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FUNDAMENTAL SOLUTIONS AND CAUCHY PROBLEMS FOR AN ODD-ORDER PARTIAL DIFFERENTIAL EQUATION WITH FRACTIONAL DERIVATIVE

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Abstract. In this article, we construct a fundamental solution of a higher-order equation with time-fractional derivative, give a representation for a solution of the Cauchy problem, and prove the uniqueness theorem in the class of functions satisfying an analogue of Tychonoff's condition.

1. Introduction

We consider the equation

$$\Big(\frac{\partial^\sigma}{\partial y^\sigma} + (-1)^n \frac{\partial^{2n+1}}{\partial x^{2n+1}}\Big) u(x,y) = f(x,y), \tag{1.1}$$

where $\sigma \in (0,1)$, and $\partial^\sigma/\partial y^\sigma$ stands for the fractional derivative of order σ with respect to the variable y with starting point at $y=0$.

Equation (1.1) refers to evolutionary equations with fractional derivative with respect to the time variable. The most studied among equations of this type are fractional diffusion and diffusion-wave equations (i.e. equations with second-order space derivative). In this connection, we refer to [4, 12, 13, 18, 21, 22, 26, 27, 28, 33, 37, 40] which present a variety of approaches to these problems and contain basic results on them.

Equations of the form (1.1) with an integer σ are also intensively investigated. The main approaches to the study of higher (odd) order equations and comprehensive bibliography on the problem can be found in [1, 2, 5, 6, 8, 9, 15, 16, 19, 20, 25].

Equation (1.1) with non-integer σ was considered in [3] from a probabilistic point of view, in particular, a fundamental solution in terms of the Laplace transform images was constructed. We also note papers [7, 29, 30, 31] in which equations in the form (1.1) were studied in the cases $n=0$ and $n=1$.

It is worth pointing out that fractional differential equations arise in modern physics and mechanics and display many effective applications. In this regard, we refer to books [14, 23, 32, 34, 36] which provide insight into the developments of the topic.

In this article, we construct a fundamental solution of equation (1.1) and give a representation for a solution of the Cauchy problem, we also prove the uniqueness

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theorem in the class of functions satisfying an analogue of Tychonoff's condition [35].

The fractional differentiation is given by the Dzhrbashyan-Nersesyan operator [10] (see also [28]) associated with ordered pair $\{\alpha, \beta\}$, i.e.

$$\frac{\partial^\sigma}{\partial y^\sigma} = D_{0y}^{\{\alpha,\beta\}} = D_{0y}^{\beta-1} D_{0y}^{\alpha}, \quad \alpha, \beta \in (0,1], \quad \sigma = \alpha + \beta - 1, \tag{1.2}$$

where $D_{0y}^{\beta-1}$ and D_{0y}^{α} are the Riemann-Liouville fractional integral and fractional derivative, respectively, defined by the formulas [24, p. 11], [17, Sec. 2.1]

$$D_{ty}^{-\varepsilon} g(y) = \frac{\operatorname{sign}(y-t)}{\Gamma(\varepsilon)} \int_t^y g(s)|y-s|^{\varepsilon-1} ds, \quad \varepsilon > 0; \quad D_{ty}^0 g(y) = g(y); \tag{1.3}$$

$$D_{ty}^{\delta} g(y) = \operatorname{sign}^q(y-t) \frac{\partial^q}{\partial y^q} D_{ty}^{\delta-q} g(y), \quad \delta \in (q-1, q], \quad q \in \mathbb{N}.$$

For the pairs $\{\sigma, 1\}$ and $\{1, \sigma\}$, operator (1.2) coincides with the Riemann-Liouville derivative $D_{0y}^{\{\sigma,1\}} = D_{0y}^{\sigma}$ and with the Caputo derivative $D_{0y}^{\{1,\sigma\}} = \partial_{0y}^{\sigma}$, respectively. Therefore, the equation under consideration covers both the special cases of the equation with the Riemann-Liouville and Caputo derivatives.

The paper has the following structure. In Section 2, we gathered information concerning fractional differentiation operators and the Wright function, which is necessary for our considerations. In Section 3, we introduce a special function that is used to represent the fundamental solution, and prove some of its properties. We formulate the Cauchy problem and give the representation of the solution in Section 4. Section 5 presents the uniqueness theorem for the solution in the class of exponential growth functions.

2. Auxiliary assertions

In what follows, the symbol C denotes positive constants, which are different in different cases, indicating in parentheses the parameters on which it can depend if necessary: $C = C(\alpha, \beta, \dots)$; we write $\arg z$ for the principal value of the argument, which is taken to lie in the range $(-\pi, \pi]$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We begin by recalling some properties of fractional integrals and derivatives ([10]–[17]):

$$\int_\eta^y h(t) D_{\eta t}^{-\delta} g(t)\, dt = \int_\eta^y g(t) D_{yt}^{-\delta} h(t)\, dt, \quad \delta > 0; \tag{2.1}$$

$$D_{\eta y}^{\varepsilon} D_{\eta y}^{\delta} h(y) = D_{\eta y}^{\varepsilon+\delta} h(y) - \sum_{j=1}^{p} \frac{|y-\eta|^{-\varepsilon-j}}{\Gamma(1-\varepsilon-j)} [D_{\eta y}^{\delta-j} h(y)]_{y=\eta}, \tag{2.2}$$

if $\varepsilon \in \mathbb{R}$ and $\delta \in (p-1, p]$, $p \in \mathbb{N}$; and

$$D_{\eta y}^{\varepsilon} \frac{|y-\eta|^{\delta-1}}{\Gamma(\delta)} = \frac{|y-\eta|^{\delta-\varepsilon-1}}{\Gamma(\delta-\varepsilon)} \tag{2.3}$$

if $\varepsilon \in \mathbb{R}$ and $\delta > 0$ or $\varepsilon \in \mathbb{N}$ and $\delta \in \mathbb{R}$.

From (1.2) and (2.2), it follows that

$$D_{\eta y}^{\{\alpha,\beta\}} h(y) = D_{\eta y}^{\sigma} h(y) - \frac{|y-\eta|^{-\beta}}{\Gamma(1-\beta)} [D_{\eta y}^{\alpha-1} h(y)]_{y=\eta}. \tag{2.4}$$

The Wright function is defined by the series [38]

$$\phi(\delta,\varepsilon;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\delta k+\varepsilon)}\quad(\delta>-1). \tag{2.5}$$

The Wright function may be written as [39]

$$\phi(\delta,\varepsilon;z)=\frac{1}{2\pi i}\int_{H(r,\omega\pi)}p^{-\varepsilon}\exp\left(p+zp^{-\delta}\right)dp,\quad(\delta>-1), \tag{2.6}$$

where $H(r,\omega\pi)=\{p:|p|=r,\,|\arg p|\le\omega\pi\}\cup\{p:|p|\ge r,\,\arg p=\pm\omega\pi\}$ is the Hankel contour, which is traversed in the direction of non-decreasing $\arg p$, and $\omega\in(1/2,1]$.

For the function $\phi(\delta,\varepsilon;z)$, it holds the differentiation formula [38], [39]

$$\frac{d}{dz}\phi\left(\delta,\varepsilon;z\right)=\phi(\delta,\varepsilon+\delta;z)\quad(\delta>-1). \tag{2.7}$$

From the asymptotic expansion of the Wright function (see [39]), an inequality follows:

$$|\phi(-\delta,\varepsilon;z)|\le C\exp\left(-\nu|z|^{\frac{1}{1-\delta}}\right),\quad C=C(\delta,\varepsilon,\nu), \tag{2.8}$$

where $\delta\in(0,1)$, $\varepsilon\in\mathbb{R}$, and

$$\nu<(1-\delta)\delta^{\frac{\delta}{1-\delta}}\cos\frac{\pi-|\arg z|}{1-\delta},\quad\pi\ge|\arg z|>\frac{1+\delta}{2}\pi.$$

In addition, the equality

$$\lim_{z\to0}\frac{1}{z}\phi\left(-\delta,\varepsilon;z\right)=\frac{1}{\Gamma(\varepsilon-\delta)} \tag{2.9}$$

holds for all integer non-positive ε. So, for $\delta\in(0,1)$, $\varepsilon\in\mathbb{R}$, $x>0$, $y>0$ and λ satisfying the condition

$$\frac{1+\delta}{2}\pi<|\arg\lambda|\le\pi, \tag{2.10}$$

it follows from (2.8) and (2.9) that

$$|y^{\varepsilon-1}\phi(-\delta,\varepsilon;\lambda xy^{-\delta})|\le Cx^{-\theta}y^{\varepsilon+\delta\theta-1},\quad\theta\ge\begin{cases}0,&(-\varepsilon)\notin\mathbb{N}_0,\\-1,&(-\varepsilon)\in\mathbb{N}_0,\end{cases} \tag{2.11}$$

where $C=C(\varepsilon,\delta,\theta,\lambda)$.

Formulas (2.7) and (2.8) yield

$$\int_0^\infty\phi\left(-\delta,\varepsilon;\lambda x\right)dx=-\frac{1}{\lambda\Gamma(\delta+\varepsilon)}, \tag{2.12}$$

for every λ satisfying (2.10).

The following Lemma establishes the fractional differentiation formula for the Wright function with a complex argument.

Lemma 2.1. *Let $\delta\in(0,1)$, ε, $\nu\in\mathbb{R}$ and the inequality (2.10) hold. Then*

$$D_{0y}^{\nu}y^{\varepsilon-1}\phi\left(-\delta,\varepsilon;\lambda y^{-\delta}\right)=y^{\varepsilon-\nu-1}\phi\left(-\delta,\varepsilon-\nu;\lambda y^{-\delta}\right). \tag{2.13}$$

Proof. First observe that representation (2.6) yields

$$y^{\varepsilon-1}\phi\left(-\delta,\varepsilon;\lambda y^{-\delta}\right)=\frac{1}{2\pi i}\int_{H(r,\omega\pi)}p^{-\varepsilon}e^{py+\lambda p^{\delta}}\,dp, \tag{2.14}$$

and formula (2.3) gives

$$D_{0y}^{\nu}e^{py}=y^{-\nu}E_{1,1-\nu}(py), \tag{2.15}$$

where

$$E_{\xi,\eta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\xi k+\eta)}$$

is the Mittag-Leffler function.

Combining (2.14) and (2.15) we obtain

$$\begin{aligned} &D_{0y}^{\nu}y^{\varepsilon-1}\phi\left(-\delta,\varepsilon;\lambda y^{-\delta}\right)\\ &=\frac{y^{-\nu}}{2\pi i}\int_{H(r,\omega\pi)}p^{-\varepsilon}E_{1,1-\nu}(py)e^{\lambda p^{\delta}}\,dp\\ &=\frac{y^{-\nu}}{2\pi i}\int_{H(r,\omega\pi)}p^{-\varepsilon}\left[E_{1,1-\nu}(py)-(py)^{\nu}e^{py}\right]e^{\lambda p^{\delta}}\,dp\\ &\quad+y^{\varepsilon-\nu-1}\phi\left(-\delta,\varepsilon-\nu;\lambda y^{-\delta}\right). \end{aligned} \tag{2.16}$$

Let us consider the integral on the right-hand side of the last equality. Set

$$C_R=\{p:|p|=R,|\arg p|\le\omega\pi\},\quad H_R=\{p\in H(r,\omega\pi):|p|<R\}.$$

It is evident that the integrand in the integral under consideration has no singularities in the domain bounded by the curves C_R and H_R. In addition, the asymptotic behavior of the Mittag-Leffler function (see [11, Ch. 3], [17, §1.8]) yields

$$|E_{1,1-\nu}(py)-(py)^{\nu}e^{py}|\le\frac{C}{yR}\quad\text{for every } p\in C_R.$$

Also, for ω close to $1/2$ and $p\in C_R$, we have

$$\frac{\pi}{2}<|\arg\lambda p^{\delta}|\le\pi.$$

Thus, from the above it follows that

$$\begin{aligned} &\int_{H(r,\omega\pi)}p^{-\varepsilon}[E_{1,1-\nu}(py)-(py)^{\nu}e^{py}]e^{\lambda p^{\delta}}\,dp\\ &=\lim_{R\to\infty}\int_{H_R}p^{-\varepsilon}\left[E_{1,1-\nu}(py)-(py)^{\nu}e^{py}\right]e^{\lambda p^{\delta}}\,dp\\ &=-\lim_{R\to\infty}\int_{C_R}p^{-\varepsilon}\left[E_{1,1-\nu}(py)-(py)^{\nu}e^{py}\right]e^{\lambda p^{\delta}}\,dp=0. \end{aligned}$$

The last equality with (2.16) proves (2.13).

We note that for $\arg\lambda=\pi$, formula (2.13) was proved in [29]. □

3. Fundamental solution

We consider the function

$$G^{\mu}_{\sigma,n}(x,y)=\frac{y^{\mu-1}}{2n+1}\times\begin{cases}\sum_{k=0}^{n-1}\lambda_k\phi\left(-\gamma,\mu;\lambda_k xy^{-\gamma}\right) & \text{for } x<0,\\ -\sum_{k=n}^{2n}\lambda_k\phi\left(-\gamma,\mu;\lambda_k xy^{-\gamma}\right) & \text{for } x>0.\end{cases}\tag{3.1}$$

Here and in what follows we use

$$\gamma=\frac{\sigma}{2n+1},\quad \lambda_k=e^{\frac{2k-n+1}{2n+1}\pi i}.\tag{3.2}$$

We also regarding the arguments and parameters of the function (3.1) have the following ranges:

$$x\in\mathbb{R},\quad y\in(0,\infty),\quad \sigma\in(0,1),\quad n\in\mathbb{N}_0,\quad \mu\in\mathbb{R}.\tag{3.3}$$

Remark 3.1. It is easy to check that the function (3.1) is real-valued. Indeed, by (2.5), we have

$$\sum_{k=p}^{q}\lambda_k\phi\left(-\gamma,\mu;\lambda_k z\right)=\sum_{m=0}^{\infty}\frac{z^m}{m!\Gamma(\mu-\gamma m)}\sum_{k=p}^{q}\lambda_k^{m+1}\quad (p,q\in\mathbb{N}_0,\quad p\ge q).\tag{3.4}$$

Let

$$\xi=\frac{m+1}{2n+1}\pi.$$

If $m\equiv 2n \pmod{(2n+1)}$ (i.e. $m=(2n+1)j+2n$ for some $j\in\mathbb{N}_0$) and, therefore, $e^{2\xi i}=1$, then

$$\sum_{k=p}^{q}\lambda_k^{m+1}=e^{(1-n)\xi i}\sum_{k=p}^{q}e^{2\xi ki}=(-1)^{(n-1)(m+1)}(q-p+1).\tag{3.5}$$

For $e^{2\xi i}\neq 1$, we have

$$\begin{aligned}\sum_{k=p}^{q}\lambda_k^{m+1}&=e^{(1-n)\xi i}\sum_{k=p}^{q}e^{2\xi ki}=\frac{e^{2\xi pi}-e^{2\xi(q+1)i}}{1-e^{2\xi ki}}e^{(1-n)\xi i}\\&=\frac{\sin\xi(q-p+1)}{\sin\xi}e^{(p+q-n+1)\xi i}.\end{aligned}$$

In particular, this implies

$$\sum_{k=0}^{n-1}\lambda_k^{m+1}=\frac{\sin\xi n}{\sin\xi},\quad \sum_{k=n}^{2n}\lambda_k^{m+1}=(-1)^{m+1}\frac{\sin\xi(n+1)}{\sin\xi}=-\frac{\sin\xi n}{\sin\xi}.\tag{3.6}$$

Equalities (3.4), (3.5) and (3.6) prove that the function $G^{\mu}_{\sigma,n}(x,y)$ is real-valued, with arguments and parameters satisfying (3.3).

Remark 3.2. In the case $n=0$, there are no summands in the sum that gives the value of (3.1) for $x<0$. That is $G^{\mu}_{\sigma,0}(x,y)=0$ for all $x<0$.

Lemma 3.3. *Let $m\in\mathbb{N}_0$. Then*

$$\begin{aligned}&\frac{\partial^m}{\partial x^m}G^{\mu}_{\sigma,n}(0-,y)-\frac{\partial^m}{\partial x^m}G^{\mu}_{\sigma,n}(0+,y)\\&=(-1)^{(m+1)(n-1)}\frac{y^{\mu-\gamma m-1}}{\Gamma\left(\mu-\gamma m\right)}\times\begin{cases}1 & \text{if } m\equiv 2n \pmod{(2n+1)},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}\tag{3.7}$$

Proof. By (2.7) and (2.9), we have

$$\frac{\partial^m}{\partial x^m} G^\mu_{\sigma,n}(0-,y) - \frac{\partial^m}{\partial x^m} G^\mu_{\sigma,n}(0+,y) = \frac{1}{2n+1} \frac{y^{\mu-\gamma m-1}}{\Gamma(\mu-\gamma m)} \sum_{k=0}^{2n} \lambda_k^{m+1}.$$

From (3.5) and (3.6), it follows that

$$\sum_{k=0}^{2n} \lambda_k^{m+1} = \begin{cases} (-1)^{(n-1)(m+1)}(2n+1) & \text{if } m \equiv 2n \pmod{(2n+1)}, \\ 0 & \text{otherwise,} \end{cases}$$

This leads to (3.7). □

Lemma 3.4. *Let $m \in \mathbb{N}_0$ and $\nu \in \mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} G^\mu_{\sigma,n}(x,y)\,dx = \frac{y^{\mu+\gamma-1}}{\Gamma(\mu+\gamma)}, \tag{3.8}$$

$$D^\nu_{0y} G^\mu_{\sigma,n}(x,y) = G^{\mu-\nu}_{\sigma,n}(x,y), \quad \frac{\partial^{2n+1}}{\partial x^{2n+1}} G^\mu_{\sigma,n}(x,y) = (-1)^{n-1} G^{\mu-\sigma}_{\sigma,n}(x,y), \tag{3.9}$$

$$\Big(D^{\{\alpha,\beta\}}_{0y} + (-1)^n \frac{\partial^{2n+1}}{\partial x^{2n+1}}\Big) G^\mu_{\sigma,n}(x,y) = 0. \tag{3.10}$$

Proof. First observe that each λ_k satisfy the inequalities

$$\begin{gathered} \frac{1+\gamma}{2}\pi < |\arg(-\lambda_k)| \le \pi, \quad k = 0,1,\dots n-1, \\ \frac{1+\gamma}{2}\pi < |\arg \lambda_k| \le \pi, \quad k = n, n+1, \dots 2n, \end{gathered}$$

we recall that we choose the branch of $\arg z$ with the range $(-\pi,\pi]$. This makes it possible to apply (2.12) to calculate the integral in (3.8),

$$\begin{aligned} &\int_{-\infty}^{\infty} G^\mu_{\sigma,n}(x,y)\,dx \\ &= \frac{y^{\mu-1}}{2n+1} \sum_{k=0}^{n-1} \lambda_k \int_{-\infty}^{0} \phi\left(-\gamma,\mu;\lambda_k x y^{-\gamma}\right) dx \\ &\quad - \frac{y^{\mu-1}}{2n+1} \sum_{k=n}^{2n} \lambda_k \int_{0}^{\infty} \phi\left(-\gamma,\mu;\lambda_k x y^{-\gamma}\right) dx \\ &= \frac{y^{\mu+\gamma-1}}{(2n+1)\Gamma(\mu+\gamma)} \sum_{k=0}^{2n} 1. \end{aligned}$$

Further, the equalities in (3.9) follow from (2.7) and (2.13), and (3.10) is a consequence of (3.9). □

Lemma 3.5. *Let $m \in \mathbb{N}_0$. Then*

$$\Big|\frac{\partial^m}{\partial x^m} G^\mu_{\sigma,n}(x,y)\Big| \le C|x|^{-\theta} y^{\mu+\gamma(\theta-m)-1}, \quad \theta \ge \begin{cases} 0, & (-\mu) \notin \mathbb{N}_0, \\ -1, & (-\mu) \in \mathbb{N}_0, \end{cases} \tag{3.11}$$

and $C = C(\sigma,\mu,\theta,m)$; and

$$\lim_{z\to\pm\infty} y^{1-\mu+\gamma m} \frac{\partial^m}{\partial x^m} G^\mu_{\sigma,n}(x,y) \exp\Big(\nu_\pm |x|^{\frac{1}{1-\gamma}} y^{-\frac{\gamma}{1-\gamma}}\Big) = 0, \quad z = xy^{-\gamma}, \tag{3.12}$$

for any ν_- *and* ν_+ *such that*

$$\nu_- < (1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}}\cos\frac{(n-1)\pi}{2n+1-\sigma}, \quad \nu_+ < (1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}}\cos\frac{n\pi}{2n+1-\sigma}. \tag{3.13}$$

Proof. By (2.7) and (3.1), we have

$$\begin{aligned}&\Big|\frac{\partial^m}{\partial x^m}G^{\mu}_{\sigma,n}(x,y)\Big|\\&\le Cy^{\mu-\gamma m-1}\begin{cases}\max_{0\le k\le n-1}|\phi(-\gamma,\mu-\gamma m;\lambda_k xy^{-\gamma})| & \text{for } x<0,\\ \max_{n\le k\le 2n}|\phi(-\gamma,\mu-\gamma m;\lambda_k xy^{-\gamma})| & \text{for } x>0.\end{cases}\end{aligned}$$

Combining this and (2.11) we obtain (3.11). Furthermore, the equalities

$$\begin{gathered}\min_{0\le k\le n-1}\cos\frac{\pi-|\arg(-\lambda_k)|}{1-\gamma}=\cos\frac{(n-1)\pi}{2n+1-\sigma},\\ \min_{n\le k\le 2n}\cos\frac{\pi-|\arg\lambda_k|}{1-\gamma}=\cos\frac{n\pi}{2n+1-\sigma},\end{gathered}$$

and the inequality (2.8) lead to (3.12). $\square$

Lemma 3.6. *Let* $\tau(x)\in C(\mathbb{R})$,

$$\lim_{x\to\pm\infty}\tau(x)\exp\Big(-\nu_\pm|x|^{\frac{1}{1-\gamma}}T^{-\frac{\gamma}{1-\gamma}}\Big)=0 \tag{3.14}$$

for some ν_- *and* ν_+ *satisfying* (3.13), *and*

$$v(x,y)=\int_{-\infty}^{\infty}\tau(s)G^{\alpha-\gamma}_{\sigma,n}(x-s,y)\,ds.$$

Then

$$\lim_{y\to 0}D^{\alpha-1}_{0y}v(x,y)=\tau(x),\quad x\in\mathbb{R}, \tag{3.15}$$

$$\Big(D^{\{\alpha,\beta\}}_{0y}+(-1)^n\frac{\partial^{2n+1}}{\partial x^{2n+1}}\Big)v(x,y)=0,\quad (x,y)\in\mathbb{R}\times(0,T). \tag{3.16}$$

Proof. By (3.8) and (3.9), for any $\varepsilon>0$, we have

$$\begin{aligned}D^{\alpha-1}_{0y}v(x,y)&=\Big(\int_{-\infty}^{x-\varepsilon}+\int_{x+\varepsilon}^{\infty}\Big)[\tau(s)-\tau(x)]\,G^{1-\gamma}_{\sigma,n}(x-s,y)\,ds\\&\quad+\int_{x-\varepsilon}^{x+\varepsilon}[\tau(s)-\tau(x)]\,G^{1-\gamma}_{\sigma,n}(x-s,y)\,ds\\&\quad+\tau(x)\int_{-\infty}^{\infty}G^{1-\gamma}_{\sigma,n}(x-s,y)\,ds\\&=I_1+I_2+\tau(x).\end{aligned}$$

Taking into account (3.12) and (3.14), we obtain

$$\lim_{y\to 0}I_1=0,\quad |I_2|\le C(\sigma)\sup_{s\in(x-\varepsilon,x+\varepsilon)}|\tau(s)-\tau(x)|\,.$$

The continuity of the function $\tau(x)$ and an arbitrary choice of ε imply (3.15).

Let us prove (3.16). By (2.4), (3.9) and (3.15), we have

$$D^{\{\alpha,\beta\}}_{0y}v(x,y)=\int_{-\infty}^{\infty}\tau(s)G^{1-\beta-\gamma}_{\sigma,n}(x-s,y)\,ds-\frac{y^{-\beta}\tau(x)}{\Gamma(1-\beta)}.$$

From (3.7) and (3.9), it follows that

$$\begin{aligned}\frac{\partial^{2n+1}}{\partial x^{2n+1}}v(x,y) &= \frac{\partial}{\partial x}\int_{-\infty}^{\infty}\tau(s)\frac{\partial^{2n}}{\partial x^{2n}}G^{\alpha-\gamma}_{\sigma,n}(x-s,y)\,ds\\ &= (-1)^{n-1}\int_{-\infty}^{\infty}\tau(s)G^{1-\beta-\gamma}_{\sigma,n}(x-s,y)\,ds + (-1)^n\frac{y^{-\beta}\tau(x)}{\Gamma(1-\beta)}.\end{aligned}$$

Combining the last two equalities, we obtain (3.16). □

Lemma 3.7. *Let $f(x,y)$ be representable in the form $f(x,y) = D^{-\varepsilon}_{0y}g(x,y)$, where $y^{1-\delta}g(x,y) \in C(\overline{D})$, $\varepsilon > 0$, $\delta > 0$ and $\varepsilon + \delta > 1 - \beta$,*

$$\lim_{x\to\pm\infty} y^{1-\delta}g(x,y)\exp\Big(-\nu_{\pm}|x|^{\frac{1}{1-\gamma}}T^{-\frac{\gamma}{1-\gamma}}\Big) = 0, \tag{3.17}$$

for some ν_- and ν_+ satisfying (3.13), and

$$w(x,y) = \int_0^y\int_{-\infty}^{\infty} f(s,t)G^{\sigma-\gamma}_{\sigma,n}(x-s,y-t)\,ds\,dt.$$

Then

$$\lim_{y\to 0} D^{\alpha-1}_{0y}w(x,y) = 0, \quad x\in\mathbb{R}, \tag{3.18}$$

$$\Big(D^{\{\alpha,\beta\}}_{0y} + (-1)^n\frac{\partial^{2n+1}}{\partial x^{2n+1}}\Big)w(x,y) = f(x,y), \quad (x,y)\in\mathbb{R}\times(0,T). \tag{3.19}$$

Proof. The conditions imposed on the function $f(x,y)$, and the relations (3.11) and (3.12) imply

$$|w(x,y)| \le Cy^{\varepsilon+\delta+\gamma(2n+\theta)-1},$$

where $C = C(\sigma,\theta)$, and θ can be chosen in $[0,1)$ sufficiently close to 1. Therefore, by (2.3) and (3.2), we obtain

$$\big|D^{\alpha-1}_{0y}w(x,y)\big| \le Cy^{\varepsilon+\delta+\sigma\frac{2n+\theta}{2n+1}-\alpha}.$$

By the conditions imposed on ε and δ, and equality $\sigma = \alpha+\beta-1$, this gives (3.18).

By (2.1), (3.7) and (3.9), we have

$$\begin{aligned}\frac{\partial^{2n+1}}{\partial x^{2n+1}}w(x,y) &= \frac{\partial}{\partial x}\int_0^y\int_{-\infty}^{\infty}g(s,t)\frac{\partial^{2n}}{\partial x^{2n}}G^{\sigma-\gamma+\varepsilon}_{\sigma,n}(x-s,y-t)\,ds\,dt\\ &= (-1)^n\int_0^y g(x,t)\frac{(y-t)^{\varepsilon-1}}{\Gamma(\varepsilon)}dt\\ &\quad + (-1)^{n-1}\int_0^y\int_{-\infty}^{\infty}g(s,t)G^{\varepsilon-\gamma}_{\sigma,n}(x-s,y-t)\,ds\,dt.\end{aligned} \tag{3.20}$$

Note that the first summand on the right-hand side of the last equality is equal to $D^{-\varepsilon}_{0y}g(x,y) = f(x,y)$. Then, taking into account (3.18), we obtain

$$\begin{aligned}D^{\{\alpha,\beta\}}_{0y}w(x,y) &= \lim_{t\to y}\int_{-\infty}^{\infty}g(s,t)G^{\varepsilon-\gamma+1}_{\sigma,n}(x-s,y-t)\,ds\\ &\quad + \int_0^y\int_{-\infty}^{\infty}g(s,t)G^{\varepsilon-\gamma}_{\sigma,n}(x-s,y-t)\,ds\,dt.\end{aligned}$$

It follows from (3.11) that the first summand equals 0. Combining this with (3.20), we obtain (3.19). □

Remark 3.8. For $n = 0$, it suffices to require convergence of the corresponding limits in (3.14) and (3.17) only for $x \to -\infty$, because $G^{\mu}_{\sigma,0}(x, y) = 0$ for $x < 0$ (see Remark 3.2).

The properties of the function $G^{\mu}_{\sigma,n}$ (in particular, formula (3.10), Lemmas 3.6 and 3.7) proved above imply that the function

$$\Gamma_{\sigma,n}(x, y; s, t) = G^{\sigma-\gamma}_{\sigma,n}(x - s, y - t) \tag{3.21}$$

is a *fundamental solution* of equation (1.1).

4. Cauchy problem

Let $D = \mathbb{R} \times (0, T)$ and $D_0 = \mathbb{R} \times [0, T)$. A function $u(x, y)$ is called a *regular solution* of equation (1.1) in the domain D if $u(x, y)$ has continuous derivatives with respect to x till the order $2n + 1$ in D, $y^{1-\mu}u(x, y) \in C(D_0)$ for some $\mu > 0$, $D^{\alpha-1}_{0y}u(x, y) \in C(D_0)$, $D^{\alpha-1}_{0y}u(x, y)$ is absolutely continuous as a function of the variable y in the half-closed interval $[0, T)$ for every fixed $x \in \mathbb{R}$, and $u(x, y)$ satisfies (1.1) for all $(x, y) \in D$. We pose the Cauchy problem:

find a regular solution of equation (1.1) in the domain D such that

$$\lim_{y\to 0} D^{\alpha-1}_{0y}u(x, y) = \tau(x), \quad x \in \mathbb{R}, \tag{4.1}$$

where $\tau(x)$ is a given function.

Theorem 4.1. *Let the functions $\tau(x)$ and $f(x, y)$ satisfy the conditions of Lemmas 3.6 and 3.7. Then the function*

$$u(x, y) = \int_{-\infty}^{\infty} \tau(s)D^{\beta-1}_{0y}\Gamma_{\sigma,n}(x, y; s, 0)\, ds + \int_0^y \int_{-\infty}^{\infty} f(s, t)\Gamma_{\sigma,n}(x, y; s, t)\, ds\, dt$$

is a regular solution of equation (1.1) *and satisfies the condition* (4.1)*.*

Proof. By (3.9) and (3.21), the statement of the theorem is a consequence of Lemmas 3.6 and 3.7. □

5. Uniqueness

Here we prove the uniqueness theorem for the problem (1.1), (4.1) in the class of fast-growing functions satisfying an analogue of Tychonoff's condition [35].

Theorem 5.1. *There is at most one regular solution of problem* (1.1)*,* (4.1) *in the class of functions that satisfy the condition*

$$\lim_{|x|\to\infty} y^{1-\delta}u(x, y) \exp\Big(-\nu|x|^{\frac{1}{1-\gamma}}\Big) = 0, \tag{5.1}$$

for some positive constants δ and ν.

Proof. We consider the function

$$W_{\mu,r}(x, y) = G^{\mu}_{\sigma,n}(x, y)\, h_r(x),$$

where

$$h_r(x) = \begin{cases} 1 & \text{if } x \in (-r, r), \\ \frac{(4n+3)!}{[(2n+1)!]^2} \int_{|x|}^{r+1} (z - r)^{2n+1}(r + 1 - z)^{2n+1} dz & \text{if } |x| \in [r, r + 1), \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that

$$h_r(x) \in C^{2n+1}(\mathbb{R}); \quad 0 \le h_r(x) \le 1; \quad |h_r^{(k)}(x)| \le \text{const.}; \tag{5.2}$$

$$h_r^{(k)}(x) = 0, \quad \text{for } |x| \notin (r, r+1); \; k = 1, 2, \dots, 2n+1. \tag{5.3}$$

Put

$$\mathbf{L} = \Big(D_{0t}^{\{\alpha,\beta\}} + (-1)^n \frac{\partial^{2n+1}}{\partial s^{2n+1}}\Big), \quad \mathbf{L}^* = \Big(D_{yt}^{\{\beta,\alpha\}} - (-1)^n \frac{\partial^{2n+1}}{\partial s^{2n+1}}\Big).$$

Let $u(x,y)$ be a regular solution of the homogeneous problem (1.1), (4.1) (i.e. $\tau(x) \equiv 0$ and $f(x,y) \equiv 0$) satisfying condition (5.1). Moreover, let (x,y) be a fixed point in $\mathbb{R} \times (0, T_0)$, where

$$T_0 = \min\Big\{T, \gamma\Big(\frac{1-\gamma}{\nu}\cos\frac{\pi n}{2n+1-\sigma}\Big)^{\frac{1}{\gamma}-1}\Big\},$$

and μ, ε and r are positive numbers such that $r > |x| + \varepsilon$ and $\mu > 2n\gamma$. By the formulas (2.1), (3.9) and (3.10), and the relation

$$\begin{aligned}
&W_{\mu,r}(x-s, y-t)\frac{\partial^{2n+1}}{\partial s^{2n+1}}u(s,t) + u(s,t)\frac{\partial^{2n+1}}{\partial s^{2n+1}}W_{\mu,r}(x-s,y-t) \\
&= \frac{\partial}{\partial s}\sum_{k=0}^{2n}(-1)^k \frac{\partial^k}{\partial s^k}W_{\mu,r}(x-s,y-t)\frac{\partial^{2n-k}}{\partial s^{2n-k}}u(s,t),
\end{aligned}$$

we obtain

$$\begin{aligned}
0 &= \int_0^y \Big(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\Big) W_{\mu,r}(x-s,y-t)\mathbf{L}u(s,t)\,ds\,dt \\
&= \int_0^y \Big(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\Big) u(s,t)\mathbf{L}^* W_{\mu,r}(x-s,y-t)\,ds\,dt \\
&\quad + \int_0^y \sum_{k=0}^{2n}(-1)^{n+k}\Big[\frac{\partial^k}{\partial s^k}W_{\mu,r}(x-s,y-t)\frac{\partial^{2n-k}}{\partial s^{2n-k}}u(s,t)\Big]_{s=x+\varepsilon}^{s=x-\varepsilon} dt \\
&= I_1 + I_2,
\end{aligned} \tag{5.4}$$

where

$$I_1 = \int_0^y \Big(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\Big) u(s,t)\mathbf{L}^* W_{\mu,r}(x-s,y-t)\,ds\,dt,$$

$$I_2 = \int_0^y \sum_{k=0}^{2n}(-1)^{n+k}\Big[\frac{\partial^k}{\partial s^k}W_{\mu,r}(x-s,y-t)\frac{\partial^{2n-k}}{\partial s^{2n-k}}u(s,t)\Big]_{s=x+\varepsilon}^{s=x-\varepsilon} dt.$$

It follows from (3.10) that

$$\begin{aligned}
&\mathbf{L}^* W_{\mu,r}(x-s,y-t) \\
&= (-1)^{n-1}\sum_{k=0}^{2n}\frac{(2n+1)!}{k!(2n-k+1)!}\frac{\partial^k}{\partial s^k}G^{\mu}_{\sigma,n}(x-s,y-t)\frac{\partial^{2n-k+1}}{\partial s^{2n-k+1}}h_r(x-s).
\end{aligned}$$

Combining this with (5.2) and (5.3), we have

$$|I_1| \le C\int_0^y \Big(\int_{-r-1}^{r} + \int_r^{r+1}\Big)|u(s,t)|\sum_{k=0}^{2n}\Big|\frac{\partial^k}{\partial s^k}G^{\mu}_{\sigma,n}(x-s,y-t)\Big|\,ds\,dt.$$

By (3.12) and (5.1), the last inequality implies $\lim_{r\to\infty} I_1 = 0$. Further, by Lemma 3.3 and the definition (1.3), we obtain

$$\lim_{\varepsilon\to 0} I_2 = -D_{0y}^{2\gamma n-\mu} u(x,y).$$

Thus, letting $\varepsilon \to 0$ and $r \to \infty$ in (5.4), we obtain

$$D_{0y}^{2\gamma n-\mu} u(x,y) = 0.$$

Since $2\gamma n - \mu < 0$, this implies $u(x,y) \equiv 0$ for all $(x,y) \in \mathbb{R} \times (0,T_0)$.

Let us now prove that $u(x,y) = 0$ for any $y > 0$. Suppose the contrary, and put $y_0 = \inf\{y : u(x,y) \neq 0 \text{ for some } x \in \mathbb{R}\}$. It is obvious that $y_0 \geq T_0$. Consider the function $\tilde{u}(x,y) = u(x, y+y_0)$. For every positive ε, our assumption and the definition of y_0 enables us to find a number $x \in \mathbb{R}$ such that

$$\tilde{u}(x,\varepsilon) \neq 0. \tag{5.5}$$

The equalities

$$D_{0,y+y_0}^{\{\alpha,\beta\}} u(x,y) = D_{0y}^{\{\alpha,\beta\}} \tilde{u}(x,y), \quad \lim_{y\to 0} D_{0y}^{\alpha-1} \tilde{u}(x,y) = 0$$

imply that the function $\tilde{u}(x,y)$ is a solution of the homogeneous equation (1.1) and satisfies the null initial condition (4.1). By the above, this means that $\tilde{u}(x,y) = 0$ at least for $y \in (0,T_0)$. This contradicts (5.5). Hence, $u(x,y) = 0$ for all $x \in \mathbb{R}$ and $y > 0$. □

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