{-1}G^{-1}(F(u))$ satisfies condition (A) with $\mu = \frac{(p-1)s - q + 1}{q}$.

2. Keller-Osserman a priori sub-estimate. Proof of Theorem 1.3

2.1. **Auxiliary statements and local energy estimates.** First of all we prove the following auxiliary statements, which will be used for further investigations.

Lemma 2.1. *Let $\{y_j\}_{j\in N}$ be a sequence of nonnegative numbers such that the following inequalities*

$$y_{j+1} \le C b^j y_j^{1+\varepsilon}$$

hold for $j=0,1,2,\dots$ with positive constants $\varepsilon, C>0, b>1$. Then

$$y_j \le C^{\frac{(1+\varepsilon)^j-1}{\varepsilon}} b^{\frac{(1+\varepsilon)^j-1}{\varepsilon^2}-\frac{j}{\varepsilon}} y_0^{(1+\varepsilon)^j}.$$

In particular, if $y_0 \le C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$, then $\lim_{j\to\infty} y_j = 0$.

We denote by the γ some constant depending only on n, p, q, ν_1, ν_2 which may vary from line to line. Let $B_r(\bar{x}) \subset \Omega$ be a ball in Ω, then we denote by the ζ some nonnegative piecewise smooth truncated function vanishing on the boundary of the ball $B_r(\bar{x})$.

Lemma 2.2. *Let u be a nonnegative weak solution to the equation* (1.1) *and let conditions* (1.2) *and* (1.3) *hold. Then for every $B_r(\bar{x}) \subset \Omega$ and for every $k>0$*

$$\begin{aligned} &\int_{A_{k,r}} f(u)\, G(|\nabla u|)\zeta^q\, dx + \int_{A_{k,r}} (F(u)-k)_+ f(u)\zeta^q\, dx \\ &\le \gamma \int_{A_{k,r}} (F(u)-k)_+ g(\delta(u)|\nabla\zeta|)|\nabla\zeta| dx, \end{aligned} \tag{2.1}$$

where $A_{k,r} = \{x \in B_r(\bar{x}) : F(u) > k\}$.

Proof. Testing integral equality (1.4) by the $\varphi = (F(u)-k)_+\zeta^q$. Using conditions (1.2) and (1.3) we obtain

$$\begin{aligned} &\int_{A_{k,r}} f(u)\, G(|\nabla u|)\zeta^q\, dx + \int_{A_{k,r}} (F(u)-k)_+ f(u)\zeta^q\, dx \\ &\le \gamma \int_{A_{k,r}} (F(u)-k)_+ g(|\nabla u|)|\nabla\zeta|\zeta^{q-1} dx. \end{aligned}$$

Let us note that the next inequality is evident

$$g(a)b \le \varepsilon g(a)a + g\Big(\frac{b}{\varepsilon}\Big)b, \quad a, b, \varepsilon > 0. \tag{2.2}$$

We use this inequality with $a = |\nabla u|, b = \gamma(F(u)-k)_+\frac{|\nabla\zeta|}{\zeta}, \varepsilon = \frac{1}{2}f(u)$ and arrive to the required inequality (2.1) $\square$

2.2. **Proof of Theorem 1.3.** Consider a ball $B_\rho(x_0)$ and for fixed $\sigma\in(0,1)$ let $\bar{x}$ be an arbitrary point in ball $B_{\sigma\rho}(x_0)$. Further we set

$$\rho_j = \frac{1-\sigma}{4}\rho(1+2^{-j}), \quad B_j = B_{\rho_j}(\bar{x}), A_{k_j,j} = \{x\in B_j : F(u) > k_j\}, \quad j=0,1,\dots$$
$$\zeta_j \in C_0^\infty(B_j), \quad 0\le\zeta_j\le 1, \quad |\nabla\zeta_j| \le \gamma(1-\sigma)^{-1}2^{-j}\rho^{-1}$$

and $\zeta_j \equiv 1$ in B_{j+1}. By the embedding theorem and Hölder inequality we obtain

$$
\begin{aligned}
&\int_{A_{k_{j+1},j+1}} (F(u)-k_{j+1})_+ dx \\
&\le \Big(\int_{A_{k_{j+1},j}} ((F(u)-k_{j+1})_+ \zeta_j^q)^{\frac{n}{n-1}} dx \Big)^{\frac{n-1}{n}} |A_{k_{j+1},j+1}|^{1/n} \\
&\le \gamma \int_{A_{k_{j+1},j}} |\nabla((F(u)-k_{j+1})_+\zeta_j^q)| |A_{k_{j+1},j}|^{1/n} \\
&\le \gamma \Big(\int_{A_{k_{j+1},j}} f(u)|\nabla u| \zeta_j^q dx \\
&\quad + \int_{A_{k_{j+1},j}} (F(u)-k_{j+1})_+ |\nabla \zeta_j| \zeta_j^{q-1} dx \Big) |A_{k_{j+1},j}|^{1/n}.
\end{aligned} \tag{2.3}
$$

Let $\ell = \delta(\rho)/\rho$. Using inequality (2.2) with $a = \ell, b = |\nabla u|, \varepsilon = 1$ and the evident inequality $(F(u) - k_{j+1})_+ \ge \frac{k}{2^{j+1}}$ on $A_{k_{j+1},j}$, we estimate the first term in the right-hand side of (2.3) as follows

$$
\begin{aligned}
&\int_{A_{k_{j+1},j}} f(u)|\nabla u|\zeta_j^q dx \\
&= \frac{1}{g(\ell)} \int_{A_{k_{j+1},j}} f(u) g(\ell) |\nabla u| \zeta_j^q dx \\
&\le \ell \int_{A_{k_{j+1},j}} f(u)\zeta_j^q dx + \frac{1}{g(\ell)} \int_{A_{k_{j+1},j}} f(u) G(|\nabla u|)\zeta_j^q dx \\
&\le 2^j \frac{\ell}{k} \int_{A_{k_{j+1},j}} (F(u)-k_{j+1})_+ f(u)\zeta_j^q dx + \frac{1}{g(\ell)} \int_{A_{k_{j+1},j}} f(u) G(|\nabla u|)\zeta_j^q dx.
\end{aligned} \tag{2.4}
$$

From the previous inequality and Lemma 2.2 it follows that

$$
\begin{aligned}
&\int_{A_{k_{j+1},j}} f(u)|\nabla u|\zeta_j^q dx \\
&\le \gamma(1-\sigma)^{-\gamma} 2^{j\gamma} \big(\frac{\ell}{k} + \frac{1}{g(\ell)}\big) \rho^{-1} g\big(\frac{\delta(\rho)}{\rho}\big) \int_{A_{k_j,j}} (F(u)-k_j)_+ dx.
\end{aligned} \tag{2.5}
$$

Choosing k such that

$$
k \ge G(\ell) = G\left(\frac{\delta(\rho)}{\rho}\right), \tag{2.6}
$$

from inequalities (2.3) and (2.4) we obtain

$$
y_{j+1} = \int_{A_{k_{j+1},j+1}} (F(u)-k_{j+1}) dx \le \gamma(1-\sigma)^{-\gamma} 2^{j\gamma} \rho^{-1} k^{-\frac{1}{n}} y_j^{1+\frac{1}{n}}. \tag{2.7}
$$

from Lemma 2.1 it follows that $y_j \to 0$ as $j \to \infty$, provided k is chosen to satisfy

$$
k \ge \gamma(1-\sigma)^{-\gamma} \rho^{-n} \int_{B_{\frac{1-\sigma}{2}\rho}(\bar{x})} F(u) dx. \tag{2.8}
$$

Inequalities (2.5) and (2.6) imply that

$$F(u(\bar{x})) \le \gamma(1-\sigma)^{-\gamma} G\left(\frac{\delta(\rho)}{\rho}\right) + \gamma(1-\sigma)^{-\gamma}\rho^{-n}\int_{B_{\frac{1-\sigma}{2}\rho}(\bar{x})} F(u)dx. \tag{2.9}$$

Let $\xi \in C_0^\infty(B_{(1-\sigma)\rho}(\bar{x})), 0 \le \xi \le 1, \xi \equiv 1$ in $B_{\frac{1-\sigma}{2}\rho}(\bar{x})$ and $|\nabla\xi| \le 2(1-\sigma)^{-1}\rho^{-1}$. To estimate the integral in the right-hand side of the (2.9) we test (1.4) by $\varphi = \xi^q$. Using conditions (1.2), (1.3) we obtain

$$\begin{aligned}\int_{B_{\frac{1-\sigma}{2}\rho}(\bar{x})} F(u)dx &\le \delta(\rho)\int_{B_{(1-\sigma)\rho}(\bar{x})} f(u)\xi^q dx\\ &\le \gamma(1-\sigma)^{-1}\frac{\delta(\rho)}{\rho}\int_{B_{(1-\sigma)\rho}(\bar{x})} g(|\nabla u|)\xi^{q-1}dx.\end{aligned}$$

We use inequality (2.2) with $a = |\nabla u|, b = \xi^{-1}$ to obtain

$$\begin{aligned}\int_{B_{\frac{1-\sigma}{2}\rho}(\bar{x})} F(u)dx &\le \gamma(1-\sigma)^{-1}\frac{\delta(\rho)}{M(\rho)}\int_{B_{(1-\sigma)\rho}(\bar{x})} G(|\nabla u|)\xi^q dx\\ &\quad + \gamma(1-\sigma)^{-1}\frac{\delta(\rho)}{\rho} g\left(\frac{M(\rho)}{\rho}\right)\rho^n.\end{aligned} \tag{2.10}$$

Test (1.4) by the function $\varphi = u\xi^q$. Using (1.2) and (1.3) we obtain

$$\int_{B_{(1-\sigma)\rho}(\bar{x})} G(|\nabla u|)\xi^q dx \le \gamma(1-\sigma)^{-\gamma} G\left(\frac{M(\rho)}{\rho}\right)\rho^n. \tag{2.11}$$

Combining (2.10) and (2.11) we arrive at

$$\int_{B_{(1-\sigma)\rho}(\bar{x})} F(u)dx \le \gamma(1-\sigma)^{-\gamma}\frac{\delta(\rho)}{\rho} g\left(\frac{M(\rho)}{\rho}\right)\rho^n. \tag{2.12}$$

Since $\bar{x}$ is an arbitrary point in $B_{\sigma\rho}(x_0)$, from (2.8) and (2.12) we obtain the required inequality (1.5). So, Theorem 1.3 is proved.

2.3. **Proof of Theorem 1.7.** For $j = 1, 2, \dots$, let us define the sequences $\{\sigma_j\}$, $\{\rho_j\}$, $\{M_j\}$ such that

$$\sigma_j = \frac{1-2^{-j-1}}{1-2^{-j-2}}, \quad \rho_j = \rho\big(1 + \frac{1}{2} + \dots + \frac{1}{2^j}\big), \quad M_j = \sup_{B_{\rho_j(x_0)}} u.$$

Rewrite inequality (1.5) for the pair of balls $B_{\rho_{j+1}}(x_0), B_{\rho_j}(x_0)$:

$$G^{-1}(F(M_j)) \le \gamma 2^{\gamma j}\rho^{-1} M_{j+1}.$$

If $\varepsilon > 0$, we obtain

$$\begin{aligned}\psi(M_j) &\le \psi(\varepsilon M_{j+1}) + \frac{1}{\varepsilon}\frac{\psi(M_j)M_j}{M_{j+1}}\\ &\le \psi_a(\varepsilon M_{j+1}) + \varepsilon^{-1}\gamma 2^{\gamma j}\rho^{-1}.\end{aligned}$$

Using condition (A) we arrive at following recursive inequalities

$$\psi(M_j) \le \varepsilon^\mu \psi(M_{j+1}) + \varepsilon^{-1}\gamma 2^{\gamma j}\rho^{-1},$$

$j = 0, 1, 2, \dots$, or

$$\psi(M_0) \le \varepsilon^{j\mu}\psi(M_j) + \varepsilon^{-1}\gamma\rho^{-1}\sum_{k=0}^{j-1}\varepsilon^{k\mu}2^{kj}.$$

We chose $\varepsilon^{\mu} = 2^{-\gamma-1}$ so that the sum on the previous inequality can be majorized by convergent series. Let $j \to \infty$. Then

$$\psi(u(x_0)) \le \psi(M_0) \le \gamma\rho^{-1}.$$

This proves the Theorem 1.7.

3. Harnack inequality. Proof of Theorem 1.8

Let x_0 be some inner point in Ω and $B_{8\rho}(x_0) \subset \Omega$. Fix $\bar{x} \in B_\rho(x_0), \sigma \in (0,1), 0 < r \le \rho$. Let also $\xi \in C_0^\infty(B_r(\bar{x})), 0 \le \xi \le 1, \xi \equiv 1$ in $B_{\sigma\rho}(\bar{x})$ and $|\nabla\xi| \le (1-\sigma)^{-1}r^{-1}$. The following lemma is an auxiliary result for proving Harnack inequality (Theorem 1.8).

Lemma 3.1. *Let the conditions of Theorem 1.8 be fulfilled. Then for every $0 < k < \sup_{B_{2\rho}(x_0)} u$, the next inequalities hold*

$$\int_{B_r(\bar{x})} G(|\nabla(u-k)_+|)\xi^q dx \le \gamma G\Big(\frac{||(u-k)_+||_{L^\infty(B_r(\bar{x}))}}{(1-\sigma)r}\Big)|A^+_{k,r}|, \tag{3.1}$$

$$\int_{B_r(\bar{x})} G(|\nabla(k-u)_+|)\xi^q dx \le \gamma G\Big(\frac{k}{(1-\sigma)r}\Big)|A^-_{k,r}|, \tag{3.2}$$

where $A^{\pm}_{k,r} = B_r(\bar{x}) \cap \{(u-k)_\pm > 0\}$.

Proof. Testing (1.4) by $\varphi = (u-k)_+\xi^q$ and using (1.2), (1.3), (2.2) we arrive at (3.1). To prove (3.2) we test (1.4) by the function $\varphi = (k-u)_+\xi^q$, using (2.2) we obtain

$$\int_{B_r(\bar{x})} G(|\nabla(k-u)_+|)\xi^q dx \le \gamma G\Big(\frac{k}{(1-\sigma)r}\Big)|A^-_{k,r}| + \gamma\int_{B_r(\bar{x})} f(u)(k-u)_+\xi^q dx.$$

The last term of the previous inequality can be estimated in the following way

$$\begin{aligned}\int_{B_r(\bar{x})} f(u)(k-u)_+\xi^q dx &= \int_{B_r(\bar{x})} f(u)\chi(u<k)\int_u^k ds\,\xi^q dx \\ &\le \int_{B_r(\bar{x})} \chi(u<k)\int_u^k f(s)ds\,\xi^q dx \\ &\le \int_0^k f(s)ds|A^-_{k,r}| = F(k)|A^-_{k,r}|.\end{aligned}$$

By Theorem 1.7 we obtain

$$\frac{F(k)}{G(\frac{k}{\rho})} \le \frac{F\big(\sup_{B_{2\rho}(x_0)} u\big)}{G\big(\frac{1}{\rho}\sup_{B_{2\rho}(x_0)} u\big)} \le \gamma.$$

The above inequality proves (3.2), and completes the proof. □

The following lemma is an expansion of positivity result, analogue in formulation as well as in the proof to [14, Lemmas 6.3, 6.4].

Lemma 3.2. *Let the conditions of Theorem 1.8 be fulfilled. Assume that for some $\bar{x} \in \Omega$, some $r > 0, N > 0$ and some $\alpha \in (0,1)$,*

$$|\{x \in B_r(\bar{x}) : u(x) \leq N\}| \leq (1-\alpha)|B_r(\bar{x})|.$$

Then for any $\varepsilon \in (0,1)$ there exists constant $\delta \in (0,1/2)$ depending only on $n, p, q, \nu_1, \nu_2, \alpha$ and ε such that

$$|\{x \in B_{4r}(\bar{x}) : u(x) \leq 2\delta N\}| \leq \varepsilon |B_{4r}(\bar{x})|,$$

and furthermore $u(x) \geq \delta N$ for almost all $x \in B_{2r}(\bar{x})$.

The next lemma is a De Giorgi type lemma, the proof of which is similar to that of [14, Lemma 6.4].

Lemma 3.3. *Let the conditions of Theorem 1.8 be fulfilled, $\bar{x} \in \Omega$, fix $r > 0, \xi, a \in (0,1)$. There exists number $\varepsilon_0 \in (0,1)$ depending only on $n, p, q, \nu_1, \nu_2, \xi$ and a such that if*

$$|\{x \in B_r(\bar{x}) : u(x) \leq M(1-\xi)\}| \leq \varepsilon_0 |B_r(\bar{x})|,$$

with some $M \geq \sup\limits_{B_r(\bar{x})} u$, then

$$u(x) \leq M(1 - a\xi) \, for \, a.a. \, x \in B_{2r}(\bar{x}).$$

Because of Lemmas 3.2 and 3.3, the rest of the arguments do not differ from the corresponding result in [4] and [14]. This completes the proof Theorem 1.8.

Conclusion. In the paper there was studied quasilinear double-phase elliptic equations with absorption term

$$-\operatorname{div}\Big(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) + f(u) = 0, \quad u \geq 0.$$

Despite the lack of comparison principle, we proved a priori estimate of Keller-Osserman type. Particularly, under some natural assumptions on the functions g, f for nonnegative solutions we proved an estimate of the form

$$\int_0^{u(x)} f(s)\, ds \leq c\frac{u(x)}{r} g\Big(\frac{u(x)}{r}\Big), \quad x \in \Omega, \; B_{8r}(x) \subset \Omega,$$

with constant c, independent on $u(x)$. Using this estimate we presented a simple proof of the Harnack inequality.

Acknowledgments. This research was supported by grants of Ministry of Education and Science of Ukraine (project numbers are 0118U003138, 0116U004691) and by SFFR of Ukraine (grant No. 116U007160).

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