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FREE BOUNDARY PROBLEMS WITH NEUMAN BOUNDARY CONDITION

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Abstract. In this work, we study the continuity of a free boundary in a class of elliptic problems, with a Neuman boundary condition. The main idea is to use a change of variable that reduces the problem to the one studied in [16].

1. Preliminaries and statement of the problem

Saadi [16] studied the problem

$$
\begin{aligned}
&\text{Find } (u,\chi) \in H^1(\Omega)\times L^\infty(\Omega) \text{ such that}\\
&\text{(i) } u \ge 0,\ 0\le \chi \le 1,\ u(1-\chi)=0 \text{ a.e. in } \Omega,\\
&\text{(ii)}\\
&\qquad \int_\Omega \big(A(x)\nabla u + \chi h(x)\mathbf{e}\big)\cdot\nabla\xi\,dx \le \int_\Gamma \beta(x,\varphi-u)\xi\,d\sigma(x)\\
&\text{for all } \xi\in H^1(\Omega),\ \xi\ge 0 \text{ on } \partial\Omega\setminus\Gamma,
\end{aligned}
\tag{1.1}
$$

where $\Omega = \{(x_1,x_2)\in\mathbb{R}^2 : x_1\in(a_0,b_0),\ d_0 < x_2 < \gamma(x_1)\}$, with $\gamma\in C^{0,1}(a_0,b_0)$, $\Gamma = \{(x_1,\gamma(x_1)) : x_1\in(a_0,b_0)\}$, $A(x)=[a_{ij}(x)]$ is a 2×2 matrix, $\mathbf{e}=(0,1)$, h is a nonnegative function defined in Ω, and φ is a nonnegative Lipschitz continuous function on Γ.

When h is non-decreasing with respect to the variable x_2, it is well known (see [3, 6, 16]) that the function χ is non-increasing with respect to x_2, which forces the free boundary i.e. the interface between the two sets $\{u=0\}$ and $\{u>0\}$, to be the graph of a function $\Phi(x_1)$. Moreover, under suitable assumptions (see [3, 6, 16]), it was proven that Φ is continuous for both Dirichlet and Neuman conditions.

In this article, we consider a more general class of free boundary problems in the spirit of [4], namely we replace the particular vector function $h(x)\mathbf{e}$ in (1.1) by a

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general vector function $\mathbf{H}$.

Find $(u,\chi) \in H^1(\Omega) \times L^\infty(\Omega)$ such that
(i) $u \ge 0$, $0 \le \chi \le 1$, $u(1-\chi) = 0$ a.e. in Ω,
(ii) $u = 0$ on Γ_2,
(iii)

$$\int_\Omega \big(A(x)\nabla u + \chi \mathbf{H}(x)\big) \cdot \nabla \xi \, dx \le \int_{\Gamma_3} \beta(x, \varphi - u)\xi \, d\sigma(x) \tag{1.2}$$

for all $\xi \in H^1(\Omega)$, $\xi \ge 0$ on Γ_2,

where Ω is a bounded domain of $\mathbb{R}^2$ with a C^1 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where Γ_1, Γ_2 and Γ_3 are disjoint nonempty sets, with Γ_3 relatively open in $\partial\Omega$.

Here $A(x) = [a_{ij}(x)]$ is a 2×2 matrix such that for two positive constants λ and Λ, we have

$$|a_{ij}(x)| \le \Lambda, \quad \text{a.e. } x \in \Omega, \ \forall i, j = 1, 2, \tag{1.3}$$

$$A(x)\xi \cdot \xi \ge \lambda |\xi|^2 \quad \text{a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^2 \tag{1.4}$$

$\mathbf{H} = (H_1, H_2)$ is a vector function that for some positive constants $\overline{h} > \underline{h}$ satisfies

$$H_1, H_2 \in C^1(\overline{\Omega}), \tag{1.5}$$

$$|H_1(x)| \le \overline{h}, \quad |H_2(x)| \le \overline{h} \quad \text{in } \Omega, \tag{1.6}$$

$$H_2(x) \ge \underline{h} \quad \text{in } \Omega, \tag{1.7}$$

$$\operatorname{div}(\mathbf{H}(x)) \ge 0, \quad \text{in } \Omega, \tag{1.8}$$

$$\mathbf{H}(x) \cdot \boldsymbol{\nu} > 0 \quad \text{on } \Gamma_3. \tag{1.9}$$

The functions φ and β satisfy

$$\varphi \text{ is a nonnegative Lipschitz continuous function on } \Gamma_3, \tag{1.10}$$

$$\beta(x, \cdot) \text{ is continuous for a.e. } x \in \Gamma_3, \tag{1.11}$$

$$\beta(x, 0) = 0 \quad \text{a.e. } x \in \Gamma_3, \tag{1.12}$$

$$\beta(x, \cdot) \text{ is non-decreasing a.e. } x \in \Gamma_3. \tag{1.13}$$

In this article, we replace the Dirichlet condition $u = \varphi$ on Γ_3 (see [4]) by the following Neuman condition, in the weak sense,

$$A(x)\nabla u \cdot \boldsymbol{\nu} = \beta(x, \varphi - u) - \chi \mathbf{H}(x) \cdot \boldsymbol{\nu} \quad \text{on } \Gamma_3$$

We observe that if $u > 0$ in a neighbourhood of $\Gamma_3' \subset \Gamma_3$, the condition becomes

$$A(x)\nabla u \cdot \boldsymbol{\nu} = \beta(x, \varphi - u) - \mathbf{H}(x) \cdot \boldsymbol{\nu} \quad \text{on } \Gamma_3'$$

Among the free boundary problems that fit in the above setting, we can mention the dam problem with leaky boundary condition on each reservoir bottom (see [7, 8, 10, 11, 12, 13]), in which case φ is the water pressure on the reservoirs bottoms. Another application arises from the thermoelectrical modeling of aluminum electrolytic cells (see [1]), in which case u is the temperature in the electrolytic bath and φ is the solidification temperature.

In problem (1.2), the free boundary is defined as the set $\partial\{u > 0\} \cap \Omega$ that separates the two regions $\{u = 0\}$ and $\{u > 0\}$. In particular, in the case of the dam problem, it represents the interface between wet and dry parts of the porous medium, and in the aluminium electrolysis problem, it is the interface between liquid and solid aluminium inside an aluminium electrolytic cell section.

In this article, we prove that the free boundary is represented locally by a family of continuous functions. Our strategy consists in using a change of variable to transform problem (1.2) locally to a problem similar to (1.1) studied in [16]. We observe that as we are concerned with the regularity of the free boundary, which is a local problem, we may assume, shrinking if necessary, that $\Gamma_1 = \Gamma_2 = \emptyset$ and that Γ_3 is connected.

We would like to point out that one of the important consequences of the regularity of the free boundary is its key role in the uniqueness proof of the solution (see [7, 11, 12, 13]).

Remark 1.1. Under assumptions (1.3)–(1.5) and (1.10)–(1.13), one can prove existence of a solution for problem (1.2) as in [7]. For a more general situation, we refer the reader to [10].

We begin with the following proposition that can be obtained as in [4].

Proposition 1.2. *For any solution (u,χ) of (1.2), we have*

(i) $\operatorname{div}(A(x)\nabla u) = -\operatorname{div}(\chi \mathbf{H}(x))$ *in* $\mathcal{D}'(\Omega)$.
(ii) $\operatorname{div}(\chi\mathbf{H}(x)) - \chi_{\{u>0\}}\operatorname{div}(\mathbf{H}(x)) \leq 0$ *in* $\mathcal{D}'(\Omega)$.

Remark 1.3. As a consequence of Proposition 1.2(i), we have

(i) $u \in C^{0,\delta}_{\rm loc}(\Omega\cup\Gamma_2)$ for some $\delta \in (0,1)$ (see [9]). In particular the set $\{u > 0\}$ is open.
(ii) If $A \in C^{0,\alpha}_{\rm loc}(\Omega)$ $(0 < \alpha < 1)$, then $u \in C^{1,\alpha}_{\rm loc}(\{u > 0\})$ (see [9]).
(iii) If $A \in C^{0,\alpha}_{\rm loc}(\Omega\cup\Gamma_2)$ $(0 < \alpha < 1)$ and for some constant c_0, $div(A(x)(x-y)) \leq c_0$ in $\mathcal{D}'(\Omega)$, for all y in Ω, then $u \in C^{0,1}_{\rm loc}(\Omega\cup\Gamma_2)$ (see [3, 15]).

Following [2, 4], for $h \in \Pi_{x_2}(\Omega)$ and $w \in \Pi_{x_1}(\Omega \cap \{x_2 = h\})$, where Π_{x_1} is the orthogonal projection on the x_1-axis, we introduce the differential equation

$$\begin{aligned} \mathbf{X}'(t,w,h) &= \mathbf{H}(\mathbf{X}(t,w,h)) \\ \mathbf{X}(0,w,h) &= (w,h)\,, \end{aligned} \tag{1.14}$$

where $\mathbf{X} = (X_1, X_2)$.

This equation has a unique solution $\mathbf{X}(\cdot,w,h)$ which is defined in a maximal interval $(\alpha_-(w,h), \alpha_+(w,h))$ and is continuous in the open set (see [17, Chp. 3]),

$$\big\{(t,w,h) : \alpha_-(w,h) < t < \alpha_+(w,h), h \in \Pi_{x_2}(\Omega), w \in \Pi_{x_1}(\Omega\cap\{x_2 = h\})\big\}.$$

Moreover by (1.7), we have $\frac{\partial X_2}{\partial t} = H_2(\mathbf{X}(t,w,h)) > 0$, which leads to

$$\mathbf{X}(\alpha_-(w,h),w,h) \in \partial\Omega\cap\{x_2 < h\} \quad \text{and} \quad \mathbf{X}(\alpha_+(w,h),w,h) \in \partial\Omega\cap\{x_2 > h\}$$

We will drop the dependence on h on the functions $\mathbf{X}(t,w,h), \alpha_-(w,h)$, and $\alpha_+(w,h)$, and will simply write $\mathbf{X}(t,w), \alpha_-(w)$ and $\alpha_+(w)$.

The function α_- (resp. α_+) is upper (resp. lower) semi-continuous [17, Theorem 3.5 page 29]. The next result gives more regularity for α_+.

Theorem 1.4. *For every $h \in \Pi_{x_2}(\Omega)$, the function α_+ is continuously differentiable at each $w_0 \in \Pi_{x_1}(\Omega\cap\{x_2 = h\})$ such that $x_0 = (x_{0,1}, x_{0,2}) = \mathbf{X}(\alpha_+(w_0), w_0) \in \Gamma_3$.*

Proof. Since $\partial\Omega$ is a C^1 curve, there exists an open set $U \subset \mathbb{R}^2$ that contains x_0 and a C^1-diffeomorphism $\mathbf{\Upsilon} = (\Upsilon_1, \Upsilon_2) : U \to B_1$ such that

$$\mathbf{\Upsilon}(U\cap\Omega) = B_1 \cap \{y_2 > 0\} \quad \text{and} \quad \mathbf{\Upsilon}(U \cap \partial\Omega) = B_1\cap\{y_2 = 0\}, \tag{1.15}$$

where B_1 is the unit ball.

Let $x_0^- \in (U \cap \partial\Omega) \setminus \{x_0\}$ such that $(x_0^- - x_0) \cdot \boldsymbol{\tau}(x_0) < 0$, where $\boldsymbol{\tau}(x_0)$ is the unit tangent vector to $\partial\Omega$ at x_0.

Since $\mathbf{H} \in C^1(\overline{\Omega})$, there exists an open set Ω^* and an extension $\mathbf{H}^*$ of $\mathbf{H}$ such that $\bar{\Omega} \subset \Omega^*$ and $\mathbf{H}^* \in C^1(\Omega^*)$. Then we consider the unique maximal solution $\mathbf{Z}(t)$ of the differential equation

$$\begin{gathered} \mathbf{Z}'(t) = \mathbf{H}^*(\mathbf{Z}(t)) \\ \mathbf{Z}(0) = x_0^- \end{gathered}$$

defined in a maximal open interval (γ, δ).

Taking into account (1.9), we can see that $\mathbf{Z}(t) \in \Omega$ for all $t \in (\gamma, 0)$. Now if we assume that h is close enough to $x_{0,2}$, and denote by t_h the real number for which the curve $\mathbf{Z}(t)$ intersects the line $x_2 = h$, then there exists $w_0^- \in \Pi_{x_1}(\Omega \cap \{x_2 = h\})$ such that $\mathbf{Z}(t_h) = (w_0^-, h)$. Moreover, it is easy to see that

$$\begin{gathered} \mathbf{X}(t) = \mathbf{Z}(t_h - t) \quad \forall t \in (\alpha_-(w_0), \alpha_+(w_0)) \\ \mathbf{X}(0) = (w_0^-, h) \end{gathered}$$

Since $(x_0^- - x_0) \cdot \boldsymbol{\tau}(x_0) < 0$, we must have $w_0^- < w_0$. Furthermore, for each $w_0^- < w < w_0$, the curve $\mathbf{X}(t, w)$ lies between the curves $\mathbf{X}(t, w_0)$ and $\mathbf{X}(t, w_0^-)$. Therefore

$$\mathbf{X}(\alpha_+(w), w) \in U \cap \partial\Omega \quad \forall w \in (w_0^-, w_0). \tag{1.16}$$

Now let $x_0^+ \in (U \cap \partial\Omega) \setminus \{x_0\}$ be such that $(x_0^+ - x_0) \cdot \boldsymbol{\tau}(x_0) > 0$. Arguing as above, we can prove that there exists $w_0^+ \in \Pi_{x_1}(\Omega \cap \{x_2 = h\})$ such that

$$\mathbf{X}(\alpha_+(w), w) \in U \cap \partial\Omega \quad \forall w \in (w_0, w_0^+) \tag{1.17}$$

Taking into account (1.15)–(1.17), we see that there exists $\eta > 0$ small enough such that

$$\Upsilon_2(\mathbf{X}(\alpha_+(w), w)) = 0 \quad \forall w \in (w_0 - \eta, w_0 + \eta) \tag{1.18}$$

For each $\omega \in \Pi_{x_1}(\Omega^* \cap \{x_2 = h\})$, let $\mathbf{X}^*(t, w)$ be the unique maximal solution of the differential equation

$$\begin{gathered} (\mathbf{X}^*)'(t, w) = \mathbf{H}^*(\mathbf{X}^*(t, w)) \\ \mathbf{X}^*(0, w) = (w, h), \end{gathered}$$

where $\mathbf{X}^*(t, w)$ is defined on the interval $(\alpha_-^*(w), \alpha_+^*(w))$, and we obviously have $\mathbf{X}^*_{|(\alpha_-(w), \alpha_+(w))} = \mathbf{X}$. Moreover, we have $\alpha_-^*(w) < \alpha_-(w)$ and $\alpha_+(w) < \alpha_+^*(w)$.

Let $D^* = \{(t, w) : w \in (w_0 - \eta, w_0 + \eta),\ t \in (\alpha_-^*(w), \alpha_+^*(w))\}$. Since $\mathbf{X}^* \in C^1(D^*)$ and $\Upsilon_2 \in C^1(U)$, the function $F^* = \Upsilon_2 \circ \mathbf{X}^*$ is in $C^1(D^*)$. In addition, F^* is an extension of $F = \Upsilon_2 \circ \mathbf{X}$ to D^* and we have

$$\begin{aligned} \frac{\partial F^*}{\partial t}(t, w) &= \nabla \Upsilon_2(\mathbf{X}^*(t, w)) \cdot (\mathbf{X}^*)'(t, w) \\ &= \nabla \Upsilon_2(\mathbf{X}^*(t, w)) \cdot \mathbf{H}^*(\mathbf{X}^*(t, w)) . \end{aligned}$$

In particular, from (1.9) and (1.15)(ii) we obtain

$$\frac{\partial F^*}{\partial t}(\alpha_+(w_0), w_0) = \nabla \Upsilon_2(\mathbf{X}(\alpha_+(w_0), w_0)) \cdot \mathbf{H}(\mathbf{X}(\alpha_+(w_0), w_0)) \neq 0$$

Therefore by the implicit function theorem, there exists $\delta \in (0,\eta)$ and a unique function $f:(w_0-\delta,w_0+\delta)\to\mathbb{R}$ such that

$$\begin{gathered}F^*(t,\omega)=0 \quad \text{if and only if} \quad t=f(\omega)\\ f(w_0)=\alpha_+(w_0) \quad \text{and} \quad f\in C^1(w_0-\delta,w_0+\delta).\end{gathered}$$

Taking into account (1.18), we obtain $\alpha_+(w)=f(w)$ for all $w\in(w_0-\delta,w_0+\delta)$. We conclude that $\alpha_+\in C^1(\Pi_{x_1}(\Omega\cap\{x_2=h\}))$. □

Following [2, 4], for $h\in\Pi_{x_2}(\Omega)$, we define the set

$$D_h=\{(t,w): w\in\Pi_{x_1}(\Omega\cap\{x_2=h\}),\ t\in(\alpha_-(w),\alpha_+(w))\}$$

and the mapping $\mathbf{T}_h: D_h\to\mathbf{T}_h(D_h)$ by

$$\mathbf{T}_h(t,w)=\mathbf{X}(t,w)$$

The next proposition was established in [4] when $\mathbf{H}\in C^{0,1}(\overline{\Omega})$. For completeness, we provide a simpler and shorter proof when $\mathbf{H}\in C^1(\overline{\Omega})$.

Proposition 1.5. (i) *D_h is an open set.*

(ii) *$\mathbf{T}_h$ is a C^1-diffeomorphism from D_h to $\mathbf{T}_h(D_h)$ with Jacobian determinant*

$$\mathcal{J}\mathbf{T}_h(t,w)=Y_h(t,w)=-H_2(w,h)\exp\Big[\int_0^t \operatorname{div}(\mathbf{H})(\mathbf{X}(s,w))ds\Big]$$

Proof. (i) Let $(t_0,w_0)\in D_h$. We will show that there exists $\eta>0$ such that $B_\eta(t_0,w_0)\subset D_h$, where $B_\eta(t_0,w_0)$ is the open ball of center (t_0,w_0) and radius η. Since $\alpha_-(w_0)<t_0<\alpha_+(w_0)$, we can find a positive number ϵ such that $\epsilon<\min(t_0-\alpha_-(w_0),\alpha_+(w_0)-t_0)$. Given that $\alpha_-(w)$ is upper semi-continuous and $\alpha_+(w)$ is lower semi-continuous [17, Theorem 3.5 page 29], there exists $\eta>0$ such that

$$|w-w_0|<\eta \ \Rightarrow\ \alpha_-(w)<\alpha_-(w_0)+\epsilon \quad \text{and} \quad \alpha_+(w_0)-\epsilon<\alpha_+(w) \tag{1.19}$$

Since $w_0\in\Pi_{x_1}(\Omega\cap\{x_2=h\})$, we can assume without loss of generality that η is small enough so that $w\in\Pi_{x_1}(\Omega\cap\{x_2=h\})$ for $|w-w_0|<\eta$. We can also choose η such that $\eta<\min(t_0-\alpha_-(w_0)-\epsilon,\alpha_+(w_0)-t_0-\epsilon)$. Then we claim that $B_\eta(t_0,w_0)\subset D_h$. Indeed, we observe that if $(t,w)\in B_\eta(t_0,w_0)$, then we have $|t-t_0|<\eta$ and $|w-w_0|<\eta$, and therefore from (1.19) we obtain

$$\begin{gathered}\alpha_-(w)<\alpha_-(w_0)+\epsilon<\alpha_-(w_0)+t_0-\alpha_-(w_0)-\eta=t_0-\eta<t\\ t<t_0+\eta<t_0+\alpha_+(w_0)-t_0-\epsilon=\alpha_+(w_0)-\epsilon<\alpha_+(w)\end{gathered}$$

Hence (i) holds.

(ii) Since $\mathbf{H}\in C^1(\overline{\Omega})$, we know that the solution $\mathbf{X}(t,w)$ of (1.14) is C^1 in the open set [17, Theorem 6.1 page 89]

$$\{(t,w,h):\alpha_-(w,h)<t<\alpha_+(w,h), h\in\Pi_{x_2}(\Omega), w\in\Pi_{x_1}(\Omega\cap\{x_2=h\})\}$$

In particular, $\mathbf{T}_h(t,w)=\mathbf{X}(t,w)$ is C^1 in the open set D_h with

$$D\mathbf{T}_h(t,w)=\begin{bmatrix}\frac{\partial T_{h,1}}{\partial t} & \frac{\partial T_{h,1}}{\partial w}\\ \frac{\partial T_{h,2}}{\partial t} & \frac{\partial T_{h,2}}{\partial w}\end{bmatrix}=\begin{bmatrix}H_1(\mathbf{X}(t,w)) & \frac{\partial X_1}{\partial w}\\ H_2(\mathbf{X}(t,w)) & \frac{\partial X_2}{\partial w}\end{bmatrix}$$

and therefore the determinant of the Jacobian of $\mathbf{T_h}$ is

$$Y_h(t,w)=H_1(\mathbf{X}(t,w))\frac{\partial X_2}{\partial w}-H_2(\mathbf{X}(t,w))\frac{\partial X_1}{\partial w} \tag{1.20}$$

Differentiating, we obtain

$$\begin{aligned} \frac{\partial Y_h}{\partial t}(t,w) = DH_1(\mathbf{X}(t,w)) \cdot \mathbf{X}'(t,w)\frac{\partial X_2}{\partial w} + H_1(\mathbf{X}(t,w))\frac{\partial^2 X_2}{\partial t \partial w} \\ - DH_2(\mathbf{X}(t,w)) \cdot \mathbf{X}'(t,w)\frac{\partial X_1}{\partial w} - H_2(\mathbf{X}(t,w))\frac{\partial^2 X_1}{\partial t \partial w} \end{aligned} \tag{1.21}$$

Using that $\mathbf{X}(t,w) = (w,h) + \int_0^t \mathbf{H}(\mathbf{X}(s,w))ds$, we obtain

$$\frac{\partial \mathbf{X}}{\partial w}(t,w) = (1,0) + \int_0^t D\mathbf{H}(\mathbf{X}(s,w)) \cdot \frac{\partial \mathbf{X}}{\partial w}(s,w)ds \tag{1.22}$$

$$\frac{\partial^2 \mathbf{X}}{\partial t \partial w} = D\mathbf{H}(\mathbf{X}(t,w)) \cdot \frac{\partial \mathbf{X}}{\partial w}(t,w) \tag{1.23}$$

Combining (1.21) and (1.23), we obtain

$$\begin{aligned} &\frac{\partial Y_h}{\partial t}(t,w) \\ &= DH_1(\mathbf{X}(t,w)) \cdot \mathbf{H}(\mathbf{X}(t,w)) \cdot \frac{\partial X_2}{\partial w} + H_1(\mathbf{X}(t,w))DH_2(\mathbf{X}(t,w)) \cdot \frac{\partial \mathbf{X}}{\partial w}(t,w) \\ &\quad - DH_2(\mathbf{X}(t,w)) \cdot \mathbf{H}(\mathbf{X}(t,w)) \cdot \frac{\partial X_1}{\partial w} - H_2(\mathbf{X}(t,w))DH_1(\mathbf{X}(t,w)) \cdot \frac{\partial \mathbf{X}}{\partial w}(t,w) \end{aligned}$$

which leads to

$$\begin{aligned} &\frac{\partial Y_h}{\partial t}(t,w) \\ &= H_1\frac{\partial H_1}{\partial x_1}\frac{\partial X_2}{\partial w} + H_2\frac{\partial H_1}{\partial x_2} \cdot \frac{\partial X_2}{\partial w} + H_1\frac{\partial H_2}{\partial x_1}\frac{\partial X_1}{\partial w} + H_1\frac{\partial H_2}{\partial x_2}\frac{\partial X_2}{\partial w} \\ &\quad - H_1\frac{\partial H_2}{\partial x_1}\frac{\partial X_1}{\partial w} - H_2\frac{\partial H_2}{\partial x_2}\frac{\partial X_1}{\partial w} - H_2\frac{\partial H_1}{\partial x_1}\frac{\partial X_1}{\partial w} - H_2\frac{\partial H_1}{\partial x_2}\frac{\partial X_2}{\partial w} \\ &= H_1\frac{\partial H_1}{\partial x_1}\frac{\partial X_2}{\partial w} + H_1\frac{\partial H_2}{\partial x_2}\frac{\partial X_2}{\partial w} - H_2\frac{\partial H_2}{\partial x_2}\frac{\partial X_1}{\partial w} - H_2\frac{\partial H_1}{\partial x_1}\frac{\partial X_1}{\partial w} \\ &= \frac{\partial H_1}{\partial x_1}\Big[H_1\frac{\partial X_2}{\partial w} - H_2\frac{\partial X_1}{\partial w}\Big] + \frac{\partial H_2}{\partial x_2}\Big[H_1\frac{\partial X_2}{\partial w} - H_2\frac{\partial X_1}{\partial w}\Big] \\ &= \Big[\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2}\Big]\Big[H_1\frac{\partial X_2}{\partial w} - H_2\frac{\partial X_1}{\partial w}\Big] \end{aligned}$$

Taking into account this equality and (1.20), we arrive at

$$\frac{\partial Y_h}{\partial t}(t,w) = \operatorname{div}(\mathbf{H})(\mathbf{X}(t,w))Y_h(t,w) \tag{1.24}$$

Using (1.20) and (1.22), we see that $Y_h(0,w) = -H_2(w,h)$. Hence from (1.24) we infer that

$$Y_h(t,w) = -H_2(w,h)\exp\Big[\int_0^t \operatorname{div}(\mathbf{H})(\mathbf{X}(s,w))ds\Big] \tag{1.25}$$

Since by (1.7) and (1.25), $Y_h(t,w) < 0$ for all $(t,w) \in D_h$, we conclude that $\mathbf{T}_h$ is a C^1-diffeomorphism from D_h to $\mathbf{T}_h(D_h)$. This completes the proof of the proposition. □

Remark 1.6. It is not difficult to see that $\Omega = \sqcup_{h \in \Pi_{x_2}(\Omega)} \mathbf{T}_h(D_h)$ (see [2, 4]). In Section 2, we will use the C^1-diffeomorphism $\mathbf{T}_h$ as a change of variable to transform the problem (1.2) locally to a problem of type (1.1). As a consequence, we obtain from [16] that the free boundary is represented locally by graphs of a family of continuous functions.

2. Parametrization of the free boundary

For each $h \in \Pi_{x_2}(\Omega)$ and each function f defined in Ω, we shall denote the function $f \circ \mathbf{T}_h$ by $\widetilde{f}$. The first result of this section is the monotonicity of $\widetilde{\chi}$ with respect to t, which translates into the fact that χ is non-increasing along the orbits of the differential equation (1.14).

Proposition 2.1. *Let (u, χ) be a solution of (1.2). Then we have for each $h \in \Pi_{x_2}(\Omega)$,*

$$\frac{\partial \widetilde{\chi}}{\partial t} \leq 0 \quad \text{in } \mathcal{D}'(D_h)$$

A proof of the above proposition can be found in [4, Theorem 2.1]. The next proposition is a consequence of the monotonicity of $\widetilde{\chi}$ and the continuity of $\widetilde{u}$.

Proposition 2.2. *Let (u, χ) be a solution of (1.2) and $(t_0, w_0) \in D_h$.*

(i) *If $\widetilde{u}(t_0, w_0) > 0$, then there exists $\epsilon > 0$ such that*
$$\widetilde{u}(t, w) > 0 \quad \forall (t, w) \in \mathcal{C}_\epsilon = \{(t, w) \in D_h : |w - w_0| < \epsilon,\ t < t_0 + \epsilon\}$$

(ii) *If $\widetilde{u}(t_0, w_0) = 0$, then*
$$\widetilde{u}(t, w_0) = 0, \quad \forall t \geq t_0$$

A proof of the above proposition can be found in [4, Proposition 3.1]. Thanks to Proposition 2.2, for each $h \in \Pi_{x_2}(\Omega)$, we define the following function in $\Pi_{x_1}(\Omega \cap \{x_2 = h\})$,

$$\Phi_h(w) = \begin{cases} \sup\{t : (t, w) \in D_h : \widetilde{u}(t, w) > 0\} & \text{if this set is not empty} \\ \alpha_-(w) & \text{otherwise} \end{cases}$$

Arguing as in [2], we can see that Φ_h satisfies the following.

Proposition 2.3. *Φ_h is lower semi-continuous in $\Pi_{x_1}(\Omega \cap \{x_2 = h\})$ and*

$$\{\widetilde{u} > 0\} \cap D_h = \{t < \Phi_h(w)\}$$

Remark 2.4. If the functions Φ_h are smooth enough, then the family of functions $\{\Phi_h\}$ provides a local parametrization of the free boundary $\partial\{u > 0\} \cap \Omega$.

The next result describes the function χ in the interior of the set $\{u = 0\}$.

Theorem 2.5. *Let (u, χ) be a solution of (1.2), $(x_{01}, x_{02}) = \mathbf{T}_h(t_0, w_0) \in \mathbf{T}_h(D_h)$, $Z_0 = \big((t_0, \infty) \times (w_0 - r, w_0 + r)\big) \cap D_h$ and $C_r = Z_0 \cup B_r(t_0, w_0)$. If $\widetilde{u} = 0$ in $B_r(t_0, w_0) \subset D_h$, then $\widetilde{u} = 0$ in C_r. Moreover*

(i) *If $\overline{\mathbf{T}_h(Z_0)} \cap \Gamma_3 = \emptyset$, then $\widetilde{\chi} = 0$ in C_r.*

(ii) *If $\overline{\mathbf{T}_h(Z_0)} \cap \Gamma_2 = \emptyset$, then*
$$\widetilde{\chi}(t, w) = \frac{Y_h(\alpha_+(w), w)}{Y_h(t, w)} \frac{\beta(\cdot, \varphi(\cdot))}{\mathbf{H} \cdot \boldsymbol{\nu}}(\mathbf{X}(\alpha_+(w), w)) \quad \text{in } C_r$$

To prove the above theorem, we need two lemmas.

Lemma 2.6. *For each $x_0 \in \Gamma_3$, there exists $\eta > 0$ small enough and a C^1 function σ such that one of the following conditions holds*

(i) $\Gamma_3 \cap B_\eta(x_0) \subset \{(x_1, \sigma(x_1))\}$,
(ii) $\Gamma_3 \cap B_\eta(x_0) \subset \{(\sigma(x_2), x_2)\}$.

Proof. Since Γ_3 is a C^1-curve, there exists an open set $U \subset \mathbb{R}^2$ that contains the point $x_0 = (x_{01}, x_{02})$ and a C^1-diffeomorphism $\boldsymbol{\Upsilon}: U \to B_1$ such that $\boldsymbol{\Upsilon}(U \cap \Omega) = B_1 \cap \{y_2 > 0\}$ and $\boldsymbol{\Upsilon}(U \cap \Gamma_3) = B_1 \cap \{y_2 = 0\}$.

If $\boldsymbol{\Upsilon} = (\Upsilon_1, \Upsilon_2)$, then

$$\Upsilon_2(x) = 0 \quad \forall x \in U \cap \Gamma_3$$

Because of (1.9), we have $\nabla \Upsilon_2(x_0) \neq 0$. Therefore either $\frac{\partial \Upsilon_2}{\partial x_1}(x_0) \neq 0$, or $\frac{\partial \Upsilon_2}{\partial x_2}(x_0) \neq 0$.

Assume for example that we have $\frac{\partial \Upsilon_2}{\partial x_2}(x_0) \neq 0$. Then by the implicit function theorem, there exists $\delta > 0$ small enough and a unique C^1-function $\sigma : (x_{01} - \delta, x_{01} + \delta) \to \mathbb{R}$ such that

$$\Upsilon_2(x_1, x_2) = 0 \quad \text{if and only if} \quad x_2 = \sigma(x_1)$$

for all $x_1 \in (x_{01} - \delta, x_{01} + \delta)$. So (i) holds.

If $\frac{\partial \Upsilon_2}{\partial x_1}(x_0) \neq 0$, we can show in a same fashion that (ii) holds. $\square$

Lemma 2.7. *Let w_1, w_2 such that $w_1 < w_2$, and for all $w \in [w_1, w_2]$,*

$$(w, h) \in \Omega \quad \text{and} \quad \mathbf{T}_h(\alpha_+(w), w) \in \Gamma_3 .$$

Then

$$\int_Z \big(B(t,w)\nabla\tilde{u} + \tilde{\chi}k(t,w)\mathbf{e}_t\big) \cdot \nabla\xi \, dt\, dw = \int_{\tilde{\Gamma}_3} \lambda(\cdot, \tilde{\varphi} - \tilde{u})\xi d\tilde{\sigma}$$

for all $\xi \in H^1(Z)$ with $\xi = 0$ on $\partial Z \cap D_h$, where

$$\begin{gathered}
Z = \{(t,w) : w_1 < w < w_2 \text{ and } h < t < \alpha_+(w)\}, \\
\tilde{\Gamma}_3 = \{(\alpha_+(w), w) : w_1 < w < w_2\} \\
\lambda((t,w), z) = \mu(w)\beta(\mathbf{T}_h(t,w), z), \\
\mu(w) = \frac{|Y_h|(\alpha_+(w), w)}{\sqrt{1 + {\alpha'_+}^2(w)}\,(\mathbf{H}\cdot\boldsymbol{\nu})(\mathbf{T}_h(\alpha_+(w), w))}, \\
k(t,w) = |Y_h(t,w)|, \quad e_t = (1,0), \\
B(t,w) = |Y_h(t,w)| P(t,w) \cdot A(\mathbf{X}(t,w)) \cdot P^T(t,w)
\end{gathered}$$

with

$$P = (\mathcal{D}\mathbf{T}_h)^{-1} = \frac{1}{Y_h(t,w)} \begin{pmatrix} \frac{\partial X_2}{\partial \omega}(t,w) & -\frac{\partial X_1}{\partial \omega}(t,w) \\ -H_2(\mathbf{X}(t,w)) & H_1(\mathbf{X}(t,w)) \end{pmatrix}.$$

Proof. Let $\xi \in H^1(Z)$ such that $\xi = 0$ on $\partial Z \cap D_h$. Then $\pm\xi \circ \mathbf{T}_h^{-1}\chi(\mathbf{T}_h(Z))$ are test functions for (1.2) and we have

$$\int_{\mathbf{T}_h(Z)} (A(x)\nabla u + \chi\mathbf{H}(x)) \cdot \nabla(\xi \circ \mathbf{T}_h^{-1})\, dx = \int_{\Gamma_3 \cap \mathbf{T}_h(\partial Z)} \beta(x, \varphi - u)\xi \circ \mathbf{T}_h^{-1}\, d\sigma(x) \tag{2.1}$$

To handle the left-hand side we use the change of variable $\mathbf{T}_h$ as in [4],

$$\begin{aligned}
&\int_{\mathbf{T}_h(Z)} (A(x)\nabla u + \chi \mathbf{H}(x)) \cdot \nabla(\xi \circ \mathbf{T}_h^{-1})\,dx \\
&= \int_Z |\mathcal{J}\mathbf{T}_h|\,(A \circ \mathbf{T}_h.(\nabla u)o\mathbf{T}_h + (\chi o\mathbf{T}_h).(\mathbf{H}o\mathbf{T}_h)) \cdot \nabla(\xi \circ \mathbf{T}_h^{-1})o\mathbf{T}_h\,dt\,dw \\
&= \int_Z |Y_h|\Big(A \circ \mathbf{T}_h \nabla(uo\mathbf{T}_h) \cdot (D\mathbf{T}_h)^{-1} \\
&\quad + (\chi o\mathbf{T}_h) \cdot (\mathbf{H}o\mathbf{T}_h)\Big) \cdot (\nabla\xi \cdot (D\mathbf{T}_h)^{-1})\,dt\,dw \\
&= \int_Z |Y_h|\Big(A \circ \mathbf{T}_h \cdot ((D\mathbf{T}_h)^{-1})^T \cdot \nabla(uo\mathbf{T}_h) \\
&\quad + (\chi o\mathbf{T}_h)(\mathbf{H}o\mathbf{T}_h)\Big) \cdot (((D\mathbf{T}_h)^{-1})^T \cdot \nabla\xi)\,dt\,dw \\
&= \int_Z \Big(|Y_h|(D\mathbf{T}_h)^{-1} \cdot A \circ \mathbf{T}_h \cdot ((D\mathbf{T}_h)^{-1})^T \cdot \nabla\widetilde{u} \\
&\quad + \widetilde{\chi}|Y_h|(D\mathbf{T}_h)^{-1} \cdot (\mathbf{H}o\mathbf{T}_h)\Big) \cdot \nabla\xi\,dt\,dw
\end{aligned} \tag{2.2}$$

Since

$$D\mathbf{T}_h = \begin{pmatrix} \frac{\partial X_1}{\partial t} & \frac{\partial X_1}{\partial w} \\ \frac{\partial X_2}{\partial t} & \frac{\partial X_2}{\partial w} \end{pmatrix} = \begin{pmatrix} H_1 & \frac{\partial X_1}{\partial w} \\ H_2 & \frac{\partial X_2}{\partial w} \end{pmatrix},$$

we obtain

$$\begin{aligned}
(D\mathbf{T}_h)^{-1}.(\mathbf{H}o\mathbf{T}_h) &= \frac{1}{Y_h}\begin{pmatrix} \frac{\partial X_2}{\partial w} & -\frac{\partial X_1}{\partial w} \\ -H_2o\mathbf{T}_h & H_1o\mathbf{T}_h \end{pmatrix}\begin{pmatrix} H_1o\mathbf{T}_h \\ H_2o\mathbf{T}_h \end{pmatrix} \\
&= \frac{1}{Y_h}\begin{pmatrix} H_1o\mathbf{T}_h\frac{\partial X_2}{\partial w} - H_2o\mathbf{T}_h\frac{\partial X_1}{\partial w} \\ -H_2o\mathbf{T}_h \cdot H_1o\mathbf{T}_h + h_1o\mathbf{T}_h \cdot H_2o\mathbf{T}_h \end{pmatrix} \\
&= \frac{1}{Y_h}\begin{pmatrix} Y_h \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned} \tag{2.3}$$

From (2.2) and (2.3) we obtain

$$\begin{aligned}
&\int_{\mathbf{T}_h(Z)} (A(x)\nabla u + \chi\mathbf{H}(x)) \cdot \nabla(\xi \circ \mathbf{T}_h^{-1})\,dx \\
&= \int_{D_h} (B(t,\omega)\nabla\widetilde{u} + \widetilde{\chi}k(t,\omega)e_t) \cdot \nabla\xi\,dt\,dw
\end{aligned} \tag{2.4}$$

where the matrix B and the function k are as defined in this Lemma.

To handle the right-hand side of (2.1), we first observe that

$$\{\mathbf{T}_h(\alpha^+(w), w), w_1 < w < w_2\} = \Gamma_3 \cap \mathbf{T}_h(\partial Z) \tag{2.5}$$

Shrinking it if necessary, we assume by Lemma 2.6, that there exists a C^1-function σ such that one of the following conditions hold

(i) $\sigma(X_1(\alpha_+(w), w)) = X_2(\alpha_+(w), w)$ for all $w \in (w_1, w_2)$,
(ii) $\sigma(X_2(\alpha_+(w), w)) = X_1(\alpha_+(w), w)$ for all $\omega \in (w_1, w_2)$.

Assume that (i) holds. Case (ii) can be treated in the same way. Since $x_1 \to (x_1, \sigma(x_1))$ is a C^1-parametrization of $\Gamma_3 \cap \partial(\mathbf{T}_h(Z))$, the integral in the right hand side of (2.1) can be written as

$$\begin{aligned}
&\int_{\Gamma_3\cap \mathbf{T}_h(\partial Z)} \beta(x,\varphi-u)\xi\circ \mathbf{T}_h^{-1}\, d\sigma(x)\\
&= \int_{\Pi_{x_1}(\Gamma_3\cap\partial(\mathbf{T}_h(Z))} \beta((x_1,\sigma(x_1)),(\varphi-u)(x,\sigma(x)))\xi\circ \mathbf{T}_h^{-1}(x_1,\sigma(x_1)) \\
&\quad\times \sqrt{1+(\sigma')^2(x_1)}dx_1
\end{aligned} \tag{2.6}$$

Now observe that $(x_1,\sigma(x_1)) = \mathbf{T}_h(\alpha_+(w),w)$ for $w\in(w_1,w_2)$, and let $\theta(w) = x_1 = \mathbf{T}_h^1(\alpha_+(w),w)$. Then θ is a C^1-function and $\theta'(w) = \alpha_+'(w)H_1(\mathbf{X}(\alpha_+(w),w))+ \frac{\partial X_1}{\partial w}$. Using Theorem 1.4, we can show via implicit differentiation in the equation $\sigma(\mathbf{X}_1(\alpha_+(w),w)) = \mathbf{X}_2(\alpha_+(w),w)$ that

$$\alpha_+'(\omega) = \frac{\sigma'(X_1(\alpha_+(w),w))\partial X_1/\partial w(\alpha_+(w),w) - \partial X_2/\partial w(\alpha_+(w),w)}{H_2(\mathbf{X}(\alpha_+(\omega),w)) - \sigma'(X_1(\alpha_+(w),w))H_1(\mathbf{X}(\alpha_+(w),w))}$$

which leads to

$$\begin{aligned}
\theta'(w) &= \frac{-Y_h(\alpha_+(w),w)}{H_2(\mathbf{X}(w_+(w),w)) - \sigma'(X_1(\alpha_+(w),w))h_1(\mathbf{X}(\alpha_+(w),w))}\\
&= \frac{|Y_h|(\alpha_+(w),w)(1+\sigma'^2(x_1))^{-1/2}}{\mathbf{H}(\mathbf{X}(\alpha_+(w),w),e)\cdot \boldsymbol{\nu}(\mathbf{X}(\alpha_+(w),w))}
\end{aligned}$$

where $\boldsymbol{\nu}(x) = \frac{(-\sigma'(x_1),1)}{\sqrt{1+\sigma'^2(x_1)}}$ is the outward unit normal to Γ_3.

Lastly we apply the change of variable θ to (2.6), taking into account (2.5),

$$\begin{aligned}
&\int_{\Gamma_3\cap \mathbf{T}_h(\partial Z)} \beta(x,\varphi-u)\xi\circ \mathbf{T}_h^{-1}\, d\sigma(x)\\
&= \int_{w_1}^{w_2} \frac{\beta((\mathbf{T}_h(\alpha_+(w),w))),(\varphi-u)(\mathbf{T}_h(\alpha_+(w),w)))|Y_h|(\alpha_+(w),w)}{\mathbf{H}(\mathbf{T}_h(\alpha_+(w),w))\cdot\boldsymbol{\nu}(\mathbf{T}_h(\alpha_+(w),w))}\\
&\quad\times \xi(\alpha_+(w),w))dw\\
&= \int_{w_1}^{w_2} \frac{\beta((\mathbf{T}_h(\alpha_+(w),w))),(\varphi-u)(\mathbf{T}_h(\alpha_+(w),w))|Y_h|(\alpha_+(w),w)}{\sqrt{1+\alpha_+'^2(w)}\mathbf{h}(\mathbf{T}_h(\alpha_+(w),w))\cdot\boldsymbol{\nu}(\mathbf{T}_h(\alpha_+(w),w))}\\
&\quad\times \xi(\alpha_+(w),w))\, d\sigma(w)\\
&= \int_{\widetilde{\Gamma}_3} \lambda((\alpha^+(w),w),\widetilde{\varphi}-\widetilde{u})\xi\, d\sigma(w)
\end{aligned} \tag{2.7}$$

Combining (2.1), (2.4) and (2.7), the result follows. $\square$

Proof of Theorem 2.5. First, we observe that we have from Proposition 2.2 (ii), $\widetilde{u}=0$ in C_r, and that statement (i) can be established as in [4].

Next, we assume that $\overline{\mathbf{T}_h(Z_0)}\cap\Gamma_2=\emptyset$. From [16, Lemma 2.2 and Prop. 2.4], we obtain for all (t,w) in C_r,

$$\begin{aligned}
\widetilde{\chi}(t,w) &= \frac{\lambda((\alpha_+(w),w),\widetilde{\varphi}(\alpha_+(w),w))}{k(t,w)\boldsymbol{\nu}_2(\alpha_+(w),w)}\\
&= \frac{|Y_h|(\alpha_+(w),w)}{\sqrt{1+\alpha_+'^2(w)}\mathbf{H}(\mathbf{T}_h(\alpha_+(w),w))\cdot\boldsymbol{\nu}(\mathbf{T}_h(\alpha_+(w),w))}
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\beta(\mathbf{X}(\alpha_+(w),w),\varphi(\mathbf{X}(\alpha_+(w),w)))}{|Y_h(t,w)|\cdot \boldsymbol{\nu}_2(\alpha_+(w),w)}\\
&= \frac{|Y_h|(\alpha_+(w),w)}{|Y_h(t,w)|}\frac{\beta(.\cdot,\varphi)}{\mathbf{H}\cdot\boldsymbol{\nu}}(\mathbf{X}(\alpha_+(w),w))
\end{aligned}
$$

Hence statement (ii) follows. □

3. Continuity of the free boundary

Besides the assumptions of Section 1, we assume that:

$$\mathbf{H}\in C^{1,1}_{\rm loc}(\Omega\cup\Gamma_3) \tag{3.1}$$

$$\exists\alpha\in(0,1) \text{ such that } A\in C^{0,\alpha}_{\rm loc}(\Omega\cup\Gamma_3) \tag{3.2}$$

$$\exists c_0\in\mathbb{R} \text{ such that for all } y\in\Omega,\ \operatorname{div}(A(x)(x-y))\le c_0 \text{ in } \mathcal{D}'(\Omega) \tag{3.3}$$

$$\Gamma_3 \text{ in } C^{1,\alpha}_{\rm loc} \tag{3.4}$$

$$\beta(x,u) \text{ is continuous in } \Gamma_3\times\mathbb{R} \tag{3.5}$$

Here is the main result of this article.

Theorem 3.1. *Let $w_0\in\Pi_{x_1}(\Omega\cap\{x_2=h\})$ such that $(w_0,\Phi_h(w_0))$ is in D_h, $\mathbf{T}_h(\alpha_+(w_0),w_0))$ is in Γ_3 and*

$$\Big[\frac{|Y_h|\beta(x,\varphi)}{\mathbf{H}\cdot\boldsymbol{\nu}}\Big](\mathbf{X}(\alpha_+(w_0),w_0)<Y_h(\mathbf{X}(w_0,\Phi(w_0))) \tag{3.6}$$

Then Φ_h is continuous at w_0.

Proof. Since $\mathbf{T}_h(\alpha_+(w),w))$ is continuous at w_0 and Γ_3 is relatively open in $\partial\Omega$, there exists $w_1<w_0$ and $w_2>w_0$ such that

$$\mathbf{T}_h(\alpha_+(w),w))\in\Gamma_3 \quad \text{for all } w\in(w_1,w_2)$$

From Lemma 2.7, we know that $(\widetilde{u},\widetilde{\chi})$ is a solution on the domain

$$Z=\{(t,w):\ w_1<w<w_2 \text{ and } h<t<\alpha_+(w)\}$$

of a similar problem to (1.1). Therefore it is sufficient to check that the assumptions of [16, Theorem 4.1] are satisfied.

First, we deduce from Proposition 1.5 and (1.5)–(1.7) that the function k satisfies for some positive constant C:

$$
\begin{gathered}
0<\underline{h}\le k(t,\omega)\le C\bar{h} \quad \forall(t,\omega)\in D_h\\
0\le k_t(t,\omega)\le C\bar{h} \quad \forall(t,\omega)\in D_h.
\end{gathered}
$$

Next, it is easy to see from (3.1)-(3.2) that $B\in C^{0,\alpha}(Z\cup\widetilde{\Gamma}_3)$. Then by arguing as in [4], we can show that for some positive constants c_0,C_0 we have

$$
\begin{gathered}
|B(t,\omega)|\le C_0\\
B(t,\omega)\xi\cdot\xi\ge c_0|Y_h||\xi|^2\ge c_0|\xi|^2 \quad \forall(t,w)\in D_h,\ \forall\xi\in\mathbb{R}^2
\end{gathered}
$$

Moreover we have

$$
\begin{aligned}
\lambda(\cdot,\widetilde{\varphi})-k\boldsymbol{\nu}_2 &= \frac{|Y_h|}{\sqrt{1+{\alpha'_+}^2(w)}}\frac{\beta(\cdot,\varphi)(\mathbf{T}_h(\alpha_+(w),w))}{\mathbf{H}\cdot\boldsymbol{\nu}(\mathbf{T}_h(\alpha_+(w),w))}-|Y_h|(\alpha_+(w),w)\boldsymbol{\nu}_2\\
&= |Y_h|\Big[\frac{\beta(\cdot,\varphi)}{\mathbf{H}\cdot\boldsymbol{\nu}}-1\Big](\mathbf{T}_h(\alpha_+(w),w))\boldsymbol{\nu}_2
\end{aligned}
\tag{3.7}
$$

Taking into account (3.1), (3.4), (3.5), and (3.7), we see that the function $\lambda(\cdot,\widetilde{\varphi}) - k\boldsymbol{\nu}_2$ is continuous on $\widetilde{\Gamma_3}$.

Finally, arguing as in the proof of Theorem 2.5 and taking into account (3.6), one can show that

$$\frac{\lambda((\alpha_+(w_0),w_0),\widetilde{\varphi}(\alpha_+(w_0),w_0))}{k(\Phi_h(w_0),w_0)\boldsymbol{\nu}_2(\alpha_+(w_0),w_0)} = \frac{|Y_h|\beta(\cdot,\varphi)(\mathbf{T}_h(\alpha_+(w_0),w_0))(\alpha_+(w_0),w_0)}{|Y_h|(\Phi_h(w_0),w_0)\mathbf{H}\cdot\boldsymbol{\nu}(\mathbf{T}_h(\alpha_+(w_0),w_0))} < 1$$

We conclude that the function Φ_h is continuous at w_0. □

Remark 3.2. Assumption (3.3) is needed only to guarantee the Lipschitz continuity of u (see [3, 15]). A proof that does not require (3.3) is provided in [14].

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