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NON-SIMULTANEOUS QUENCHING IN A SEMILINEAR PARABOLIC SYSTEM WITH MULTI-SINGULAR REACTION TERMS

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ABSTRACT. This article concerns quenching properties of solutions for a semilinear parabolic system with multi-singular reaction terms. We obtain sufficient conditions for the existence of finite time quenching of global solutions. The blow up of time-derivatives at the quenching point is verified. In addition, we identify simultaneous and non-simultaneous quenching, and provide a classification of parameters for the simultaneous quenching rates.

1. Introduction

In this article, we consider the semilinear parabolic system

$$
\begin{gathered}
u_t = \Delta u + (1-u)^{-p_1} + (1-v)^{-q_1}, \quad x \in \Omega,\ t > 0, \\
v_t = \Delta v + (1-u)^{-p_2} + (1-v)^{-q_2}, \quad x \in \Omega,\ t > 0, \\
u(x,t) = 0, \quad v(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \bar{\Omega},
\end{gathered} \tag{1.1}
$$

where $p_1, p_2 \geq 0$, $q_1, q_2 > 0$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. In addition, $u_0(x), v_0(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ are sufficiently smooth functions satisfying the compatibility conditions and $0 \leq u_0(x),\ v_0(x) < 1$ in $\bar{\Omega}$. This problem can be considered as the classical non-Newtonian filtration system that incorporates the effects of singular boundary outflux and nonlinear reaction sources. The quenching behavior represents an interesting phenomenon where the solution tends to a constant but the time derivative approaches infinity as (x,t) tends to some point in the spatial-time space.

Definition 1.1. We say that the solution (u,v) to problem (1.1) quenches in finite time, if there exists $0 < T < \infty$ such that

$$
\lim_{t \to T^-} \max_{x \in \bar{\Omega}} \{u(x,t), v(x,t)\} = 1.
$$

From now on, we denote by $T (0 < T < \infty)$ the quenching time of problem (1.1).

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The study of the quenching behavior began with the work by Kawarada [1] who first introduced the quenching behavior of the semilinear heat equation $u_t = u_{xx} + (1-u)^{-1}$ at level $u = 1$, and obtained that the reaction term and the time derivative blow up as u reached this level. Since then, many researchers have worked on the quenching properties of solutions for different kinds of parabolic equations (see [2]-[13] and the references therein). In particular, Zhi and Mu [9] considered the quenching properties for the semilinear equation

$$\begin{gathered} u_t = u_{xx} + (1-u)^{-p}, \quad 0 < x < 1,\ t > 0 \\ u_x(0,t) = u^{-q}(0,t), \quad u_x(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 < x < 1, \end{gathered} \tag{1.2}$$

and studied solution quenching in finite time, blow-up of time-derivatives and bounds of quenching rates. Later, Wang et al [11] investigated the following parabolic equation with localized reaction term,

$$\begin{gathered} u_t = \Delta u + (1-u(x,t))^{-p} + (1-u(x^*,t))^{-q}, \quad x \in B,\ t > 0 \\ u(x,t) = 0, \quad x \in \partial B, t > 0, \\ u(x,0) = u_0(x), \quad x \in B, \end{gathered} \tag{1.3}$$

where $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$, $x^* \in B$. They obtained the existence of the unique classical solution and proved the solution quenched in a finite time. In addition, when $x^* = 0$, they also gave bounds for the quenching rate.

There are two evident gaps in [11]: (a) the existence of classical solution in $\Omega \subset \mathbb{R}^n$; (b) the bounds of the quenching rate for any $x^* \in \Omega$. This article explore these two questions and extend the results for equation (1.3) to the system (1.1). Also we try obtain non-simultaneous quenching results.

Recently, some papers considered the non-simultaneous quenching behavior of solutions reaching the level $u = 0$ for parabolic systems (see [14]–[20]). For instance, Zheng and Wang [19] studied quenching properties for the nonlinear parabolic system

$$\begin{gathered} u_t = \Delta u - v^{-p}, \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v - u^{-q}, \quad x \in \Omega,\ t > 0 \\ , u = v = 1, \quad x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \bar{\Omega}. \end{gathered} \tag{1.4}$$

They obtained a solution quenching in finite time, and time-derivative blow up at the quenching point, under proper conditions. In addition, when $\Omega = B_R$, they studied sufficient conditions for non-simultaneous and simultaneous quenching. Later, Ji, Zhou and Zheng [17] studied the quenching behavior of solutions for heat system

$$u_t = u_{xx} - u^{-m} - v^{-p}, v_t = v_{xx} - u^{-q} - v^{-n},$$

with Neumann boundary conditions, They identified non-simultaneous and simultaneous quenching and described four possible simultaneous quenching rates via a characteristic algebraic system. However, there are very few papers in nonsimultaneous quenching for solutions reaching the level $u = 1$, which motivates us to consider the problem in this article.

This article is organized as follows. In Section 2, we obtain the global existence result for Ω small enough and finite time quenching for Ω large enough. Also we deduce the blow up of time-derivatives at the quenching point. In Section 3, we consider the non-simultaneous quenching of solutions for (1.1) with $\Omega = B_R(x^*)$. We will prove if $p_2 \geq p_1+1$ and $q_1 \geq q_2+1$, then quenching is always simultaneous; while $p_2 \geq p_1 + 1$ and $q_1 < 1$, then quenching is always non-simultaneous. If $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$), then the non-simultaneous quenching may occur; and if $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$, then both non-simultaneous and simultaneous quenching also may occur for proper initial data. In Section 4, we give a precise classification of parameters for the simultaneous quenching rates.

In this article we use the hypothesis

$$\begin{aligned} &\Delta u_0 + (1-u_0)^{-p_1} + (1-v_0)^{-q_1} > 0, \\ &\Delta v_0 + (1-u_0)^{-p_2} + (1-v_0)^{-q_2} > 0. \end{aligned} \tag{1.5}$$

2. Finite time quenching and blow up of time derivatives

Let λ_1 and φ_1 denote the first eigenvalue and the first eigenfunction of the problem

$$\begin{gathered} \Delta\varphi + \lambda\varphi = 0, \quad \text{in } \Omega, \\ \varphi = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

and choose $\varphi_1(x)$ to satisfy

$$\varphi_1(x) > 0, \quad \text{in } \Omega, \quad \int_\Omega \varphi dx = 1.$$

Theorem 2.1. *If $\lambda_1 < \min\{p_1+p_2, q_1+q_2\}+2$, then there exists a finite time T, such that the solution of* (1.1) *quenches at this time.*

Proof. By the maximum principle, we have $0 < u, v < 1$ in $\Omega \times (0,T)$. Assume that $p_1+p_2 \geq q_1+q_2$. Let $F(t) = \int_\Omega u\varphi dx$, $G(t) = \int_\Omega v\varphi dx$, and $\Phi(t) = F(t)+G(t)$ for $t \in [0,T)$. By Jensen's inequality,

$$\begin{aligned} F'(t) &= \int_\Omega \Delta u\varphi dx + \int_\Omega (1-u)^{-p_1}\varphi dx + \int_\Omega (1-v)^{-q_1}\varphi dx \\ &\geq -\int_\Omega \lambda_1 u\varphi dx + p_1\int_\Omega u\varphi dx + q_1\int_\Omega v\varphi dx + 2 \\ &= (p_1-\lambda_1)F(t) + q_1 G(t) + 2. \end{aligned} \tag{2.1}$$

Similarly, we have

$$G'(t) \geq (q_2-\lambda_1)G(t) + p_2F(t) + 2, \tag{2.2}$$

so we have

$$\begin{aligned} \Phi'(t) &\geq (p_1+p_2-\lambda_1)F(t) + (q_1+q_2-\lambda_1)G(t) + 4 \\ &\geq (p_1+p_2-\lambda_1)\Phi(t) + 4. \end{aligned} \tag{2.3}$$

Since $\lambda_1 < \min\{p_1+p_2, q_1+q_2\}+2$ and $0 < F, G < 1$, we have $(p_1+p_2-\lambda_1)\Phi(t)+4 > 0$ for $t \in [0,T)$. Integrating (2.3) from 0 to t, we have

$$t \leq \begin{cases} \frac{1}{p_1+p_2-\lambda_1} \ln \frac{(p_1+p_2-\lambda_1)\Phi(t)+4}{(p_1+p_2-\lambda_1)\Phi(0)+4}, & \lambda_1 \neq p_1+p_2, \\ \frac{1}{4}[\Phi(t)-\Phi(0)], & \lambda_1 = p_1+p_2, \end{cases} \tag{2.4}$$

Since $\lim_{t\to T^-} \Phi(t) \leq 2$, so we have the upper bound for quenching time T:

$$T \leq \begin{cases} \frac{1}{p_1+p_2-\lambda_1} \ln \frac{2(p_1+p_2-\lambda_1)+4}{(p_1+p_2-\lambda_1)\Phi(0)+4}, & \lambda_1 \neq p_1+p_2, \\ \frac{1}{4}[2-\Phi(0)], & \lambda_1 = p_1+p_2, \end{cases} \tag{2.5}$$

it is easy to see the right-hand side of (2.5) is greater than 0, so the solution of (1.1) quenches in finite time. □

We note that λ_1 decreases when the domain size increases, so Theorem 2.1 says that the solution of (1.1) will quench in finite time for Ω large enough. Next, we obtain the existence of a global solution for Ω small enough, which can be proved by adapting methods that are established in [19].

Theorem 2.2. *Assume that $u_0, v_0 \le \sigma_0 < 1$ in $\bar{\Omega}$ and the diameter of Ω is small enough. Then the solutions of* (1.1) *exist globally.*

Proof. Consider the auxiliary problem

$$\begin{gathered}
\bar{u}_t = \Delta\bar{u} + (1-\bar{u})^{-p_1} + (1-\bar{v})^{-q_1}, \quad (x,t) \in \Omega \times [0,T), \\
\bar{v}_t = \Delta\bar{v} + (1-\bar{u})^{-p_2} + (1-\bar{v})^{-q_2}, \quad (x,t) \in \Omega \times [0,T), \\
\bar{u}(x,t) = \sigma_0, \quad \bar{v}(x,t) = \sigma_0, \quad x \in \partial\Omega,\ t > 0, \\
\bar{u}(x,0) = \sigma_0, \quad \bar{v}(x,0) = \sigma_0, \quad x \in \Omega.
\end{gathered} \tag{2.6}$$

It is easy to see the solution of (2.6) is an upper-solution of (1.1). By the comparison principle, we have $u \le \bar{u}, v \le \bar{v}$, it suffices to prove that $(\bar{u}, \bar{v})$ is global. Let ϕ satisfy

$$\begin{gathered}
\Delta\phi - C_0 = 0, \quad x \in B_R(x^*), \\
\phi = \sigma_0, \qquad x \in \partial B_R(x^*),
\end{gathered} \tag{2.7}$$

where $B_R(x^*) = \{x \in \Omega : |x - x^*| \le R\}$ and

$$C_0 < \min\{-(1-\sigma_0)^{-p_1} - (1-\sigma_0)^{-q_1}, -(1-\sigma_0)^{-p_2} - (1-\sigma_0)^{-q_2}\} < 0,$$

hence

$$\phi(x) = \frac{C_0(|x - x^*|^2 - R^2)}{2N} + \sigma_0 \tag{2.8}$$

with $\max_{\bar{B}_R(x^*)} \phi(\cdot) = \sigma_0 - \frac{C_0 R^2}{2N}$. Taking R small enough such that

$$C_0 < \min_{\bar{B}_R(x^*)} \big\{ -(1-\phi)^{-p_1} - (1-\phi)^{-q_1}, -(1-\phi)^{-p_2} - (1-\phi)^{-q_2} \big\},$$

so (ϕ, ϕ) is a time-independent upper-solution of (2.6) for $\Omega \subset B_R(x^*)$, which implies the global solutions of (1.1) exist for the diameter of Ω small enough. □

Now we consider the blow up of time derivatives.

Lemma 2.3. *If* (1.5) *holds, then $u_t, v_t > 0$ for $(x,t) \in \Omega \times [0,T)$. Moreover, for any $\eta > 0$, there exists $c > 0$ such that*

$$u_t(x,t), v_t(x,t) \ge c, \quad \forall (x,t) \in \bar{\Omega}^\eta \times [0,T),$$

with $\Omega^\eta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \eta\}$.

Proof. Let $\Phi = u_t(x,t)$, $\Psi = v_t(x,t)$, since (1.5) holds, we have

$$\begin{gathered}
\Phi_t - \Delta\Phi = p_1(1-u)^{-p_1-1}\Phi + q_1(1-v)^{-q_1-1}\Psi, \quad (x,t) \in \Omega \times [0,T), \\
\Psi_t - \Delta\Psi = q_2(1-v)^{-q_2-1}\Psi + p_2(1-u)^{-p_2-1}\Phi, \quad (x,t) \in \Omega \times [0,T), \\
\Phi(x,t) = \Psi(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T), \\
\Phi(x,0) = \Delta u_0 + (1-u_0)^{-p_1} + (1-v_0)^{-q_1} > 0, \quad x \in \bar{\Omega}, \\
\Psi(x,0) = \Delta v_0 + (1-u_0)^{-p_2} + (1-v_0)^{-q_2} > 0, \quad x \in \bar{\Omega},
\end{gathered} \tag{2.9}$$

so by the maximum principle, $\Phi = u_t(x,t) > 0$, $\Psi = v_t(x,t) > 0$ for $(x,t) \in \Omega \times [0,T)$.

Let (u^*, v^*) be the solution for the auxiliary problem

$$\begin{gathered} u_t^* = \Delta u^* + (1-u_0)^{-p_1} + (1-v_0)^{-q_1}, \quad x \in \Omega,\ t > 0, \\ v_t^* = \Delta v^* + (1-u_0)^{-p_2} + (1-v_0)^{-q_2}, \quad x \in \Omega,\ t > 0, \\ u^*(x,t) = 0, \quad v^*(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ u^*(x,0) = u_0(x), \quad v^*(x,0) = v_0(x), \quad x \in \Omega\,. \end{gathered} \tag{2.10}$$

Let $\Phi^* = u_t^*(x,t)$, $\Psi^* = v_t^*(x,t)$, Then by the abovewe deduce that $u_t^*, v_t^* > 0$.

Next, let $w = u - u^*, z = v - v^*$ and $\widehat{\Phi} = w_t, \widehat{\Psi} = z_t$. It is easy to obtain

$$\begin{gathered} \widehat{\Phi}_t - \Delta\widehat{\Phi} \geq 0, \quad (x,t) \in \Omega \times [0,T), \\ \widehat{\Psi}_t - \Delta\widehat{\Psi} \geq 0, \quad (x,t) \in \Omega \times [0,T), \\ \widehat{\Phi}(x,t) = \widehat{\Psi}(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T), \\ \widehat{\Phi}(x,0) = \widehat{\Psi}(x,0) = 0, \quad x \in \bar{\Omega}, \end{gathered}$$

so that $u_t \geq u_t^*, v_t \geq v_t^*$ in $\Omega \times [0,T)$. Taking

$$c = \min\big\{ \min_{\bar{\Omega}^\eta \times [\eta,T)} |u_t^*|, \min_{\bar{\Omega}^\eta \times [\eta,T)} |v_t^*| \big\},$$

we have $u_t, v_t \geq c$ in $\bar{\Omega}^\eta \times [\eta, T)$. □

Lemma 2.4. *Assume that Ω is a convex domain and (1.5) holds, then for any η. Then there exists a positive constant ζ such that*

$$\begin{gathered} u_t \geq \zeta[(1-u)^{-p_1} + (1-v)^{-q_1}], \quad \text{in } \Omega^\eta \times (\eta,T), \\ v_t \geq \zeta[(1-u)^{-p_2} + (1-v)^{-q_2}], \quad \text{in } \Omega^\eta \times (\eta,T). \end{gathered} \tag{2.11}$$

Proof. Let

$$\begin{gathered} I = u_t - \zeta[(1-u)^{-p_1} + (1-v)^{-q_1}], \quad (x,t) \in \Omega^\eta \times (\eta,T), \\ J = v_t - \zeta[(1-u)^{-p_2} + (1-v)^{-q_2}], \quad (x,t) \in \Omega^\eta \times (\eta,T)\,. \end{gathered} \tag{2.12}$$

Then we have

$$\begin{aligned} I_t - \Delta I &= (u_t - \Delta u)_t - \zeta p_1(1-u)^{-p_1-1}(u_t - \Delta u) - \zeta q_1(1-v)^{-q_1-1}(v_t - \Delta v) \\ &\quad + \zeta p_1(p_1+1)(1-u)^{-p_1-2}|\nabla u|^2 + \zeta q_1(q_1+1)(1-v)^{-q_1-2}|\nabla v|^2 \\ &\geq p_1(1-u)^{-p_1-1} I + q_1(1-v)^{-q_1-1} J\,. \end{aligned}$$

Similarly,

$$J_t - \Delta J \geq q_2(1-v)^{-q_2-1} J + p_2(1-u)^{-p_2-1} I. \tag{2.13}$$

In addition, by Lemma 2.3 and taking ζ small enough, we have

$$\begin{gathered} I(x,t) = u_t - \zeta[(1-u)^{-p_1} + (1-v)^{-q_1}] \geq 0, \quad (x,t) \in \partial\Omega^\eta \times (0,T), \\ J(x,t) = v_t - \zeta[(1-u)^{-p_2} + (1-v)^{-q_2}] \geq 0, \quad (x,t) \in \partial\Omega^\eta \times (0,T), \end{gathered} \tag{2.14}$$

and the initial data

$$I(x,0),\ J(x,0) \geq 0 \quad x \in \Omega^\eta, \tag{2.15}$$

By the maximum principle, we have $I(x,t), J(x,t) \geq 0$ for $(x,t) \in \Omega^\eta \times (0,T)$. □

As a direct consequence of Lemma 2.4, we deduce time-derivatives blow up at the quenching point.

Theorem 2.5. *If Ω is a convex domain and* (1.5) *holds, then (u_t, v_t) blows up at the quenching point.*

3. Simultaneous and non-simultaneous quenching

In this section, we deal with radial solutions of (1.1) with $\Omega = B_R(x^*) = \{x \in \mathbb{R}^N : |x - x^*| < R\}$, and non-increasing initial data satisfying (1.5). By the maximum principle [11, Lemma 3.2], we have $u_r(r,t), v_r(r,t) \le 0$. At first, we give the sufficient condition for finite-time quenching of radical solutions in $\bar{B}_R(x^*) \times (0,T)$.

Lemma 3.1. *Assume (u,v) is the global solution of* (1.1) *with $(u_0, v_0) \equiv (0,0)$, in other words, there exists a constant $c \in [0,1)$ such that $u, v \le c < 1$ on $\bar{B}_R(x^*) \times [0,\infty)$. Then (u,v) approaches uniformly from below to a solution (U,V) of the steady-state problem*

$$\begin{gathered}
\Delta U = -(1-U)^{-p_1} - (1-V)^{-q_1}, \quad x \in B_R(x^*), \\
\Delta V = -(1-U)^{-p_2} - (1-V)^{-q_2}, \quad x \in B_R(x^*), \\
U = V = 0, \quad x \in \partial B_R(x^*).
\end{gathered} \tag{3.1}$$

Proof. By [19, Lemma 4.1], we define

$$W(x,t) = \int_{B_R(x^*)} G(x,y)u(y,t)dy, \quad Z(x,t) = \int_{B_R(x^*)} G(x,y)v(y,t)dy,$$

for $(x,t) \in \bar{B}_R(x^*) \times [0,\infty)$, where $G(x,y)$ is Green's function associated with the operator $-\Delta$ on $B_R(x^*)$ under Dirichlet boundary conditions. then

$$W_t(x,t) = 1 - u(x,t) + \int_{B_R(x^*)} G(x,y)(1-u)^{-p_1}dy + \int_{B_R(x^*)} G(x,y)(1-v)^{-q_1}dy,$$

$$Z_t(x,t) = 1 - v(x,t) + \int_{B_R(x^*)} G(x,y)(1-u)^{-p_2}dy + \int_{B_R(x^*)} G(x,y)(1-v)^{-q_2}dy.$$

Combining Lemma 2.3 and the monotone convergence theorem, we have

$$\begin{aligned}
&\lim_{t\to\infty} W_t(x,t) \\
&= 1 - U(x) + \int_{B_R(x^*)} G(x,y)(1-U)^{-p_1}dy + \int_{B_R(x^*)} G(x,y)(1-V)^{-q_1}dy, \\
&\lim_{t\to\infty} Z_t(x,t) \\
&= 1 - V(x) + \int_{B_R(x^*)} G(x,y)(1-U)^{-p_2}dy + \int_{B_R(x^*)} G(x,y)(1-V)^{-q_2}dy,
\end{aligned}$$

where $c \ge U(x) = \lim_{t\to\infty} u(x,t)$, $c \ge V(x) = \lim_{t\to\infty} v(x,t)$. In addition, since W, Z are bounded and $W_t, Z_t \ge 0$, we have

$$\lim_{t\to\infty} W_t(x,t) = 0, \quad \lim_{t\to\infty} Z_t(x,t) = 0, \tag{3.2}$$

which imply

$$\begin{aligned}
U(x) &= 1 + \int_{B_R(x^*)} G(x,y)(1-U)^{-p_1}dy + \int_{B_R(x^*)} G(x,y)(1-V)^{-q_1}dy, \\
V(x) &= 1 + \int_{B_R(x^*)} G(x,y)(1-U)^{-p_2}dy + \int_{B_R(x^*)} G(x,y)(1-V)^{-q_2}dy,
\end{aligned} \tag{3.3}$$

which is the solution of (3.1), and by Dini's theorem, we can get the uniform convergence. □

Inspired by [20, Theorem 1.3], with Lemma 3.1 at hand, we obtain the following theorem.

Theorem 3.2. *If $R \geq \sqrt{N}$, then the radial solution of* (1.1) *will quench in finite time for any initial data.*

Proof. Considering the auxiliary system

$$\begin{gathered} \underline{u}_t = \Delta \underline{u} + (1-\underline{u})^{-p_1} + (1-\underline{v})^{-q_1}, \quad (x,t) \in B_R(x^*) \times [0,T), \\ \underline{v}_t = \Delta \underline{v} + (1-\underline{u})^{-p_2} + (1-\underline{v})^{-q_2}, \quad (x,t) \in B_R(x^*) \times [0,T). \\ \underline{u}(x,t) = 0, \quad \underline{v}(x,t) = 0, \quad x \in \partial B_R(x^*),\ t > 0, \\ \underline{u}(x,0) = 0, \quad \underline{v}(x,0) = 0, \quad x \in \bar{B_R}(x^*), \end{gathered} \tag{3.4}$$

by the comparison principle, we have $u \geq \underline{u}, v \geq \underline{v}$. Now we introduce the problem

$$\begin{gathered} -\Delta \underline{u}^* = 2, \quad -\Delta \underline{v}^* = 2, \quad r \in\ B_R(x^*), \\ \underline{u}^* = \underline{v}^* = 0, \quad r \in \partial B_R(x^*), \end{gathered} \tag{3.5}$$

with solution denoted as

$$\underline{u}^* = \frac{-2(|x-x^*|^2 - R^2)}{2N}, \quad \underline{v}^* = \frac{-2(|x-x^*|^2 - R^2)}{2N}. \tag{3.6}$$

So we have $\max\{\underline{u}^*, \underline{v}^*\} = R^2/N$. Clearly, $(\underline{u}^*, \underline{v}^*)$ is a sub-solution of (1.1). By Lemma 3.1, the solution (u,v) is global only if $\underline{u}^*, \underline{v}^* < 1$. Therefore, if $\underline{u}^*$ or $\underline{v}^* \geq 1$, namely $R \geq \sqrt{N}$, then the solution of (1.1) quenches in finite time for any initial data. □

Remark 3.3. Theorem 3.2 indicates that the solution quenches in finite time for $R \geq \sqrt{N}$. However, for radical solutions of (1.1) with $\Omega = B_R = \{x \in \mathbb{R}^N : \|x\| < R\}$ and assuming (1.5) and that $u_0'(r), v_0'(r) \leq 0$, by [20], we can obtain that the solution quenches in finite time without the condition $R \geq \sqrt{N}$. Also we obtain that $r = 0$ is the only quenching point.

Next, we will focus on the simultaneous and non-simultaneous quenching quenching of solutions for (1.1). To simplify our work, we deal with the radical solutions of (1.1) with $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$, and assume that (1.5) holds and $u_0'(r), v_0'(r) \leq 0$. It is easy to see that $\max_{0 \leq r \leq R} u(r,t) = u(0,t)$, $\max_{0 \leq r \leq R} v(r,t) = v(0,t)$ by Remark 3.3. In addition, c, c_i, C, C_i denote positive constants independents of t, which are different from line to line. First, we give a necessary condition for the non-simultaneous quenching.

Theorem 3.4. *If $v(0,t) \leq c < 1$ for $t \in [0,T)$, then $p_2 < p_1 + 1$.*

Proof. Since $u_r, v_r \leq 0$, by the Hopf's lemma, we can see that $u_{rr}(0,t), v_{rr}(0,t) \leq 0$. Then by Lemma 2.4, we have

$$\begin{gathered} \zeta((1-u)^{-p_1} + (1-v)^{-q_1})(0,t) \leq u_t(0,t) \leq (1-u)^{-p_1} + (1-v)^{-q_1}(0,t), \\ \zeta((1-u)^{-p_2} + (1-v)^{-q_2})(0,t) \leq v_t(0,t) \leq (1-u)^{-p_2} + (1-v)^{-q_2}(0,t). \end{gathered} \tag{3.7}$$

Combing (**??**) with $v(0,t) \leq c < 1$, we have

$$u_t(0,t) \leq C(1-u)^{-p_1}(0,t). \tag{3.8}$$

Integrating on (t, T) gives

$$1 - u(0,t) \le C(T-t)^{\frac{1}{p_1+1}}. \tag{3.9}$$

So by Lemma 2.4 and (**??**), we have

$$v_t(0,t) \ge \zeta(1-u(0,t))^{-p_2} \ge C(T-t)^{-\frac{p_2}{p_1+1}}.$$

Integrating on $(0, T)$, we have

$$v(0,T) - v(0,0) \ge C \int_0^T (T-t)^{-\frac{p_2}{p_1+1}} dt. \tag{3.10}$$

If $p_2 \ge p_1 + 1$, this integral diverges. The proof is complete. $\square$

Corollary 3.5. *If $p_2 \ge p_1 + 1$ and $q_1 \ge q_2 + 1$, then quenching is simultaneous.*

Next, we give a sufficient condition for non-simultaneous quenching.

Theorem 3.6. *If $p_2 \ge p_1 + 1$, $q_1 < 1$, then $u(0,t) \le c < 1$ for $t \in [0, T]$.*

Proof. Define $(\widetilde{u}(t), \widetilde{v}(t)) := (u(0,t), v(0,t))$. By (**??**), there exist two positive constants c_0, c_1 such that

$$\begin{aligned} c_0[(1-\widetilde{u})^{-p_1} + (1-\widetilde{v})^{-q_1}]\widetilde{v}' &\le \widetilde{u}'[(1-\widetilde{u})^{-p_2} + (1-\widetilde{v})^{-q_2}] \\ &\le c_1[(1-\widetilde{u})^{-p_1} + (1-\widetilde{v})^{-q_1}]\widetilde{v}', \end{aligned} \tag{3.11}$$

Multiplying the second inequality by $(1-\widetilde{u})^{p_1}(1-\widetilde{v})^{q_1}$, we have

$$\widetilde{u}'(1-\widetilde{u})^{-p_2+p_1} \le c\widetilde{v}(1-\widetilde{v})^{-q_1}. \tag{3.12}$$

Integrating on $(0, T)$, if $p_2 > p_1 + 1, q_1 < 1$, we have

$$(1-\widetilde{u}(T))^{1-p_2+p_1} \le c_0 - c(1-\widetilde{v}(T))^{1-q_1}, \tag{3.13}$$

if $p_2 = p_1 + 1, q_1 < 1$, we have

$$-\ln(1-\widetilde{u}(T)) \le c_0 - c(1-\widetilde{v}(T))^{1-q_1},$$

a contradiction, if u quenches. $\square$

Theorem 3.7. *If $p_2 < p_1 + 1$ ($q_1 < q_2 + 1$), then there exist the initial data such that $u(v)$ quenches while $v(u) \le c_0 < 1$.*

Proof. By Lemma 2.4, we have

$$u_t(0,t) \ge \zeta(1-u(0,t))^{-p_1}, \tag{3.14}$$

Integrating (**??**) on (t, T), we have there exists a positive constant C such that

$$1 - u(0,t) \ge C(T-t)^{\frac{1}{p_1+1}}. \tag{3.15}$$

Similarly,

$$1 - v(0,t) \ge C(T-t)^{\frac{1}{q_2+1}}. \tag{3.16}$$

Combining (**??**), (**??**) and (**??**), we obtain

$$v_t(0,t) \le C(T-t)^{-\frac{p_2}{1+p_1}} + C(T-t)^{-\frac{q_2}{1+q_2}}. \tag{3.17}$$

Integrating on $(0, T)$, we obtain

$$v(0,T) \le v(0,0) + c_1 T^{\frac{1+p_1-p_2}{1+p_1}} + c_2 T^{\frac{1}{1+q_2}}. \tag{3.18}$$

By Lemma 2.3, we have $u_t, v_t \ge c$. By integrating on $(0, t)$ and letting $t \to T^-$, we have $T \le \frac{1}{c}\min\{1-u_0(0), 1-v_0(0)\}$. We take $u_0(x) = 1 - \epsilon$, then $T \le \frac{1}{c}\epsilon$. If ϵ, and hence T, are small enough, we can conclude from (**??**) that $v(0,T) \le c_0 < 1$. The proof is complete. $\square$

Next we show that if $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$, then both non-simultaneous and simultaneous quenching also may occur for proper initial data. At first, we give the following lemma.

Lemma 3.8 ([19, Lemma 4.5])**.** *If $p_2 < p_1 + 1$, $q_1 < q_2 + 1$, then the set of initial data such that one of the components quenching alone is open.*

Theorem 3.9. *If $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$, then both simultaneous and non-simultaneous quenching may occur for proper initial data.*

Proof. Step I. We prove non-simultaneous quenching. Assume for contradiction that u and v quenches simultaneously for every initial data. Since $u_t(0,t) \le (1 - u(0,t))^{-p_1} + (1 - v(0,t))^{-q_1}$ by (**??**), integrating on $(0,t)$ gives

$$v(0,t) \le v_0(0) + \int_0^t (1 - u(0,s))^{-p_1} + (1 - v(0,s))^{-q_1}\,ds, \tag{3.19}$$

introducing (**??**) and (**??**) in (3.8), letting $t \to T^-$, we obtain that

$$v(0,T) \le v_0(0) + T^{\frac{1}{q_2+1}} + T^{\frac{p_1-p_2+1}{p_1+1}}. \tag{3.20}$$

As in Theorem 3.7. We take $v_0(x) = 1 - \epsilon$, then $T \le \frac{1}{C}\epsilon$. if ϵ, and hence T, are small enough, we can conclude from (3.9) that $v(0,T) \le c < 1$, a contradiction.

Step II. We prove simultaneous quenching. Since $p_2 < p_1 + 1,\ q_1 < q_2 + 1$, From (**??**), we have

$$v(0,T) \le v(0,0) + c_1 T^{\frac{1+p_1-p_2}{1+p_1}} + c_2 T^{\frac{1}{1+q_2}}. \tag{3.21}$$

Similarly,

$$u(0,T) \le u(0,0) + c_3 T^{\frac{1+q_2-q_1}{1+q_2}} + c_4 T^{\frac{1}{1+p_1}}. \tag{3.22}$$

Denote (u_α, v_α) as a solution of (1.1) with initial data $(1 - \alpha u_0, 1 - (1-\alpha)v_0)$, where $\alpha \in (0,1)$. Let T_α be the quenching time, we have $u_\alpha(0,T) \le c < 1$ for $\alpha \to 1$ and $v_\alpha(0,T) \le c < 1$ for $\alpha \to 0$. Define $\Psi_u = \{\alpha \in (0,1) : u_\alpha(0,T) < 1\}$, $\Psi_v = \{\alpha \in (0,1) : v_\alpha(0,T) < 1\}$, it is easy to see that

$$\Phi_u \cap \Psi_v = \emptyset,$$

however by Lemma 3.8, we have that Φ_u and Psi_v are open. Hence u, v quench simultaneously for some initial data. The proof complete. $\square$

4. Simultaneous and non-simultaneous quenching rates

The notation $f \sim g$ means that there exist positive constants c_1, c_2 such that $c_1 g \le f \le c_2 g$. At first, we give a lemma which needs two additional assumptions.

- (H1) $p_2 \ge p_1 + 1$, $q_1 \ge q_2 + 1$, $q_1 \ge q_2$, and $\xi(1-u_0)^{p_2-1} \ge (1-v_0)^{q_1-1}$ with $\xi > \frac{p_2-1}{q_1-p_2}$;
- (H2) $p_2 \ge p_1 + 1$, $q_1 \ge q_2 + 1$, $q_1 \le q_2$ and $\eta(1-u_0)^{p_2-1} \le (1-v_0)^{q_1-1}$ with $\eta < \frac{p_2-1}{q_1-p_2}$.

Lemma 4.1. *Let (u,v) be the solution of problem* (1.1)*. Then $\xi(1-u)^{p_2-1} \ge (1-v)^{q_1-1}$ under assumption* (H1)*, and $\eta(1-u)^{p_2-1} \le (1-v)^{q_1-1}$ under assumption* (H2)*, for $(r,t) \in (0,R) \times (0,T)$.*

Proof. Let $\varphi = \xi(1-u)^{p_2-1} - (1-v)^{q_1-1}$, $\psi = \eta(1-u)^{p_2-1} - (1-v)^{q_1-1}$. We have

$$\begin{aligned}
&\varphi_t - \varphi_{rr} - h\varphi_r + l\varphi \\
&= -\xi(p_2-1)(1-u)^{p_2-p_1-2} + \xi(q_1-1)(1-u)^{-1}(1-v)^{-1} \\
&\quad + (q_1-1)(1-v)^{q_1-q_2-2} - \xi(p_2-1)(1-u)^{p_2-2}(1-v)^{-q_1} \\
&\quad + (q_1-p_2)(1-u)^{-1}(1-v)^{q_1-2}u_r v_r \\
&\geq \xi(q_1-p_2)(1-u)^{-1}(1-v)^{-1} - \xi(p_2-1)(1-u)^{p_2-2}(1-v)^{-q_1} \\
&\quad + (q_1-p_2)(1-u)^{-1}(1-v)^{q_1-2}u_r v_r \\
&= \xi(q_1-p_2)(1-u)^{-1}(1-v)^{-1} - (p_2-1)(1-u)^{-1}(1-v)^{-1}(1+\varphi(1-v)^{1-q_1}) \\
&\quad + (q_1-p_2)(1-u)^{-1}(1-v)^{q_1-2}u_r v_r
\end{aligned}$$

where

$$\begin{gathered}
h = \frac{N-1}{r}(q_1-2)(1-v)^{-1}v_x + (p_2-2)(1-u)^{-1}u_x, \\
l = (q_1-1)(1-u)^{-p_2}(1-v)^{-1} - (p_2-1)(q_1-2)(1-u)^{-1}(1-v)^{-1};
\end{gathered} \tag{4.1}$$

so

$$\begin{aligned}
&\varphi_t - \varphi_{rr} - h\varphi_r + (l + (p_2-1)(1-u)^{-1}(1-v)^{-q_1})\varphi \\
&\geq (\xi(q_1-p_2) - p_2 + 1)(1-u)^{-1}(1-v)^{-1} \\
&\quad + (q_1-p_2)(1-u)^{-1}(1-v)^{q_1-2}u_r v_r
\end{aligned} \tag{4.2}$$

Since $\xi > \frac{p_2-1}{q_1-p_2}$, we have

$$\varphi_t - \varphi_{rr} - h\varphi_r + (l + (p_2-1)(1-u)^{-1}(1-v)^{-q_1})\varphi \geq 0. \tag{4.3}$$

In addition,

$$\begin{gathered}
\varphi(r,0) = \xi(1-u_0)^{p_2-1} - (1-v_0)^{q_1-1} \geq 0, \quad r \in [0,R], \\
\varphi_r(0,t) = \varphi_r(R,t) = 0, \quad t \in (0,T)
\end{gathered} \tag{4.4}$$

By the maximum principle,

$$\varphi = \xi(1-u)^{p_2-1} - (1-v)^{q_1-1} \geq 0 \tag{4.5}$$

Similarly, if (H2) holds, we can obtain $\psi = \eta(1-u)^{p_2-1} - (1-v)^{q_1-1} \leq 0$. The proof is complete. □

Next, we give bounds for the non-simultaneous quenching rate.

Theorem 4.2. *If quenching is non-simultaneous and u is the quenching component, then for $t \to T^-$, we have*

$$1 - u(0,t) \sim (T-t)^{\frac{1}{1+p_1}}.$$

The proof of the above theorem is a direct consequence of (**??**) and (**??**). Next, we give bounds for the simultaneous quenching rate.

Theorem 4.3. *Assume that* (H1) *or* (H2) *hold. Then quenching is simultaneous, and for $t \to T^-$,*

$$1 - u(0,t) \sim (T-t)^{\frac{q_1-1}{p_2 q_1 - 1}}, \quad 1 - v(0,t) \sim (T-t)^{\frac{p_1-1}{p_2 q_1 - 1}}.$$

Proof. Without loss of generality, consider the case of (H1) only. Since $\xi(1-u)^{p_2-1} \geq (1-v)^{q_1-1}$, by (**??**), we obtain

$$\begin{aligned} v_t(0,t) &\leq (1-u(0,t))^{-p_2} + (1-v(0,t))^{-q_2} \\ &\leq (1-v(0,t))^{\frac{-p_2(q_1-1)}{p_2-1}} + (1-v(0,t))^{-q_2} \\ &\leq c(1-v(0,t))^{\frac{-p_2(q_1-1)}{p_2-1}}, \end{aligned} \tag{4.6}$$

by $p_2 \geq p_1+1$ and $q_1 \geq q_2+1$. Integrating (4.6) on $(0,T)$, we have

$$1-v(0,t) \leq C(T-t)^{\frac{p_2-1}{p_2q_1-1}}. \tag{4.7}$$

By Lemma 2.4, we have

$$u_t(0,t) \geq \zeta(1-v)^{-q_1}(0,t) \geq c(T-t)^{\frac{-q_1(p_2-1)}{p_2q_1-1}}. \tag{4.8}$$

Integrating on $(0,T)$, we have

$$1-u(0,t) \geq C(T-t)^{\frac{q_1-1}{p_2q_1-1}}, \tag{4.9}$$

by Lemma 2.4 again, we have

$$v_t(0,t) \geq \zeta(1-u)^{-p_2}(0,t). \tag{4.10}$$

Integrating on (t,T) we have

$$1-v(0,t) \geq C\int_t^T (1-u(0,\eta))^{-p_2} dt \geq c(1-u(0,t))^{-p_2}(T-t), \tag{4.11}$$

by (**??**), we have

$$u_t(0,t) \leq (1-u(0,t))^{-p_1} + C(1-u(0,t))^{p_2q_1}(T-t)^{-q_1} \tag{4.12}$$

combining (**??**) and (**??**), we have

$$u_t(0,t) \leq C(1-u(0,t))^{p_2q_1}(T-t)^{-q_1}. \tag{4.13}$$

Integrating (**??**) on (t,T), we have

$$1-u(0,t) \geq C(T-t)^{\frac{q_1-1}{p_2q_1-1}}, \tag{4.14}$$

from Lemma 2.4, we have

$$v_t(0,t) \geq \zeta(1-u)^{-p_2} \geq C(T-t)^{\frac{-p_2(q_1-1)}{1-p_2q_1}}. \tag{4.15}$$

Integrating on (t,T), we have

$$1-v(0,t) \geq C(T-t)^{\frac{p_2-1}{p_2q_1-1}}. \tag{4.16}$$

□

Theorem 4.4. *Assume $p_2 < p_1+1$, $q_1 < q_2+1$. Then quenching is simultaneous, and for $t \to T^-$,*

$$1-u(0,t) \sim (T-t)^{1-\frac{q_1}{q_2+1}}, 1-v(0,t) \sim (T-t)^{\frac{1}{q_2+1}},$$
$$\frac{p_1(q_2+1)}{p_1+1} \leq q_1 < q_2+1, p_2 \leq \frac{q_2(p_1+1)}{q_2+1},$$
$$1-u(0,t) \sim (T-t)^{1-\frac{q_1}{q_2+1}}, 1-v(0,t) \sim (T-t)^{\frac{1}{q_2+1}}, q_1 < q_2+1,$$
$$\frac{q_2(p_1+1)}{q_2+1} \leq p_2 \leq \frac{q_2}{q_2+1-q_1},$$

$$
\begin{gathered}
1-u(0,t)\sim (T-t)^{\frac{1}{p_1+1}}, 1-v(0,t)\sim (T-t)^{1-\frac{p_2}{p_1+1}},\\
\frac{q_2(p_1+1)}{q_2+1}\le p_2<\frac{q_2}{q_2+1-q_1}, q_1\le \frac{p_1(q_2+1)}{p_1+1},\\
1-u(0,t)\sim (T-t)^{\frac{1}{p_1+1}}, 1-v(0,t)\sim (T-t)^{1-\frac{p_2}{p_1+1}},\\
p_2<p_1+1, \frac{p_1(q_2+1)}{p_1+1}\le q_1\le \frac{p_1}{p_1+1-p_2},\\
1-u(0,t)\sim (T-t)^{\frac{1}{p_1+1}}, 1-v(0,t)\sim (T-t)^{\frac{1}{q_2+1}},\\
q_1\le \frac{p_1(q_2+1)}{p_1+1}, p_2\le \frac{q_2(p_1+1)}{q_2+1}.
\end{gathered}
$$

Note that Theorem 4.3 gives the simultaneous quenching rate under $p_2\ge p_1+1$ and $q_1\ge q_2+1$, while Theorem 4.4 gives the simultaneous quenching rate under $p_2<p_1+1$ and $q_1<q_2+1$. The proof is similar to [17], so we omit it.

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