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# STABLE MANIFOLDS FOR IMPULSIVE DELAY EQUATIONS AND PARAMETER DEPENDENCE

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Abstract. In this article, we establish the existence of Lipschitz stable invariant manifolds for the semiflows generated by the delay differential equation $x' = L(t)x_t + f(t, x_t, \lambda)$ with impulses at times $\{\tau_i\}_{i=1}^{\infty}$, assuming that the perturbation $f(t, x_t, \lambda)$ as well as the impulses are small and the corresponding linear delay differential equation admits a nonuniform exponential dichotomy. We also show that the obtained manifolds are Lipschitz in the parameter $\lambda$.

## 1. Introduction

In the modern theory of dynamical systems, the invariant manifold theory plays an essential role in the study of qualitative behavior of the dynamics. According to the classical formulation, the existence of a uniform exponential dichotomy, introduced by Perron [11], for linear differential equations is sufficient to ensure the existence of invariant manifolds for semiflows generated by the linear differential equations. In 1976, Pesin [12] obtained a smooth invariant manifold for uniformly hyperbolic trajectories. It is important to note that the existence of uniform exponential dichotomy neglects the possibility of the dependence of norm of the solution on the initial time. While this dependency is quite natural as almost all trajectories with nonzero Lyapunov exponents of a smooth dynamical system preserving a finite invariant measure have a linear variational equation for which such dependency occurs. Therefore Oseledets [10], Pesin [13] as well as other experts in this field, were considering the more general type of hyperbolic behavior. Luis Barreira and Claudia Valls introduced the notion of *nonuniform exponential dichotomy* in [3] and they obtained a smooth invariant manifold for a nonautonomous equation. Their further work can be seen in their book on stability of nonautonomous differential equation [4].

Differential equations involving impulsive effects are seen as a natural description of observed evolution phenomenon of several real world problems. In recent times, impulsive dynamical systems have also described many applications to the real world as elaborated in [8]. Similarly, delay differential equations have also been used to interpret many physical models. Continuing the work started by Luis Barreira and Claudia Valls in [1, 7], we are considering an impulsive delay differential

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equation,

$$x' = L(t)x_t + f(t, x_t, \lambda) \quad t \neq \tau_i, \quad x_s = \phi, \quad \Delta x(\tau_i) = I_i(x_{\tau_i}, \lambda), \tag{1.1}$$

with impulses at times $s < \tau_i < \tau_{i+1} < t$ for $i \in \mathbb{N}$. We will construct a Lipschitz stable invariant manifold for the semiflows generated by the differential equation (1.1), assuming the existence of a nonuniform exponential dichotomy for the corresponding linear differential equation. We also establish that the stable manifold is Lipschitz in the parameter.

According to the theory of nonuniform hyperbolic dynamics for a dynamical system, if the Lyapunov exponents are nonzero then the system follows nonuniform exponential behavior (showed by Y. Pesin) [2]. Therefore our work applies to all such impulsive systems and can also be considered as a contribution to the nonuniform hyperbolic dynamics. Our result helps in the further development of the geometric theory of impulsive dynamical systems. Primarily this article is inspired by [1, 7, 6] in which author obtained Lipschitz stable manifolds for the differential equation $x' = L(t)x_t + f(t, x_t, \lambda)$ with the assumption of a nonuniform exponential dichotomy. Due to the existence of infinite impulses, our main task is to control the bounds of the solutions disturbed by the impulses. We are going to do so by assuming sufficiently fast decay of the impulses in time.

This article is divided into three sections. In the first section, some preliminary results are given. In the second section, we are proving the stability result for the solution of the impulsive system with delay by assuming contraction on the solution operator and in the final section, we establish invariant manifolds under the assumption of a nonuniform exponential dichotomy.

## 2. Preliminaries

Given $r > 0$, let $\mathcal{B} = C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous functions, $\phi : [-r, 0] \to \mathbb{R}^n$ with the norm

$$\|\phi\| = \sup_{\theta \in [-r,0]} |\phi(\theta)|, \tag{2.1}$$

where $|\cdot|$ is the norm in $\mathbb{R}^n$. Now let us consider a set $\hat{\mathcal{B}}$ of all functions, $\phi : [-r, 0] \to \mathbb{R}^n$, such that for each $s \in (-r, 0)$ the limits $\lim_{\theta \to s^-} \phi(\theta)$ and $\lim_{\theta \to s^+} \phi(\theta)$ exist and $\lim_{\theta \to s^+} \phi(\theta) = \phi(s)$. One can easily show that $\hat{\mathcal{B}}$ is a Banach space with the norm $\| \cdot \|$ given in (2.1).

For $s \geq 0$ and $\phi \in \mathcal{B}$, consider the initial value problem

$$x' = L(t)x_t, \quad x_s = \phi, \tag{2.2}$$

where $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$, $t \geq s$, and $L(t) : \mathcal{B} \to \mathbb{R}^n$ are linear operators for each $t \geq 0$. We assume that the map $(t, x) \mapsto L(t)x$ is continuous and there exists $k > 0$ such that

$$\int_t^{t+r} \|L(\tau)\| d\tau \leq k(1 + t),$$

for all $t \geq 0$.

Given $a > s$, a continuous function $x : [s - r, a] \to \mathbb{R}^n$ is said to be a weak solution of (2.2) if $x_s = \phi$ and the integral

$$x(t) = \phi(0) + \int_s^t L(\tau)x_\tau d\tau, \tag{2.3}$$

holds for $t \in [s, a)$. With above assumptions on the operators $L(t)$, for each $(s, \phi) \in \mathbb{R}^+ \times \mathcal{B}$, there exists a unique global solution $[s-r, +\infty) \ni t \mapsto x_t(\cdot, s, \phi)$ of the initial value problem (2.2)[9]. Now let $T(t,s) : \mathcal{B} \to \mathcal{B}$ be the evolution operator associated with equation (2.2) defined by

$$T(t,s)\phi = x_t(\cdot, s, \phi), \quad t \geq s. \tag{2.4}$$

To extend $T(t,s)$ to the space $\hat{\mathcal{B}}$, we write $L(t)$ in the form

$$L(t)\phi = \int_{-r}^{0} d_\theta[\eta(t,\theta)]\phi(\theta), \tag{2.5}$$

for some $n \times n$ matrices $\eta(\theta)$ that are measurable in $(t, \theta) \in \mathbb{R}^+ \times [-r, 0]$ and continuous from the left in $\theta$. Setting $m_\eta(t) = \mathrm{Var}\ \eta(t)$, where Var denotes the total variation in $[-r, 0]$ and we have $\|L(t)\| = m_\eta(t)$. Each linear operator $L(t)$ can be extended to $\hat{\mathcal{B}}$ using the integral in (2.5), provided that the Reimann-Stieltjes sums take the value $[\eta(t,b) - \eta(t,a)]\phi(b^-)$ for each sub-interval $[a,b] \subset [-r, 0]$. (so that the points at which both $\phi$ and $\eta(t, \cdot)$ have discontinuities cause no problems)

Moreover, for each $(s, \phi) \in \mathbb{R}^+ \times \hat{\mathcal{B}}$, there is a unique solution $t \mapsto x_t(\cdot, s, \phi) \subset \hat{\mathcal{B}}$ of the integral equation (2.3) with $x_s = \phi$ [9]. The corresponding evolution operator $\hat{T}(t,s) : \hat{\mathcal{B}} \to \hat{\mathcal{B}}$ is defined by

$$\hat{T}(t,s)\phi = x_t(\cdot, s, \phi), \quad t \geq s.$$

We note that $\hat{T}(t,s)\big|_{\mathcal{B}} = T(t,s)$ and $\hat{T}(t,s)\big|_{\hat{\mathcal{B}}} \subset \mathcal{B}$ for any $t \geq s + r$. The extension $\hat{T}(t,s)$ of the evolution operator $T(t,s)$ to the space $\hat{\mathcal{B}}$ is needed to write the variation of parameters formula in the space $\hat{\mathcal{B}}$.

We consider the impulsive functional differential equation with state dependent delay,

$$\begin{gathered} x' = L(t)x_t + f(t, x_t, \lambda); \quad x_s = \phi; \\ \Delta x(\tau_i) = I_i(x_{\tau_i}, \lambda), \quad s \leq \tau_1 < \cdots < \tau_i < \tau_{i+1} \dots, \end{gathered} \tag{2.6}$$

for some perturbation $f : \mathbb{R}^+ \times \hat{\mathcal{B}} \times Y \to \mathbb{R}^n$, where $Y = (Y, |\cdot|)$ is an open subset of $\mathbb{R}$ (the parameter space), $\phi \in \hat{\mathcal{B}}$. $I_i : \hat{\mathcal{B}} \times Y \to \mathbb{R}^n$, for $i \in \mathbb{N}$ are appropriate functions and the symbol $\Delta\xi(t)$ represents the jump of the function $\xi$ at $t$, which is defined by $\Delta\xi = \xi(t^+) - \xi(t^-)$. Impulses are introduced at times $\{\tau_i\}_{i=1}^{\infty}$ satisfying $s < \tau_i \leq \tau_{i+1}$ and $\lim_{i \to +\infty} \tau_i \to +\infty$, with

$$p := \sup_{t > s > 0} \frac{\mathrm{card}\{i \in \mathbb{N} : s \leq \tau_i < t\}}{t - s} < \infty. \tag{2.7}$$

Further we assume that

(i) $f(t, 0, \lambda) = 0$ for $t \geq 0$ and $\lambda \in Y$;
(ii) there exist constants $c, q > 0$ such that for $\phi, \psi \in \hat{B}$ and $\lambda, \mu \in Y$,

$$|f(t,\phi,\lambda) - f(t,\psi,\lambda)| \leq c\|\phi - \psi\| \left(\|\phi\|^q + \|\psi\|^q\right), \tag{2.8}$$

$$|f(t,\phi,\lambda) - f(t,\phi,\mu)| \leq c|\lambda - \mu|\, \|\phi\|^{q+1}. \tag{2.9}$$

Similarly,

(iii) $I_i(0, \lambda) = 0$ for $\lambda \in$Y and $i = 1, 2, \dots$.
(iv) Given $\epsilon > 0$ there exists a positive constant $\delta$ such that for each $t \geq 0, i \in \mathbb{N}$, $\xi_1, \xi_2 \in \hat{\mathcal{B}}$ and $\lambda, \mu \in Y$,

$$|I_i(\xi_1, \lambda) - I_i(\xi_2, \lambda)| \leq \delta e^{-3\epsilon\tau_i}\|\xi_1 - \xi_2\|, \tag{2.10}$$

$$|I_i(\xi,\lambda)-I_i(\xi,\mu)| \le \delta e^{-3\epsilon\tau_i}|\lambda-\mu|\ \|\xi\|. \tag{2.11}$$

## 3. Stability with nonuniform contractions

In this section we study the persistence of the stability when there is a *nonuniform contraction*. We say that equation (2.2) admits a nonuniform exponential contraction if there are constants $a<0, \epsilon \ge 0$ and $K>1$ such that

$$\|\hat{T}(t,s)\| \le Ke^{a(t-s)+\epsilon s}, \tag{3.1}$$

for every $t \ge s \ge 0$. Moreover, equation (2.2) is said to admit a uniform exponential contraction if it admits a nonuniform exponential contraction with $\epsilon = 0$.

The following is a result for contractions. Let $\beta = \epsilon(1+\frac{1}{q})$ and for a given $\nu > 0$, we set

$$r_\nu = \sup_{s>0} \sum_{\tau_i \ge s} e^{-\nu(\tau_i - s)}.$$

**Theorem 3.1.** *Let $K' > K$ be fixed. Let equation* (2.2) *admits a nonuniform exponential contraction with $aq+\epsilon<0$ and $r_\epsilon < \infty$. Also we suppose that the constants $c$ and $\delta$ in* (2.8) *and* (2.10) *are sufficiently small. Then, for each $(s,\phi) \in \mathbb{R}^+ \times \hat{\mathcal{B}}$ with $\|\phi\| \le e^{-\beta s}$, the solution $x_t^\lambda$ of differential equation* (2.6) *with impulse satisfies,*

$$\|x_t^\lambda - x_t^\mu\| \le K' e^{a(t-s)+\epsilon s}|\lambda-\mu|\ \|\phi\|, \tag{3.2}$$

*for $t \ge s$ and $\lambda, \mu \in$ Y.*

*Proof.* For $t \in [s, t_1)$, the solution of the equation (2.6) is given by [9]

$$x_t = \hat{T}(t,s)\phi + \int_s^t \hat{T}(t,\tau)X_0 f(\tau, x_\tau, \lambda)d\tau, \tag{3.3}$$

where

$$X_0(\theta) = \begin{cases} 0, & -r \le \theta < 0, \\ \mathrm{Id}, & \theta = 0. \end{cases}$$

For each $\lambda \in$ Y, the solution given above can also be written as

$$x(t) = \hat{T}(t,s)\,\phi(0) + \int_s^t \hat{T}(t,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(0)\,d\tau, \quad t \in [s,\tau_1).$$

Therefore,

$$x(\tau_1^-) = \hat{T}(\tau_1,s)\phi(0) + \int_s^{\tau_1} \hat{T}(\tau_1,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(0)d\tau.$$

Using the condition $x(\tau_1^+) = x(\tau_1^-) + I_1(x_{\tau_1},\lambda)$, for $t \in [\tau_1,\tau_2)$, we have

$$\begin{aligned} x(t) &= \hat{T}(t,\tau_1)x(\tau_1^+) + \int_{\tau_1}^t \hat{T}(t,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(0)d\tau \\ &= \hat{T}(t,\tau_1)[x(\tau_1^-) + I_1(x_{\tau_1},\lambda)] + \int_{\tau_1}^t \hat{T}(t,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(0)d\tau \\ &= \hat{T}(t,\tau_1)\Big[\hat{T}(\tau_1,s)\phi(0) + \int_s^{\tau_1} \hat{T}(\tau_1,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(0)d\tau + I_1(x_{\tau_1},\lambda)\Big] \\ &\quad + \int_{\tau_1}^t \hat{T}(t,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(0)d\tau \end{aligned}$$

$$
\begin{aligned}
&= \hat{T}(t,s)\phi(0) + \hat{T}(t,\tau_1)\Big[\int_s^{\tau_1} \hat{T}(\tau_1,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(0)d\tau\Big] \\
&\quad + \hat{T}(t,\tau_1)I_1(x_{\tau_1},\lambda) + \int_{\tau_1}^t \hat{T}(t,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(0)d\tau \\
&= \hat{T}(t,s)\phi(0) + \int_s^t \hat{T}(t,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(0)\,d\tau + \hat{T}(t,\tau_1)I_1(x_{\tau_1},\lambda).
\end{aligned}
$$

By repeating this process we obtain

$$
x_t = \hat{T}(t,s)\phi + \int_s^t \hat{T}(t,\tau)X_0 f(\tau,x_\tau,\lambda)d\tau + \sum_{s<\tau_i<t} \hat{T}(t,\tau_i)I_i(x_{\tau_i},\lambda), \quad t \ge s. \tag{3.4}
$$

Let $\Omega$ be the set of all functions $x : [s-r,+\infty) \to \mathbb{R}^n$ such that $x_s = \phi,\ x_t \in \hat{\mathcal{B}}$ for every $t \ge s$, and $\|x\|_* \le e^{-\beta s}$, where

$$
\|x\|_* = \frac{1}{2K}\sup\Big\{\frac{\|x_t\|}{e^{a(t-s)+\epsilon s}} : t \ge s\Big\}. \tag{3.5}
$$

One can easily verify that $\Omega$ is a complete metric space with the norm given by (3.5). Now for each $\lambda \in \mathrm{Y}$, define an operator $S^\lambda$ on $\Omega$ by

$$
\begin{aligned}
S^\lambda x(t+\theta) &= [\hat{T}(t,s)\phi](\theta) + \int_s^t \hat{T}(t,\tau)\left(X_0 f(\tau,x_\tau,\lambda)\right)(\theta)d\tau \\
&\quad + \sum_{s<\tau_i<t} \hat{T}(t,\tau_i)I_i(x_{\tau_i},\lambda)(\theta).
\end{aligned}
$$

Note that for each $\phi \in \Omega$ we have $(S^\lambda x)_s = \phi$ and $(S^\lambda x)_t \in \hat{\mathcal{B}}$ for every $t \ge s$. Using (2.8),(2.10) and (3.1) in the last equation, we have

$$
\begin{aligned}
&\|S^\lambda x(t+\theta)\| \\
&\le \|\hat{T}(t,s)\phi\| + \int_s^t \|\hat{T}(t,\tau)\|\,|f(\tau,x_\tau,\lambda)|d\tau + \sum_{s<\tau_i<t} \|\hat{T}(t,\tau_i)\|\ |I_i(x_{\tau_i},\lambda)| \\
&\le Ke^{a(t-s)+\epsilon s}\|\phi\| + cK\int_s^t e^{a(t-\tau)+\epsilon\tau}\|x_\tau\|^{q+1}d\tau \\
&\quad + \sum_{s<\tau_i<t} Ke^{a(t-\tau_i)+\epsilon\tau_i}\delta e^{-3\epsilon\tau_i}\|x_{\tau_i}\| \\
&\le Ke^{a(t-s)+\epsilon s}\|\phi\| + 2K^2\delta\|x\|_* \sum_{s<\tau_i<t} e^{a(t-s)+\epsilon s}e^{-2\epsilon\tau_i} \\
&\quad + 2^{q+1}cK^{q+2}\|x\|_*^{q+1}\int_s^t e^{a(t-\tau)+\epsilon\tau}e^{a(q+1)(\tau-s)}e^{\epsilon(q+1)s}d\tau \\
&\le Ke^{a(t-s)+\epsilon s}\|\phi\| + \{2Ke^{a(t-s)+\epsilon s}\}\delta K\|x\|_* \sum_{s<\tau_i<t} e^{-2\epsilon\tau_i} \\
&\quad + 2^{q+1}cK^{q+2}\|x\|_*^{q+1}e^{a(t-s)+\epsilon s}e^{(-a+\epsilon)sq}\int_s^t e^{(\epsilon+aq)\tau}d\tau,
\end{aligned}
$$

$$
\|(S^\lambda x)_t\| \le 2Ke^{a(t-s)+\epsilon s}\Big\{\frac{1}{2}\|\phi\| + \frac{1}{2}ve^{\epsilon(q+1)s}\|x\|_*^{q+1} + \delta K\|x\|_* \sum_{s<\tau_i<t} e^{-2\epsilon\tau_i}\Big\},
$$

where $v = c(2K)^{q+1}/|aq+\epsilon|$. Hence,

$$\begin{aligned} \|S^\lambda x\|_* &\le \frac{1}{2}\Big(\|\phi\| + ve^{(\epsilon q+\epsilon-\beta q)s}\|x\|_* + 2\delta K r_{2\epsilon}\|x\|_*\Big) \\ &\le \frac{1}{2}\left(1 + v + 2\delta K r_{2\epsilon}\right) e^{-\beta s}, \end{aligned} \tag{3.6}$$

provided that $c$ and $\delta$ are sufficiently small, therefore $v + 2\delta K r_{2\epsilon} < 1$. Hence we conclude that $S^\lambda(\Omega) \subset \Omega$. Furthermore, for each $x, y \in \Omega$, we have

$$\begin{aligned} &\|(S^\lambda x)_t - (S^\lambda y)_t\| \\ &\le \int_s^t \|\hat{T}(t,\tau)\||f(\tau,x_\tau,\lambda) - f(\tau,y_\tau,\lambda)|d\tau + \sum_{s<\tau_i<t} \|\hat{T}(t,\tau_i)\||I_i(x_{\tau_i},\lambda) - I_i(y_{\tau_i},\lambda)| \\ &\le cK\int_s^t e^{a(t-\tau)+\epsilon s}\|x_\tau - y_\tau\|(\|x_\tau\|^q + \|y_\tau\|^q)d\tau \\ &\quad + \sum_{s<\tau_i<t} Ke^{a(t-\tau_i)+\epsilon\tau_i}\delta e^{-3\epsilon\tau_i}\|x_{\tau_i} - y_{\tau_i}\| \\ &\le c(2K)^{q+1}\int_s^t e^{a(t-\tau)+\epsilon\tau}e^{aq(\tau-s)}e^{(\epsilon q-\beta q)s}\|x_\tau - y_\tau\|d\tau \\ &\quad + \sum_{s<\tau_i<t} Ke^{a(t-\tau_i)+\epsilon\tau_i}\delta e^{-3\epsilon\tau_i}\|x_{\tau_i} - y_{\tau_i}\| \\ &\le c(2K)^{q+2}e^{a(t-s)}e^{(\epsilon+\epsilon q-\beta q-aq)s}\|x-y\|_*\int_s^t e^{(\epsilon+aq)\tau}d\tau \\ &\quad + 2\delta K^2\sum_{s<\tau_i<t} e^{a(t-\tau_i)+\epsilon\tau_i}e^{a(\tau_i-s)+\epsilon s}e^{-3\epsilon\tau_i}\|x-y\|_* \\ &\le 2Kve^{a(t-s)+\epsilon s}e^{(\epsilon+\epsilon q-\beta q)s}\|x-y\|_* + 2\delta K^2e^{a(t-s)+\epsilon s}\|x-y\|_*\sum_{s<\tau_i<t} e^{-2\epsilon\tau_i} \\ &\le 2Ke^{a(t-s)+\epsilon s}\left\{v + \delta K r_{2\epsilon}\right\}\|x-y\|_*. \end{aligned}$$

This gives

$$\|S^\lambda x - S^\lambda y\|_* \le \{v + \delta K r_{2\epsilon}\}\|x-y\|_*. \tag{3.7}$$

Since $v + \delta K r_{2\epsilon} < 1$, $S^\lambda$ is a contraction map in $\Omega$. Therefore for each $\lambda \in \mathrm{Y}$, there exists a unique function $x = x^\lambda \in \Omega$ such that $S^\lambda x = x$. Moreover, using (3.6) we have

$$\begin{aligned} \|x\|_* &\le \frac{1}{2}\|\phi\| + \frac{v}{2}e^{(\epsilon q+\epsilon-\beta q)s}\|x\|_* + \delta K r_{2\epsilon}\|x\|_* \\ &\le \frac{1}{2}\|\phi\| + \Big(\frac{v + 2\delta K r_{2\epsilon}}{2}\Big)\|x\|_*. \end{aligned}$$

Since $v + 2\delta K r_{2\epsilon} \le 1$, we obtain

$$\|x_t^\lambda\| \le 2Ke^{a(t-s)+\epsilon s}\|\phi\|, \quad t \ge s. \tag{3.8}$$

Now we establish (3.2), using (2.8),(2.9) and (3.8). First we obtain

$$
\begin{aligned}
&|f(\tau, x_\tau^\lambda, \lambda) - f(\tau, x_\tau^\mu, \mu)| \\
&\le |f(\tau, x_\tau^\lambda, \lambda) - f(\tau, x_\tau^\lambda, \mu)| + |f(\tau, x_\tau^\lambda, \mu) - f(\tau, x_\tau^\mu, \mu)| \\
&\le c|\lambda - \mu|\, \|x_\tau^\lambda\|^{q+1} + c\|x_\tau^\lambda - x_\tau^\mu\| \left(\|x_\tau^\lambda\|^q + \|x_\tau^\mu\|^q\right) \\
&\le c(2K)^{q+1} e^{a(q+1)(\tau - s)} e^{[q(\epsilon-\beta)+\epsilon]s} |\lambda - \mu|\, \|\phi\| \\
&\quad + 2c(2K)^q e^{aq(\tau-s)} e^{q(\epsilon-\beta)s} \|x_\tau^\lambda - x_\tau^\mu\| \\
&\le c(2K)^{q+1} e^{a(q+1)(\tau - s)} e^{[q(\epsilon-\beta)+\epsilon]s} \left\{|\lambda - \mu|\, \|\phi\| + 2\|x^\lambda - x^\mu\|_*\right\}.
\end{aligned}
\tag{3.9}
$$

Similarly,

$$
\begin{aligned}
&|I_i(x_{\tau_i}^\lambda, \lambda) - I_i(x_{\tau_i}^\mu, \mu)| \\
&\le |I_i(x_{\tau_i}^\lambda, \lambda) - I_i(x_{\tau_i}^\lambda, \mu)| + |I_i(x_{\tau_i}^\mu, \mu) - I_i(x_{\tau_i}^\lambda, \mu)| \\
&\le \delta e^{-3\epsilon\tau_i} |\lambda - \mu| \|x_{\tau_i}^\lambda\| + \delta e^{-3\epsilon\tau_i} \|x_{\tau_i}^\lambda - x_{\tau_i}^\mu\| \\
&\le 2\delta K e^{-3\epsilon\tau_i} e^{a(\tau_i - s)+\epsilon s} \left(|\lambda - \mu| \|x^\lambda\|_* + \|x^\lambda - x^\mu\|_*\right).
\end{aligned}
\tag{3.10}
$$

Using the inequalities given above, we have

$$
\begin{aligned}
&\|x_t^\lambda - x_t^\mu\| \\
&\le \int_s^t \|\hat{T}(t,\tau)\|\, |f(\tau, x_\tau^\lambda, \lambda) - f(\tau, x_\tau^\mu, \mu)| d\,\tau \\
&\quad + \sum_{s<\tau_i<t} \|\hat{T}(t,\tau_i)\|\, |I_i(x_{\tau_i}^\lambda, \lambda) - I_i(x_{\tau_i}^\mu, \mu)| \\
&\le c(2K)^{q+1} \{|\lambda - \mu|\, \|\phi\| + 2\|x^\lambda - x^\mu\|_*\} \\
&\quad \times \int_s^t K e^{a(t-\tau)+\epsilon\tau} e^{a(q+1)(\tau-s)} e^{[q(\epsilon-\beta)+\epsilon]s} d\tau \\
&\quad + 2\delta K \left(|\lambda - \mu| \|x^\lambda\|_* + \|x^\lambda - x^\mu\|_*\right) \sum_{s<\tau_i<t} K e^{a(t-\tau_i)-2\epsilon\tau_i} e^{a(\tau_i - s)+\epsilon s}
\end{aligned}
$$

$$
\begin{aligned}
\|x^\lambda - x^\mu\|_* &\le \frac{v}{2} |\lambda - \mu|\, \|\phi\| + v\|x^\lambda - x^\mu\|_* + \delta K r_{2\epsilon} \left(|\lambda - \mu|\, \|x^\lambda\|_* + \|x^\lambda - x^\mu\|_*\right) \\
&\le \left(\frac{v}{2} + \delta K r_{2\epsilon}\right) |\lambda - \mu|\, \|\phi\| + \|x^\lambda - x^\mu\|_* (v + \delta K r_{2\epsilon}),
\end{aligned}
$$

$$
\|x^\lambda - x^\mu\|_* \le \Big[\frac{v/2 + \delta K r_{2\epsilon}}{1 - (v + \delta K r_{2\epsilon})}\Big] |\lambda - \mu|\, \|\phi\|,
$$

where $K' = 2K\left[\frac{v/2+\delta K r_{2\epsilon}}{1-(v+\delta K r_{2\epsilon})}\right]$. By making the constants $c$ and $\delta$ sufficiently small, $K'$ can be made arbitrarily possible and hence we obtain the desired result (3.2). $\square$

## 4. Existence of stable manifolds

In this section we establish the existence of stable invariant manifolds under sufficiently small nonlinear perturbations in equation (2.2). We say that equation (2.2) admits a *nonuniform exponential dichotomy* if there are projections $P(t) : \hat{\mathcal{B}} \to \hat{\mathcal{B}}$ for $t \ge 0$ and constants $a < 0 \le b, \epsilon \ge 0$ and $K > 1$, such that for every $t \ge s \ge 0$:

(i) $P(t)\hat{T}(t,s) = \hat{T}(t,s)P(s)$;

(ii) Let $I$ be the identity operator on $\hat{\mathcal{B}}$. Setting $Q(t) = I - P(t)$, the map $\hat{T}_Q(t,s) := \hat{T}(t,s)|Q(s)\hat{\mathcal{B}} : Q(s)\hat{\mathcal{B}} \to Q(t)\hat{\mathcal{B}}$ is invertible;

(iii)

$$\|\hat{T}(t,s)P(s)\| \le Ke^{a(t-s)+\epsilon s}, \quad \|\hat{T}_Q(t,s)^{-1}Q(t)\| \le Ke^{-b(t-s)+\epsilon t}. \tag{4.1}$$

Now, we define the *stable* and *unstable* subspaces by

$$E(t) = P(t)(\hat{\mathcal{B}}) \quad \text{and} \quad F(t) = Q(t)(\hat{\mathcal{B}}),$$

for each $t \ge 0$. Moreover, we say that (2.2) admits a *uniform exponential dichotomy* if it admits a *nonuniform exponential dichotomy* with $\epsilon = 0$.

For each $s \ge 0$, let $B_s(\delta) \subset E(s)$ be the open ball of radius $\delta$ centered at zero. Given $\eta > 0$ and $\beta = \epsilon(1 + \frac{2}{q}) + \frac{1}{q}$, we consider the set of initial conditions

$$Z_\beta(\eta) = \big\{(s,\phi) : s \ge 0, \phi \in B_s\big(\frac{e^{-\beta s}}{\eta}\big)\big\}$$

and we set $Z_\beta = Z_\beta(1)$. We denote by $\chi$ the space of all continuous functions $\Phi : Z_\beta \to \hat{\mathcal{B}}$ having at most discontinuities of the first kind in the first variable such that for each $s \ge 0$:

(1) $\Phi(s,0) = 0$ and $\Phi(s, B_s(e^{-\beta s})) \subset F(s)$;
(2) $\|\Phi(s,\phi) - \Phi(s,\psi)\| \le \|\phi - \psi\|$ for every $\phi, \psi \in B_s(e^{-\beta s})$.

One can easily verify that $\chi$ is a Banach space with the norm

$$|\Phi|' := \sup\big\{\frac{\|\Phi(s,\phi)\|}{\|\phi\|} : s \ge 0 \text{ and } \phi \in B_s(e^{-\beta s}) \setminus \{0\}\big\}.$$

Given $\Phi \in \chi$ and $\lambda \in$Y, we consider the graph

$$W_\lambda = \{(s, \phi, \Phi(s,\phi)) : (s,\phi) \in Z_\beta\}. \tag{4.2}$$

Furthermore, for each $\lambda \in$Y, let $\Psi_\kappa^\lambda$ be the semiflow defined by the autonomous system

$$t' = 1, \quad x' = L(t)x_t + f(t, x_t, \lambda).$$

Given $\kappa = t - s \ge 0$ and $(s, u_s, v_s) \in \mathbb{R}^+ \times E(s) \times F(s)$, we have

$$\Psi_\kappa^\lambda(s, u_s, v_s) = (s + \kappa, u_{s+\kappa}, v_{s+\kappa}),$$

where $u_t$ and $v_t$ are solutions of equation (2.6) on stable and unstable spaces given by

$$\begin{aligned} u_t = \hat{T}(t,s)u_s &+ \int_s^t P(t)\hat{T}(t,\tau)X_0 f(\tau, u_\tau, v_\tau, \lambda)d\tau \\ &+ \sum_{s \le \tau_i < t} P(t)\hat{T}(t,\tau_i^+)I_i(u_{\tau_i}, v_{\tau_i}, \lambda), \end{aligned} \tag{4.3}$$

$$\begin{aligned} v_t = \hat{T}(t,s)v_s &+ \int_s^t Q(t)\hat{T}(t,\tau)X_0 f(\tau, u_\tau, v_\tau, \lambda)d\tau \\ &+ \sum_{s \le \tau_i < t} Q(t)\hat{T}(t,\tau_i^+)I_i(u_{\tau_i}, v_{\tau_i}, \lambda). \end{aligned} \tag{4.4}$$

The following is our main result. It establishes the existence of a stable invariant manifold for differential equation (2.6).

**Theorem 4.1.** *Let equation* (2.2) *admit a nonuniform exponential dichotomy with* $a+\beta<0$ *and* $r_\epsilon<\infty$. *Also provided that the constants* $c$ *and* $\delta$ *in* (2.8) *and* (2.10) *are sufficiently small, then for each* $\lambda \in Y$, *there exists a unique function* $\Phi=\Phi^\lambda \in \chi^*$ *such that the set* $W_\lambda$ *is forward invariant under the semiflow* $\Psi_\kappa^\lambda$, *in the sense that*

$$\Psi_\kappa^\lambda(s,\phi,\Phi(s,\phi)) \in W_\lambda, \quad \text{for every } (s,\phi)\in Z_{\beta+\epsilon}(2K) \text{ and } \kappa \geq 0. \tag{4.5}$$

*Furthermore, for every* $\kappa=t-s\geq 0$; $\phi,\psi \in B(e^{-(\beta+\epsilon)}/(2K))$ *and* $\lambda,\mu\in Y$, *there exists* $D>0$ *such that*

$$\|\Psi_\kappa^\lambda(s,\phi,\Phi(s,\phi))-\Psi_\kappa^\lambda(s,\psi,\Phi(s,\psi))\| \leq De^{a(t-s)+\epsilon s}\|\phi-\psi\|, \tag{4.6}$$

$$\|\Psi_\kappa^\lambda(s,\phi,\Phi^\lambda(s,\phi))-\Psi_\kappa^\mu(s,\phi,\Phi^\mu(s,\phi))\| \leq De^{a(t-s)+\epsilon s}|\lambda-\mu|\ \|\phi\|. \tag{4.7}$$

*Proof.* The proof is obtained in several steps. Let $\chi^*$ be the space of all functions $\Phi:\mathbb{R}_0^+\times\hat{\mathcal{B}}\to\hat{\mathcal{B}}$ such that $\Phi\big|_{Z_\beta}\in\chi$ and

$$\Phi(s,\phi)=\Phi\left(s,e^{-\beta s}\phi/\|\phi\|\right) \quad \text{for every } (s,\phi)\notin Z_\beta.$$

We note that the functions in $\chi^*$ have at most discontinuities of the first kind as the functions in $\chi$ and also there is a one-to-one correspondence between functions in $\chi$ and functions in $\chi^*$. Clearly, $\chi^*$ is a Banach space with the norm $\chi^*\ni\Phi\mapsto|\ \Phi|_{Z_\beta}|'$. It is not difficult to show that for each $\Phi\in\chi^*$, we have

$$\|\Phi(s,\phi)-\Phi(s,\psi)\|\leq 2\|\phi-\psi\|, \tag{4.8}$$

for every $s\geq 0$ and $\phi,\psi\in E(s)$ [5]. For the rest of the paper we assume that $r_\epsilon<\infty$. $\square$

Since we want our stable manifold as a graph of some Lipschitz function $\Phi$ which is invariant under the semiflows. Therefore the form of the solution in the manifold must be $x(t)=(u(t),\Phi(t,u(t)))\in E(t)\times F(t)$ with the initial condition $(u_s,v_s)\in E(s)\times F(s)$ and hence the expression of the solution in the manifold is given by

$$\begin{aligned} u_t=\hat{T}(t,s)u_s&+\int_s^t P(t)\hat{T}(t,\tau)X_0f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau \\ &+\sum_{s\leq\tau_i<t}P(t)\hat{T}(t,\tau_i^+)I_i(u_{\tau_i},\Phi(\tau,u_{\tau_i}),\lambda), \end{aligned} \tag{4.9}$$

$$\begin{aligned} \Phi(t,u_t)=\hat{T}(t,s)\Phi(s,\phi)&+\int_s^t \hat{T}(t,\tau)Q(\tau)X_0f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau \\ &+\sum_{s\leq\tau_i<t}\hat{T}(t,\tau_i^+)Q(\tau_i^+)I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i},\lambda)). \end{aligned} \tag{4.10}$$

We first proof the existence of stable solution given by (4.9) and establish bounds for it.

**Lemma 4.2.** *Given the constants* $c,\delta>0$ *sufficiently small, and* $(s,\phi,\lambda,\Phi)\in Z_\beta\times Y\times\chi^*$, *there exists a unique function* $u:[s-r,\infty)\to\mathbb{R}^n$ *with* $u_s=\phi$ *such that* $u_t\in E(t)$ *and* (4.9) *holds for every* $t\geq s$. *Furthermore:*

(1) *$u$ is continuous in $t$ with at most discontinuities of the first kind at the points $\tau_i$;*

(2) *for each $t \geq s$, with $(s,\phi) \in Z_\beta$ and $\lambda, \mu \in Y$ we have*

$$\|u_t\| \leq 2Ke^{a(t-s)+\epsilon s}\|\phi\|, \tag{4.11}$$

$$\|u_t^\lambda - u_t^\mu\| \leq 2L'e^{a(t-s)+\epsilon s}\|\phi\||\lambda - \mu|, \tag{4.12}$$

*for some constant $L'$.*

*Proof.* Let $\Omega'$ be the space of all continuous functions $u : [s-r,\infty) \to \mathbb{R}^n$ with at most discontinuities of the first kind at the points $\{\tau_i\}$ such that $u_s = \phi$, $u_t \in E(t)$ for every $t \geq s$ with $(s,\phi) \in Z_\beta$, and $\|u\|_* \leq e^{-\beta s}$, where $\|\cdot\|_*$ is given by (3.5). $\Omega'$ is a complete metric space with the norm $\|\cdot\|_*$. Given $(s,\phi) \in Z_\beta$ and $\Phi \in \chi^*$, for each $\lambda \in$Y we define an operator $L^\lambda$ in $\Omega'$ by

$$\begin{aligned}(L^\lambda u)_t &= \hat{T}(t,s)\phi + \int_s^t \hat{T}(t,\tau)P(\tau)X_0 f(\tau, u_\tau, \Phi(\tau,u_\tau),\lambda)d\tau \\ &\quad + \sum_{s\leq\tau_i<t} \hat{T}(t,\tau_i)P(\tau_i^+)I_i(u_{\tau_i}, \Phi(\tau_i,u_{\tau_i}),\lambda).\end{aligned}$$

We note that $(L^\lambda u)_s = \phi$ and $(L^\lambda u)_t \in E(t)$ for every $t \geq s$. By using (2.8), (4.1), and (4.8),

$$\begin{aligned}|f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)| &\leq c\,\|(u,\Phi(\tau,u_\tau))\|^{q+1} \\ &\leq c\,(\|u_\tau\| + \|\Phi(\tau,u_\tau)\|)^{q+1} \\ &\leq c\,(\|u_\tau\| + 2\|u_\tau\|)^{q+1} = 3^{q+1}c\|u_\tau\|^{q+1}.\end{aligned}$$

Similarly, by using equations (2.9), (4.1), and (4.8),

$$\begin{aligned}|I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda)| &\leq \delta e^{-3\epsilon\tau_i}\|(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}))\| \\ &\leq \delta e^{-3\epsilon\tau_i}\,(\|u_{\tau_i}\| + \|\Phi(\tau,u_{\tau_i})\|) \\ &\leq 3\delta e^{-3\epsilon\tau_i}\|u_{\tau_i}\|.\end{aligned}$$

Now using the relations given above and (4.1),

$$\begin{aligned}\|(L^\lambda u)_t\| &\leq \|\hat{T}(t,s)\phi\| + \int_s^t \|\hat{T}(t,\tau)P(\tau)\|\,|f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)|d\tau \\ &\quad + \sum_{s\leq\tau_i<t}\|\hat{T}(t,\tau_i)P(\tau_i^+)\||I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda)| \\ &\leq Ke^{a(t-s)+\epsilon s}\|\phi\| + \int_s^t Ke^{a(t-\tau)+\epsilon\tau}3^{q+1}c\|u_\tau\|^{q+1}d\tau \\ &\quad + \sum_{s\leq\tau_i<t} Ke^{a(t-\tau_i)+\epsilon\tau_i}3\delta e^{-3\epsilon\tau_i}\|u_{\tau_i}\|.\end{aligned}$$

Denoting $v' = 3^{q+1}v$, we have

$$\begin{aligned}\|(L^\lambda u)_t\| &\leq Ke^{a(t-s)+\epsilon s}\|\phi\| + Kv'e^{a(t-s)+\epsilon s}e^{\epsilon(q+1)s}\|u\|_*^{q+1} \\ &\quad + 6\delta K^2 e^{a(t-s)+\epsilon s}\|u\|_* \sum_{s\leq\tau_i<t} e^{-2\epsilon\tau_i} \\ &\leq 2Ke^{a(t-s)+\epsilon s}\Big(\frac{1}{2}\|\phi\| + \frac{v'}{2}e^{\epsilon(q+1)s}\|u\|_*^{q+1} + 3K\delta r_{2\epsilon}\|u\|_*\Big) \\ &\leq \frac{1}{2}\Big(1 + v'e^{\{\epsilon(q+1)-\beta q\}s} + 6K\delta r_{2\epsilon}\Big)e^{-\beta s}.\end{aligned}$$

Provided that the constants $c$ and $\delta$ are sufficiently small so that $(v'+6K\delta r_{2\epsilon})<1$, we obtain $L^\lambda(\Omega')\subset\Omega'$. Furthermore, for each, $u,v\in\Omega'$,

$$\begin{aligned}
&\|(L^\lambda u)_t-(L^\lambda v)_t\|\\
&\le\int_s^t\|\hat{T}(t,\tau)P(\tau)\||f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)-f(\tau,v_\tau,\Phi(\tau,v_\tau),\lambda)|d\tau\\
&\quad+\sum_{s\le\tau_i<t}\|\hat{T}(t,\tau_i)P(\tau_i^+)\|\,|I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda)-I_i(v_{\tau_i},\Phi(\tau_i,v_{\tau_i}),\lambda)|\\
&\le c3^{q+1}K\int_s^t e^{a(t-\tau)+\epsilon\tau}\|u_\tau-v_\tau\|\left(\|u_\tau\|^q+\|v_\tau\|^q\right)\\
&\quad+3\delta K\sum_{s\le\tau_i<t}e^{a(t-\tau_i)+\epsilon\tau_i}e^{-3\epsilon\tau_i}\|u_{\tau_i}-v_{\tau_i}\|\\
&\le 2Kv'e^{a(t-s)+\epsilon s}e^{(\epsilon q+\epsilon)s-\beta qs}\|u-v\|_*+6\delta K^2r_{2\epsilon}e^{a(t-s)+\epsilon s}\|u-v\|_*
\end{aligned}$$

$$\|L^\lambda u-L^\lambda v\|_*\le(v'+3K\delta r_{2\epsilon})\,\|u-v\|_*.$$

The choices of $c$ and $\delta$ are such that, $(v'+3K\delta r_{2\epsilon})<1$, therefore map $L^\lambda$ is a contraction in $\Omega'$. Hence there exists a unique function $u=u^\lambda\in\Omega'$ such that $L^\lambda u=u$. Similar to (3.6), we obtain

$$\|u\|_*\le\Big(\frac12\|\phi\|+\frac{v'}{2}\|u\|_*+3K\delta r_{2\epsilon}\|u\|_*\Big)\le\frac12\|\phi\|+\Big(\frac{v'}{2}+3K\delta r_{2\epsilon}\Big)\|u\|_*.$$

Since $1/\big(1-(\frac{v'}{2}+3K\delta r_{2\epsilon})\big)<2$, (4.11) holds. Similarly, using (3.9) and (3.10), we have

$$\begin{aligned}
\|u_t^\lambda-u_t^\mu\|&\le\int_s^t\|\hat{T}(t,\tau)P(\tau)\|\,|f(\tau,u_\tau^\lambda,\Phi(\tau,u_\tau^\lambda),\lambda)-f(\tau,u_\tau^\mu,\Phi(\tau,u_\tau^\mu),\mu)|\,d\tau\\
&\quad+\sum_{s<\tau_i<t}\|\hat{T}(t,\tau_i)P(\tau_i)\|\,|I_i(u_{\tau_i}^\lambda,\Phi(\tau,u_{\tau_i}^\lambda),\lambda)-I_i(u_{\tau_i}^\mu,\Phi(\tau,u_{\tau_i}^\mu),\mu)|\\
&\le 3^{q+1}c(2K)^{q+1}\{|\lambda-\mu|\;\|\phi\|+2\|u^\lambda-u^\mu\|_*\}\\
&\quad\times\int_s^t Ke^{a(t-\tau)+\epsilon\tau}e^{a(q+1)(\tau-s)}e^{[q(\epsilon-\beta)+\epsilon]s}\\
&\quad+6\delta K\{|\lambda-\mu|\|u^\lambda\|_*+\|u^\lambda-u^\mu\|_*\}\\
&\sum_{s<\tau_i<t}Ke^{a(t-\tau_i)+\epsilon\tau_i}e^{-3\epsilon\tau_i}e^{a(\tau_i-s)+\epsilon s}\\
&\le c3^{q+1}(2K)^{q+1}\{|\lambda-\mu|\;\|\phi\|+2\|u^\lambda-u^\mu\|_*\}\\
&\quad\times Ke^{a(t-s)+\epsilon s}e^{q(\epsilon-\beta-a)s}\int_s^t e^{\tau(\epsilon+qa)}d\tau\\
&\quad+6K^2\delta e^{a(t-s)+\epsilon s}\left\{|\lambda-\mu|\|u^\lambda\|_*+\|u^\lambda-u^\mu\|_*\right\}\sum_{s\le\tau_i<t}e^{-2\epsilon\tau_i};
\end{aligned}$$

$$\begin{aligned}
\|u^\lambda-u^\mu\|_*&\le\frac{v'}{2}|\lambda-\mu|\,\|\phi\|+v'\,\|u^\lambda-u^\mu\|_*+3\delta Kr_{2\epsilon}\left(|\lambda-\mu|\,\|\phi\|+\|u^\lambda-u^\mu\|_*\right)\\
&=\Big(\frac{v'}{2}+3\delta Kr_{2\epsilon}\Big)|\lambda-\mu|\,\|\phi\|+\|u^\lambda-u^\mu\|_*\left(v'+3\delta Kr_{2\epsilon}\right);
\end{aligned}$$

$$\|u^\lambda - u^\mu\|_* \le \left[\frac{v'/2 + 3\delta K r_{2\epsilon}}{1-(v' + 3\delta K r_{2\epsilon})}\right] |\lambda - \mu| \, \|\phi\|.$$

This gives the desired result (4.12). □

In the next two lemmas, we establish some auxiliary properties of the stable solution obtained in Lemma 4.2.

**Lemma 4.3.** *Provided that $c, \delta$ are sufficiently small, there exists $K_1 > 0$ such that*

$$\|u_t - v_t\| \le K_1 e^{a(t-s)+\epsilon s} \|\phi - \psi\|, \tag{4.13}$$

*for every $\lambda \in Y$; $\Phi \in \chi^*$; $(s,\phi), (s,\psi) \in Z_\beta$ and $t \ge s$, where $u$ and $v$ are the functions given by Lemma 4.2 respectively for $(s,\phi,\lambda,\Phi)$ and $(s,\psi,\lambda,\Phi)$.*

*Proof.* Using equations (2.8),(4.1) and (4.8) in the form of stable solution (4.9), we have

$$\begin{aligned}
&\|u_t - v_t\| \\
&\le \|\hat{T}(t,s)(\phi - \psi)\| \\
&\quad + \int_s^t \|P(t)\hat{T}(t,\tau)\| \, |f(\tau, u_\tau, \Phi(\tau,u_\tau),\lambda) - f(\tau, v_\tau, \Phi(\tau, v_\tau), \lambda)| d\tau \\
&\quad + \sum_{s \le \tau_i < t} \|P(t)\hat{T}(t,\tau_i^+)\| \, |I_i(u_{\tau_i}, \Phi(\tau, u_{\tau_i}), \lambda) - I_i(v_{\tau_i}, \Phi(\tau, v_{\tau_i}), \lambda)| \\
&\le K e^{a(t-s)+\epsilon s} \|\phi - \psi\| + 3^{q+1} cK \int_s^t e^{a(t-\tau)+\epsilon\tau} \|u_\tau - v_\tau\| (\|u_\tau\|^q + \|v_\tau\|^q) d\tau \\
&\quad + 3\delta K \sum_{s \le \tau_i < t} e^{a(t-\tau_i)+\epsilon\tau_i} e^{-3\epsilon\tau_i} \|u_{\tau_i} - v_{\tau_i}\| \\
&\le K e^{a(t-s)+\epsilon s} \|\phi - \psi\| + 2Kv' e^{a(t-s)+\epsilon s} \|u - v\|_* \\
&\quad + 6\delta r_{2\epsilon} K^2 e^{a(t-s)+\epsilon s} \|u - v\|_* \\
&\le K e^{a(t-s)+\epsilon s} \|\phi - \psi\| + (v' + 3\delta K r_{2\epsilon}) \, 2K e^{a(t-s)+\epsilon s} \|u - v\|_*
\end{aligned}$$

We can also write it as,

$$\|u - v\|_* \le \frac{1}{2} \|\phi - \psi\| + (v' + 3\delta K r_{2\epsilon}) \|u - v\|_*.$$

This establishes the desired result for $K_1 = K/\{1 - (v' + 3\delta K r_{2\epsilon})\}$. □

**Lemma 4.4.** *Let the constants $c, \delta$ be sufficiently small, there exists $K_2 > 0$ such that*

$$\|u_t - v_t\| \le K_2 \, e^{a(t-s)} \|\phi\| \, |\Phi - \Psi|', \tag{4.14}$$

*for every $\lambda \in Y$, $\Phi, \Psi \in \chi^*$, $(s,\phi) \in Z_\beta$, and $t \ge s$, where $u$ and $v$ are the functions given by Lemma 4.2 for $(s,\phi,\lambda,\Phi)$ and $(s,\phi,\lambda,\Psi)$ respectively.*

*Proof.* We note that

$$\|\Phi(\tau, u_\tau) - \Psi(\tau, v_\tau)\| \le \|u_\tau\| \, |\Phi - \Psi|' + 2\|u_\tau - v_\tau\|.$$

Using the above estimates,

$$\begin{aligned}
&\|u_t - v_t\| \\
&\le \int_s^t \|\hat{T}(t,\tau)P(\tau)\| \, |f(\tau, u_\tau, \Phi(\tau, u_\tau), \lambda) - f(\tau, v_\tau, \Psi(\tau, v_\tau), \lambda)| d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{s\le\tau_i<t} \|\hat{T}(t,\tau_i)P(\tau_i)\|\,|I_i(\tau_i,u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda) - I_i(\tau_i,v_{\tau_i},\Psi(\tau_i,v_{\tau_i}),\lambda)| \\
&\le 3^q cK \int_s^t e^{a(t-\tau)+\epsilon\tau}\left(\|u_\tau - v_\tau\| + \|\Phi(\tau,u_\tau)-\Psi(\tau,v_\tau)\|\right)\left(\|u_\tau\|^q + \|v_\tau\|^q\right)d\tau \\
&\quad + 3\delta K \sum_{s\le\tau_i<t} e^{a(t-\tau_i)+\epsilon\tau_i} e^{-3\epsilon\tau_i}\left(\|u_{\tau_i}-v_{\tau_i}\| + \|\Phi(\tau,u_{\tau_i})-\Psi(\tau,v_{\tau_i})\|\right) \\
&\le 2\cdot 6^q cK^{q+1}\int_s^t e^{a(t-\tau)}e^{(aq+\epsilon)\tau}e^{q(\epsilon-a-\beta)s}\left(3\|u_\tau-v_\tau\| + \|u_\tau\|\,|\Phi-\Psi|'\right)d\tau \\
&\quad + 3\delta K \sum_{s\le\tau_i<t} e^{a(t-\tau_i)}e^{-2\epsilon\tau_i}\left(3\|u_{\tau_i}-v_{\tau_i}\| + \|u_{\tau_i}\|\,|\Phi-\Psi|'\right) \\
&\le 4\cdot 6^q cK^{q+2}\left(3\|u-v\|_* + \|\phi\|\,|\Phi-\Psi|'\right) \\
&\quad \times \int_s^t e^{a(t-\tau)}e^{(aq+\epsilon)\tau}e^{q(\epsilon-a-\beta)s}e^{a(\tau-s)+\epsilon s}d\tau \\
&\quad + 6\delta K^2\left(3\|u-v\|_* + \|\phi\|\,|\Phi-\Psi|'\right)\sum_{s\le\tau_i<t} e^{a(t-\tau_i)}e^{-2\epsilon\tau_i}e^{a(\tau_i-s)+\epsilon s} \\
&\le \Big(2\cdot 6^q cK^{q+1}\left(3\|u-v\|_* + \|\phi\|\,|\Phi-\Psi|'\right)e^{q(\epsilon-a-\beta)s}\int_s^t e^{(aq+\epsilon)\tau} \\
&\quad + 3\delta K\left(3\|u-v\|_* + \|\phi\|\,|\Phi-\Psi|'\right)\sum_{s\le\tau_i<t} e^{-2\epsilon\tau_i}\Big)2Ke^{a(t-s)+\epsilon s}
\end{aligned}
$$

$$
\begin{aligned}
\|u-v\|_* &\le \Big\{\frac{2\cdot 6^q cK^{q+1}}{|aq+\epsilon|}e^{(q\epsilon-q\beta+\epsilon)s} + 3\delta K r_{2\epsilon}\Big\}\left(3\|u-v\|_* + \|\phi\|\,|\Phi-\Psi|'\right) \\
&\le \Big\{\frac{2\cdot 6^q cK^{q+1}}{|aq+\epsilon|} + 3\delta K r_{2\epsilon}\Big\}\left(3\|u-v\|_* + \|\phi\|\,|\Phi-\Psi|'\right).
\end{aligned}
$$

By adjusting the constant we obtain the desired result. □

Next we give the existence of a function $\Phi \in \chi^*$ such that the graph of $\Phi$ will be our stable manifold.

**Lemma 4.5.** *Given $\lambda \in Y$ and $\Phi \in \chi^*$ and denoting by $u$, the unique function obtained by Lemma 4.2, the following properties holds:*

(i) *If*

$$
\begin{aligned}
\Phi(t,u_t) &= \hat{T}(t,s)\Phi(s,\phi) + \int_s^t \hat{T}(t,\tau)Q(\tau)X_0 f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau \\
&\quad + \sum_{s\le\tau_i<t}\hat{T}(t,\tau_i^+)Q(\tau_i^+)I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i},\lambda)),
\end{aligned} \tag{4.15}
$$

*for $t \ge s$, $(s,\phi) \in Z_\beta$ and $\lambda \in Y$, then*

$$
\begin{aligned}
\Phi(s,\phi) &= -\int_s^\infty \hat{T}_Q(\tau,s)^{-1}Q(\tau)X_0 f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau \\
&\quad - \sum_{s\le\tau_i}\hat{T}_Q(\tau_i^+,s)^{-1}Q(\tau_i^+)I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda),
\end{aligned} \tag{4.16}
$$

*for every $t \ge s$, $(s,\phi) \in Z_\beta$ and $\lambda \in Y$;*

(ii) *If identity* (4.16) *holds for every* $(s,\phi) \in Z_\beta$*, then* (4.15) *holds for all* $(s,\phi) \in Z_{\beta+\epsilon}(2K)$.

*Proof.* First we show that the integral in (4.15) is well defined.

$$
\begin{aligned}
&\int_s^\infty \|\hat{T}_Q(\tau,s)^{-1}Q(\tau)\|\,|f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)|d\tau\\
&\le \int_s^t Ke^{-b(\tau-s)+\epsilon\tau}c\|\,(u_\tau,\Phi(\tau,u_\tau))\,\|^{q+1}d\tau\\
&\le \int_s^t Ke^{-b(\tau-s)+\epsilon\tau}c3^{q+1}\|u_\tau\|^{q+1}d\tau\\
&\le 6^{q+1}cK^{q+2}\int_s^t e^{-b(\tau-s)+\epsilon\tau}e^{\{a(\tau-s)+\epsilon s\}(q+1)}\|\phi\|^{q+1}d\tau\\
&\le (6K)^{q+1}cKe^{[b-(q+1)(a-\epsilon+\beta)]s}\int_s^\infty e^{[-b+\epsilon+a(q+1)]\tau}d\tau\\
&\le \frac{(6K)^{q+1}cK}{|-b+\epsilon+a(q+1)|}e^{[\epsilon-(q+1)(\beta-\epsilon)]s}\\
&\le \frac{(6K)^{q+1}cK}{|-b+\epsilon+a(q+1)|}e^{-(1+\beta)s}<\infty,
\end{aligned}
$$

Since

$$-b+\epsilon+a(q+1)=(-b+a)+aq+\epsilon<a+\beta<0\,.$$

Similarly,

$$
\begin{aligned}
&\sum_{s\le\tau_i}\|\hat{T}_Q(\tau_i^+,s)^{-1}Q(\tau_i^+)\|\,|I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda)|\\
&\le \sum_{s\le\tau_i}Ke^{-b(\tau_i-s)+\epsilon\tau_i}\delta e^{-3\epsilon\tau_i}\|(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}))\|\\
&\le 3\delta K\sum_{s\le\tau_i}e^{-b(\tau_i-s)+\epsilon\tau_i}e^{-3\epsilon\tau_i}\|u_{\tau_i}\|\\
&\le 6\delta K^2\sum_{s\le\tau_i}e^{s(b-a+\epsilon)}e^{(-b-2\epsilon+a)\tau_i}\|\phi\|\\
&= 6\delta K^2e^{s(b-a+\epsilon)}\|\phi\|\sum_{s\le\tau_i}e^{(-b-2\epsilon+a)\tau_i}<\infty,
\end{aligned}
$$

since $(-b-2\epsilon+a)<0$. Therefore (4.15) is well-defined.

If (4.15) holds for every $(s,\phi)\in Z_\beta$ and $t\ge s$, then

$$
\begin{aligned}
\Phi(s,\phi)=\hat{T}_Q(t,s)^{-1}\Phi(t,u_t)&-\int_s^t\hat{T}_Q(\tau,s)^{-1}Q(\tau)X_0f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau\\
&-\sum_{s\le\tau_i<t}\hat{T}_Q(\tau_i^+,s)^{-1}Q(\tau_i^+)I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda).
\end{aligned}
\tag{4.17}
$$

Now it follows from (4.1), (4.8) and (4.11) that

$$
\begin{aligned}
\|\hat{T}_Q(t,s)^{-1}\Phi(t,u_t)\|&\le 4K^2e^{-b(t-s)+\epsilon t}e^{a(t-s)+\epsilon s}e^{-\beta s}\\
&=4K^2e^{(a-b+\epsilon)t}e^{(-a+b+\epsilon-\beta)s}.
\end{aligned}
$$

Since $a+\epsilon<a+\beta<0$, letting $t\to\infty$ in (4.17), we obtain the identity (4.16).

Now we assume that equation (4.16) holds for every $(s,\phi)\in Z_\beta$, then for each $(s,\phi)\in Z_{\beta+\epsilon}(2K)$,

$$\|u_t\|\le 2Ke^{a(t-s)+\epsilon s}\|\phi\|\le e^{-\beta t}e^{(a+\beta)(t-s)}\le e^{-\beta t}\le e^{-\beta s}.$$

This implies that $(t,u_t)\in Z_\beta$ for every $t\ge s$. Now we apply $\hat{T}(t,s)$ to equation (4.16) to get

$$\begin{aligned}
\hat{T}(t,s)\Phi(s,\phi) &= -\int_s^\infty \hat{T}(t,\tau)Q(\tau)X_0 f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau\\
&\quad -\sum_{s\le\tau_i}\hat{T}(t,\tau_i)Q(\tau_i^+)I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda)\\
&= -\int_s^t \hat{T}(t,\tau)Q(\tau)X_0 f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau\\
&\quad -\sum_{s\le\tau_i<t}\hat{T}(t,\tau_i)Q(\tau_i^+)I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda)\\
&\quad -\int_t^\infty \hat{T}(t,\tau)Q(\tau)X_0 f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau\\
&\quad -\sum_{t\le\tau_i}\hat{T}(t,\tau_i)Q(\tau_i^+)I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda);\\
\hat{T}(t,s)\Phi(s,\phi) &= -\int_s^t \hat{T}(\tau,t)^{-1}Q(\tau)X_0 f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau\\
&\quad -\sum_{s\le\tau_i<t}\hat{T}(\tau_i,t)^{-1}Q(\tau_i^+)I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda)+\Phi(t,u_t).
\end{aligned}$$

This completes the proof of lemma. □

Now we establish the existence of a unique function $\Phi=\Phi^\lambda$ satisfying equation (4.16) for each $\lambda\in Y$.

**Lemma 4.6.** *Provided that the constants $c,\delta$ are sufficiently small, for each $\lambda\in Y$ there exists a unique function $\Phi=\Phi^\lambda\in\chi^*$ such that* (4.16) *holds for every $(s,\phi)\in Z_\beta$.*

*Proof.* Given $\lambda\in$Y, for each $\Phi\in\chi^*$ we define an operator $J^\lambda$ by

$$\begin{aligned}
(J^\lambda\Phi)(s,\phi) &= -\int_s^\infty \hat{T}_Q(\tau,s)^{-1}Q(\tau)X_0 f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)d\tau\\
&\quad -\sum_{s\le\tau_i}\hat{T}_Q(\tau_i^+,s)^{-1}Q(\tau_i^+)I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda),
\end{aligned}$$

for each $(s,\phi)\in Z_\beta$, where $u$ is a unique function given by Lemma 4.2 for $(s,\phi,\lambda,\Phi)$. We note that $(J^\lambda\Phi)(s,0)=0$ for every $s\ge0$. Moreover, for each $(s,\phi),(s,\psi)\in Z_\beta$, using the properties of $u$ and $v$ proved in Lemma 4.3 and by the equations (4.1), (4.11), we have

$$\begin{aligned}
&\|(J^\lambda\Phi)(s,\phi)-(J^\lambda\Phi)(s,\psi)\|\\
&\le \int_s^\infty \|\hat{T}_Q(\tau,s)^{-1}Q(\tau)\|\,|f(\tau,u_\tau,\Phi(\tau,u_\tau),\lambda)-f(\tau,v_\tau,\Phi(\tau,v_\tau),\lambda)|d\tau\\
&\quad +\sum_{s\le\tau_i}\|\hat{T}_Q(\tau_i^+,s)^{-1}Q(\tau_i^+)\|\,|I_i(u_{\tau_i},\Phi(\tau_i,u_{\tau_i}),\lambda)-I_i(v_{\tau_i},\Phi(\tau_i,v_{\tau_i}),\lambda)|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_s^\infty K e^{-b(\tau-s)+\epsilon\tau} 3^{q+1} c \|u_\tau - v_\tau\| \left(\|u_\tau\|^q + \|v_\tau\|^q\right) d\tau \\
&\quad + \sum_{s\le\tau_i} K e^{-b(\tau_i-s)+\epsilon\tau_i} 3\delta e^{-3\epsilon\tau_i} \|u_{\tau_i} - v_{\tau_i}\| \\
&\le (6K)^{q+1} c K_1 \int_s^\infty e^{-b(\tau-s)+\epsilon\tau} e^{a(q+1)(\tau-s)+\epsilon(q+1)s-\beta qs} \|\phi-\psi\| d\tau \\
&\quad + 3\delta K K_1 \sum_{s\le\tau_i} e^{-b(\tau_i-s)+\epsilon\tau_i} e^{-3\epsilon\tau_i} e^{a(\tau_i-s)+\epsilon s} \|\phi-\psi\| \\
&\le (6K)^{q+1} c K_1 e^{\{\epsilon(q+1)-\beta q - a(q+1)+b)\}s} \|\phi-\psi\| \int_s^\infty e^{\{a(q+1)-b+\epsilon\}\tau} d\tau \\
&\quad + 3\delta K K_1 e^{(b-a+\epsilon)s} \sum_{s\le\tau_i} e^{(a-b-2\epsilon)\tau_i} \|\phi-\psi\| \\
&\le \frac{(6K)^{q+1} c K_1}{|-b+a(q+1)+\epsilon|} \|\phi-\psi\| e^{\{\epsilon(q+1)-\beta q - a(q+1)+b)\}s} e^{s\{-b+a(q+1)+\epsilon\}} \\
&\quad + 3\delta K K_1 e^{(b-a+\epsilon)s} r_{2\epsilon} \|\phi-\psi\| \\
&\le \Big\{ \frac{(6K)^{q+1} c K_1 e^{-(1+\epsilon)s}}{|-b+a(q+1)+\epsilon|} + 3\delta K K_1 e^{(b-a+\epsilon)s} r_{2\epsilon} \Big\} \|\phi-\psi\|.
\end{aligned}
$$

Since the constants $c, \delta$ are sufficiently small, we obtain

$$
\|(J^\lambda \Phi)(s,\phi) - (J^\lambda \Phi)(s,\psi)\| \le \|\phi-\psi\|.
$$

Therefore, extending $J^\lambda \Phi$ to $\mathbb{R}^+ \times \hat{\mathcal{B}}$ by

$$
(J^\lambda \Phi)(s,\phi) = (J^\lambda \Phi)(s, e^{-\beta s}\phi/\|\phi\|),
$$

for every $(s,\phi) \notin Z_\beta$, we have $J^\lambda(\chi^*) \subset \chi^*$.

Now we show that $J^\lambda$ is a contraction map. Given $(s,\phi) \in Z_\beta$, for each $\Phi, \Psi \in \chi^*$ and using the notation $u$ and $v$ as in Lemma 4.4, it follows from equations (4.1), (4.8), (4.11) and (4.14),

$$
\begin{aligned}
&\|(J^\lambda \Phi)(s,\phi) - (J^\lambda \Psi)(s,\phi)\| \\
&\le \int_s^\infty \|\hat{T}_Q(\tau,s)^{-1} Q(\tau)\| \, |f(\tau, u_\tau, \Phi(\tau,u_\tau),\lambda) - f(\tau, v_\tau, \Psi(\tau, v_\tau),\lambda)| d\tau \\
&\quad + \sum_{s\le\tau_i} \|\hat{T}_Q(\tau_i^+, s)^{-1} Q(\tau_i^+)\| \, |I_i(u_{\tau_i}, \Phi(\tau_i, u_{\tau_i}),\lambda) - I_i(v_{\tau_i}, \Psi(\tau_i, v_{\tau_i}),\lambda)| \\
&\le \int_s^\infty K e^{-b(\tau-s)+\epsilon\tau} 3^q c \left(3\|u_\tau - v_\tau\| + \|u_\tau\| \, |\Phi-\Psi|'\right) \left(\|u_\tau\|^q + \|v_\tau\|^q\right) d\tau \\
&\quad + \sum_{s\le\tau_i} K e^{-b(\tau_i-s)+\epsilon\tau_i} \delta e^{-3\epsilon\tau_i} \left(3\|u_{\tau_i} - v_{\tau_i}\| + \|u_{\tau_i}\| \, |\Phi-\Psi|'\right) \\
&\le (6K)^q 2cK \left(3K_2 + 2K\right) \|\phi\| \, |\Phi-\Psi|' \\
&\quad \times \int_s^\infty e^{-b(\tau-s)+\epsilon\tau} \left(e^{aq(\tau-s)+q\epsilon s} e^{-\beta qs}\right) e^{a(\tau-s)+\epsilon s} d\tau \\
&\quad + K\delta(3K_2 + 2K)\|\phi\| \, |\Phi-\Psi|' \sum_{s\le\tau_i} e^{-b(\tau_i-s)+\epsilon\tau_i} e^{-3\epsilon\tau_i} e^{a(\tau_i-s)+\epsilon s} \\
&\le (6K)^q 2cK \left(3K_2 + 2K\right) \|\phi\| \, |\Phi-\Psi|' e^{\{b+\epsilon(q+1)-a(q+1)-q\beta\}s}
\end{aligned}
$$

$$\begin{aligned}
&\times \int_s^\infty e^{\{-b+\epsilon+a(q+1)\}\tau} \\
&+ K\delta(3K_2+2K)\|\phi\|\,|\Phi-\Psi|' e^{(b-a+\epsilon)s} \sum_{s\le\tau_i} e^{(-b-2\epsilon+a)\tau_i} \\
&\le (6K)^q 2cK\,(3K_2+2K)\,\|\phi\|\,|\Phi-\Psi|' \frac{e^{\{\epsilon(q+1)-\beta q+\epsilon\}s}}{|-b+a(q+1)+\epsilon|} \\
&\quad + K\delta(3K_2+2K)\,\|\phi\|\,|\Phi-\Psi|' e^{(b-a+\epsilon)s} r_{2\epsilon} \\
&\le \Big\{\frac{(6K)^q 2cKe^{-s}}{|-b+a(q+1)+\epsilon|} + K\delta e^{(b-a+\epsilon)s} r_{2\epsilon}\Big\}(3K_2+2K)\|\phi\|\,|\Phi-\Psi|' \\
&\le \omega\,|\Phi-\Psi|'.
\end{aligned}$$

By taking the constants $c,\delta$ sufficiently small we can make $\omega < 1$. Therefore, for each $\lambda \in$Y there exists a unique function $\Phi = \Phi^\lambda \in \chi^*$ such that $J^\lambda\Phi = \Phi$, which completes the proof. □

Here we are going to obtain the parameter dependent estimate for the stable part of the solution obtained in Lemma 4.2.

**Lemma 4.7.** *Assume that the constants $c,\delta$ are sufficiently small, there exists $K_3 > 0$ such that*

$$\|u_t - u_t'\| \le K_3 e^{a(t-s)+\epsilon s}|\lambda-\mu|\,\|\phi\|, \tag{4.18}$$

*for every $\lambda \in Y, (s,\phi) \in Z_\beta$ and $t \ge s$, where $u$ and $u'$ are the functions given by Lemma 4.2 respectively for $(s,\phi,\lambda,\Phi^\lambda)$ and $(s,\phi,\mu,\Phi^\mu)$.*

*Proof.* First we obtain some estimates using equation (4.1), (4.11) and (4.12), we have

$$\begin{aligned}
&\|\Phi^\lambda(s,\phi) - \Phi^\mu(s,\phi)\| \\
&\le \int_s^\infty \|\hat T_Q(\tau,s)^{-1}Q(\tau)\|\,|f(\tau,u_\tau,\Phi^\lambda(\tau,u_\tau),\lambda) - f(\tau,u'_\tau,\Phi^\mu(\tau,u'_\tau),\mu)|\,d\tau \\
&\quad + \sum_{s\le\tau_i} \|\hat T_Q(\tau_i^+,s)^{-1}Q(\tau_i^+)\|\,|I_i(u_{\tau_i},\Phi^\lambda(\tau_i,u_{\tau_i}),\lambda) - I_i(u'_{\tau_i},\Phi^\mu(\tau_i,u'_{\tau_i}),\mu)| \\
&\le \int_s^\infty Ke^{-b(\tau-s)+\epsilon\tau}\Big\{c|\lambda-\mu|\,3^{q+1}\|u_\tau\|^{q+1} \\
&\quad + 3^q c\Big(\|u_\tau - u'_\tau\| + \|\Phi^\lambda(\tau,u_\tau) - \Phi^\mu(\tau,u'_\tau)\|\Big)\,(\|u_\tau\|^q + \|u'_\tau\|^q)\Big\}d\tau \\
&\quad + \sum_{s\le\tau_i} Ke^{-b(\tau_i-s)+\epsilon\tau_i}\delta e^{-3\epsilon\tau_i}\Big(3|\lambda-\mu|\|u_{\tau_i}\| + \|u_{\tau_i} - u'_{\tau_i}\| \\
&\quad + \|\Phi^\lambda(\tau_i,u_{\tau_i}) - \Phi^\mu(\tau_i,u'_{\tau_i})\|\Big) \\
&\le \int_s^\infty Ke^{-b(\tau-s)+\epsilon\tau}\Big\{c|\lambda-\mu|\,3^{q+1}\|u_\tau\|^{q+1} \\
&\quad + 3^q c\left(3\|u_\tau - u'_\tau\| + |\Phi^\lambda - \Phi^\mu|\|u'_\tau\|\right)(\|u_\tau\|^q + \|u'_\tau\|^q)\Big\}d\tau \\
&\quad + \sum_{s\le\tau_i} Ke^{-b(\tau_i-s)+\epsilon\tau_i}\delta e^{-3\epsilon\tau_i}\left(3|\lambda-\mu|\,\|u_{\tau_i}\| + 3\|u_{\tau_i} - u'_{\tau_i}\| + |\Phi^\lambda - \Phi^\mu|\,\|u'_{\tau_i}\|\right) \\
&\le \int_s^\infty Ke^{-b(\tau-s)+\epsilon\tau}(2K)^{q+1}e^{a(q+1)(\tau-s)+\epsilon(q+1)s}\|\phi\|^q 3^q c
\end{aligned}$$

$$
\begin{aligned}
&\Big\{3|\lambda-\mu|\,\|\phi\|+2\left(3\|\phi\||\lambda-\mu|+|\Phi^{\lambda}-\Phi^{\mu}|\|\phi\|\right)\Big\}d\tau\\
&+\sum_{s\le\tau_i}Ke^{(a-b)(\tau_i-s)}\delta e^{-2\epsilon\tau_i+\epsilon s}2K\left(6|\lambda-\mu|\,\|\phi\|+|\Phi^{\lambda}-\Phi^{\mu}|\,\|\phi\|\right)\\
\le{}& 3^qcK(2K)^{q+1}\left(9|\lambda-\mu|\|\phi\|+2\|\phi\||\Phi^{\lambda}-\Phi^{\mu}|\right)\\
&\times e^{(b+(\epsilon-a)(q+1)-\beta q)s}\int_s^{\infty}e^{(-b+\epsilon+a(q+1))\tau}d\tau\\
&+2K^2\delta\left(6|\lambda-\mu|\,\|\phi\|+\|\phi\||\Phi^{\lambda}-\Phi^{\mu}|\right)e^{(b-a+\epsilon)s}\sum_{s\le\tau_i}e^{(-b-2\epsilon+a)\tau_i}\\
\le{}& L_1(c,\delta)|\lambda-\mu|\|\phi\|+L_2(c,\delta)\|\phi\||\Phi^{\lambda}-\Phi^{\mu}|,
\end{aligned}
$$

for some $L_1,L_2>0$. Therefore taking $c,\delta$ sufficiently small, for each $(s,\phi)\in Z_\beta$ we have

$$
\|\Phi^{\lambda}(s,\phi)-\Phi^{\mu}(s,\phi)\|\le|\lambda-\mu|\|\phi\|. \tag{4.19}
$$

Now observe that

$$
\begin{aligned}
&|f(\tau,u_\tau,\Phi^{\lambda}(\tau,u_\tau),\lambda)-f(\tau,u'_\tau,\Phi^{\mu}(\tau,u'_\tau),\mu)|\\
&\le c|\lambda-\mu|\,\|(u_\tau,\Phi^{\lambda}(\tau,u_\tau))\|^{q+1}+c\|(u_\tau,\Phi^{\lambda}(\tau,u_\tau))-(u'_\tau,\Phi^{\mu}(\tau,u'_\tau))\|\\
&\quad\times\left(\|(u_\tau,\Phi^{\lambda}(\tau,u_\tau))\|^q+\|(u'_\tau,\Phi^{\mu}(\tau,u'_\tau))\|^q\right)\\
&\le 3^{q+1}c|\lambda-\mu|\,\|u_\tau\|^{q+1}+3^qc\left(3\|u_\tau-u'_\tau\|+\|\Phi^{\lambda}(\tau,u'_\tau)-\Phi^{\mu}(\tau,u'_\tau)\|\right)\\
&\quad\times(\|u_\tau\|^q+\|u'_\tau\|^q)\\
&\le 3^{q+1}c|\lambda-\mu|\,\|u_\tau\|^{q+1}+3^qc\,(3\|u_\tau-u'_\tau\|+|\lambda-\mu|\,\|u'_\tau\|)\,(\|u_\tau\|^q+\|u'_\tau\|^q)\\
&\le(6K)^{q+1}ce^{a(q+1)(\tau-s)+\epsilon(q+1)s}e^{-\beta qs}\|\phi\||\lambda-\mu|\\
&\quad+2(6K)^qce^{aq(\tau-s)+\epsilon qs}e^{-\beta qs}\Big(3\|u_\tau-u'_\tau\|+2Ke^{a(\tau-s)+\epsilon s}\|\phi\||\lambda-\mu|\Big)\\
&\le 2K(6K)^qce^{a(q+1)(\tau-s)}e^{\{\epsilon(q+1)-\beta q\}s}\left(5\|\phi\||\lambda-\mu|+6\|u-u'\|_*\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|I_i(u_{\tau_i},\Phi^{\lambda}(\tau_i,u_{\tau_i}),\lambda)-I_i(u'_{\tau_i},\Phi^{\mu}(\tau_i,u'_{\tau_i}),\mu)|\\
&\le\delta e^{-3\epsilon\tau_i}\left(\|u_{\tau_i}-u'_{\tau_i}\|+\|\Phi^{\lambda}(\tau_i,u_{\tau_i})-\Phi^{\mu}(\tau_i,u'_{\tau_i})\|\right)\\
&\quad+\delta e^{-3\epsilon\tau_i}|\lambda-\mu|\|(u'_{\tau_i},\Phi^{\mu}(\tau_i,u'_{\tau_i}))\|\\
&\le\delta e^{-3\epsilon\tau_i}\left(3\|u_{\tau_i}-u'_{\tau_i}\|+\|\Phi^{\lambda}(\tau_i,u'_{\tau_i})-\Phi^{\mu}(\tau_i,u'_{\tau_i})\|+3|\lambda-\mu|\|u'_{\tau_i}\|\right)\\
&\le\delta e^{-3\epsilon\tau_i}\left(3\|u_{\tau_i}-u'_{\tau_i}\|+|\lambda-\mu|\,\|u'_{\tau_i}\|\right)+3\delta e^{-3\epsilon\tau_i}|\lambda-\mu|\|u'_{\tau_i}\|\\
&\le 2K\delta e^{a(\tau_i-s)+\epsilon s-3\epsilon\tau_i}\left(4|\lambda-\mu|\,\|\phi\|+3\|u-u'\|_*\right).
\end{aligned}
$$

Now using the above estimates, we obtain

$$
\begin{aligned}
\|u_t-u'_t\|&\le\int_s^t\|\hat{T}(t,\tau)P(\tau)\|\,|f(\tau,u_\tau,\Phi^{\lambda}(\tau,u_\tau),\lambda)-f(\tau,u'_\tau,\Phi^{\mu}(\tau,u'_\tau),\mu)|\,d\tau\\
&\quad+\sum_{s\le\tau_i<t}\|\hat{T}(t,\tau_i)P(\tau_i)\||I_i(u_{\tau_i},\Phi^{\lambda}(\tau_i,u_{\tau_i}),\lambda)-I_i(u'_{\tau_i},\Phi^{\mu}(\tau_i,u'_{\tau_i}),\mu)|\\
&\le\int_s^tKe^{a(t-\tau)+\epsilon\tau}2K(6K)^qce^{a(q+1)(\tau-s)}e^{\{\epsilon(q+1)-\beta q\}s}\\
&\quad\times(5\|\phi\||\lambda-\mu|+6\|u-u'\|_*)\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{s\le\tau_i<t} Ke^{a(t-\tau_i)+\epsilon\tau_i} 2K\delta e^{a(\tau_i-s)+\epsilon s-3\epsilon\tau_i} \left(4|\lambda-\mu|\,\|\phi\| + 3\|u-u'\|_*\right) \\
&\le \Big( 2K(6K)^q c \left(5\|\phi\||\lambda-\mu| + 6\|u-u'\|_*\right) e^{(-aq-\epsilon)s} \int_s^t e^{(\epsilon+aq)\tau} d\tau \\
&+ 2K\delta \left(4|\lambda-\mu|\,\|\phi\| + 3\|u-u'\|_*\right) \sum_{s\le\tau_i<t} e^{-2\epsilon\tau_i} \Big) 2Ke^{a(t-s)+\epsilon s}
\end{aligned}
$$

$$
\begin{aligned}
\|u-u'\|_* \le\ & \frac{2K(6K)^q c}{|aq+\epsilon|} \left(5\|\phi\||\lambda-\mu| + 6\|u-u'\|_*\right) \\
&+ 2K\delta r_{2\epsilon} \left(4|\lambda-\mu|\,\|\phi\| + 3\|u-u'\|_*\right).
\end{aligned}
$$

Adjusting the coefficients we obtain the desired result,

$$
\|u_t - u'_t\| \le K_3 e^{a(t-s)+\epsilon s} |\lambda-\mu|\,\|\phi\|.
$$

□

*Proof of Theorem 4.1.* In the view of the required forward invariance property (4.5), to give the existence of a stable manifold, it is equivalent to find a function $\Phi \in \chi^*$ satisfying the equations (4.9) and (4.15). It follows from Lemma 4.2 that for each $(s,\phi,\lambda,\Phi) \in Z_\beta \times Y \times \chi^*$ there exists a unique function $u = u_\phi$ satisfying (4.9). Now using the function obtained from Lemma 4.2 we have the existence of $\Phi \in \chi^*$ satisfying (4.15) from Lemma 4.5 and Lemma 4.6. More precisely, we showed that for each $s \ge 0$, $\lambda \in Y$ and $\phi \in Z_{\beta+\epsilon}(2K)$, there exists a unique function $\Phi = \Phi^\lambda$ satisfying (4.15). To verify forward invariance of stable manifold, it is sufficient to note that by (4.11), if $(s,\phi) \in Z_{\beta+\epsilon}(2K)$ then

$$
\begin{aligned}
\|u_t\| \le 2Ke^{a(t-s)+\epsilon s}\|\phi\| &\le 2Ke^{a(t-s)+\epsilon s} \frac{1}{2K} e^{-(\beta+\epsilon)s} = e^{a(t-s)-\beta s} \\
&\le e^{(a+\beta)(t-s)-\beta t} \le e^{-\beta t}.
\end{aligned}
$$

Therefore it shows that $(s,u_t) \in Z_\beta$ for every $t \ge s$ and hence by (4.2),

$$
\Psi^\lambda_{t-s}(s,\phi,\Phi(s,\phi)) = (t,u_t,\Phi(t,u_t)) \in W_\lambda, \tag{4.20}
$$

for all $t > 0$. Identity (4.20) follows from equation (4.9) in Lemma 4.2, Lemma 4.3 and equation (4.15) in Lemma 4.5, which, up to the replacement of $Z_\beta$ by $Z_{\beta+\epsilon}(2K)$ shows that (4.5) holds.

Now for each $(s,\phi),(s,\psi) \in Z_{\beta+\epsilon}(2K); \lambda,\mu \in Y$ and $\kappa = t-s \ge 0$, by Lemma 4.3 and Lemma 4.7 we have

$$
\begin{aligned}
&\|\Psi^\lambda_\kappa(s,\phi,\Phi^\lambda(s,\phi)) - \Psi^\lambda_\kappa(s,\psi,\Phi^\lambda(s,\psi))\| \\
&\le \|(t,u_t^{\lambda,\phi},\Phi^\lambda(t,u_t^{\lambda,\phi})) - (t,u_t^{\lambda,\psi},\Phi^\lambda(t,u_t^{\lambda,\psi}))\| \\
&\le 3\|u_t^{\lambda,\phi} - u_t^{\lambda,\psi}\| \\
&\le 3K_1 e^{a(t-s)+\epsilon s}\|\phi-\psi\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\Psi^\lambda_\kappa(s,\phi,\Phi^\lambda(s,\phi)) - \Psi^\mu_\kappa(s,\phi,\Phi^\mu(s,\phi))\| \\
&\le \|(t,u_t^{\lambda,\phi},\Phi^\lambda(t,u_t^{\lambda,\phi})) - (t,u_t^{\mu,\phi},\Phi^\mu(t,u_t^{\mu,\phi}))\| \\
&\le 3\|u_t^{\lambda,\phi} - u_t^{\mu,\phi}\| + \|\Phi^\lambda(t,u_t^{\lambda,\phi}) - \Phi^\mu(t,u_t^{\lambda,\phi})\| \\
&\le De^{a(t-s)+\epsilon s}|\lambda-\mu|\,\|\phi\|.
\end{aligned}
$$

This completes the proof. □

**Example 4.8** ([6]). Consider the differential equation

$$x' = (-\omega + \epsilon\, t \cos t)x, \tag{4.21}$$

for constants $\omega > \epsilon > 0$. The solution is $x(t) = T(t,s)x(s)$ where

$$T(t,s) = e^{-\omega(t-s)+\epsilon(t \sin t - s \sin s + \cos t - \cos s)}.$$

Note that for every $t \ge s \ge 0$,

$$T(t,s) \le e^{2\epsilon}\, e^{(-\omega+\epsilon)(t-s)+2\epsilon s},$$

this means that the solution satisfies a *nonuniform exponential dichotomy.*

Now we introduce impulses in the above equation. Consider the differential equation (4.21) with impulses at times $\{\tau_i\}_{i=1}^{\infty}$ where $\{\tau_i\}_{i=1}^{\infty}$ is strictly increasing with $\lim_{i\to\infty} \tau_i \to \infty$ and the jumps satisfy (2.7). For this problem, let $\omega > \epsilon + p\log(1+e^{-5\epsilon})$. The solution to this problem is

$$x(t) = T(t,s)(1+e^{-5\epsilon})^{\operatorname{card}\{i\in\mathbb{N}:\tau_i<t\}} x(s).$$

We denote the solution operator $\hat{T}(t,s) = T(t,s)(1+e^{-5\epsilon})^{\operatorname{card}\{i\in\mathbb{N}:\tau_i<t\}}$. Now using (2.7), we have

$$\begin{aligned}
\|\hat{T}(t,s)\| &= \|T(t,s)\|(1+e^{-5\epsilon})^{\operatorname{card}\{i\in\mathbb{N}:\tau_i<t\}} \\
&\le Ce^{2\epsilon}\, e^{-\omega(t-s)+2\epsilon s} e^{p\log(1+e^{-5\epsilon})t} \\
&\le C_1 e^{-(\omega-\epsilon-p\log(1+e^{-5\epsilon}))(t-s)} e^{\left(2\epsilon+p\log(1+e^{-5\epsilon})\right)s}.
\end{aligned}$$

This shows that the differential equation (4.21) with impulses also satisfies *nonuniform exponential dichotomy.* Next, we consider the delay in the nonhomogeneous term.

**Example 4.9.** Consider the delay impulsive equation

$$\begin{aligned}
x' &= (-\omega + \epsilon t \cos t)x + (\lambda_1 + 2\sin t)y^2(t-1), \\
y' &= (\omega - \epsilon t \sin t)y + (\lambda_2 + \cos t)x^2(t-1),
\end{aligned}$$

with impulses

$$\Delta x(\tau_i) = \lambda_1\, x_{\tau_i} e^{-5\epsilon\tau_i}, \quad \Delta y(\tau_i) = \lambda_2\, y_{\tau_i} e^{-7\epsilon\tau_i},$$

where $\lambda = (\lambda_1, \lambda_2) \in [-1,1]\times[-1,1]$ and $\omega > \epsilon > 0$. For each $\phi = (\phi_1, \phi_2) \in \hat{\mathcal{B}}$, let

$$\begin{aligned}
f(t,\phi,\lambda) &= \big(\, (\lambda_1 + 2\sin t)\, \phi_2^2(t-1), (\lambda_2 + \cos t)\, \phi_1^2(t-1)\, \big), \\
I_i(u_{\tau_i}, \lambda) &= (\lambda_1\, x_{\tau_i} e^{-5\epsilon\tau_i}, \lambda_2\, y_{\tau_i} e^{-7\epsilon\tau_i}).
\end{aligned}$$

The Evolution Operator $T(t,s)$ associated with above problem is

$$T(t,s) = \begin{bmatrix} U(t,s) & 0 \\ 0 & V(t,s) \end{bmatrix},$$

where

$$\begin{aligned}
U(t,s) &= e^{-\omega(t-s)+\epsilon\,(t\sin t - s\sin s + \cos t - \cos s)}, \\
V(t,s) &= e^{\omega(t-s)-\epsilon(-t\cos t + s\cos s + \sin t - \sin s)}.
\end{aligned}$$

Now let $P(t)(x,y) = (x,0)$. It is easy to verify that

$$\|T(t,s)P(t)\| = \|U(t,s)\| \le e^{2\epsilon} e^{(-\omega+\epsilon)(t-s)+2\epsilon s},$$

$$\|T(t,s)^{-1}Q(t)\| = \|V(t,s)^{-1}\| \le e^{2\epsilon} e^{(-\omega-\epsilon)(t-s)+2\epsilon t}$$

for $t \ge s \ge 0$. This shows that the corresponding linear equation admits a nonuniform exponential dichotomy with $a = (-\omega + \epsilon) < 0,\ b = \omega + \epsilon > 0$. Now, when we consider the impulses, the corresponding solution operator $\hat{T}(t,s)$ also satisfies nonuniform exponential dichotomy.

Moreover, the perturbation $f(t,\phi,\lambda)$ satisfies the conditions (2.8), (2.9) for $q = 1$ and $f(t,0,\lambda) = 0$. Also the impulse function $I_i(x_{\tau_i},\lambda)$, satisfies the conditions $I_i(0,\lambda) = 0$, (2.10) and (2.11). For this example we choose $\omega > 4\epsilon + 1$ so that the condition $aq + \epsilon < 0$ in Theorem 3.1 is satisfied and also for Theorem 4.1 $a + \beta = -\omega + 4\epsilon + 1 < 0$ is satisfied.

It follows from Theorem (4.1) that Example 4.9 has a Lipschitz stable invariant manifold.

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