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UPPER AND LOWER SOLUTIONS METHODS FOR IMPULSIVE CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this article we use the method of lower and upper solutions combined with the fixed point theorem by Bohnnenblust-Karlin to show the existence of solutions for initial-value problems of impulsive Caputo-Hadamard fractional differential inclusions of order $\alpha \in (0,1)$.

1. Introduction

In this article we study the initial value problem (IVP for short) for the α-th order fractional differential inclusion with impulse,

$$ {}^{CH}D^{\alpha} y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J = [a,T],\ t \neq t_k,\ k = 1,\dots,m, \tag{1.1} $$

$$ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{1.2} $$

$$ y(a) = y_a, \tag{1.3} $$

where $0 < \alpha < 1$, $a > 0$, ${}^{CH}D^{\alpha}$ is the Caputo-Hadamard fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1,\dots,m$, are continuous functions, $a = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{\varepsilon \to 0^+} y(t_k + \varepsilon)$ and $y(t_k^-) = \lim_{\varepsilon \to 0^-} y(t_k + \varepsilon)$ represent the right and left limits of y at $t = t_k$, $k = 1,\dots,m$. We apply the method of lower and upper solutions combined with the fixed point theorem of Bohnenblust-Karlin to establish the existence of solutions to this problem.

Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (such as blood flow phenomena), economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetism, and so on. They are commonly viewed as better tools for description of hereditary properties of various materials and processes than the corresponding integer order differential equations; see, for example [9, 21, 22, 28, 32, 39, 44, 45, 48, 49, 50, 51, 54, 55].

In addition, there is high research activity in the theoretical development of fractional calculus, fractional ordinary differential equations and fractional partial

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differential equations; see for example the recent textbook by Goodrich and Peterson [29], as well as the monographs [33, 35, 39, 46, 51]. These monographs and papers such as [6, 10] and their references therein, reflect the large number of papers devoted to fractional research. There are numerous definitions of a fractional derivative, with the most common being the Riemann-Liouville type fractional derivative, and the Caputo type fractional derivative. Podlubny's book [51] and papers [34, 52] are good references for geometric and physical interpretations of both types of fractional derivative. One major difference between the two types of fractional derivatives is that the Caputo fractional derivative incorporates initial values "at zero", while the Riemann-Liouville is independent of any such initial values. Hadamard's [30] fractional derivative, introduced in 1892, differs significantly from both the Riemann-Liouville type and the Caputo type. In particular, the integral's kernel in the definition of Hadamard's fractional derivative contains a logarithmic function of so-called *arbitrary exponent*. Good overviews and applications of where the Hadamard derivative and the Hadamard integral arise can be found in the papers by Butzer, Kilbas and Trujillo [15, 16, 17]. Other important results dealing with Hadamard fractional calculus and Hadamard differential equations can be found in [7, 38, 41, 58], as well as in the monograph by Kilbas *et al.* [39].

This article involves a recent Caputo-type modification of the Hadamard fractional derivative, which was first studied by Jarad *et al.* [36]. This derivative is now known as the *Caputo-Hadamard fractional derivative*, and for a couple of other papers dealing with Caputo-Hadamard calculus and Caputo-Hadamard differential equations, we cite [5, 25].

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems, such as engineering and control theory [18, 27, 37, 53], population dynamics [20, 47, 57, 59] and medicine [19, 24, 26]. For general aspects of impulsive differential equations, we suggest the classical monographs [8, 43, 56, 60] and the more recent monograph [12]. Benchohra and Slimani [11] initiated the study of fractional differential equations with impulses.

The method of lower and upper solutions is among the most common (yet classical) techniques employed to establish existence of solutions for nonlinear differential equations; see the classical monographs [31, 42] and the paper [23]. Recently, the lower and upper solutions method was applied to obtain solutions for fractional differential equations and fractional differential inclusions in the above cited monograph [12], as well as in the papers [1, 2, 3, 4, 13].

2. Preliminaries

In this section, we introduce definitions and preliminary facts that are used in the remainder of this article.

Let $[a, b]$ be a compact interval, and $C([a, b], \mathbb{R})$ be the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : a \le t \le b\},$$

and we denote by $L^1([a,b],\mathbb{R})$ the Banach space of functions $y:[a,b]\to\mathbb{R}$ that are Lebesgue integrable, with norm

$$\|y\|_{L^1}=\int_a^b |y(t)|dt.$$

$AC([a,b],\mathbb{R})$ is the space of functions $y:[a,b]\to\mathbb{R}$, which are absolutely continuous. Let $(X,\|\cdot\|)$ be a Banach space. Let $P_{cl}(X)=\{Y\in\mathcal{P}(X): Y \text{ closed}\}$, $P_b(X)=\{Y\in\mathcal{P}(X): Y \text{ bounded}\}$, $P_{cp}(X)=\{Y\in\mathcal{P}(X): Y \text{ compact}\}$ and $P_{cp,c}(X)=\{Y\in\mathcal{P}(X): Y \text{ compact and convex}\}$.

A multivalued map $G: X\to\mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x\in X$. G is bounded on bounded sets if $G(B)=\cup_{x\in B}G(x)$ is bounded in X for all $B\in P_b(X)$, i.e.

$$\sup_{x\in B}\{\sup\{|y|: y\in G(x)\}\}<\infty.$$

G is called upper semi-continuous (u.s.c.) on X, if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0)\subseteq N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B\in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n\to x_*, y_n\to y_*, y_n\in G(x_n)$ imply $y_*\in G(x_*)$). G has a fixed point if there is $x\in X$ such that $x\in G(x)$. The fixed point set of the multivalued operator G will be denote by $FixG$. A multivalued map $G: J\to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y\in\mathbb{R}$, the function

$$t\mapsto d(y,G(t))=\inf\{|y-z|: z\in G(t)\}$$

is measurable.

Definition 2.1. A function $F:[a,b]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if

(1) $t\to F(t,u)$ is measurable for each $u\in\mathbb{R}$;
(2) $u\to F(t,u)$ is upper semicontinuous for almost all $t\in[a,b]$;
(3) for each $q>0$, there exists $\varphi_q\in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}}=\sup\{|v|: v\in F(t,u)\}\le\varphi_q \quad \text{for all } |u|\le q \text{ and a.e. } t\in J.$$

For each $y\in C([a,b],\mathbb{R})$, define the set of selections of F by

$$S_{F,y}=\{v\in L^1([a,b],\mathbb{R}): v(t)\in F(t,y(t)) \text{ a.e. } t\in[a,b]\}.$$

Let (X,d) be a metric space induced from the normed space $(X,\|\cdot\|)$. Consider $H_d:\mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}_+\cup\{\infty\}$ given by

$$H_d(A,B)=\max\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\},$$

where $d(A,b)=\inf_{a\in A}d(a,b)$, $d(a,B)=\inf_{b\in B}d(a,b)$. Then $(P_{b,cl}(X),H_d)$ is a metric space and $(P_{cl}(X),H_d)$ is a generalized metric space (see [40]).

Lemma 2.2 (Bohnenblust-Karlin [14]). *Let X be a Banach space, $K\in P_{cl,c}(X)$. Suppose that the operator $G: K\to P_{cl,c}(K)$ is upper semicontinuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

Definition 2.3 ([39])**.** The Hadamard fractional integral of order α for a function $h:[a,b]\to\mathbb{R}$ where $a,b\geq 0$ is defined by

$$I_a^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\big(\ln\frac{t}{s}\big)^{\alpha-1}\frac{h(s)}{s}ds,\quad \alpha>0,$$

provided the integral exists.

Definition 2.4 ([36])**.** Let $AC_\delta^n[a,b]=\{g:[a,b]\to\mathbb{C}\mid \delta^{n-1}g\in AC[a,b]\}$ where $\delta=t\frac{d}{dt}$, $0<a<b<\infty$ and let $\alpha\in\mathbb{C}$, such that $\operatorname{Re}(\alpha)\geq 0$. For a function $g\in AC_\delta^n[a,b]$ the Caputo type Hadamard derivative of fractional order α is defined as follows:

(i) if $\alpha\notin\mathbb{N}$, then for $n-1<[\operatorname{Re}(\alpha)]<n$, where $[\operatorname{Re}(\alpha)]$ denotes the integer part of $\operatorname{Re}(\alpha)$,

$$({}^{CH}D_a^\alpha g)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\big(\ln\frac{t}{s}\big)^{n-\alpha-1}\delta^n g(s)\frac{1}{s}\,ds;$$

(ii) if $\alpha\in\mathbb{N}$, then $({}^{CH}D_a^\alpha g)(t)=\delta^n g(t)$,

where $\ln(\cdot)=\log_e(\cdot)$.

Lemma 2.5 ([36])**.** *Let $y\in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$, and let $\alpha\in\mathbb{C}$. Then*

$$I_a^\alpha({}^{CH}D_a^\alpha y)(t)=y(t)-\sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\big(\ln\frac{t}{a}\big)^k. \tag{2.1}$$

Example 2.6 ([36])**.** Let $\operatorname{Re}(\alpha)>0$, $n=\operatorname{Re}(\alpha)+1$ and $g\in C[a,b]$.

(i) If $\operatorname{Re}(\alpha)\neq 0$ or $\alpha\in\mathbb{N}$, then

$${}^{CH}D_a^\alpha(I_a^\alpha g)(t)=g(t)\quad {}^{CH}D_b^\alpha(I_b^\alpha g)(t)=g(t)\,.$$

(ii) If $\operatorname{Re}(\alpha)\in\mathbb{N}$ and $\operatorname{Re}(\alpha)\neq 0$ then

$${}^{CH}D_a^\alpha(I_a^\alpha g)(t)=g(t)-\frac{I_a^{\alpha+1-n}g(a)}{\Gamma(n-\alpha)}\Big(\ln\frac{t}{a}\Big)^{n-\alpha}$$

3. Main results

Consider the space

$$\begin{aligned}PC(J,\mathbb{R})=\Big\{&y:J\to\mathbb{R}:y\in C((t_k,t_{k+1}],\mathbb{R}),\ k=0,\dots,m+1,\\ &\text{there exist } y(t_k^+) \text{ and } y(t_k^-) \text{ with } y(t_k^-)=y(t_k) \text{ for } k=1,\dots,m\Big\}.\end{aligned}$$

This set is a Banach space with the norm $\|y\|_{PC}=\sup_{t\in J}|y(t)|$. Set $J'=J\setminus\{t_1,\dots,t_m\}$.

Definition 3.1. A function $y\in PC(J,\mathbb{R})\cap\cup_{k=0}^m AC((t_k,t_{k+1}],\mathbb{R})$ is a solution of (1.1)-(1.3) if there exists a function $v\in L^1([a,T],\mathbb{R})$ such that $v(t)\in F(t,y(t))$ a.e. $t\in J$, for which ${}^{CH}D^\alpha y(t)=v(t)$ on J', and y also satisfies the impulsive conditions

$$\Delta y|_{t=t_k}=I_k(y(t_k^-)),\quad k=1,\dots,m,$$

and the initial condition $y(a)=y_a$.

Definition 3.2. A function $u \in PC(J,\mathbb{R}) \cap \cup_{k=0}^{m} AC((t_k,t_{k+1}],\mathbb{R})$ is said to be a lower solution of (1.1)-(1.3) if there exists a function $v_1 \in L^1([a,T],\mathbb{R})$ such that $v_1(t) \in F(t,u(t))$ a.e. $t \in J$, for which ${}^{CH}D^{\alpha}u(t) \le v_1(t)$ on J', and u also satisfies the conditions $\Delta u|_{t=t_k} \le I_k(u(t_k^-))$, $k = 1,\dots,m$, and $u(a) \le y_a$.

Similarly, a function $w \in PC(J,\mathbb{R}) \cap \cup_{k=0}^{m} AC((t_k,t_{k+1}],\mathbb{R})$ is said to be an upper solution of(1.1)-(1.3) if there exists a function $v_2 \in L^1([a,T],\mathbb{R})$ such that $v_2(t) \in F(t,w(t))$ a.e. $t \in J$, for which ${}^{CH}D^{\alpha}w(t) \ge v_2(t)$ on J' and w also satisfies the conditions $\Delta w|_{t=t_k} \ge I_k(w(t_k^-))$, $k = 1,\dots,m$, and $w(a) \ge y_a$.

To prove the existence of solutions to (1.1)-(1.3), we need the following auxiliary lemma.

Lemma 3.3. *Let $0 < \alpha < 1$ and let $\rho \in AC(J',\mathbb{R})$. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_a + \frac{1}{\Gamma(\alpha)} \int_a^t \big(\ln \frac{t}{s}\big)^{\alpha-1} \rho(s) \frac{1}{s}\, ds, & \text{if } t \in [a,t_1] \\ y_a + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \big(\ln \frac{t_i}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds \\ +\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \big(\ln \frac{t}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds + \sum_{i=1}^{k} I_i(y(t_i^-)), \\ \qquad \text{if } t \in (t_k,t_{k+1}],\ k = 1,\dots,m, \end{cases} \tag{3.1}$$

if and only if y is a solution of the impulsive fractional IVP

$$ {}^{CH}D^{\alpha}y(t) = \rho(t), \quad \text{for each } t \in J', \tag{3.2}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{3.3}$$

$$y(a) = y_a. \tag{3.4}$$

Proof. Let y be a solution of (3.2)-(3.4). Applying the Hadamard fractional integral of order α to both sides of (3.2), using conditions (3.3), (3.4) and Lemma 2.5 we obtain: If $t \in [a,t_1]$, then

$$y(t) = y_a + \frac{1}{\Gamma(\alpha)} \int_a^t \big(\ln \frac{t}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds.$$

If $t \in (t_1,t_2]$, then

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \big(\ln \frac{t}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds \\ &= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \big(\ln \frac{t}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds \\ &= y_a + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \big(\ln \frac{t_1}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \big(\ln \frac{t}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds. \end{aligned}$$

If $t \in (t_2,t_3]$, then

$$\begin{aligned} y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t \big(\ln \frac{t}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds \\ &= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t \big(\ln \frac{t}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds \\ &= y_a + I_2(y(t_2^-)) + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \big(\ln \frac{t_1}{s}\big)^{\alpha-1} \frac{\rho(s)}{s}\, ds \end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\big(\ln\frac{t_2}{s}\big)^{\alpha-1}\frac{\rho(s)}{s}\,ds+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}\big(\ln\frac{t}{s}\big)^{\alpha-1}\frac{\rho(s)}{s}\,ds.$$

If $t\in(t_k,t_{k+1}]$, by induction we obtain (3.1).

Conversely, assume that y satisfies the impulsive fractional integral equation (3.1). If $t\in[a,t_1]$, then $y(a)=y_a$ and using the fact that ${}^{CH}D_a^{\alpha}$ is the left inverse of I_a^{α}, we obtain

$$ {}^{CH}D^{\alpha}y(t)=\rho(t),\quad \text{for all } t\in[a,t_1]. $$

Next, let $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$. We have ${}^{CH}D_a^{\alpha}\kappa=0$, for any constant κ, then

$$ {}^{CH}D^{\alpha}y(t)=\rho(t),\quad \text{or all } t\in(t_k,t_{k+1}]. $$

Also, it is straightforward that $\Delta y|_{t=t_k}=I_k(y(t_k^-)$ for $k=1,\dots,m$. □

For the next theorem we use the following hypotheses:

(H1) $F:J\times\mathbb{R}\to\mathcal{P}_{cp,c}(\mathbb{R})$ is an L^1 Carathéodory multi-valued map.

(H2) There exist u and $w\in PC(J,\mathbb{R})\cap AC((t_k,t_{k+1}],\mathbb{R})$, $k=0,\dots,m$, lower and upper solutions, respectively, for problem (1.1)-(1.3) such that $u(t)\le w(t)$ for each $t\in J$.

(H3)

$$u(t_k^+)\le\min_{y\in[u(t_k^-),w(t_k^-)]}I_k(y)\le\max_{y\in[u(t_k^-),w(t_k^-)]}I_k(y)\le w(t_k^+),\ k=1,\dots,m.$$

(H4) There exists $l\in L^1(J,\mathbb{R}^+)$, such that

$$\begin{gathered}H_d(F(t,u),F(t,\bar u))\le l(t)|u-\bar u|\quad\forall u,\bar u\in\mathbb{R},\\ d(0,F(t,0))\le l(t)\quad\text{a.e. } t\in J.\end{gathered}$$

Theorem 3.4. *Under assumptions* (H1)–(H4), *problem* (1.1)-(1.3) *has at least one solution y such that*

$$u(t)\le y(t)\le w(t)\quad\forall t\in J.$$

Proof. We transform problem (1.1)-(1.3) into a fixed point problem. For $0<\alpha<1$ and $a>0$, we consider the modified problem

$$ {}^{CH}D^{\alpha}y(t)\in F(t,\tau(y(t))),\quad\text{for a.e. } t\in J=[a,T],\ t\neq t_k,\ k=1,\dots,m, \tag{3.5}$$

$$\Delta y|_{t=t_k}=I_k(\tau(t_k^-,y(t_k^-))),\quad k=1,\dots,m, \tag{3.6}$$

$$y(a)=y_a, \tag{3.7}$$

where $\tau:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is the truncation operator defined by

$$(\tau y)(t)=\begin{cases}u(t), & y(t)<u(t),\\ y(t), & u(t)\le y(t)\le w(t),\\ w(t), & y(t)>w(t).\end{cases}$$

A solution to (3.5)-(3.7) is a fixed point of the operator $N:PC(J,\mathbb{R})\to\mathcal{P}(PC(J,\mathbb{R}))$ defined by

$$\begin{aligned}N(y)=\Big\{h\in PC(J,\mathbb{R}):h(t)&=y_a+\frac{1}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\big(\ln\frac{t_k}{s}\big)^{\alpha-1}\nu(s)\frac{1}{s}\,ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\big(\ln\frac{t}{s}\big)^{\alpha-1}\nu(s)\frac{1}{s}\,ds+\sum_{a<t_k<t}I_k(\tau(t_k^-,y(t_k^-)))\Big\},\end{aligned}$$

where

$$\begin{gathered}
\nu \in \widetilde{S}^1_{F,\tau y}\{v \in S^1_{F,\tau y} : v(t) \geq v_1(t) \text{ on } A_1 \text{ and } v(t) \leq v_2(t) \text{ on } A_2\},\\
S^1_{F,\tau y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,(\tau y)(t)) \text{ for } t \in J\},\\
A_1 = \{t \in J : y(t) < u(t) \leq w(t)\},\\
A_2 = \{t \in J : u(t) \leq w(t) < y(t)\}.
\end{gathered}$$

Remark 3.5. (1) For each $y \in PC(J,\mathbb{R})$, the set $\widetilde{S}^1_{F,\tau y}$ is nonempty. In fact, (H_1) implies that there exists $v_3 \in S^1_{F,\tau y}$, so we set

$$v = v_1\chi_{A_1} + v_2\chi_{A_2} + v_3\chi_{A_3},$$

where $A_3 = \{t \in J : u(t) \leq y(t) \leq w(t)\}$. Then, by decomposability, $v \in \widetilde{S}^1_{F,\tau y}$.

(2) By the definition of τ it is clear that $F(\cdot,\tau y(\cdot))$ is an L^1- Carathéodory multi-valued map with compact convex values and there exists $\phi_1 \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,\tau y(t))\|_{\mathcal{P}} \leq \phi_1(t) \quad \text{for each } y \in \mathbb{R}.$$

(3) By the definition of τ and from (H3) we have

$$u(t_k^+) \leq I_k(\tau(t_k, y(t_k))) \leq w(t_k^+), \quad k = 1,\dots,m.$$

Set

$$\begin{gathered}
R = |y_a| + \frac{\|\phi_1\|_{L_1}}{\Gamma(\alpha+1)}\sum_{k=1}^m \big(\ln\frac{t_k}{t_{k-1}}\big)^\alpha + \frac{\|\phi_1\|_{L_1}}{\Gamma(\alpha+1)}\big(\ln\frac{T}{a}\big)^\alpha\\
+ \sum_{k=1}^m \max\{|u(t_k^+)|,|w(t_k^+|\}\\
B = \{y \in PC(J,\mathbb{R}) : \|y\|_{PC} \leq R\}.
\end{gathered}$$

Clearly B is a closed convex subset of $PC(J,\mathbb{R})$ and that N maps B into B. We shall show that B satisfies the assumptions of Lemma 2.2. The proof will be given in several steps.

Step 1: $N(y)$ is convex for each $y \in B$. Let h_1, h_2 belong to $N(y)$. Then there exist $\nu_1, \nu_2 \in \widetilde{S}^1_{F,\tau y}$ such that for each $t \in J$ we have

$$\begin{aligned}
h_i(t) &= y_a + \frac{1}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\big(\ln\frac{t_k}{s}\big)^{\alpha-1}\nu_i(s)\frac{1}{s}\,ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t\big(\ln\frac{t}{s}\big)^{\alpha-1}\nu_i(s)\frac{1}{s}\,ds + \sum_{a<t_k<t} I_k(\tau t_k^-, y(t_k^-))), \quad i = 1,2.
\end{aligned}$$

Let $0 \leq \lambda \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned}
&(\lambda h_1 + (1-\lambda)h_2)(t)\\
&= y_a + \frac{1}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\big(\ln\frac{t_k}{s}\big)^{\alpha-1}[\lambda\nu_1 + (1-\lambda)\nu_2](s)\frac{1}{s}\,ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t\big(\ln\frac{t}{s}\big)^{\alpha-1}[\lambda\nu_1 + (1-\lambda)\nu_2](s)\frac{1}{s}\,ds + \sum_{a<t_k<t} I_k(\tau t_k^-, y(t_k^-)))
\end{aligned}$$

Since $\widetilde{S}_{F,y}$ is convex (because F has convex values), we have $\lambda h_1 + (1-\lambda)h_2 \in N(y)$.

Step 2: $N(B)$ **is bounded.** For each $h \in N(y)$ and $y \in B$, there exists $\nu \in \widetilde{S}^1_{F,y}$ such that

$$\begin{aligned} h(t) &= y_a + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \big(\ln \frac{t_k}{s}\big)^{\alpha-1} \nu(s) \frac{1}{s}\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \big(\ln \frac{t}{s}\big)^{\alpha-1} \nu(s) \frac{1}{s}\, ds + \sum_{a<t_k<t} I_k(\tau t_k^-, y(t_k^-))). \end{aligned}$$

By (H1)–(H3), for each $t \in J$, we have

$$\begin{aligned} |h(t)| &\le |y_a| + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \big(\ln \frac{t_k}{s}\big)^{\alpha-1} |\nu(s)| \frac{1}{s}\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \big(\ln \frac{t}{s}\big)^{\alpha-1} |\nu(s)| \frac{1}{s}\, ds + \sum_{a<t_k<t} |I_k(\tau t_k^-, y(t_k^-)))| \\ &\le |y_a| + \frac{\|\phi_1\|_{L_1}}{\Gamma(\alpha+1)} \sum_{k=1}^{m} \big(\ln \frac{t_k}{t_{k-1}}\big)^{\alpha} + \frac{\|\phi_1\|_{L_1}}{\Gamma(\alpha+1)} \big(\ln \frac{T}{a}\big)^{\alpha} \\ &\quad + \sum_{k=1}^{m} \max\{|u(t_k^+)|, |w(t_k^+|\}. \end{aligned}$$

Therefore $\|h\|_\infty \le R$.

Step 3: $N(B)$ **is equicontinuous.** Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, Let $y \in B$ and $h \in N(y)$. Then

$$\begin{aligned} &|h(t_2) - h(t_1)| \\ &= \frac{1}{\Gamma(\alpha)} \sum_{\tau_1<t_k<\tau_2} \int_{t_{k-1}}^{t_k} \big(\ln \frac{t_k}{s}\big)^{\alpha-1} |\nu(s) \frac{1}{s}\, ds + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \big(\ln \frac{\tau_2}{s}\big)^{\alpha-1} |\nu(s)| \frac{1}{s}\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\tau_1} \big[\big(\ln \frac{\tau_2}{s}\big)^{\alpha-1} - \big(\ln \frac{\tau_1}{s}\big)^{\alpha-1}\big] |\nu(s)| \frac{1}{s}\, ds \\ &\quad + \sum_{\tau_1<t_k<\tau_2} |I_k \tau y(t_k^-, t_k^-)))| \\ &\le \frac{\|\phi_1\|_{L_1}}{\Gamma(\alpha)} \sum_{\tau_1<t_k<\tau_2} \int_{t_{k-1}}^{t_k} \big(\ln \frac{t_k}{s}\big)^{\alpha-1} \frac{1}{s}\, ds + \frac{\|\phi_1\|_{L_1}}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \big(\ln \frac{\tau_2}{s}\big)^{\alpha-1} \frac{1}{s}\, ds \\ &\quad + \frac{\|\phi_1\|_{L_1}}{\Gamma(\alpha)} \int_{t_k}^{\tau_1} \big[\big(\ln \frac{\tau_2}{s}\big)^{\alpha-1} - \big(\ln \frac{\tau_1}{s}\big)^{\alpha-1}\big] \frac{1}{s}\, ds \\ &\quad + \sum_{\tau_1<t_k<\tau_2} I_k |\tau(t_k^-, y(t_k^-))))|. \end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : B \to \mathcal{P}(B)$ is completely continuous.

Step 4: N **has a closed graph.** Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $\nu_n \in \widetilde{S}^1_{F,\tau y_n}$, such that, for each $t \in J$,

$$\begin{aligned} h_n(t) &= y_a + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \big(\ln \frac{t_k}{s}\big)^{\alpha-1} \nu_n(s) \frac{1}{s}\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \big(\ln \frac{t}{s}\big)^{\alpha-1} \nu_n(s) \frac{1}{s}\, ds + \sum_{a<t_k<t} I_k \tau(t_k^-, y(t_k^-)))). \end{aligned}$$

We must show that there exists $\nu_* \in \widetilde{S}^1_{F,\tau y_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) &= y_a + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \big(\ln \frac{t_k}{s}\big)^{\alpha-1} \nu_*(s) \frac{1}{s}\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \big(\ln \frac{t}{s}\big)^{\alpha-1} \nu_*(s) \frac{1}{s}\, ds + \sum_{a<t_k<t} I_k \tau(t_k^-, y(t_k^-)))). \end{aligned}$$

Since $F(t,\cdot)$ is upper semi-continuous, for every $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n \geq n_0$, we have

$$\nu_n(t) \in F(t, \tau y_*(t)) + \epsilon B(0,1), \quad , \text{a.e. } t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence $\nu_{n_m}(\cdot)$ such that

$$\begin{gathered} \nu_{n_m}(\cdot) \to \nu_*(\cdot) \quad \text{as } m \to \infty, \\ \nu_*(t) \in F(t, \tau y_*(t)), \quad \text{a.e. } t \in J. \end{gathered}$$

For every $w \in F(t, \tau y_*(t))$, we have

$$|\nu_{n_m}(t) - \nu_*(t)| \leq |\nu_{n_m}(t) - w| + |w - \nu_*(t)|.$$

Then $|\nu_{n_m}(t) - \nu_*(t)| \leq d(\nu_{n_m}(t), F(t, \tau y_*(t)).$

We obtain an analogous relation by interchanging the roles of ν_{n_m} and ν_*, and it follows that

$$|\nu_{n_m}(t) - \nu_*(t)| \leq H_d(F(t, \tau y_n(t)), F(t, \tau y_*(t))) \leq l(t) \|y_n - y_*\|_\infty.$$

Then

$$\begin{aligned} |h_{n_m}(t) - h_*(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \big(\ln \frac{t_k}{s}\big)^{\alpha-1} |\nu_{n_m}(s) - \nu_*(s)| \frac{1}{s}\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \big(\ln \frac{t}{s}\big)^{\alpha-1} |\nu_{n_m}(s) - \nu_*(s)| \frac{1}{s}\, ds \\ &\quad + \sum_{a<t_k<t} |I_k(\tau(t_k^-, y_{n_m}(t_k^-)))) - I_k(\tau(t_k^-, (y_*(t_k^-)))| \\ &\leq \frac{m}{\Gamma(\alpha+1)} \big(\ln \frac{T}{a}\big)^{\alpha} \int_a^T l(s) ds \|y_{n_m} - y_*\|_\infty \\ &\quad + \frac{1}{\Gamma(\alpha+1)} \big(\ln \frac{T}{a}\big)^{\alpha} \int_a^T l(s) ds \|y_{n_m} - y_*\|_\infty \\ &\quad + \sum_{a<t_k<t} |I_k(\tau(t_k^-, y_{n_m}(t_k^-)))) - I_k(\tau(t_k^-, y_*(t_k^-)))|. \end{aligned}$$

Hence

$$\|h_{n_m}(t) - h_*(t)\|_\infty \leq \frac{m+1}{\Gamma(\alpha+1)} \big(\ln \frac{T}{a}\big)^{\alpha} \int_a^T l(s) ds \|y_{n_m} - y_*\|_\infty$$

$$+\sum_{a<t_k<t}|I_k(\tau(t_k^-,y_{n_m}(t_k^-)))) - I_k(y_*\tau(t_k^-,t_k^-)))| \to 0.$$

as $m\to\infty$.

Step 5: The solution y of (3.5)-(3.7) satisfies $u(t)\le y(t)\le w(t)$ for all $t\in J$. Let y be the above solution to (3.5)-(3.7). We prove that

$$y(t)\le w(t)\quad \forall t\in J.$$

Assume that $y-w$ attains a positive maximum on $[t_k^+,t_{k+1}^-]$ at $\bar{t}_k\in[t_k^+,t_{k+1}^-]$ for some $k=0,\dots,m$; that is

$$(y-w)(\bar{t}_k)=\max\{y(t)-w(t): t\in[t_k^+,t_{k+1}^-]\}>0,\ \text{for some } k=0,\dots,m.$$

We distinguish the following cases.

Case 1. If $\bar{t}_k\in(t_k^+,t_{k+1}^-)$ then there exists $t_k^*\in(t_k^+,t_{k+1}^-)$ such that

$$y(t_k^*)-w(t_k^*)\le 0, \tag{3.8}$$

$$y(t)-w(t)>0,\quad \forall t\in(t_k^*,\bar{t}_k]. \tag{3.9}$$

From the definition of τ, we have

$${}^{CH}D^\alpha y(t)\in F(t,w(t))\quad \forall t\in[t_k^*,\bar{t}_k].$$

An integration on $[t_k^*,\bar{t}]$ yields

$$y(t)-y(t_k^*)=\frac{1}{\Gamma(\alpha)}\int_{t_k^*}^t\big(\ln\frac{t}{s}\big)^{\alpha-1}\nu(s)\frac{1}{s}\,ds, \tag{3.10}$$

where $\nu(t)\in F(t,w(t))$. From (3.10) and using the fact that w is an upper solution to (1.1)-(1.3) we obtain

$$y(t)-y(t_k^*)\le w(t)-w(t_k^*). \tag{3.11}$$

From (3.8),(3.9) and (3.11) we obtain the contradiction

$$0<y(t)-w(t)\le y((t_k^*)-w((t_k^*)\le 0,\quad \forall t\in[t_k^*,\bar{t}_k].$$

Case 2. If $\bar{t}_k=t_k^+$, $k=1,\dots,m$, then

$$w(t_k^+)<I_k(\tau(t_k^-,y(t_k^-)))<w(t_k^+),$$

which is a contradiction. Thus $y(t)\le w(t)$ for all $t\in[a,T]$.

Analogously, we can prove that $y(t)\ge u(t)$ for all $t\in[a,T]$. This shows that problem (3.5)-(3.7) has a solution in the interval $[u,w]$ which is a solution of (1.1)-(1.3). ☐

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