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STABILITY ANALYSIS OF A WEIGHTED DIFFERENCE SCHEME FOR TWO-DIMENSIONAL HYPERBOLIC EQUATIONS WITH INTEGRAL CONDITIONS

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Abstract. We consider two-dimensional hyperbolic equations with nonlocal purely integral conditions. We analyze the spectral properties of the finite difference scheme for the two-dimensional hyperbolic problem. To analyze the stability of a weighted difference scheme, we investigate the spectrum of a finite difference operator, subject to integral conditions.

1. Introduction

In this article, we consider the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad (x,y) \in \Omega,\ t \in (0,T], \tag{1.1}$$

where $\Omega = (0,1) \times (0,1)$, with initial conditions

$$u(x,y,0) = \phi(x,y), \quad \frac{\partial u(x,y,0)}{\partial t} = \psi(x,y), \quad x \in [0,1] \tag{1.2}$$

and the nonlocal integral conditions

$$\int_0^1 u(x,y,t)\,dx = g_1(y,t), \quad \int_0^1 xu(x,y,t)\,dx = g_2(y,t), \tag{1.3}$$

$$\int_0^1 u(x,y,t)\,dy = g_3(x,t), \quad \int_0^1 yu(x,y,t)\,dy = g_4(x,t), \tag{1.4}$$

where $x \in [0,1]$, $y \in [0,1]$, and $t \in [0,T]$.

The mathematical modelling of modern physical problems requires defining appropriate nonlocal boundary conditions. Such conditions are used when it is impossible to determine the boundary values of unknown function and its derivatives. Nonlocal integral conditions represent averaged data and are often used in practice, for example some recent articles in noise control and suppression problems [14], diffusion processes [2] and complex dynamical systems [1]. We also notice, that a broad list of literature on differential equations subject to nonlocal conditions can be found in [36].

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The uniqueness and existence of a solution for one-dimensional hyperbolic equation with nonlocal integral conditions were considered by many authors [5, 6, 8, 24, 27]. Nonlocal problem for two- or n-dimensional hyperbolic equation was a topic in [7, 19, 28].

The solution for two-dimensional hyperbolic integro-differential equation subject to nonlocal integral conditions (1.3)–(1.4) was presented in [23]. Integral conditions of the type (1.3)–(1.4) are commonly called purely integral conditions. Such boundary conditions in various dynamic problems represent moments (of the zero and first order), and can be found in different nonlocal problems (not necessarily hyperbolic) [10, 9, 22].

In the mathematical sense purely integral conditions (1.3)–(1.4) are of a practical interest for the reason, that the eigenspectrum of the simplest differential and difference operators with these conditions has special properties: all eigenvalues are strictly positive, eigenvectors are linearly independent (see e.g. [17]). The eigenspectrum structure of the problems with other type nonlocal conditions can be complex [33].

Motivated by previous works, the aim of this paper is to extend our previous results in [16, 25, 26] by applying the eigenspectrum analysis methods to the two-dimensional hyperbolic problem (1.1)–(1.4) with nonlocal integral conditions. The stability results in these papers are proved using the analysis of non selfadjoint operators of the three-layer finite difference scheme [30]. The stability of high-accuracy finite difference scheme for one-dimensional Klein–Gordon equation with integral conditions is studied in [21].

To the authors' knowledge, the stability analysis of the finite difference schemes for the two-dimensional hyperbolic equations with nonlocal integral conditions, using spectral properties of difference operators, is investigated for the first time. Another methods of investigating finite difference schemes for hyperbolic equations with integral conditions can be found in [3, 4].

The paper is organized as follows. In Section 2 notation and definitions used in the paper are stated. In Sections 3 and 4 the finite difference problem is formulated and an eigenvalue problem for a finite difference operator is stated and certain spectral properties of this operator are investigated. The detailed eigenspectrum and stability analysis of the three-layer finite difference scheme is provided in Section 5.

2. Notation

We introduce uniform grids

$$\begin{gathered}
\overline{\omega}_x^h := \{x_i \colon x_i = ih,\ i = \overline{0,N}\}, \quad \overline{\omega}_y^h = \{y_j \colon y_j = jh, \quad j = \overline{0,N}\},\ h = 1/N, \\
\overline{\omega}^\tau := \{t^n \colon t^n = n\tau, n = \overline{0,M}\}, \quad \tau = T/M, \quad \widetilde{\omega}^\tau := \{t^1, \dots, t^M\}, \\
\omega_x^h := \{x_1, \dots, x_{N-1}\}, \quad \omega_y^h := \{y_1, \dots, y_{N-1}\}, \quad \omega^\tau := \{t^1, \dots, t^{M-1}\}, \\
\overline{\omega}^h := \overline{\omega}_x^h \times \overline{\omega}_y^h, \quad \omega^h := \omega_x^h \times \omega_y^h,
\end{gathered}$$

where $N+1$ is the number of grid points for x and y directions, $M+1$ is the number of grid points for t direction, and $N, M \geq 2$.

Remark 2.1. We use a unit square domain $\overline{\omega}$ (Ω for the differential case) for simplicity. The results are valid on any extended rectangular domain. The grid steps h for x and y directions are also used for simplicity.

We use the notation $U_{ij}^n := U(x_i, y_j, t^n)$ for the function defined on the grid (or parts of the grid) $\overline{\omega}^h \times \overline{\omega}^\tau$. We denote $\check{U} := U^{n-1}$ and $\widehat{U} := U^{n+1}$ on grids $\widetilde{\omega}^\tau$ and $\omega^\tau \cup \{t_0\}$ respectively. We define space grid operators:

$$\delta_x^2 \colon \overline{\omega}^h \to \omega^h, \quad \left(\delta_x^2 U\right)_{ij} := \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h^2},$$
$$\delta_y^2 \colon \overline{\omega}^h \to \omega^h, \quad \left(\delta_y^2 U\right)_{ij} := \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{h^2},$$

and time grid operators

$$\overline{\partial}_t \colon \overline{\omega}^\tau \to \widetilde{\omega}^\tau, \quad \overline{\partial}_t U := \frac{U - \check{U}}{\tau},$$
$$\partial_t^2 \colon \overline{\omega}^\tau \to \omega^\tau, \quad \partial_t^2 U := \frac{\widehat{U} - 2U + \check{U}}{\tau^2},$$

We consider weight $\sigma \in \mathbb{R}$ in the finite difference scheme

$$U^{(\sigma)} = \sigma \widehat{U} + (1 - 2\sigma) U + \sigma \check{U}.$$

Let $\overline{H}$ and H be spaces of real grid functions on $\overline{\omega}^h$ and ω^h, respectively. Functions $U \in H$ can be represented as vectors $\mathbf{U} := (U_{\cdot 1}, \dots, U_{\cdot, N-1})^\top$, $U_{\cdot j} := (U_{1j}, \dots, U_{N-1,j})$, $j = \overline{1, N-1}$. Let U and V be the grid functions. We use the following notation

$$[U, V]_{x,j} := U_{0j} V_{0j} h/2 + (U, V)_{x,j} + U_{Nj} V_{Nj} h/2, \quad U, V \in \overline{H}, \ \forall j = \overline{0, N},$$
$$[U, V]_{y,i} := U_{i0} V_{i0} h/2 + (U, V)_{y,j} + U_{iN} V_{iN} h/2, \quad U, V \in \overline{H}, \ \forall i = \overline{0, N},$$
$$(U, V)_{x,j} := \sum_{i=1}^{N-1} U_{ij} V_{ij} h, \quad U, V \in H, \quad \forall j = \overline{1, N-1},$$
$$(U, V)_{y,i} := \sum_{j=1}^{N-1} U_{ij} V_{ij} h, \quad U, V \in H, \quad \forall i = \overline{1, N-1}.$$

Let $\mathbf{P}$ be a nonsingular matrix ($\det \mathbf{P} \neq 0$); we define the norm of any $m \times m$ matrix $\mathbf{M}$ as follows:

$$\|\mathbf{M}\|_* = \|\mathbf{P}^{-1} \mathbf{M} \mathbf{P}\|_2,$$

where $\|\mathbf{M}\|_2 = (\max_{1 \le i \le m} \lambda_i(\mathbf{M}^* \mathbf{M}))^{1/2}$ is the classical matrix norm and $\mathbf{M}^*$ is the adjoint matrix. We define the associated vector norm by the formula

$$\|\mathbf{V}\|_* = \|\mathbf{P}^{-1} \mathbf{V}\|_2 = \Big(\sum_{i=1}^m |\tilde{V}_i|^2 \Big)^{1/2}, \tag{2.1}$$

where $\tilde{V}_i$, $i = \overline{1, m}$ are the coordinates of the vector $\mathbf{P}^{-1} \mathbf{V}$.

If a nonsymmetric $(m \times m)$ matrix $\mathbf{S}$ has linearly independent eigenvector system $\mathbf{V}^1, \mathbf{V}^2, \dots, \mathbf{V}^m$, then the matrix $\mathbf{T} = (\mathbf{V}^1, \mathbf{V}^2, \dots, \mathbf{V}^m)$ is nonsingular and we have a relation

$$\|\mathbf{S}\|_* = \|\mathbf{T}^{-1} \mathbf{S} \mathbf{T}\|_2 = \|\mathbf{J}\|_2 = \max_{1 \le i \le m} \|\mu_i(\mathbf{S})\| = \rho(\mathbf{S}), \tag{2.2}$$

where $\mathbf{J} = \operatorname{diag}(\mu^1, \dots, \mu^m)$, μ^i, $i = \overline{1, m}$ are the eigenvalues of matrix $\mathbf{S}$ and $\rho(\mathbf{S})$ is the spectral radius of matrix $\mathbf{S}$.

The vector norm associated with the matrix norm (2.2) is defined by identity (2.1) with $\mathbf{P} = \mathbf{T}$.

3. Finite difference scheme

We state a finite difference scheme for the two-dimensional differential problem (1.1)–(1.4)

$$\partial_t^2 U - \left(\delta_x^2 + \delta_y^2\right) U^{(\sigma)} = F, \quad (x_i, y_j, t^n) \in \omega^h \times \omega^\tau, \tag{3.1}$$

where σ is a scheme weight parameter. The initial conditions are approximated as follows

$$U^0 = \Phi, \quad (x_i, y_j) \in \overline{\omega}^h, \tag{3.2}$$

$$\overline{\partial}_t U^1 = \Psi, \quad (x_i, y_j) \in \overline{\omega}^h, \tag{3.3}$$

and the boundary conditions

$$[1, U]_x = G_1, \quad (y_j, t^n) \in \overline{\omega}_y^h \times \overline{\omega}^\tau, \tag{3.4}$$

$$[x, U]_x = G_2, \quad (y_j, t^n) \in \overline{\omega}_y^h \times \overline{\omega}^\tau, \tag{3.5}$$

$$[1, U]_y = G_3, \quad (x_i, t^n) \in \overline{\omega}_x^h \times \overline{\omega}^\tau, \tag{3.6}$$

$$[y, U]_y = G_4, \quad (x_i, t^n) \in \overline{\omega}_x^h \times \overline{\omega}^\tau. \tag{3.7}$$

Functions f, ϕ, ψ, g_1, g_2, g_3, and g_4 in the above stated problem (3.1)–(3.7) are approximated by grid functions F, Φ, Ψ, and G_1, G_2, G_3, and G_4, accordingly.

If the solution u of problem (1.1)-(1.3) is smooth enough $u \in C^4(\Omega \times [0, T])$, then scheme (3.1) approximates equation (1.1) at the point (x_i, y_j, t^n) with an accuracy $\mathcal{O}(h^2+\tau^2)$ (see e.g. [12]). The initial condition (3.2) is approximated exactly, and initial condition (3.3) with accuracy $\mathcal{O}(h^2)$ if $\Psi = \psi(x_i, y_j) + \frac{\tau}{2}((\delta_x^2 + \delta_y^2)U^0 + f(x_i, y_j, t^0))$. The approximation order of trapezoid formulas (3.4)–(3.7) is $\mathcal{O}(h^2)$. So, finite difference scheme (3.1)–(3.7) approximates differential problem (1.1)-(1.3) with accuracy $\mathcal{O}(h^2 + \tau^2)$.

Equations (3.4)–(3.7) can be considered as a system of linear equations for unknowns U_{0j}, U_{Nj}, U_{i0}, and U_{iN}. We express these unknowns via inner points U_{ij}, $i, j = \overline{1, N-1}$, and obtain

$$U_{0j} = 2\left(x - 1, U\right)_{x,j} + (\widetilde{G}_1)_j, \tag{3.8}$$

$$U_{Nj} = -2\left(x, U\right)_{x,j} + (\widetilde{G}_2)_j, \tag{3.9}$$

$$U_{i0} = 2\left(y - 1, U\right)_{y,i} + (\widetilde{G}_3)_i, \tag{3.10}$$

$$U_{iN} = -2\left(y, U\right)_{y,i} + (\widetilde{G}_4)_i, \tag{3.11}$$

where $\widetilde{G}_1 = 2h^{-1}\left(G_1 - G_2\right)$, $\widetilde{G}_2 = 2h^{-1} G_2$, $\widetilde{G}_3 = 2h^{-1}\left(G_3 - G_4\right)$, $\widetilde{G}_4 = 2h^{-1} G_4$.

We substitute expressions (3.8)–(3.11) into (3.1) for $i = 1$, $i = N - 1$ and $j = 1$, $j = N - 1$ and rewrite it in the matrix form

$$\mathbf{A}\widehat{\mathbf{U}} + \mathbf{B}\mathbf{U} + \mathbf{A}\check{\mathbf{U}} = \tau^2 \mathbf{F}, \quad \mathbf{F} = \left(F_{\cdot 1}, \dots, F_{\cdot, N-1}\right)^\top, \tag{3.12}$$

$$\mathbf{A} = \mathbf{I} + \tau^2 \sigma \mathbf{\Lambda}, \ \mathbf{B} = -2\mathbf{I} + \tau^2 (1 - 2\sigma) \mathbf{\Lambda}, \quad \mathbf{\Lambda} := \mathbf{\Lambda}_1 + \mathbf{\Lambda}_2, \tag{3.13}$$

where $F_{\cdot j} = \left(\widetilde{F}_{1j}, \dots, \widetilde{F}_{N-1,j}\right)$, $\widetilde{F}_{1j} = \widetilde{F}_{1j}(F_{1j}, G_1, G_2, G_3, G_4)$, $\widetilde{F}_{ij} = F_{ij}$, $i, j = \overline{2, N-2}$, $\widetilde{F}_{N-1,j} = \widetilde{F}_{N-1,j}(F_{N-1,j}, G_1, G_2, G_3, G_4)$,

$$\mathbf{\Lambda}_1 = \frac{1}{h^2}\begin{pmatrix} \mathbf{\Lambda}_x & & & & \\ & \mathbf{\Lambda}_x & & & \\ & & \ddots & & \\ & & & \mathbf{\Lambda}_x & \\ & & & & \mathbf{\Lambda}_x \end{pmatrix},$$

$$\mathbf{\Lambda}_2 = \frac{1}{h^2}\begin{pmatrix} (2-\alpha_1)\mathbf{I} & -(1+\alpha_2)\mathbf{I} & -\alpha_3\mathbf{I} & \dots & -\alpha_{N-2}\mathbf{I} & -\alpha_{N-1}\mathbf{I} \\ -\mathbf{I} & 2\mathbf{I} & -\mathbf{I} & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\mathbf{I} & 2\mathbf{I} & -\mathbf{I} \\ -\beta_1\mathbf{I} & -\beta_2\mathbf{I} & -\beta_3\mathbf{I} & \dots & -(1+\beta_{N-2})\mathbf{I} & (2-\beta_{N-1})\mathbf{I} \end{pmatrix},$$

are $(N-1)^2 \times (N-1)^2$ block matrices. In (3.13) the identity matrix $\mathbf{I}$ is $(N-1)^2 \times (N-1)^2$ matrix, too. The indentity matrix $\mathbf{I}$ in matrix $\mathbf{\Lambda}_2$ is $(N-1) \times (N-1)$ matrix. $\mathbf{\Lambda}_x$ is $(N-1) \times (N-1)$ matrix of the form

$$\mathbf{\Lambda}_x = \begin{pmatrix} 2-\alpha_1 & -1-\alpha_2 & -\alpha_3 & \cdots & -\alpha_{N-2} & -\alpha_{N-1} \\ -1 & 2 & -1 & \ddots & 0 & 0 \\ 0 & -1 & 2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -\beta_1 & -\beta_2 & -\beta_3 & \cdots & -1-\beta_{N-2} & 2-\beta_{N-1} \end{pmatrix},$$

where $\alpha_i = 2 - 2ih$, $\beta_i = -2ih$, $i = \overline{1, N-1}$.

Remark 3.1. Suppose all eigenvalues of matrix $\mathbf{\Lambda}$ are positive. In this case, if

$$\sigma > -\frac{1}{\tau^2 \lambda_{\max}}, \tag{3.14}$$

then $\det \mathbf{A} > 0$. Matrix $\mathbf{A}^{-1}$ exists for such σ.

4. Discrete eigenvalue problem

Now we investigate the eigenspectrum of the matrix $\mathbf{\Lambda}$. We consider the finite difference eigenvalue problem

$$\left(\delta_x^2 + \delta_y^2\right) U + \lambda U = 0, \quad (x_i, y_j) \in \omega^h, \tag{4.1}$$

$$[1, U]_x = 0, \quad [x, U]_x = 0, \quad y_j \in \overline{\omega}_y^h \tag{4.2}$$

$$[1, U]_y = 0, \quad [y, U]_y = 0, \quad x_i \in \overline{\omega}_i^h. \tag{4.3}$$

Remark 4.1. Eigenvalue problem (4.1)–(4.3) is equivalent to the algebraic eigenvalue problem

$$\mathbf{\Lambda U} = \lambda \mathbf{U}.$$

Theorem 4.2. *All the eigenvalues λ of the matrix $\mathbf{\Lambda}$ are positive and all the eigenvectors $\mathbf{U}$ are linearly independent for all $h > 0$.*

Proof. Using the Fourier method, we separate variables

$$U_{ij} = X_i Y_j, \quad x_i \in \overline{\omega}_x^h, \quad y_j \in \overline{\omega}_y^h. \tag{4.4}$$

By substituting (4.4) into eigenvalue problem (4.1)–(4.3) we obtain two one-dimensional problems

$$\delta_x^2 X + \xi X = 0, \quad x_i \in \omega_x^h, \tag{4.5}$$
$$[1, X]_x = 0, \tag{4.6}$$
$$[x, X]_x = 0, \tag{4.7}$$

and

$$\delta_y^2 Y + \eta Y = 0, \tag{4.8}$$
$$[1, Y]_y = 0, \tag{4.9}$$
$$[y, Y]_y = 0, \tag{4.10}$$

where $[U, V]_x := U_0V_0h/2 + (U, V)_x + U_NV_Nh/2$ for U, V defined on the grid $\overline{\omega}_x^h$, and $[U, V]_y := U_0V_0h/2 + (U, V)_y + U_NV_Nh/2$ for U, V defined on the grid $\overline{\omega}_y^h$, $(U, V)_x := \sum_{i=1}^{N-1} U_iV_ih$ and $(U, V)_y := \sum_{j=1}^{N-1} U_jV_jh$. The eigenvalues of the problem (4.1)–(4.3) are of the form

$$\lambda^{kl} = \xi^k + \eta^l.$$

The eigenfunctions of the first problem (4.5)–(4.7) can be found from the corresponding algebraic problem $\mathbf{\Lambda}_x\mathbf{X} = \xi\mathbf{X}$, $\mathbf{X} = (X_1, \dots, X_{N-1})^\top$. After we found the eigenvectors $\mathbf{X}^k = \big(X_1^k, \dots, X_{N-1}^k\big)$, we can reconstruct eigenfunctions $\big(X_0^k, X_1^k, \dots, X_N^k\big)$ using relations $X_0^k = 2(x-1, X^k)_x$ and $X_N^k = -2(x, X^k)_x$. Analogously, the corresponding algebraic problem for (4.8)–(4.10) is $\mathbf{\Lambda}_x\mathbf{Y} = \eta\mathbf{Y}$, $\mathbf{Y} = (Y_1, \dots, Y_{N-1})^\top$, and the eigenfunctions $\big(Y_0^l, Y_1^l, \dots, Y_N^l\big)$ can be reconstructed using relations $Y_0^l = 2(y-1, Y^l)_y$ and $Y_N^l = -2(y, Y^l)_y$.

Now, using the results of [17] we can analyze two one-dimensional problems (4.5)–(4.7) and (4.8)–(4.10). The general solution of the difference equation (4.5) is

$$X_i = c_1 \cos(\alpha i h) + c_2 \sin(\alpha i h), \quad i = \overline{0, N}. \tag{4.11}$$

By substituting this expression into nonlocal conditions (4.6)–(4.7) one gets eigenvalues (see e.g. [17])

$$\xi^k = \frac{4}{h^2} \sin^2 \frac{\alpha^k h}{2}, \quad k = \overline{1, N-1}, \tag{4.12}$$

where α^k are either roots of the equation

$$\sin \frac{\alpha}{2} = 0, \tag{4.13}$$

or of the equation

$$\tan \frac{\alpha}{2} = \frac{N}{2} \sin(\alpha h). \tag{4.14}$$

Equation (4.13) implies, that

$$\alpha^{2k-1} = 2k\pi, \quad k = \overline{1, k_1}, \quad k_1 = \begin{cases} N/2, & N \text{ is even}, \\ (N-1)/2, & N \text{ is odd}. \end{cases} \tag{4.15}$$

Analogously, (4.14) implies

$$\alpha^{2k} \in (2k\pi,(2k+1)\pi), \quad k=\overline{1,k_2}, \quad k_2 = \begin{cases} N/2-1, & N \text{ is even}, \\ (N-1)/2, & N \text{ is odd}. \end{cases} \tag{4.16}$$

Eigenvalues ξ^k are simple. The number of roots is $N-1$. Therefore, formula (4.12) defines $N-1$ real, positive and distinct eigenvalues of the eigenvalue problem (4.5)–(4.7). So, corresponding eigenfunctions are linearly independent.

Analogously, the eigenvalues of the problem (4.8)–(4.10) are defined by the formula

$$\eta^l = \frac{4}{h^2}\sin^2\frac{\alpha^l h}{2}, \quad l=\overline{1,N-1}, \tag{4.17}$$

where α^l are defined by the same formulas (4.15) and (4.16). Further, the eigenvalues of the problem (4.1)–(4.3) are real, positive, and of the form

$$\lambda^{kl} = \frac{4}{h^2}\Big(\sin^2\frac{\alpha^k h}{2}+\sin^2\frac{\alpha^l h}{2}\Big), \quad k,l=\overline{1,N-1}. \tag{4.18}$$

The eigenfunctions of the problem (4.1)–(4.3) are of the form

$$U^{kl}_{ij} = X^k_i \cdot Y^l_j, \quad i,j=\overline{0,N}, \quad k,l=\overline{1,N-1}. \tag{4.19}$$

Analogously as in [18], eigenfunctions U^{kl} can be defined as Kronecker (tensor) product of two one-dimensional eigenfunctions $X^k = \big(X^k_0,\dots,X^k_N\big)$ and $Y^l = \big(Y^l_0,\dots,Y^l_N\big)$

$$U^{kl} = Y^l \otimes X^k, \quad k,l=\overline{1,N-1}. \tag{4.20}$$

□

Remark 4.3. The eigenfunctions X^k_i (and Y^l_j) in (4.19) can be found by applying to the general solution (4.11) (analogously for Y^l_j) the condition (see [17])

$$\begin{gathered} c_1\frac{\sin\alpha}{\alpha}+c_2\frac{1-\cos\alpha}{\alpha}=0, \\ c_1\Big(\frac{\sin\alpha}{\alpha}-\frac{h(1-\cos\alpha)}{\alpha\sin(\alpha h)}\Big)+c_2\Big(\frac{h\sin\alpha}{\alpha\sin(\alpha h)}-\frac{\cos\alpha}{\alpha}\Big)=0. \end{gathered} \tag{4.21}$$

For the case $\sin(\alpha/2)\neq 0$ from $(4.21)_1$ we have

$$c_2 = -c_1\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}. \tag{4.22}$$

Substituting (4.22) into (4.11) we obtain the eigenfunctions

$$X^k_i = \sin(\alpha^k/2)\cos(\alpha^k ih)-\cos(\alpha^k/2)\sin(\alpha^k ih) = \sin\big(\alpha^k(1/2-ih)\big), \quad \text{for even } k, \tag{4.23}$$

where $i=\overline{0,N}$. For the case $\sin(\alpha/2)=0$ we use $(4.21)_2$ (as $(4.21)_1$ gives $0=0$), and obtain $c_2=0$ and $0\cdot c_1=0$. For this case the form of eigenfunction is

$$X^k_i = \cos(\alpha^k ih), \quad \text{for odd } k. \tag{4.24}$$

Since eigenfunctions X^k and Y^l are linearly independent, the eigenfunctions (4.20) are linearly independent [18, 34].

5. Eigenspectrum structure

We represent the three-layer scheme (3.12) as an equivalent two-layer scheme (see e.g. [16, 30])

$$\widehat{\mathbf{W}} = \mathbf{S}\mathbf{W} + \mathbf{G}, \tag{5.1}$$

where

$$\widehat{\mathbf{W}} = \begin{pmatrix} \widehat{\mathbf{U}} \\ \mathbf{U} \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} \mathbf{U} \\ \check{\mathbf{U}} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \tau^2\mathbf{A}^{-1}\mathbf{F} \\ \mathbf{0} \end{pmatrix}.$$

According to [29, 13], one can study the stability conditions for the two-layer difference scheme (3.12) by analyzing the spectrum of the matrix $\mathbf{S}$. Note that the matrices $\mathbf{S}$ and $\mathbf{\Lambda}$ are nonsymmetric.

First, we note one important property of the three-layer scheme (3.12) with $(N-1)^2 \times (N-1)^2$ matrices $\mathbf{A}$ and $\mathbf{B}$ defined by (3.13). We use notation $\lambda^k(\mathbf{A})$ and $\lambda^k(\mathbf{B})$ for the k-th eigenvalue of matrix $\mathbf{A}$ and $\mathbf{B}$ accordingly. We investigate the case of the complete $(N-1)^2$ order eigenvector system $\{\mathbf{V}_1, \dots, \mathbf{V}_{(N-1)^2}\}$.

Lemma 5.1. *If matrix $\mathbf{\Lambda}$ has complete eigenvector system, then the matrices $\mathbf{A}$ and $\mathbf{B}$ have a common system of eigenvectors. More precisely, the eigenvectors of the matrix $\mathbf{\Lambda}$ are the eigenvectors of the matrices $\mathbf{A}$ and $\mathbf{B}$.*

Proof. The eigenvectors of the matrix $\mathbf{\Lambda}$ are also the eigenvectors of the unit matrix $\mathbf{I}$. So, since $\mathbf{A}$ and $\mathbf{B}$ are the linear combination of matrices $\mathbf{I}$ and $\mathbf{\Lambda}$, the formulated lemma is valid. □

Let μ be the eigenvalue of the $2(N-1)^2$ order matrix $\mathbf{S}$ (see (5.1)). We consider the eigenvalue problem

$$\begin{aligned}\det(\mathbf{S}-\mu\mathbf{I}) &= \det\begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B}-\mu\mathbf{I} & -\mathbf{I} \\ \mathbf{I} & -\mu\mathbf{I} \end{pmatrix} \\ &= \det\begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B}-\mu\mathbf{I} & -\mu^2\mathbf{I}-\mathbf{A}^{-1}\mathbf{B}\mu-\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \\ &= \det(\mathbf{A}\mu^2+\mathbf{B}\mu+\mathbf{A})\det(\mathbf{A}^{-1}) = 0.\end{aligned} \tag{5.2}$$

We rearrange determinant in(5.2) and get a characteristic equation for the eigenvalues of the generalized nonlinear eigenvalue problem

$$(\mu^2\mathbf{A}+\mu\mathbf{B}+\mathbf{A})\mathbf{U} = 0, \quad \mathbf{U} \neq \mathbf{0}. \tag{5.3}$$

Problem (5.3) is rather well studied for the case of symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ (e.g., see [20]). We note that the eigenvalues μ of the matrix $\mathbf{S}$ coincide with the eigenvalues of the generalized nonlinear eigenvalue problem (5.3). The number of eigenvalues of problem (5.3) is $2(N-1)^2$. Let us clarify the relationship between the eigenvalues μ of the matrix $\mathbf{S}$ and the eigenvalues λ of the matrix $\mathbf{\Lambda}$.

By substituting an eigenvector $\mathbf{V}^k$ of matrix $\mathbf{\Lambda}$, into (5.3) we obtain

$$\big(\mu^2\mathbf{A}+\mu\mathbf{B}+\mathbf{A}\big)\mathbf{V}^k = \big(\mu^2\lambda^k(\mathbf{A})+\mu\lambda^k(\mathbf{B})+\lambda^k(\mathbf{A})\big)\mathbf{V}^k = 0. \tag{5.4}$$

So, eigenvalues of the matrix $\mathbf{S}$ satisfy the quadratic equation

$$\mu^2\lambda^k(\mathbf{A})+\mu\lambda^k(\mathbf{B})+\lambda^k(\mathbf{A}) = 0, \quad k = \overline{1,(N-1)^2}. \tag{5.5}$$

Remark 5.2. Note, that $\mu = 0$ is not the root of Eq. (5.5) for all $\lambda^k > 0$.

The root condition. A polynomial satisfies the *root condition* if all the roots of polynomial

$$A\mu^2 + B\mu + C, \quad A \neq 0, \quad B, C \in \mathbb{C}, \tag{5.6}$$

are in the closed unit disc of the complex plane and roots of magnitude 1 are simple [15, 12]. For polynomial of the second order (5.6) the following statement is valid. The roots of the second order polynomial are in the closed unit disc of the complex plane and those roots of magnitude 1 are simple if

$$|C|^2 + |\overline{A}B - \overline{B}C| \le |A|^2, \tag{5.7a}$$

$$|B| < 2|A|. \tag{5.7b}$$

Remark 5.3. In the case $A = C$ condition (5.7b) guarantee, that we have two complex roots $\mu_1 \neq \mu_2$ and $|\mu_{1,2}| \le 1$. Using Vieta's theorem $\mu_1 \cdot \mu_2 = 1$. So, $|\mu_1| = |\mu_2| = 1$.

Now we prove the main result of this paper.

Theorem 5.4. *If*

$$\sigma > \frac{1}{4} - \frac{1}{\tau^2 \lambda_{\max}}, \tag{5.8}$$

then $\rho(\mathbf{S}) = 1$ *and finite difference scheme* (3.1)–(3.7) *is stable.*

Proof. To prove the theorem, we show, that conditions (5.7a) and (5.7b) are satisfied for polynomial (5.5). First, we rewrite polynomial in a form

$$p(\mu) := a\mu^2 - 2(a - \eta)\mu + a = 0, \tag{5.9}$$

where $a = 1 + \tau^2\sigma\lambda \in \mathbb{R}$, $\eta = \tau^2\lambda/2 \in \mathbb{R}$. For this real polynomial $p(\mu)$, inequality (5.7a) is trivial. The strong inequality (5.7b) ensures that these roots are simple [35]. So, condition (5.7b) can be written as

$$|a - \eta| < |a|. \tag{5.10}$$

For $\lambda > 0$ we have $\eta > 0$. If $a \le 0$, then $a - \eta < 0$ and we can rewrite (5.10) as $\eta - a < -a$ or $\eta < 0$, which contradicts with $\eta > 0$. If $a > 0$, then from condition $-a < a - \eta < a$ follows, that $\eta < 2a$. So, we have

$$\sigma > \frac{1}{4} - \frac{1}{\tau^2 \lambda}. \tag{5.11}$$

If $\sigma > 1/4 - 1/(\tau^2 \lambda_{\max})$, then (5.11) is valid for all λ_k, $k = \overline{1, N-1}$. □

Remark 5.5. If $\sigma \ge 1/4$, then the finite difference scheme (3.1)–(3.7) is unconditionally stable. If $\sigma = 0$, then difference scheme is stable under the condition $\tau^2/h^2 \le 1/2$.

Lemma 5.6. *Each eigenvalue* $\lambda^k(\mathbf{\Lambda})$, $k = \overline{1, (N-1)^2}$ *corresponds to two distinct complex eigenvalues* μ_1^k *and* μ_2^k *of the matrix* $\mathbf{S}$*:*

$$\mu_{1,2}^k = -b^k \pm \sqrt{(b^k)^2 - 1}, \quad b^k = \frac{-1 + \tau^2(1/2 - \sigma)\lambda^k}{1 + \tau^2\sigma\lambda^k}, \quad k = \overline{1, (N-1)^2}. \tag{5.12}$$

Proof. Using relations (3.13) and Remark 5.3, we calculate $\lambda^k(\mathbf{A}) = 1 + \tau^2\sigma\lambda^k$, $\lambda^k(\mathbf{B}) = -2 + \tau^2(1 - 2\sigma)\lambda^k$. By substituting these values into (5.3), and solving the resulting equation, we obtain relations (5.12) for eigenvalues of matrix $\mathbf{S}$. □

Remark 5.7. Equation (5.12) determines the relation between eigenvalues μ_m^k and λ^k. Other properties of $\mu_{1,2}$ follow from the Remarks 5.2 and 5.3.

Lemma 5.8. *Let λ^k and $\mathbf{V}^k$ be an eigenvalue and an eigenvector of the matrix $\mathbf{\Lambda}$, respectively. Let μ_1^k and μ_2^k be the eigenvalues of matrix $\mathbf{S}$ corresponding to λ^k. Then*

$$\mathbf{W}_m^k = \begin{pmatrix} \mathbf{V}^k \\ (\mu_m^k)^{-1}\mathbf{V}^k \end{pmatrix}, \quad k = \overline{1,(N-1)^2},\ m = 1,2, \tag{5.13}$$

are linearly independent eigenvectors of the matrix $\mathbf{S}$.

Proof. Consider the eigenvalue problem $\mathbf{SW} = \mu_m \mathbf{W}$, $m = 1$ or $m = 2$. Using definition of matrix $\mathbf{S}$ (see (3.13)) we have

$$\begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix} = \mu_m \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix}, \quad m = 1,2, \tag{5.14}$$

where $\mathbf{W} = \big(\mathbf{W}_1, \mathbf{W}_2\big)^\intercal$ is an eigenvector. So, two equalities are valid

$$-\mathbf{A}^{-1}\mathbf{B}\mathbf{W}_1 - \mathbf{W}_2 = \mu_m \mathbf{W}_1, \tag{5.15}$$

$$\mathbf{W}_1 = \mu_m \mathbf{W}_2. \tag{5.16}$$

Substituting (5.16) into (5.15) and multiplying it by $\mu_m\mathbf{A}$ we get an analogue of formula (5.4): $\big((\mu_m)^2\mathbf{A} + \mu_m\mathbf{B} + \mathbf{A}\big)\mathbf{W}_1 = 0$. Every $\mathbf{V}_k$, $k = \overline{1,(N-1)^2}$, satisfies (5.4) with $\mu = \mu_m^k$. So, we can take $\mathbf{W}_1 = \mathbf{V}_k$, $k = \overline{1,(N-1)^2}$. Then, from (5.16) it follows that $\mathbf{W}_2 = \big(\mu_m^k\big)^{-1}\mathbf{V}_k$. $\square$

Remark 5.9. We have $2(N-1)^2$ linear independent eigenvectors $\mathbf{W}_m^k$, $k = \overline{1,(N-1)^2}$, $m = 1,2$ which form a complete eigenvector system. Since eigenvalues μ_m^k, $m = 1,2$ are complex, then eigenvectors $\mathbf{W}_m^k$ are also complex.

6. Conclusions

In this article, we considered the stability in an energy norm of the weighted finite difference schemes' class for the second order hyperbolic equation with nonlocal integral conditions (1.3), (1.4). The proof of stability is essentially based on two problem's properties. In more detail, all eigenvalues of the stationary difference operator, corresponding to the differential problem, are positive and all eigenfunctions are linearly independent.

Hence, the following important corollary may be formulated: the described methodology of investigating stability can also be used for the hyperbolic equation (1.1) with another type nonlocal conditions. In many cases, the stability of finite difference schemes for the nonlocal boundary problems is proved only in special energetic norms [13, 16, 17, 29, 31]. Numerical experiments prove the efficiency of such schemes. For the parabolic equations with nonlocal boundary conditions the equivalence of such energetic norms to the L_2 norms is proved. The aim of this article is to investigate stability of the class of weighted finite difference schemes according to the weight of scheme and spectrum. It is important, that the corresponding difference operator with those nonlocal conditions would have only positive eigenvalues. Such results on the properties of spectrum of the difference with nonlocal conditions are obtained in a considerable amount of literature, e.g. Bitsadze-Samarskii conditions in [31], multipoint conditions in [11], Samarskii-Ionkin conditions in [13], boundary integral conditions in [16, 26]. The existence of only positive eigenvalues for the difference operator with boundary integral conditions in the case of variable coefficients in differential equation is considered in [32].

Using methodology of this article, it is possible to investigate the stability of finite difference scheme with above mentioned nonlocal conditions.

Note that, stability statements proved in the article remain true if on the right side of equation (1.1) there is a term $-c(t)U$, $c(t) \geq 0$.

Assertions about the stability of finite difference scheme remain valid if instead of the difference equation (1.1) one has more general equation

$$\frac{\partial^2 u}{\partial t^2} = a(t)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x,y,t), \quad (x,y) \in \Omega,\ t \in (0,T],$$

where $0 < a_o \leq a(t) \leq a_1 < \infty$. In this case finite difference scheme (3.1) is of the form

$$\partial_t^2 U - a(t^n)\Big(\delta_x^2 + \delta_y^2\Big)U^{(\sigma)} = F, \quad (x_i, y_j, t^n) \in \omega^h \times \omega^\tau,$$

and matrices $\mathbf{A}$ and $\mathbf{B}$ in the scheme (3.13) contain multiplier $a(t^n)$ next to the matrix $\mathbf{\Lambda}$. In this case Theorem 5.4 remains valid with (5.8) of the form

$$\sigma > \frac{1}{4} - \frac{1}{\tau^2 a_1 \lambda_{\max}}.$$

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