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AN APPLICATION OF GLOBAL GRADIENT ESTIMATES IN LORENTZ-MORREY SPACES FOR THE EXISTENCE OF STATIONARY SOLUTIONS TO DEGENERATE DIFFUSIVE HAMILTON-JACOBI EQUATIONS

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Abstract. In mathematics and physics, the Kardar-Parisi-Zhang equation or quasilinear stationary version of a time-dependent viscous Hamilton-Jacobi equation in growing interface and universality classes is also known as the quasilinear Riccati type equation. The existence of solutions to this type of equations still remains an interesting open problem. In previous studies [36, 38], we obtained global bounds and gradient estimates for quasilinear elliptic equations with measure data. The main goal of this article is to obtain the existence of a renormalized solution to the quasilinear stationary solution for the degenerate diffusive Hamilton-Jacobi equation with finite measure data in Lorentz-Morrey spaces.

1. Introduction

This article is devoted to the existence of renormalized solution of the following stationary degenerate diffusive Hamilton-Jacobi equation, with respect to a given measure data μ,

$$\begin{gathered} -\operatorname{div}(A(x,\nabla u)) = |\nabla u|^q + \mu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

in the Lorentz-Morrey spaces $L^{s,t;\kappa}(\Omega)$ (the optimal range of s,t and κ will be clarified in our proof later). It is noticeable that our domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain whose complement satisfies a p-capacity uniform thickness condition. Specifically and precisely, in the present work, we consider for extended case, in which $p \in (\frac{3n-2}{2n-1}, n)$. Moreover, in our problem, the nonlinearity $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathédory vector valued function which satisfies growth and monotonicity conditions, i.e., there exist positive constants c_1, c_2 such that for some $p > 1$ it holds

$$\begin{gathered} |A(x,\xi)| \leq c_1|\xi|^{p-1}, \\ \langle A(x,\xi_1) - A(x,\xi_2), \xi_1 - \xi_2\rangle \geq c_2\big(|\xi_1|^2 + |\xi_2|^2\big)^{\frac{p-2}{2}}|\xi_1 - \xi_2|^2, \end{gathered}$$

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for every ξ, ξ_1, $\xi_2 \in \mathbb{R}^n \setminus \{0\}$ and $x \in \Omega$ almost everywhere.

This type of equations often appear in physical theory of surface growth, also known as the Kardar-Parisi-Zhang (KPZ) equation, where the study of this equation is still a challenge for mathematicians. It can be viewed as a quasilinear stationary version of a time-dependent viscous Hamilton-Jacobi equation, and it would be applied much in growing interface and universality classes (see [14, 17]). Specifically, for the case of $A(x,\xi) = |\xi|^{p-2}\xi$, the considered equation (1.1) is a type of standard p-Laplace equation

$$-\Delta_p u = |\nabla u|^q + \mu.$$

This equation has been studied extensively by several authors with their fine papers [3, 13, 21], in both historical view of mathematics and physics. Since then, for the general nonlinearity A, much attention has been devoted to the existence of solution also some comparison estimates, regularity theories of the problem. There have been several studies to the existence of solution to (1.1) under different assumptions, and later extended to several spaces. More precisely, it was mentioned in [3, page 13-14] about the sharp existence for the p-Laplacian problem in supercritical case. And later, in many works of Martio [22], Mengesha *et al.* [24], Phuc *et al.* (see [24, 32, 33]) and Tran *et al.* (see [37]), it is also related to the existence of renormalized solution to (1.1) under different hypotheses of domain Ω, the nonlinearity operator A and the functional spaces. Motivated by these works, we are interested in the solvability of (1.1) in Lorentz-Morrey spaces for the supercritical case $q \in (\frac{n(p-1)}{n-1}, p)$ under the p-capacity uniform thickness condition of the domain Ω.

There are several tools developed for linear and/or nonlinear potential and Calderón-Zygmund theories in recent years (see [4, 7, 9, 10, 23, 25, 27, 31, 33]). It is worth pointing out that in our study, the key ingredients were based on some local comparison estimates of renormalized solution to the quasilinear elliptic equation

$$\begin{gathered} -\operatorname{div}(A(x,\nabla u)) = \mu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

Earlier, there were a series of works by Mingione *et al.* [9, 10, 18, 19, 25, 26], Phuc *et al.* [1, 31, 32, 33], Nguyen *et al.* [27, 28, 30, 29] Tran *et al.* [36, 38], in which authors gave a local a nd global gradient estimates in Lorentz or Lorentz-Morrey spaces under various assumptions on Ω.

Using the hypothesis of p-capacity uniform thickness condition in [32], the gradient estimate of renormalized solution to (1.2) were obtained for the regular case of $p \in (2 - \frac{1}{n}, n)$. And in our previous work [38], we established the Lorentz-Morrey global bound for quasilinear elliptic equation (1.2) in the singular case of $p \in (\frac{3n-2}{2n-1}, 2 - \frac{1}{n}]$. The Morrey global bound for equation (1.2) in the singular case is also studied in [30] under hypotheses of Reifenberg domain Ω and smallness BMO of operator A. In this article, as an application of global gradient estimates studied in [38], we discuss the solvability of (1.1) in Lorentz-Morrey spaces for singular cases with only the hypothesis of p-capacity uniform thickness condition. More precisely, the domain Ω has its complement $\mathbb{R}^n \setminus \Omega$ which is uniformly p-capacity thick. However, we connect the estimates in [32] and [38] to obtain a complete existence result for both regular and singular cases, that is why we generalize our result for $p \in (\frac{3n-2}{2n-1}, n)$.

On the other hand, it is worth mentioning that in this paper, we adopt a weak assumption on domain Ω. This condition is stronger than Weiner's condition described in [16], and weaker than the usual Reifenberg flatness condition (The class of domains include all C^1-domains, Lipschitz domains with small Lipschitz constants, and domains with fractal boundaries), see [35, 8, 15] and various references therein. Moreover, the gradient estimates obtained in [38] can be proved using two facts that are the reverse Hölder's inequality (or Gehring's type inequality) and the comparison estimates. To our knowledge, the p-capacity assumption is necessary and weakest sufficient condition on the boundary of the domain in which the Gehring's type inequality hold.

The existence and uniqueness of the renormalized solution to (1.2) is classical and can be found in [2]. The authors proved that the unique solution u of (1.2) satisfies $|\nabla u|^{p-1} \in L^{\frac{n\gamma}{n-\gamma}}(\Omega)$ if provided $\mu \in L^{\gamma}(\Omega)$ for $1 < \gamma < \frac{np}{np-n+p}$ on given data. Later, for the borderline case, Mingione in [26] considered the Morrey density condition which is also a classical topic (see [5, 6, 20]), that is

$$\rho^{\kappa-n}\int_{B_\rho}|\mu|^{\gamma}dx \le C, \quad 0 \le \kappa \le n,$$

holds for all ball B_ρ in Ω. This function belongs to the Morrey space $L^{\gamma,\kappa}(\Omega)$ equipped to

$$\|\mu\|^{\gamma}_{L^{\gamma,\kappa}(\Omega)} := \sup_{B_\rho\subset\Omega}\rho^{\kappa-n}\int_{B_\rho}|\mu|^{\gamma}dx.$$

It is important to notice that $L^{\gamma,n}(\Omega) \equiv L^{\gamma}(\Omega)$ and $L^{\gamma,0}(\Omega) \equiv L^{\infty}(\Omega)$, and Mingione in his fine paper [26] also emphasized that Morrey spaces provide a scale "orthogonal" to L ebesgue spaces. And it is natural to motivate our approach with assumption that the data μ belongs to Lorentz-Morrey spaces which are more general than Morrey spaces.

We now recall the Lorentz-Morrey global bounds of renormalized solution to equation (1.2), that was proved in [32] and [38]. The following theorem is obtained by combining the gradient estimate results for the regular case in [32, Theorem 1.1] and the singular case in [38, Theorem 1.1]. We notice that the quasi-norm $\|\cdot\|_{L^{s,t;\kappa}(\Omega)}$ in Lorentz-Morrey space $L^{s,t;\kappa}(\Omega)$ will be presented in the next section.

Theorem 1.1. *Let $n \ge 2$, $p \in (\frac{3n-2}{2n-1}, n)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain whose complement satisfies a p-capacity uniform thickness condition. Assume that $\mu \in L^{\frac{s(\theta-1)}{\theta(p-1)},\frac{t(\theta-1)}{\theta(p-1)};\frac{s(\theta-1)}{p-1}}(\Omega)$ for some $s \in (0,p]$, $t \in (0,\infty]$ and $\theta \in [p,n]$. Then for any renormalized solution u to (1.2), there exists a positive constant C such that*

$$\|\nabla u\|_{L^{s,t;\frac{s(\theta-1)}{p-1}}(\Omega)} \le C\|\mu\|^{\frac{1}{p-1}}_{L^{\frac{s(\theta-1)}{\theta(p-1)},\frac{t(\theta-1)}{\theta(p-1)};\frac{s(\theta-1)}{p-1}}(\Omega)}. \tag{1.3}$$

In this article, we prove an existence result of a renormalized solution to (1.1) in Lorentz-Morrey space for both singular and regular cases $p \in (\frac{3n-2}{2n-1}, n)$ in the super-critical case $q > \frac{n(p-1)}{n-1}$. Our proof is based on applying Theorem 1.1 and the Schauder Fixed Point Theorem in [11]. The main idea of this proof comes from the proof of the existence result studied in [24]. More precisely, we consider a closed and convex set S as the form

$$S = \big\{v \in W_0^{1,1}(\Omega) : \|\,|\nabla v|^q\|_{L^{s,t;\frac{sq(\theta-1)}{p-1}}(\Omega)} \le \varepsilon\big\},$$

where the positive constant ε is chosen later. We note that the convexity of S will be obtained for $qs > 1$. For every $v \in S$, we define by $T(v) = u$ the unique renormalized solution to the equation

$$\begin{gathered} -\operatorname{div}(A(x, \nabla u)) = |\nabla v|^q + \mu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

We refer to [7] for the uniqueness of renormalized solution to above equation. By Theorem 1.1, we can prove that the mapping $T : S \to S$ is well-defined, continuous and $T(S)$ is precompact under the strong topology of $W_0^{1,1}(\Omega)$. The existence result can be obtained by the Schauder Fixed Point Theorem. Let us state our main result in the following theorem.

Theorem 1.2. *Let $n \geq 2$, $p \in (\frac{3n-2}{2n-1}, n)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain whose complement satisfies a p-capacity uniform thickness condition. Assume that*

$$\max\Big\{\frac{n(p-1)}{n-1}, p-1+\frac{1}{n}\Big\} < q < p. \tag{1.4}$$

For any $q \leq t \leq \infty$, and

$$\max\Big\{1, \frac{1}{q}\Big\} < s \leq \min\Big\{\frac{p}{q}, \frac{n}{\theta}\Big\}, \tag{1.5}$$

with $\theta = \frac{q}{q-p+1}$. There exists $\delta_0 > 0$ such that if $\|\mu\|_{L^{s,t;\theta s}(\Omega)} \leq \delta_0$ then (1.1) admits a renormalized solution u satisfying

$$\|\nabla u\|^q_{L^{qs,qt;\theta s}(\Omega)} \leq \theta\delta_0 - \|\mu\|_{L^{s,t;\theta s}(\Omega)}. \tag{1.6}$$

The rest of this article is organized as follows. In the next section, we recall the definitions of Lorentz and Lorentz-Morrey spaces. Moreover, we introduce a norm which is equivalent to the quasi-norm in Lorentz-Morrey spaces. The proof of Theorem 1.2 is given in the last section.

2. Lorentz-Morrey spaces

In this section, we give some backgrounds about the definitions of Lorentz and Lorentz-Morrey spaces equipped to an usual quasi-norm in general. The nice feature is that this quasi-norm is equivalent to a norm in these functional spaces (see [12]). In this paper, we give a simple proof for the equivalence between two norms which is useful for our proof in the next section. We assume that Ω is an open bounded subset of $\mathbb{R}^n$ with $n \geq 2$. For convenience of the reader, we first recall the definition of renormalized solution which details can be found in several papers such as [2, 7, 36].

2.1. **Renormalized solution.** For each integer $k > 0$, and for $s \in \mathbb{R}$ we firstly define the operator $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \max\{-k, \min\{k, s\}\}, \tag{2.1}$$

which belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$, and satisfies

$$-\operatorname{div} A(x, \nabla T_k(u)) = \mu_k$$

in the sense of distributions in Ω for a finite measure μ_k in Ω.

Definition 2.1. Let u be a measurable function defined on Ω which is finite almost everywhere, and satisfies $T_k(u) \in W_0^{1,1}(\Omega)$ for every $k > 0$. Then, there exists a unique measurable function $v : \Omega \to \mathbb{R}^n$ such that

$$\nabla T_k(u) = \chi_{\{|u|\leq k\}} v, \quad \text{almost everywhere in } \Omega, \text{ for every } k > 0. \tag{2.2}$$

Moreover, the function v is so-called "distributional gradient ∇u" of u.

We define $\mathfrak{M}_b(\Omega)$ as the space of all Radon measures on Ω with bounded total variation. The positive part, the negative part and total variation of a measure μ in $\mathfrak{M}_b(\Omega)$ are denoted by μ^+, μ^- and $|\mu|$ - is a bounded positive measure on Ω, respectively. For every measure μ in $\mathfrak{M}_b(\Omega)$ can be written in a unique way as $\mu = \mu_0 + \mu_s$, where μ_0 in $\mathfrak{M}_0(\Omega)$ and μ_s in $\mathfrak{M}_s(\Omega)$. The following Definition 2.2 of renormalized solution to equation (1.2) was introduced in [7], and we reproduce them herein as.

Definition 2.2. Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$, where $\mu_0 \in \mathfrak{M}_0(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. A measurable function u defined in Ω and finite almost everywhere is called a renormalized solution of (1.2) if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{n}{n-1}$, and u has the following additional property. For any $k > 0$ there exist nonnegative Radon measures $\lambda_k^+, \lambda_k^- \in \mathfrak{M}_0(\Omega)$ concentrated on the sets $u = k$ and $u = -k$, respectively, such that $\mu_k^+ \to \mu_s^+$, $\mu_k^- \to \mu_s^-$ in the narrow topology of measures and that

$$\int_{\{|u|<k\}} \langle A(x, \nabla u), \nabla \varphi \rangle dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_\Omega \varphi d\lambda_k^+ - \int_\Omega \varphi d\lambda_k^-,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

2.2. **Lorentz spaces.** For some $s \in (0, \infty)$ and $t \in (0, \infty]$, the Lorentz space $L^{s,t}(\Omega)$ is defined as the set of all Lebesgue measurable functions f on Ω such that:

$$\|f\|_{L^{s,t}(\Omega)} := \Big[s \int_0^\infty \lambda^t |\{x \in \Omega : |f(x)| > \lambda\}\Big|^{t/s} \frac{d\lambda}{\lambda}\big]^{1/2} < \infty, \tag{2.3}$$

as $t \neq \infty$ and

$$\|f\|_{L^{s,\infty}(\Omega)} := \sup_{\lambda>0} \lambda \left|\{x \in \Omega : \ |f(x)| > \lambda\}\right|^{1/s} < \infty,$$

where $|\mathcal{O}|$ denotes the n-dimensional Lebesgue measure of a set $\mathcal{O} \subset \mathbb{R}^n$. The space $L^{s,\infty}(\Omega)$ is known as the usual weak $L^s(\Omega)$ or Marcinkiewicz space with notice that $L^s(\Omega) \subset L^{s,\infty}(\Omega) \subset L^r(\Omega)$ for $1 < r < s < \infty$.

It is well known that for $t = s$, the Lorentz space $L^{s,s}(\Omega)$ in (2.3) is exactly the Lebesgue space $L^s(\Omega)$. Moreover, for some $0 < r < s < t \leq \infty$, we have the following remark, with continuous embeddings

$$L^t(\Omega) \subset L^{s,r}(\Omega) \subset L^s(\Omega) \subset L^{s,t}(\Omega) \subset L^r(\Omega).$$

In fact, the quasi-norm $\|\cdot\|_{L^{s,t}(\Omega)}$ may be defined as the other form which is given by Lemma 2.3 below. For a measure function f in Ω, the distribution function $d_f : [0, \infty) \to [0, \infty)$ of f is defined by

$$d_f(\lambda) = |\{x \in \Omega : |f(x)| > \lambda\}|.$$

The decreasing rearrangement $f^* : [0, \infty) \to [0, \infty)$ of f defines as follows

$$f^*(\lambda) = \inf\{\eta > 0 : d_f(\eta) \leq \lambda\}.$$

Lemma 2.3. *Let $s \in (0, \infty)$ and $t \in (0, \infty]$. For some $f \in L^{s,t}(\Omega)$, it holds*

$$\|f\|_{L^{s,t}(\Omega)} = \begin{cases} [\int_0^\infty (\lambda^{1/s} f^*(\lambda))^t \frac{d\lambda}{\lambda}]^{1/2}, & t < \infty, \\ \sup_{\lambda > 0} \lambda^{1/s} f^*(\lambda), & t = \infty. \end{cases} \tag{2.4}$$

The proof of this lemma can be found in [12, Proposition 1.4.9].

2.3. **A norm in the Lorentz space.** We define by $f^{**} : [0, \infty) \to [0, \infty)$ the maximal functional of f as follows

$$f^{**}(\lambda) = \frac{1}{\lambda} \int_0^\lambda f^*(\eta) d\eta, \quad \text{or } \lambda > 0 \text{ and } f^{**}(0) = f^*(0).$$

For some $s \in (1, \infty)$, $t \in [1, \infty]$ and for any $f \in L^{s,t}(\Omega)$, let us introduce

$$|||f|||_{L^{s,t}(\Omega)} := \Big[\int_0^\infty (\lambda^{1/s} f^{**}(\lambda))^t \frac{d\lambda}{\lambda} \Big]^{1/2}, \tag{2.5}$$

if $1 \le t < \infty$, and

$$|||f|||_{L^{s,\infty}(\Omega)} := \sup_{\lambda > 0} \lambda^{1/s} f^{**}(\lambda). \tag{2.6}$$

The following lemma was obtained in book authored by Rakotoson in [34], or one can find easily in [12, Exercise 1.4.3]. Here, for the convenience of the reader, we provide a brief proof.

Lemma 2.4. *Let $s \in (1, \infty)$ and $t \in [1, \infty]$. The functional $||| \cdot |||_{L^{s,t}(\Omega)}$ defined by (2.5)-(2.6) is a norm in Lorentz space $L^{s,t}(\Omega)$. Moreover, for any $f \in L^{s,t}(\Omega)$ it holds*

$$\|f\|_{L^{s,t}(\Omega)} \le |||f|||_{L^{s,t}(\Omega)} \le \frac{s}{s-1} \|f\|_{L^{s,t}(\Omega)}. \tag{2.7}$$

Proof. We prove that the functional $||| \cdot |||_{L^{s,t}(\Omega)}$ defined by (2.5)-(2.6) is a norm in Lorentz space $L^{s,t}(\Omega)$. We remark that

$$f^{**}(\lambda) = \frac{1}{\lambda} \int_0^\lambda f^*(\eta) d\eta = \frac{1}{\lambda} \sup_{|E| = \lambda} \int_E |f(x)| dx.$$

This indicate the subadditivity of the maximal functional, i.e., for any measurable function f, g and for any $\lambda > 0$, it holds

$$\begin{aligned} (f+g)^{**}(\lambda) &= \frac{1}{\lambda} \sup_{|E| = \lambda} \int_E |f(x) + g(x)| dx \\ &\le \frac{1}{\lambda} \sup_{|E| = \lambda} \int_E |f(x)| dx + \frac{1}{\lambda} \sup_{|E| = \lambda} \int_E |g(x)| dx \\ &= f^{**}(\lambda) + g^{**}(\lambda). \end{aligned}$$

By the above subadditivity and Minkowski's inequality, it follows that the functional $||| \cdot |||_{L^{s,t}(\Omega)}$ is a norm in Lorentz space $L^{s,t}(\Omega)$.

The first inequality of (2.7) is obtained from Lemma 2.3 and the fact that $f^*(\lambda) \le f^{**}(\lambda)$ for every $\lambda > 0$. We then prove the second inequality of (2.7).

For any $1 < t < \infty$, by Holder's inequality with $\frac{1}{t} + \frac{1}{t'} = 1$, we obtain

$$\begin{aligned}
\Big(\int_0^\lambda f^*(\eta)d\eta\Big)^t &= \Big(\int_0^\lambda f^*(\eta)\eta^{\frac{1}{s}-\frac{1}{ts}}\eta^{-\frac{1}{s}+\frac{1}{ts}}d\eta\Big)^t \\
&\le \Big(\int_0^\lambda (f^*(\eta))^t\eta^{\frac{t}{s}-\frac{1}{s}}d\eta\Big)\Big(\int_0^\lambda \eta^{-\frac{t}{s}+\frac{t'}{ts}}d\eta\Big)^{t/t'} \\
&= \Big(\int_0^\lambda (f^*(\eta))^t\eta^{\frac{t}{s}-\frac{1}{s}}d\eta\Big)\Big(\int_0^\lambda \eta^{-\frac{1}{s}}d\eta\Big)^{t-1} \\
&= \Big(\frac{1}{1-1/s}\Big)^{t-1}\lambda^{(t-1)(1-1/s)}\int_0^\lambda (f^*(\eta))^t\eta^{\frac{t}{s}-\frac{1}{s}}d\eta,
\end{aligned} \tag{2.8}$$

for any $\lambda > 0$. It is easy to see that the inequality (2.8) also holds for $t = 1$. By integrating both sides of (2.8) from zero to infinity and using Fubini's Theorem we obtain

$$\begin{aligned}
|||f|||_{L^{s,t}(\Omega)} &= \Big[\int_0^\infty \lambda^{\frac{t}{s}-t-1}\Big(\int_0^\lambda f^*(\eta)d\eta\Big)^t d\lambda\Big]^{1/2} \\
&\le \Big[\Big(\frac{1}{1-1/s}\Big)^{t-1}\int_0^\infty \lambda^{\frac{1}{s}-2}\int_0^\lambda (f^*(\eta))^t\eta^{\frac{t}{s}-\frac{1}{s}}d\eta d\lambda\Big]^{1/2} \\
&= \Big[\Big(\frac{s}{s-1}\Big)^{t-1}\int_0^\infty (f^*(\eta))^t\eta^{\frac{t}{s}-\frac{1}{s}}\int_\eta^\infty \lambda^{\frac{1}{s}-2}d\lambda d\eta\Big]^{1/2} \\
&= \frac{s}{s-1}\|f\|_{L^{s,t}(\Omega)},
\end{aligned}$$

which deduces the second inequality for $t \in [1, \infty)$. In the case of $t = \infty$, we also have

$$\begin{aligned}
|||f|||_{L^{s,\infty}(\Omega)} &= \sup_{\lambda>0}\lambda^{\frac{1}{s}-1}\int_0^\lambda \eta^{-\frac{1}{s}}\eta^{1/s}f^*(\eta)d\eta \\
&\le \sup_{\lambda>0}\lambda^{\frac{1}{s}-1}\Big(\int_0^\lambda \eta^{-\frac{1}{s}}d\eta\Big)\|f\|_{L^{s,\infty}(\Omega)} \\
&= \frac{s}{s-1}\|f\|_{L^{s,\infty}(\Omega)}.
\end{aligned}$$

□

2.4. **Lorentz-Morrey spaces.** Let $s \in (0, \infty)$, $t \in (0, \infty]$ and $\kappa \in (0, n]$. The Lorentz-Morrey functional spaces $L^{s,t;\kappa}(\Omega)$ is the set of all functions $g \in L^{s,t}(\Omega)$ such that

$$\|f\|_{L^{s,t;\kappa}(\Omega)} := \sup_{0<\rho\le \operatorname{diam}(\Omega);x\in\Omega} \rho^{\frac{\kappa-n}{s}}\|f\|_{L^{s,t}(B_\rho(x)\cap\Omega)} < \infty, \tag{2.9}$$

where $B_\rho(x)$ denotes the ball centered x with radius ρ in $\mathbb{R}^n$.

It is clear to see that in the case $s = t$, the Lorentz-Morrey space $L^{s,s;\kappa}(\Omega)$ is coincident to the Morrey space $L^{s;\kappa}(\Omega)$ and another case $\kappa = n$, the Lorentz-Morrey space $L^{s,t;\kappa}(\Omega)$ is exact the Lorentz space $L^{s,t}(\Omega)$. In addition, with these spaces, the functional $\|\cdot\|_{L^{s,t;\kappa}(\Omega)}$ is just a quasi-norm in general. Therefore, it is necessary to define a norm where the Lorentz-Morrey spaces are endowed with.

Let $s \in (1, \infty)$, $t \in [1, \infty]$ and $\kappa \in (0, n]$. For any $f \in L^{s,t;\kappa}(\Omega)$, let us set

$$|||f|||_{L^{s,t;\kappa}(\Omega)} := \sup_{0<\rho\le \operatorname{diam}(\Omega);x\in\Omega} \rho^{\frac{\kappa-n}{s}}|||f|||_{L^{s,t}(B_\rho(x)\cap\Omega)}. \tag{2.10}$$

The following corollary is directly obtained by definition (2.10) and Lemma 2.4. And with this norm, the set V_ε defined by (3.1) in the next section will be convex.

Corollary 2.5. *Let $s \in (1,\infty)$, $t \in [1,\infty]$ and $\kappa \in (0,n]$. The function $|||\cdot|||_{L^{s,t;\kappa}(\Omega)}$ defined by* (2.10) *is a norm in Lorentz-Morrey space $L^{s,t;\kappa}(\Omega)$. Moreover, for any $f \in L^{s,t;\kappa}(\Omega)$, it holds*

$$\|f\|_{L^{s,t;\kappa}(\Omega)} \leq |||f|||_{L^{s,t;\kappa}(\Omega)} \leq \frac{s}{s-1}\|f\|_{L^{s,t;\kappa}(\Omega)}. \tag{2.11}$$

3. Proof of main theorem

The proof is divided into four steps under the hypotheses of Theorem 1.2. the key idea of our proof is based on applying Schauder Fixed Point Theorem (see [11]) for a continuous mapping $T : V_\varepsilon \to V_\varepsilon$, where V_ε is closed, convex and $\overline{T(V_\varepsilon)}$ is a compact set under the strong topology of $W_0^{1,1}(\Omega)$.

Proof of Theorem 1.2. Let q, s, t satisfying (1.4), (1.5) and set $\theta = \frac{q}{q-p+1}$ as in Theorem 1.2. For every $\varepsilon > 0$, we consider the set

$$V_\varepsilon = \left\{u \in W_0^{1,1}(\Omega) : |||\nabla u|||_{L^{qs,qt;\theta s}(\Omega)} \leq \varepsilon\right\}. \tag{3.1}$$

We introduce the mapping $T : V_\varepsilon \to V_\varepsilon$ defined by

$$T(v) = u, \quad \text{for } v \in V_\varepsilon, \tag{3.2}$$

where u is the unique renormalized solution to the equation

$$\begin{aligned} -\operatorname{div}(A(x,\nabla u)) &= |\nabla v|^q + \mu \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

First step: V_ε is closed and convex under the strong topology of $W_0^{1,1}(\Omega)$. We first prove that V_ε is convex. Indeed, for any $u, v \in V_\varepsilon$ and $\eta \in [0,1]$, we must to show that $w = \eta u + (1-\eta)v \in V_\varepsilon$. We remark that $|||\cdot|||_{L^{s,t}(\mathcal{O})}$ is a norm in Lorentz-Morrey space $L^{s,t}(\mathcal{O})$, for any subset $\mathcal{O}$ of Ω. Therefore, for any $z \in \Omega$ and $0 < \rho \leq \operatorname{diam}(\Omega)$, we have

$$|||\nabla w|||_{L^{s,t}(B_\rho(z)\cap\Omega)} \leq \eta|||\nabla u|||_{L^{s,t}(B_\rho(z)\cap\Omega)} + (1-\eta)|||\nabla v|||_{L^{s,t}(B_\rho(z)\cap\Omega)}.$$

Multiplying both sides of this inequality by $\rho^{\frac{\kappa-n}{s}}$, we obtain

$$\begin{aligned} &\rho^{\frac{\kappa-n}{s}}|||\nabla w|||_{L^{s,t}(B_\rho(z)\cap\Omega)}\\ &\leq \eta\rho^{\frac{\kappa-n}{s}}|||\nabla u|||_{L^{s,t}(B_\rho(z)\cap\Omega)} + (1-\eta)\rho^{\frac{\kappa-n}{s}}|||\nabla v|||_{L^{s,t}(B_\rho(z)\cap\Omega)}, \end{aligned}$$

which implies

$$|||\nabla w|||_{L^{s,t;\kappa}(\Omega)} \leq \eta|||\nabla u|||_{L^{s,t;\kappa}(\Omega)} + (1-\eta)|||\nabla v|||_{L^{s,t;\kappa}(\Omega)} \leq \varepsilon,$$

which gives $w \in V_\varepsilon$.

Next we show that V_ε is closed under the strong topology of $W_0^{1,1}(\Omega)$. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in V_ε such that u_k converges strongly in $W_0^{1,1}(\Omega)$ to a function u. Let $z \in \Omega$ and $0 < \rho \leq \operatorname{diam}(\Omega)$, we note that ∇u_k converges to ∇u almost everywhere in $B_\rho(z)\cap\Omega$. By [12, Proposition 1.4.9], it follows that the sequence $(\nabla u_k)^*$ converges to $(\nabla u)^*$ in $[0,\infty)$. For any $\lambda > 0$, by Fatou's lemma, we obtain that

$$\frac{1}{\lambda}\int_0^\lambda (\nabla u)^*(\eta)d\eta \leq \limsup_{k\to\infty}\frac{1}{\lambda}\int_0^\lambda (\nabla u_k)^*(\eta)d\eta,$$

which asserts that

$$(\nabla u)^{**}(\lambda) \le \limsup_{k\to\infty} (\nabla u_k)^{**}(\lambda).$$

We thus obtain

$$\begin{aligned}\rho^{\frac{\kappa-n}{s}} \||\nabla u|\|_{L^{s,t}(B_\rho(z)\cap\Omega)} &\le \limsup_{k\to\infty} \rho^{\frac{\kappa-n}{s}} \||\nabla u_k|\|_{L^{s,t}(B_\rho(z)\cap\Omega)} \\ &\le \||\nabla u_k|\|_{L^{s,t;,\kappa}(\Omega)} \le \varepsilon.\end{aligned}$$

It follows that

$$\||\nabla u|\|_{L^{s,t;\kappa}(\Omega)} = \sup_{0<\rho\le \operatorname{diam}(\Omega),\, z\in\Omega} \rho^{\frac{\kappa-n}{s}} \||\nabla u|\|_{L^{s,t}(B_\rho(z)\cap\Omega)} \le \varepsilon,$$

which leads to $u \in V_\varepsilon$.

Second step: There exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that if $\|\mu\|_{L^{s,t;\theta s}(\Omega)} \le \delta_0$ then the mapping $T : V_{\varepsilon_0} \to V_{\varepsilon_0}$ in (3.2) is well-defined. Under the hypotheses (1.4) and (1.5), by Corollary 1.1, there exists a positive constant C such that for any renormalized solution u to equation (1.2), it holds

$$\|\nabla u\|^{p-1}_{L^{qs,qt;\theta s}(\Omega)} \le C\|\mu\|_{L^{s,t;\theta s}(\Omega)}. \tag{3.4}$$

We first prove that there exists $\delta_0 > 0$ such that if $\|\mu\|_{L^{s,t;\theta s}(\Omega)} \le \delta_0$ then there exists a positive number y_0 satisfying

$$\frac{Cs}{s-1}\Big(\frac{qs}{qs-1}\Big)^{p-1}\big(y_0 + \|\mu\|_{L^{s,t;\kappa}(\Omega)}\big) = y_0^{\frac{p-1}{q}}. \tag{3.5}$$

We consider the function $g : [0,\infty) \to \mathbb{R}$ defined by

$$g(y) = (cy + ca)^{\frac{\theta}{\theta-1}} - y, \tag{3.6}$$

with $c = \frac{Cs}{s-1}(\frac{qs}{qs-1})^{p-1}$ and $a = \|\mu\|_{L^{s,t;\theta s}(\Omega)}$. Noting that $\theta > 1$, let us choose

$$\delta_0 = \frac{1}{c\theta}\Big(\frac{\theta-1}{c\theta}\Big)^{\theta-1} > 0.$$

If $a \le \delta_0$ then the function g given by (3.6) satisfies $g(0) > 0$ and $\lim_{y\to\infty} g(y) = \infty$. Moreover, $g'(y) = \frac{\theta c}{\theta-1}(cy+ca)^{\frac{1}{\theta-1}} - 1$, thus $g'(y) = 0$ if and only if $y = y^*$ given by

$$y^* = \frac{1}{c}\Big(\frac{\theta-1}{c\theta}\Big)^{\theta-1} - a = \theta\delta_0 - a > 0.$$

It follows that the minimum value of g on $[0,\infty)$ is

$$g(y^*) = (cy^* + ca)\frac{\theta-1}{c\theta} - y^* = a - \delta_0 \le 0.$$

For this reason, we conclude that g has exactly one root $y_0 \in (0, y^*]$ which satisfies (3.5).

Let us set $\varepsilon_0 = y_0^{1/q}$. By the definition of T, for any $v \in V_{\varepsilon_0}$, $u = T(v) \in W_0^{1,1}(\Omega)$ is the unique renormalized solution to equation (3.3) (see [7] for the uniqueness of renormalized solution to (3.3)). Applying (3.4) and Corollary 2.5, we obtain

$$\|\nabla u\|^{p-1}_{L^{qs,qt;\theta s}(\Omega)} \le C\||\nabla v|^q + \mu\|_{L^{s,t;\theta s}(\Omega)} \le C\|||\nabla v|^q + \mu|\|_{L^{s,t;\theta s}(\Omega)}. \tag{3.7}$$

Combining (3.7) with the triangle inequality and Corollary 2.5, one has

$$
\begin{aligned}
|||\nabla u|||^{p-1}_{L^{qs,qt;\theta s}(\Omega)} &\leq \Big(\frac{qs}{qs-1}\Big)^{p-1}\|\nabla u\|^{p-1}_{L^{qs,qt;\theta s}(\Omega)}\\
&\leq C(\frac{qs}{qs-1})^{p-1}[|||(|\nabla v|^q)|||_{L^{s,t;\theta s}(\Omega)} + |||\mu|||_{L^{s,t;\theta s}(\Omega)}]\\
&\leq \frac{Cs}{s-1}\Big(\frac{qs}{qs-1}\Big)^{p-1}\big[\|\nabla v\|^q_{L^{qs,qt;\theta s}(\Omega)} + \|\mu\|_{L^{s,t;\theta s}(\Omega)}\big]\\
&\leq \frac{Cs}{s-1}\Big(\frac{qs}{qs-1}\Big)^{p-1}\big[|||\nabla v|||^q_{L^{qs,qt;\theta s}(\Omega)} + \|\mu\|_{L^{s,t;\theta s}(\Omega)}\big].
\end{aligned}
\tag{3.8}
$$

Note that $|||\nabla v|||^q_{L^{qs,qt;\theta s}(\Omega)} \leq y_0$, with y_0 is the root of (3.5) and $\varepsilon_0 = y_0^{1/q}$. Then, we can rewrite (3.8) as

$$
|||\nabla u|||^{p-1}_{L^{qs,qt;\theta s}(\Omega)} \leq y_0^{\frac{p-1}{q}} = \varepsilon_0^{p-1},
$$

which yields $T(v) = u \in V_{\varepsilon_0}$. We conclude that the mapping T is well-defined.

Third step: $T : V_{\varepsilon_0} \to V_{\varepsilon_0}$ is continuous, and $\overline{T(V_{\varepsilon_0})}$ is a compact set under the strong topology of $W_0^{1,1}(\Omega)$. Let us consider $\{v_k\}_{k\in\mathbb{N}}$ as a sequence in V_{ε_0} such that v_k converges strongly in Sobolev space $W_0^{1,1}(\Omega)$ to a function $v \in V_{\varepsilon_0}$. For every $k \in \mathbb{N}$, we denote by $u_k = T(v_k)$ the renormalized solution to the equation

$$
\begin{gathered}
-\operatorname{div}(A(x,\nabla u_k)) = |\nabla v_k|^q + \mu \quad \text{in } \Omega,\\
u_k = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{3.9}
$$

with

$$
\|\nabla v_k\|_{L^{qs,qt;\theta s}(\Omega)} \leq \varepsilon_0. \tag{3.10}
$$

We obtain

$$
\|\nabla v_k\|_{L^r(\Omega)} \leq \varepsilon_0, \tag{3.11}
$$

for any $q < r < qs$. Therefore there exists a subsequence $\{v_{k_j}\}_{j\in\mathbb{N}}$ of $\{v_k\}_{k\in\mathbb{N}}$ such that ∇v_{k_j} converges to ∇v almost everywhere in Ω. It follows from (3.11) and Vitali Convergence Theorem that ∇v_{k_j} converges to ∇v strongly in $L^q(\Omega)$. It can be concluded that ∇v_k converges to ∇v strongly in $L^q(\Omega)$.

By the stability result of renormalized solution in [7, Theorem 3.4], there exists a subsequence $\{u_{k_j}\}$ such that $\{u_{k_j}\}$ converges to u almost everywhere in Ω, where u is the unique renormalized solution of the equation

$$
\begin{gathered}
-\operatorname{div}(A(x,\nabla u)) = |\nabla v|^q + \mu \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

In addition, ∇u_{k_j} also converges to ∇u almost everywhere in Ω. We can do a similar way as above by applying again Vitali Convergence Theorem with the facts that $qs > 1$ and

$$
\|\nabla u_{k_j}\|_{L^{qs,qt;\theta s}(\Omega)} \leq \varepsilon_0,
$$

it follows that u_k converges strongly to u in $W_0^{1,1}(\Omega)$. It guarantees the continuity of the mapping T.

To prove the relative compactness of the set $T(V_{\varepsilon_0})$ under the strong topology of $W_0^{1,1}(\Omega)$, we can use by the same method as above. Indeed, let $\{u_m\} = \{T(v_m)\}_{m\in\mathbb{N}}$ be a sequence in $T(V_{\varepsilon_0})$ where $\{v_m\} \subset V_{\varepsilon_0}$, then we get (3.9), (3.10). Thanks to [7, Theorem 3.4], there exist a subsequence $\{u_{m_j}\}$ and a function $u \in W_0^{1,1}(\Omega)$ such that $\nabla u_{m_j} \to \nabla u$ almost everywhere in Ω. Finally, applying Vitali Convergence

Theorem again, it implies that the subsequence $\{u_{m_j}\}$ strongly converges to u in $W_0^{1,1}(\Omega)$.

Fourth step: Applying Schauder Fixed Point Theorem. By Schauder Fixed Point Theorem, the mapping $T : V_{\varepsilon_0} \to V_{\varepsilon_0}$ has a fixed point u in V_{ε_0}. This gives a solution u to equation (1.1). Moreover, applying Corollary (2.5) and the last inequality in the proof of the second step, we obtain the estimation

$$\|\nabla u\|^q_{L^{qs,qt;\theta s}(\Omega)} \leq \||\nabla u|\|^q_{L^{qs,qt;\theta s}(\Omega)} \leq y^* \leq \theta\delta_0 - \|\mu\|_{L^{s,t;\theta s}(\Omega)}.$$

The proof is complete. $\square$

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