NON-SIMULTANEOUS QUENCHING IN A SEMILINEAR PARABOLIC SYSTEM WITH MULTI-SINGULAR REACTION TERMS

ZHE JIA, ZUODONG YANG, CHANGYING WANG

Abstract. This article concerns quenching properties of solutions for a semilinear parabolic system with multi-singular reaction terms. We obtain sufficient conditions for the existence of finite time quenching of global solutions. The blow up of time-derivatives at the quenching point is verified. In addition, we identify simultaneous and non-simultaneous quenching, and provide a classification of parameters for the simultaneous quenching rates.

1. Introduction

In this article, we consider the semilinear parabolic system

\begin{align*}
    u_t &= \Delta u + (1 - u)^{-p_1} + (1 - v)^{-q_1}, \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v + (1 - u)^{-p_2} + (1 - v)^{-q_2}, \quad x \in \Omega, \ t > 0, \\
    u(x, t) &= 0, \quad v(x, t) = 0, \quad x \in \partial \Omega, \ t > 0, \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}

where \( p_1, p_2 \geq 0, \ q_1, q_2 > 0, \) and \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary. In addition, \( u_0(x), v_0(x) \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) are sufficiently smooth functions satisfying the compatibility conditions and \( 0 \leq u_0(x), \ v_0(x) < 1 \) in \( \Omega. \) This problem can be considered as the classical non-Newtonian filtration system that incorporates the effects of singular boundary outflux and nonlinear reaction sources. The quenching behavior represents an interesting phenomenon where the solution tends to a constant but the time derivative approaches infinity as \( (x, t) \) tends to some point in the spatial-time space.

Definition 1.1. We say that the solution \((u, v)\) to problem (1.1) quenches in finite time, if there exists \( 0 < T < \infty \) such that

\[
    \lim_{t \to T^-} \max_{x \in \Omega} \{u(x, t), v(x, t)\} = 1.
\]

From now on, we denote by \( T(0 < T < \infty) \) the quenching time of problem (1.1).
The study of the quenching behavior began with the work by Kawarada [1] who first introduced the quenching behavior of the semilinear heat equation \( u_t = u_{xx} + (1 - u)^{-p} \) at level \( u = 1 \), and obtained that the reaction term and the time derivative blow up as \( u \) reached this level. Since then, many researchers have worked on the quenching properties of solutions for different kinds of parabolic equations (see [2]-[13] and the references therein). In particular, Zhi and Mu [9] considered the quenching properties for the semilinear equation

\[
\begin{align*}
  u_t &= u_{xx} + (1 - u) - 1, \\
  u_x(0, t) &= u_0(x), \\
  u_x(1, t) &= 0, \\
  u(x, 0) &= u(x), \\
  0 < x < 1, \quad t > 0,
\end{align*}
\]

(1.2)

and studied solution quenching in finite time, blow-up of time-derivatives and bounds of quenching rates. Later, Wang et al [11] investigated the following parabolic equation with localized reaction term,

\[
\begin{align*}
  u_t &= \Delta u + (1 - u(x, t)) - p, \\
  u(x, t) &= 0, \\
  u(x, 0) &= u_0(x),
\end{align*}
\]

(1.3)

where \( B = \{ x \in \mathbb{R}^n : \| x \| < 1 \} \), \( x^* \in B \). They obtained the existence of the unique classical solution and proved the solution quenched in a finite time. In addition, when \( x^* = 0 \), they also gave bounds for the quenching rate.

There are two evident gaps in [11]: (a) the existence of classical solution in \( \Omega \subset \mathbb{R}^n \); (b) the bounds of the quenching rate for any \( x^* \in \Omega \). This article explores these two questions and extend the results for equation (1.3) to the system (1.1). Also we try obtain non-simultaneous quenching results.

Recently, some papers considered the non-simultaneous quenching behavior of solutions reaching the level \( u = 0 \) for parabolic systems (see [14]-[20]). For instance, Zheng and Wang [19] studied quenching properties for the nonlinear parabolic system

\[
\begin{align*}
  u_t &= \Delta u - v - p, \\
  v_t &= \Delta v - u - q, \\
  u &= v = 1, \\
  u(x, 0) &= u_0(x), \\
  v(x, 0) &= v_0(x),
\end{align*}
\]

(1.4)

They obtained a solution quenching in finite time, and time-derivative blow up at the quenching point, under proper conditions. In addition, when \( \Omega = B_R \), they studied sufficient conditions for non-simultaneous and simultaneous quenching. Later, Ji, Zhou and Zheng [17] studied the quenching behavior of solutions for heat system

\[
\begin{align*}
  u_t &= u_{xx} - u_m - v_n - p, \\
  v_t &= v_{xx} - u - q - v_n,
\end{align*}
\]

with Neumann boundary conditions, They identified non-simultaneous and simultaneous quenching and described four possible simultaneous quenching rates via a characteristic algebraic system. However, there are very few papers in nonsimultaneous quenching for solutions reaching the level \( u = 1 \), which motivates us to consider the problem in this article.

This article is organized as follows. In Section 2, we obtain the global existence result for \( \Omega \) small enough and finite time quenching for \( \Omega \) large enough. Also we deduce the blow up of time-derivatives at the quenching point. In Section 3, we
consider the non-simultaneous quenching of solutions for (1.1) with \( \Omega = B_R(x^*) \).
We will prove if \( p_2 \geq p_1 + 1 \) and \( q_1 \geq q_2 + 1 \), then quenching is always simultaneous; while \( p_2 \geq p_1 + 1 \) and \( q_1 < 1 \), then quenching is always non-simultaneous. If \( p_2 < p_1 + 1 \) and \( q_1 < q_2 + 1 \), then the non-simultaneous quenching may occur; and if \( p_2 < p_1 + 1 \) and \( q_1 < q_2 + 1 \), then both non-simultaneous and simultaneous quenching also may occur for proper initial data. In Section 4, we give a precise classification of parameters for the simultaneous quenching rates.

In this article we use the hypothesis

\[
\begin{align*}
\Delta u_0 + (1 - u_0)^{-p_1} + (1 - v_0)^{-q_1} &> 0, \\
\Delta v_0 + (1 - u_0)^{-p_2} + (1 - v_0)^{-q_2} &> 0.
\end{align*}
\]  

(1.5)

2. Finite time quenching and blow up of time derivatives

Let \( \lambda_1 \) and \( \varphi_1 \) denote the first eigenvalue and the first eigenfunction of the problem

\[
\begin{align*}
\Delta \varphi + \lambda \varphi &= 0, & \text{in } \Omega, \\
\varphi &= 0, & \text{on } \partial \Omega,
\end{align*}
\]

and choose \( \varphi_1(x) \) to satisfy

\[
\varphi_1(x) > 0, \quad \text{in } \Omega, \quad \int_\Omega \varphi dx = 1.
\]

Theorem 2.1. If \( \lambda_1 < \min\{p_1 + p_2, q_1 + q_2\} + 2 \), then there exists a finite time \( T \), such that the solution of (1.1) quenches at this time.

Proof. By the maximum principle, we have \( 0 < u, v < 1 \) in \( \Omega \times (0, T) \). Assume that \( p_1 + p_2 \geq q_1 + q_2 \). Let \( F(t) = \int_\Omega u \varphi dx \), \( G(t) = \int_\Omega v \varphi dx \), and \( \Phi(t) = F(t) + G(t) \) for \( t \in [0, T) \). By Jensen’s inequality,

\[
F'(t) = \int_\Omega \Delta u \varphi dx + \int_\Omega (1 - u)^{-p_1} \varphi dx + \int_\Omega (1 - v)^{-q_1} \varphi dx
\geq -\int_\Omega \lambda_1 u \varphi dx + p_1 \int_\Omega u \varphi dx + q_1 \int_\Omega v \varphi dx + 2
\]

\[
= (p_1 - \lambda_1) F(t) + q_1 G(t) + 2.
\]

Similarly, we have

\[
G'(t) \geq (q_2 - \lambda_1) G(t) + p_2 F(t) + 2,
\]

so we have

\[
\Phi'(t) \geq (p_1 + p_2 - \lambda_1) F(t) + (q_1 + q_2 - \lambda_1) G(t) + 4
\geq (p_1 + p_2 - \lambda_1) \Phi(t) + 4.
\]

Since \( \lambda_1 < \min\{p_1 + p_2, q_1 + q_2\} + 2 \) and \( 0 < F, G < 1 \), we have \( (p_1 + p_2 - \lambda_1) \Phi(t) + 4 > 0 \) for \( t \in [0, T) \). Integrating \( (2.3) \) from 0 to \( t \), we have

\[
t \leq \begin{cases}
\frac{1}{p_1 + p_2 - \lambda_1} \ln \left( \frac{(p_1 + p_2 - \lambda_1) \Phi(t) + 4}{(p_1 + p_2 - \lambda_1) \Phi(0) + 4} \right), & \lambda_1 \neq p_1 + p_2, \\
\frac{1}{4} \ln \left( \frac{\Phi(t) - \Phi(0)}{\Phi(t) - \Phi(0)} \right), & \lambda_1 = p_1 + p_2,
\end{cases}
\]

(2.4)

Since \( \lim_{t \to T^-} \Phi(t) \leq 2 \), so we have the upper bound for quenching time \( T \):

\[
T \leq \begin{cases}
\frac{1}{p_1 + p_2 - \lambda_1} \ln \left( \frac{2(p_1 + p_2 - \lambda_1) + 4}{(p_1 + p_2 - \lambda_1) \Phi(0) + 4} \right), & \lambda_1 \neq p_1 + p_2, \\
\frac{1}{4} \ln \left( \frac{2 - \Phi(0)}{2 - \Phi(0)} \right), & \lambda_1 = p_1 + p_2,
\end{cases}
\]

(2.5)
it is easy to see the right-hand side of (2.5) is greater than 0, so the solution of (1.1) quenches in finite time.

We note that $\lambda_1$ decreases when the domain size increases, so Theorem 2.1 says that the solution of (1.1) will quench in finite time for $\Omega$ large enough. Next, we obtain the existence of a global solution for $\Omega$ small enough, which can be proved by adapting methods that are established in [19].

**Theorem 2.2.** Assume that $u_0, v_0 < \sigma_0 < 1$ in $\bar{\Omega}$ and the diameter of $\Omega$ is small enough. Then the solutions of (1.1) exist globally.

**Proof.** Consider the auxiliary problem

$$
\begin{align*}
\bar{u}_t &= \Delta \bar{u} + (1 - \bar{u})^{-p_1} + (1 - \bar{v})^{-q_1}, \quad (x, t) \in \Omega \times [0, T), \\
\bar{v}_t &= \Delta \bar{v} + (1 - \bar{u})^{-p_2} + (1 - \bar{v})^{-q_2}, \quad (x, t) \in \Omega \times [0, T), \\
\bar{u}(x, t) &= \sigma_0, \quad \bar{v}(x, t) = \sigma_0, \quad x \in \partial \Omega, \; t > 0, \\
\bar{u}(x, 0) &= \sigma_0, \quad \bar{v}(x, 0) = \sigma_0, \quad x \in \Omega.
\end{align*}
$$

(2.6)

It is easy to see the solution of (2.6) is an upper-solution of (1.1). By the comparison principle, we have $u \leq \bar{u}, v \leq \bar{v}$, it suffices to prove that $(\bar{u}, \bar{v})$ is global. Let $\phi$ satisfy

$$
\begin{align*}
\Delta \phi - C_0 &= 0, \quad x \in B_R(x^*), \\
\phi &= \sigma_0, \quad x \in \partial B_R(x^*),
\end{align*}
$$

(2.7)

where $B_R(x^*) = \{ x \in \Omega : |x - x^*| \leq R \}$ and

$$
C_0 < \min \{ -(1 - \sigma_0)^{-p_1} - (1 - \sigma_0)^{-q_1}, -(1 - \sigma_0)^{-p_2} - (1 - \sigma_0)^{-q_2} \} < 0,
$$

hence

$$
\phi(x) = C_0(|x - x^*|^2 - R^2) + \sigma_0
$$

(2.8)

with $\max_{B_R(x^*)} \phi(x) = \sigma_0 - \frac{C_0 R^2}{2N}$. Taking $R$ small enough such that

$$
C_0 < \min_{B_R(x^*)} \left\{ -(1 - \phi)^{-p_1} - (1 - \phi)^{-q_1}, -(1 - \phi)^{-p_2} - (1 - \phi)^{-q_2} \right\},
$$

so $(\phi, \phi)$ is a time-independent upper-solution of (2.6) for $\Omega \subset B_R(x^*)$, which implies the global solutions of (1.1) exist for the diameter of $\Omega$ small enough.

Now we consider the blow up of time derivatives.

**Lemma 2.3.** If (1.5) holds, then $u_t, v_t > 0$ for $(x, t) \in \Omega \times [0, T)$. Moreover, for any $\eta > 0$, there exists $c > 0$ such that

$$
u_t(x, t), v_t(x, t) \geq c, \quad \forall (x, t) \in \bar{\Omega} \times [0, T),
$$

with $\Omega^\eta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \eta \}.$

**Proof.** Let $\Phi = u_t(x, t), \Psi = v_t(x, t)$, since (1.5) holds, we have

$$
\begin{align*}
\Phi_t - \Delta \Phi &= p_1 (1 - u)^{-p_1 - 1} \Phi + q_1 (1 - v)^{-q_1 - 1} \Psi, \quad (x, t) \in \Omega \times [0, T), \\
\Psi_t - \Delta \Psi &= q_2 (1 - u)^{-q_2 - 1} \Psi + p_2 (1 - u)^{-p_2 - 1} \Phi, \quad (x, t) \in \Omega \times [0, T), \\
\Phi(x, t) &= \Psi(x, t) = 0, \quad (x, t) \in \partial \Omega \times [0, T), \\
\Phi(x, 0) &= \Delta u_0 + (1 - u_0)^{-p_1} + (1 - v_0)^{-q_1} > 0, \quad x \in \bar{\Omega}, \\
\Psi(x, 0) &= \Delta v_0 + (1 - u_0)^{-p_2} + (1 - v_0)^{-q_2} > 0, \quad x \in \bar{\Omega},
\end{align*}
$$

(2.9)
Let \((u^*, v^*)\) be the solution for the auxiliary problem
\[
\begin{align*}
u^*_t &= \Delta u^* + (1 - u_0)^{-p_1} + (1 - v_0)^{-q_1}, \quad x \in \Omega, \ t > 0, \\
v^*_t &= \Delta v^* + (1 - u_0)^{-p_2} + (1 - v_0)^{-q_2}, \quad x \in \Omega, \ t > 0, \\
\Phi(x, t) &= \Phi(x, 0), \quad v^*(x, 0) = v_0(x), \quad x \in \Omega.
\end{align*}
\] (2.10)

Let \(\Phi^* = u^*_t(x, t), \Psi^* = v^*_t(x, t)\), Then by the above we deduce that \(u^*_t, v^*_t > 0\) .

Next, let \(w = u - u^*, z = v - v^*\) and \(\Phi = u_t, \Psi = z_t\). It is easy to obtain
\[
\begin{align*}
\Phi_t - \Delta \Phi &\geq 0, \quad (x, t) \in \Omega \times [0, T), \\
\Psi_t - \Delta \Psi &\geq 0, \quad (x, t) \in \Omega \times [0, T), \\
\Phi(x, t) &= \Phi(x, 0) = 0, \quad (x, t) \in \partial \Omega \times [0, T), \\
\Phi(x, 0) &= \Psi(x, 0) = 0, \quad x \in \bar{\Omega},
\end{align*}
\]
so that \(u_t \geq u^*_t, v_t \geq v^*_t\) in \(\Omega \times [0, T)\). Taking
\[
c = \min \left\{ \min_{\Omega \times \eta, T} |u^*_t|, \ \min_{\Omega \times \eta, T} |v^*_t| \right\},
\]
we have \(u_t, v_t \geq c\) in \(\bar{\Omega} \times [\eta, T)\). \(\square\]

**Lemma 2.4.** Assume that \(\Omega\) is a convex domain and (1.5) holds, then for any \(\eta\). Then there exists a positive constant \(\zeta\) such that
\[
\begin{align*}
u_t &\geq \zeta[(1 - u)^{-p_1} + (1 - v)^{-q_1}], \quad \eta, t \in \Omega^0 \times (\eta, T), \\
v_t &\geq \zeta[(1 - u)^{-p_2} + (1 - v)^{-q_2}], \quad \eta, t \in \Omega^0 \times (\eta, T).
\end{align*}
\] (2.11)

**Proof.** Let
\[
\begin{align*}
I_t &= u_t - \zeta[(1 - u)^{-p_1} + (1 - v)^{-q_1}], \quad (x, t) \in \Omega^0 \times (\eta, T), \\
J_t &= v_t - \zeta[(1 - u)^{-p_2} + (1 - v)^{-q_2}], \quad (x, t) \in \Omega^0 \times (\eta, T).
\end{align*}
\] (2.12)

Then we have
\[
\begin{align*}
I_t - \Delta I_t &= (u_t - \Delta u)_t - \zeta p_1 (1 - u)^{-p_1 - 1} (u_t - \Delta u) - \zeta q_1 (1 - v)^{-q_1 - 1} (v_t - \Delta v) \\
&\quad + \zeta p_1 (p_1 + 1)(1 - u)^{-p_1 - 2} |\nabla u|^2 + \zeta q_1 (q_1 + 1)(1 - v)^{-q_2 - 2} |\nabla v|^2 \\
&\geq p_1 (1 - u)^{-p_1 - 1} I + q_1 (1 - v)^{-q_1 - 1} J.
\end{align*}
\]

Similarly,
\[
J_t - \Delta J_t \geq q_2 (1 - v)^{-q_2 - 1} J + p_2 (1 - u)^{-p_2 - 1} I.
\] (2.13)

In addition, by Lemma 2.3 and taking \(\zeta\) small enough, we have
\[
\begin{align*}
I(x, t) &= u_t - \zeta[(1 - u)^{-p_1} + (1 - v)^{-q_1}] \geq 0, \quad (x, t) \in \partial \Omega^0 \times (0, T), \\
J(x, t) &= v_t - \zeta[(1 - u)^{-p_2} + (1 - v)^{-q_2}] \geq 0, \quad (x, t) \in \partial \Omega^0 \times (0, T),
\end{align*}
\] (2.14)

and the initial data
\[
I(x, 0), J(x, 0) \geq 0 \quad x \in \Omega^0, \quad (2.15)
\]

By the maximum principle, we have \(I(x, t), J(x, t) \geq 0\) for \((x, t) \in \Omega^0 \times (0, T)\). \(\square\)

As a direct consequence of Lemma 2.4, we deduce time-derivatives blow up at the quenching point.
Theorem 2.5. If $\Omega$ is a convex domain and \[1.5\] holds, then $(u_t, v_t)$ blows up at the quenching point.

3. SIMULTANEOUS AND NON-SIMULTANEOUS QUENCHING

In this section, we deal with radial solutions of \[1.1\] with $\Omega = B_R(x^*) = \{x \in \mathbb{R}^N : |x - x^*| < R\}$, and non-increasing initial data satisfying \[1.5\]. By the maximum principle \[11, Lemma 3.2\], we have $u_r(r, t), v_r(r, t) \leq 0$. At first, we give the sufficient condition for finite-time quenching of radical solutions in $B_R(x^*) \times (0, T)$.

Lemma 3.1. Assume $(u, v)$ is the global solution of \[1.1\] with $(u_0, v_0) \equiv (0, 0)$, in other words, there exists a constant $c \in [0, 1)$ such that $u, v \leq c < 1$ on $B_R(x^*) \times [0, \infty)$. Then $(u, v)$ approaches uniformly from below to a solution $(U, V)$ of the steady-state problem

$$
\Delta U = -(1 - U)^{-p_1} - (1 - V)^{-q_1}, \quad x \in B_R(x^*), \\
\Delta V = -(1 - U)^{-p_2} - (1 - V)^{-q_2}, \quad x \in B_R(x^*), \\
U = V = 0, \quad x \in \partial B_R(x^*).
$$

Proof. By \[19, Lemma 4.1\], we define

$$W(x, t) = \int_{B_R(x^*)} G(x, y)u(y, t)dy, \quad Z(x, t) = \int_{B_R(x^*)} G(x, y)v(y, t)dy,$$

for $(x, t) \in B_R(x^*) \times [0, \infty)$, where $G(x, y)$ is Green’s function associated with the operator $-\Delta$ on $B_R(x^*)$ under Dirichlet boundary conditions. Then

$$W_t(x, t) = 1 - u(x, t) + \int_{B_R(x^*)} G(x, y)(1 - u)^{-p_1}dy + \int_{B_R(x^*)} G(x, y)(1 - v)^{-q_1}dy,$$

$$Z_t(x, t) = 1 - v(x, t) + \int_{B_R(x^*)} G(x, y)(1 - u)^{-p_2}dy + \int_{B_R(x^*)} G(x, y)(1 - v)^{-q_2}dy.$$

Combining Lemma 2.3 and the monotone convergence theorem, we have

$$\lim_{t \to \infty} W_t(x, t) = 1 - U(x) + \int_{B_R(x^*)} G(x, y)(1 - U)^{-p_1}dy + \int_{B_R(x^*)} G(x, y)(1 - V)^{-q_1}dy,$$

$$\lim_{t \to \infty} Z_t(x, t) = 1 - V(x) + \int_{B_R(x^*)} G(x, y)(1 - U)^{-p_2}dy + \int_{B_R(x^*)} G(x, y)(1 - V)^{-q_2}dy,$$

where $c \geq U(x) = \lim_{t \to \infty} u(x, t), \ c \geq V(x) = \lim_{t \to \infty} v(x, t)$. In addition, since $W, Z$ are bounded and $W_t, Z_t \geq 0$, we have

$$\lim_{t \to \infty} W_t(x, t) = 0, \quad \lim_{t \to \infty} Z_t(x, t) = 0,$$

(3.2)

which imply

$$U(x) = 1 + \int_{B_R(x^*)} G(x, y)(1 - U)^{-p_1}dy + \int_{B_R(x^*)} G(x, y)(1 - V)^{-q_1}dy,$$

$$V(x) = 1 + \int_{B_R(x^*)} G(x, y)(1 - U)^{-p_2}dy + \int_{B_R(x^*)} G(x, y)(1 - V)^{-q_2}dy,$$

(3.3)
which is the solution of (3.1), and by Dini’s theorem, we can get the uniform convergence. □

Inspired by [20] Theorem 1.3, with Lemma 3.1 at hand, we obtain the following theorem.

**Theorem 3.2.** If $R \geq \sqrt{N}$, then the radial solution of (1.1) will quench in finite time for any initial data.

**Proof.** Considering the auxiliary system

\[
\begin{align*}
\dot{u} &= \Delta u + (1-u)^{-p_1} + (1-v)^{-q_1}, \quad (x,t) \in B_R(x^*) \times [0,T), \\
\dot{v} &= \Delta v + (1-u)^{-p_2} + (1-v)^{-q_2}, \quad (x,t) \in B_R(x^*) \times [0,T),
\end{align*}
\]

(3.4)

By the comparison principle, we have $u \geq u_0, v \geq v_0$. Now we introduce the problem

\[
\begin{align*}
-\Delta u^* &= 2, \quad -\Delta v^* = 2, \quad r \in B_R(x^*), \\
u^* &= v^* = 0, \quad r \in \partial B_R(x^*),
\end{align*}
\]

(3.5)

with solution denoted as

\[
u^* = \frac{-2(|x-x^*|^2 - R^2)}{2N}, \quad v^* = \frac{-2(|x-x^*|^2 - R^2)}{2N}.
\]

(3.6)

So we have $\max\{u^*, v^*\} = R^2/N$. Clearly, $(u^*, v^*)$ is a sub-solution of (1.1). By Lemma 3.1, the solution $(u,v)$ is global only if $u^*, v^* < 1$. Therefore, if $u^*$ or $v^* \geq 1$, namely $R \geq \sqrt{N}$, then the solution of (1.1) quenches in finite time for any initial data.

**Remark 3.3.** Theorem 3.2 indicates that the solution quenches in finite time for $R \geq \sqrt{N}$. However, for radical solutions of (1.1) with $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and assuming (1.5) and that $u_0'(r), v_0'(r) \leq 0$, by [20], we can obtain that the solution quenches in finite time without the condition $R \geq \sqrt{N}$. Also we obtain that $r = 0$ is the only quenching point.

Next, we will focus on the simultaneous and non-simultaneous quenching of solutions for (1.1). To simplify our work, we deal with the radical solutions of (1.1) with $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$, and assume that (1.5) holds and $u_0'(r), v_0'(r) \leq 0$. It is easy to see that $\max_{0 \leq r \leq R} u(r,t) = u(0,t)$, $\max_{0 \leq r \leq R} v(r,t) = v(0,t)$ by Remark 3.3. In addition, $c, c_i, C, C_i$ denote positive constants independent of $t$, which are different from line to line. First, we give a necessary condition for the non-simultaneous quenching.

**Theorem 3.4.** If $v(0,t) \leq c < 1$ for $t \in [0,T)$, then $p_2 < p_1 + 1$.

**Proof.** Since $u_r, v_r \leq 0$, by the Hopf’s lemma, we can see that $u_{rr}(0,t), v_{rr}(0,t) \leq 0$. Then by Lemma 2.4, we have

\[
\begin{align*}
\zeta((1-u)^{-p_1} + (1-v)^{-q_1})(0,t) &\leq u_t(0,t) \leq (1-u)^{-p_1} + (1-v)^{-q_1}(0,t), \\
\zeta((1-u)^{-p_2} + (1-v)^{-q_2})(0,t) &\leq v_t(0,t) \leq (1-u)^{-p_2} + (1-v)^{-q_2}(0,t).
\end{align*}
\]

(3.7)

Combing (3.7) with $v(0,t) \leq c < 1$, we have

\[
u_t(0,t) \leq C(1-u)^{-p_1}(0,t).
\]

(3.8)
Integrating on \((t, T)\) gives
\[
1 - u(0, t) \leq C(T - t)^\frac{1}{p_1 + 1}.
\]
(3.9)
So by Lemma 2.4 and (3.4), we have
\[
v_t(0, t) \geq \zeta (1 - u(0, t))^{-p_2} \geq C(T - t)^{-\frac{p_2}{p_1 + 1}}.
\]
Integrating on \((0, T)\), we have
\[
v(0, T) - v(0, 0) \geq C \int_0^T (T - t)^{-\frac{p_2}{p_1 + 1}} dt.
\]
(3.10)
If \(p_2 \geq p_1 + 1\), this integral diverges. The proof is complete.

**Corollary 3.5.** If \(p_2 \geq p_1 + 1\) and \(q_1 \geq q_2 + 1\), then quenching is simultaneous.

Next, we give a sufficient condition for non-simultaneous quenching.

**Theorem 3.6.** If \(p_2 \geq p_1 + 1\), \(q_1 < 1\), then \(u(0, t) \leq c < 1\) for \(t \in [0, T]\).

**Proof.** Define \((\bar{u}(t), \bar{v}(t)) := (u(0, t), v(0, t))\). By (3.4), there exist two positive constants \(c_0, c_1\) such that
\[
c_0[(1 - \bar{u})^{-p_1} + (1 - \bar{v})^{-q_1}] \leq \bar{u}'[(1 - \bar{u})^{-p_2} + (1 - \bar{v})^{-q_2}]
\leq c_1[(1 - \bar{u})^{-p_1} + (1 - \bar{v})^{-q_1}] \bar{v}',
\]
(3.11)
Multiplying the second inequality by \((1 - \bar{u})^{p_1}(1 - \bar{v})^{q_1}\), we have
\[
\bar{u}'(1 - \bar{u})^{-p_2 + p_1} \leq c \bar{v}(1 - \bar{v})^{-q_1}.
\]
(3.12)
Integrating on \((0, T)\), if \(p_2 > p_1 + 1, q_1 < 1\), we have
\[
(1 - \bar{u}(T))^{1 - p_2 + p_1} \leq c_0 - c(1 - \bar{v}(T))^{1 - q_1},
\]
(3.13)
if \(p_2 = p_1 + 1, q_1 < 1\), we have
\[
- \ln(1 - \bar{u}(T)) \leq c_0 - c(1 - \bar{v}(T))^{1 - q_1},
\]
a contradiction, if \(u\) quenches.

**Theorem 3.7.** If \(p_2 < p_1 + 1\) \((q_1 < q_2 + 1)\), then there exist the initial data such that \(u(v)\) quenches while \(v(u) \leq c_0 < 1\).

**Proof.** By Lemma 2.4, we have
\[
u_t(0, t) \geq \zeta (1 - u(0, t))^{-p_1},
\]
(3.14)
Integrating (3.4) on \((t, T)\), we have there exists a positive constant \(C\) such that
\[
1 - u(0, t) \leq C(T - t)^\frac{1}{p_1 + 1}.
\]
(3.15)
Similarly,
\[
1 - v(0, t) \geq C(T - t)^\frac{1}{p_1 + 1}.
\]
(3.16)
Combining (3.5), (3.5) and (3.4), we obtain
\[
v_t(0, t) \leq C(T - t)^{-\frac{p_2}{p_1 + 1}} + C(T - t)^{-\frac{p_2}{p_1 + 2}}.
\]
(3.17)
Integrating on \((0, T)\), we obtain
\[
v(0, T) \leq v(0, 0) + c_1 T^{1 - p_1 - \frac{p_2}{p_1 + 1}} + c_2 T^{1 - \frac{p_2}{p_1 + 2}}.
\]
(3.18)
By Lemma 2.3 we have \(u_0, v_t \geq c\). By integrating on \((0, t)\) and letting \(t \to T^-\), we have \(T \leq \frac{1}{c} \min\{1 - u_0(0), 1 - v_0(0)\}\). We take \(u_0(x) = 1 - \epsilon\), then \(T \leq \frac{1}{c} \epsilon\). If \(\epsilon\),
and hence $T$, are small enough, we can conclude from (??) that $v(0, T) \leq c_0 < 1$. The proof is complete.

Next we show that if $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$, then both non-simultaneous and simultaneous quenching also may occur for proper initial data. At first, we give the following lemma.

**Lemma 3.8 (19 Lemma 4.5).** If $p_2 < p_1 + 1$, $q_1 < q_2 + 1$, then the set of initial data such that one of the components quenching alone is open.

**Theorem 3.9.** If $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$, then both simultaneous and non-simultaneous quenching may occur for proper initial data.

**Proof.** Step I. We prove non-simultaneous quenching. Assume for contradiction that $u$ and $v$ quench simultaneously for every initial data. Since $u_1(0, t) \leq (1 - u(0, t))^{-\xi p_1} + (1 - v(0, t))^{-\eta q_1}$ by (??), integrating on $(0, t)$ gives

$$v(0, t) \leq v_0(0) + \int_0^t (1 - u(0, s))^{-\xi p_1} + (1 - v(0, s))^{-\eta q_1} ds,$$

(3.19)

introducing (??) and (??) in (3.8), letting $t \to T^-$, we obtain that

$$v(0, T) \leq v_0(0) + t^{-\frac{1}{\xi p_1}} + T^{-\frac{1}{\eta q_1}}.$$

(3.20)

As in Theorem 3.7. We take $v_0(x) = 1 - c$, then $T \leq \frac{1}{c}$. if $c$, and hence $T$, are small enough, we can conclude from (3.9) that $v(0, T) \leq c < 1$, a contradiction.

Step II. We prove simultaneous quenching. Since $p_2 < p_1 + 1, q_1 < q_2 + 1$, From (??), we have

$$v(0, T) \leq v_0(0) + c_1 T^{-\frac{1 + p_1 - p_2}{1 + p_1}} + c_2 T^{-\frac{1}{1 + \frac{q_1}{q_2}}}.$$

(3.21)

Similarly,

$$u(0, T) \leq u_0(0) + c_3 T^{-\frac{1 + q_2 - q_1}{1 + \frac{q_1}{q_2}}} + c_4 T^{-\frac{1}{1 + q_1}}.$$

(3.22)

Denote $(u_\alpha, v_\alpha)$ as a solution of (1.1) with initial data $(1 - \alpha u_0, 1 - (1 - \alpha)v_0)$, where $\alpha \in (0, 1)$. Let $T_\alpha$ be the quenching time, we have $u_\alpha(0, T) \leq c < 1$ for $\alpha \to 1$ and $v_\alpha(0, T) \leq c < 1$ for $\alpha \to 0$. Define $\Psi_u = \{\alpha \in (0, 1) : u_\alpha(0, T) < 1\}$, $\Psi_v = \{\alpha \in (0, 1) : v_\alpha(0, T) < 1\}$, it is easy to see that

$$\Phi_u \cap \Psi_v = \emptyset,$$

however by Lemma 3.8 we have that $\Phi_u$ and $P\Psi_v$ are open. Hence $u, v$ quench simultaneously for some initial data. The proof complete. \qed

4. SIMULTANEOUS AND NON-SIMULTANEOUS QUENCHING RATES

The notation $f \sim g$ means that there exist positive constants $c_1, c_2$ such that $c_1 g \leq f \leq c_2 g$. At first, we give a lemma which needs two additional assumptions.

(H1) $p_2 \geq p_1 + 1$, $q_1 \geq q_2 + 1$, $q_1 \geq q_2$, and $\xi (1 - u_0)^{p_2 - 1} \geq (1 - v_0)^{q_1 - 1}$ with $\xi > \frac{p_2 - 1}{q_1 - p_2}$;

(H2) $p_2 \geq p_1 + 1$, $q_1 \geq q_2 + 1$, $q_1 \geq q_2$ and $\eta (1 - u_0)^{p_2 - 1} \leq (1 - v_0)^{q_1 - 1}$ with $\eta < \frac{p_2 - 1}{q_1 - p_2}$.

**Lemma 4.1.** Let $(u, v)$ be the solution of problem (1.1). Then $\xi (1 - u)^{p_2 - 1} \geq (1 - v)^{q_1 - 1}$ under assumption (H1), and $\eta (1 - u)^{p_2 - 1} \leq (1 - v)^{q_1 - 1}$ under assumption (H2), for $(r, t) \in (0, R) \times (0, T)$. 

Proof. Let \( \varphi = \xi (1-u)^{p_2-1} - (1-v)^{q_1-1} \), \( \psi = \eta (1-u)^{p_2-1} - (1-v)^{q_1-1} \). We have
\[
\begin{align*}
\varphi_t - \varphi_{rr} - h \varphi_r + l \varphi &= -\xi (p_2 - 1)(1-u)^{p_2-p_1-2} + \xi (q_1 - 1)(1-u)^{-1}(1-v)^{-1} \\
&+ (q_1 - p_2)(1-u)^{-1}(1-v)^{q_1-2}u_r v_r \\
&\geq \xi (q_1 - p_2)(1-u)^{-1}(1-v)^{-1} - \xi (p_2 - 1)(1-u)^{p_2-2}(1-v)^{-q_1} \\
&+ (q_1 - p_2)(1-u)^{-1}(1-v)^{q_1-2}u_r v_r \\
&= \xi (q_1 - p_2)(1-u)^{-1}(1-v)^{-1} - (p_2 - 1)(1-u)^{-1}(1-v)^{-1}(1 + \varphi(1-v)^{1-q_1}) \\
&+ (q_1 - p_2)(1-u)^{-1}(1-v)^{q_1-2}u_r v_r
\end{align*}
\]
where
\[
h = \frac{N - 1}{r} (q_1 - 2)(1-v)^{-1}v_x + (p_2 - 2)(1-u)^{-1}u_x, \quad (4.1)
\]
\[
l = (q_1 - 1)(1-u)^{-p_2}(1-v)^{-1} - (p_2 - 1)(q_1 - 2)(1-u)^{-1}(1-v)^{-1};
\]
so
\[
\begin{align*}
\varphi_t - \varphi_{rr} - h \varphi_r + (l + (p_2 - 1)(1-u)^{-1}(1-v)^{-q_1}) \varphi \\
&\geq (\xi (q_1 - p_2) - p_2 + 1)(1-u)^{-1}(1-v)^{-1} \\
&+ (q_1 - p_2)(1-u)^{-1}(1-v)^{q_1-2}u_r v_r
\end{align*}
\]
Since \( \xi > \frac{p_2-1}{q_1-p_2} \), we have
\[
\varphi_t - \varphi_{rr} - h \varphi_r + (l + (p_2 - 1)(1-u)^{-1}(1-v)^{-q_1}) \varphi \geq 0. \quad (4.3)
\]
In addition,
\[
\begin{align*}
\varphi(r, 0) &= \xi (1-u_0)^{p_2-1} - (1-v_0)^{q_1-1} \geq 0, \quad r \in [0, R], \\
\varphi_r(0, t) &= \varphi_r(R, t) = 0, \quad t \in (0, T)
\end{align*}
\]
By the maximum principle,
\[
\varphi = \xi (1-u)^{p_2-1} - (1-v)^{q_1-1} \geq 0 \quad (4.5)
\]
Similarly, if (H2) holds, we can obtain \( \psi = \eta (1-u)^{p_2-1} - (1-v)^{q_1-1} \leq 0 \). The proof is complete. \( \square \)

Next, we give bounds for the non-simultaneous quenching rate.

**Theorem 4.2.** If quenching is non-simultaneous and \( u \) is the quenching component, then for \( t \to T^- \), we have
\[
1 - u(0, t) \sim (T-t)^{\frac{1}{q_1-1}}.
\]

The proof of the above theorem is a direct consequence of (4.1) and (4.2). Next, we give bounds for the simultaneous quenching rate.

**Theorem 4.3.** Assume that (H1) or (H2) hold. Then quenching is simultaneous, and for \( t \to T^- \),
\[
1 - u(0, t) \sim (T-t)^{\frac{q_1-1}{p_2q_1-1}}, \quad 1 - v(0, t) \sim (T-t)^{\frac{q_1-1}{p_2q_1-1}}.
\]
Proof: Without loss of generality, consider the case of (H1) only. Since $\xi(1 - u)^{p_2 - 1} \geq (1 - v)^{q_1 - 1}$, by (4.5), we obtain
\[ v_t(0, t) \leq (1 - u(0, t))^{p_2} + (1 - v(0, t))^{q_2} \]
\[ \leq (1 - v(0, t))^{\frac{p_2q_1 - 1}{p_2 - 1}} + (1 - v(0, t))^{q_2} \]
\[ \leq c(1 - v(0, t))^{\frac{p_2q_1 - 1}{p_2 - 1}}, \]
by $p_2 \geq p_1 + 1$ and $q_1 \geq q_2 + 1$. Integrating (4.6) on $(0, T)$, we have
\[ 1 - v(0, t) \leq C(T - t)^{\frac{p_2 - 1}{p_2q_1 - 1}}. \]

By Lemma 2.4, we have
\[ u_t(0, t) \geq \xi(1 - v)^{-q_1} (0, t) \geq c(T - t)^{-\frac{q_1(p_2 - 1)}{p_2q_1 - 1}}. \]

Integrating on $(0, T)$, we have
\[ 1 - u(0, t) \geq C(T - t)^{\frac{q_1 - 1}{p_2q_1 - 1}}, \]
by Lemma 2.4 again, we have
\[ v_t(0, t) \geq \xi(1 - u)^{-p_2} (0, t). \]

Integrating on $(t, T)$ we have
\[ 1 - v(0, t) \geq C \int_t^T (1 - u(0, \eta))^{-p_2} d\eta \geq c(1 - u(0, t))^{-p_2}(T - t), \]
by (4.5), we have
\[ u_t(0, t) \leq (1 - u(0, t))^{-p_1} + C(1 - u(0, t))^{p_2q_1} (T - t)^{-q_1} \]
combining (4.12) and (4.5), we have
\[ u_t(0, t) \leq C(1 - u(0, t))^{p_2q_1} (T - t)^{-q_1}. \]

Integrating (4.12) on $(t, T)$, we have
\[ 1 - u(0, t) \geq C(T - t)^{\frac{q_1 - 1}{p_2q_1 - 1}}, \]
from Lemma 2.4 we have
\[ v_t(0, t) \geq \xi(1 - u)^{-p_2} \geq C(T - t)^{-\frac{p_2q_1 - 1}{p_2q_1 - 1}}. \]

Integrating on $(t, T)$, we have
\[ 1 - v(0, t) \geq C(T - t)^{\frac{p_2 - 1}{p_2q_1 - 1}}. \]

\[ \square \]

Theorem 4.4. Assume $p_2 < p_1 + 1$, $q_1 < q_2 + 1$. Then quenching is simultaneous, and for $t \to T^-$,
\[ 1 - u(0, t) \sim (T - t)^{\frac{q_1}{q_2 - 1}}, 1 - v(0, t) \sim (T - t)^{\frac{1}{q_2}}, \]
\[ \frac{p_1(q_2 + 1)}{p_1 + 1} \leq q_1 < q_2 + 1, p_2 \leq \frac{q_2(p_1 + 1)}{q_2 + 1}, \]
\[ 1 - u(0, t) \sim (T - t)^{\frac{q_1}{q_2 - 1}}, 1 - v(0, t) \sim (T - t)^{\frac{1}{q_2}}, q_1 < q_2 + 1, \]
\[ \frac{q_2(p_1 + 1)}{q_2 + 1} \leq p_2 \leq \frac{q_2}{q_2 + 1 - q_1}, \]
Acknowledgments. This research was supported by the National Natural Science Foundation of China (grants 11371093 and 11471164).

References


Zhe Jia  
School of Mathematics Science, Nanjing Normal University, Nanjing 210023, China  
Email address: 1500516020@qq.com

Zuodong Yang (corresponding author)  
School of Teacher Education, Nanjing Normal University, Nanjing 210097, China.  
School of Teacher Education, Nanjing University of Information Science and Technology, Jiangsu Nanjing 210044, China  
Email address: zdyang_jin@263.net

Changying Wang  
School of Data Science and Software Engineering, Qingdao University, Qingdao 266071, China  
Email address: Wcing800126.com