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NONLOCAL APPROACH TO PROBLEMS ON LONGITUDINAL VIBRATION IN A SHORT BAR

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Abstract. In this article, we consider a problem with dynamic nonlocal conditions for a forth-order PDE with dominating mixed derivative. This problem is closely related to vibration problems, in particular, to longitudinal vibration in a short bar. The existence and uniqueness of a generalized solution are proved.

1. Introduction

We study a nonlocal problem for a forth-order PDE with dominating mixed derivative

$$\mathcal{L}u \equiv \sigma(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\Big(a(x)\frac{\partial u}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(b(x)\frac{\partial^3 u}{\partial t^2 \partial x}\Big) = f(x,t). \tag{1.1}$$

This equation is closely related to the problem of longitudinal vibration of a short thick bar. Vibration problems are of great importance in engineering and have been studied by many researches. The majority of works deals with second order hyperbolic equation. Initial-boundary problems for wave equation has been studied comprehensively and became classical [19].

However this model is not strictly correct for vibration of a thick short bar as is shown by Rayleigh [18]. But many machine components may be interpreted just as a thick short bar. For a more precise analysis of the longitudinal vibrations in a thick short bar we need to take into account the transverse deformations. Mathematical model of longitudinal vibration considering the effect of transverse movements in a thick short bar is called *Rayleigh bar* and is based on the equation (1.1). Some results of studying of initial-boundary problems for (1.1) can be find in [3, 6].

In this article we do a next step to make this model more precise. To this end we propose to define more exactly boundary conditions from the following reasoning. The assumption on dimension of the bar suggests that there exists certain connection between values of a required solution in different boundary points. Such effect was found by Steclov [20] for heat equation. A relation connecting values of a solution to a PDE in various boundary points is a nonlocal condition.

Thus we suggest a nonlocal approach to study longitudinal vibration of a short thick bar. Note that nonlocal approach is in agreement with survey and results of

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experiments analyzed in [2] and turn out to be often more precise in mathematical modeling. Motivated by this, we consider the problem with nonlocal dynamical boundary conditions [1, 5, 7, 8, 9, 10, 11, 12, 14, 16, 21].

Note that there is close connection between nonlocal boundary conditions of the form to be dealt with below and nonlocal integral conditions [4].

2. Statement of the problem

Consider the longitudinal vibration of a thick short bar. Suppose that the bar represents the solid of revolution around the axis Ox. Denote by $u(x,t)$ the longitudinal displacements subject to determination. Let the exciting distributed force be $f(x,t)$. Suppose that the left and right ends of the bar, $x=0$ and $x=l$, are attached to the immovable ground with the help of the point masses M_1, M_2 and springs. In addition we take into account the resistance of medium. The latter implies the presence of u_t in the boundary conditions. Lagrangian of Rayleigh bar is constructed in [17, p. 158-184]. Hamilton variational principle and elementary manipulation lead to the equation

$$\sigma(x)u_{tt} - (a(x)u_x)_x - (b(x)u_{ttx})_x = f(x,t), \tag{2.1}$$

where

$$\sigma(x) = \rho(x)A(x), \quad a(x) = A(x)E(x), \quad b(x) = \rho(x)\nu^2(x)I_p(x),$$

$A(x)$ is the cross-section area, $\rho(x)$ is the mass density of the bar, $E(x)$ is Young's modulus, $I_p(x)$ is the polar moment of inertia, ν is the Poisson coefficient.

The main object of this article is the following problem: find in $Q_T = (0,l) \times (0,T)$ a solution to (2.1) satisfying the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0 \tag{2.2}$$

and the nonlocal boundary conditions

$$\begin{gathered} a(0)u_x(0,t) + b(0)u_{xtt}(0,t) = \\ \alpha_{11}u(0,t) + \alpha_{12}u(l,t) + \beta_{11}u_t(0,t) + \beta_{12}u_t(l,t) + M_1u_{tt}(0,t), \\ a(l)u_x(l,t) + b(l)u_{xtt}(l,t) = \\ \alpha_{21}u(0,t) + \alpha_{22}u(l,t) + \beta_{21}u_t(0,t) + \beta_{22}u_t(l,t) - M_2u_{tt}(l,t). \end{gathered} \tag{2.3}$$

Some particular cases of the problem (2.1)–(2.3), namely when $\alpha_{12} = \alpha_{21} = \beta_{ij} = 0$ and for special form of coefficients of (2.1), are considered in [6]. In [3] the generalized solvability of (2.1)–(2.3) when $\alpha_{12} = \alpha_{21} = \beta_{ij} = 0$ is proved. The main goal of our paper is to determine conditions under which there exists a unique solution to the problem (2.1)–(2.3), that is to the problem with nonlocal dynamical conditions.

To prove solvability of nonlocal problem (2.1)–(2.3) we suggest an approach which enables us to use many well-known techniques. We define a notion of a weak solution for (2.1)–(2.3) and show that under some assumptions on data there exists a unique weak solution.

It is convenient here to list main assumptions on the data.

- (H1) $a, b, \sigma \in C^1[0,l]$, $a(x) \ge a_0 > 0$, $b(x) \ge b_0 > 0, \sigma(x) \ge \sigma_0 > 0$;
- (H2) $f, f_t \in C(\bar{Q}_T)$;
- (H3) $M_i > 0$, $i = 1, 2$.

Remark 2.1. Positiveness of coefficients a, b, σ and M_i is a consequence of physical significance of them.

Remark 2.2. We consider homogeneous initial conditions only to simplify calculations and without loss of generality.

Denote

$$\Gamma_0 = \{(x,t) : x = 0, t \in [0,T]\}, \quad \Gamma_l = \{(x,t) : x = l, t \in [0,T]\},$$
$$W(Q_T) = \{u : u \in W_2^1(Q_T),\ u_{xt} \in L_2(Q_T),\ u_t \in L_2(\Gamma_0 \cup \Gamma_l)\},$$
$$V(Q_T) = \{v : v \in W(Q_T),\ v(x,T) = 0\}.$$

Now we define a solution of the problem using a standard method [13, p. 92]: integrating by parts an identity $\int_0^T \int_0^l (Lu - f)v\,dx\,dt = 0$ where $u(x,t)$ satisfies (2.1)–(2.3), $v \in C^2(Q_T) \cap C^1(\bar{Q}_T)$ we obtain the equality

$$\begin{aligned}
&\int_0^T \int_0^l \left(-\sigma(x)u_t v_t + a(x)u_x v_x - b(x)u_{xt}v_{xt}\right) dx\,dt \\
&+ \int_0^T v(0,t)[\alpha_{11}u(0,t) + \alpha_{12}u(l,t) + \beta_{11}u_t(0,t) + \beta_{12}u_t(l,t)]dt \\
&- \int_0^T v(l,t)[\alpha_{21}u(0,t) + \alpha_{22}u(l,t) + \beta_{21}u_t(0,t) + \beta_{22}u_t(l,t)]dt \\
&- M_1 \int_0^T u_t(0,t)v_t(0,t)dt - M_2 \int_0^T u_t(l,t)v_t(l,t)dt \\
&= \int_0^T \int_0^l fv\,dx\,dt.
\end{aligned} \tag{2.4}$$

Note that all integrals in (2.4) exist also for $u \in W(Q_T)$, $v \in V(Q_T)$. Hence, (2.4) is suitable for a definition of a generalized solution to the problem (2.1)–(2.3).

Definition 2.3. A function $u \in W(Q_T)$ is said to be a weak solution to the problem (2.1)–(2.3) if $u(x,0) = 0$ and for every $v \in V(Q_T)$ the identity (2.4) holds.

3. Main results

Theorem 3.1. *Under assumptions* (H1)–(H3) *there exists a unique weak solution to problem* (2.1)–(2.3) *if*

$$\alpha_{11}\xi_1^2 + 2\alpha_{12}\xi_1\xi_2 - \alpha_{22}\xi_2^2 \ge 0, \quad \xi = (\xi_1, \xi_2) \in R^2.$$

Proof. Uniqueness. Let u_1, u_2 be two weak solutions of (2.1)–(2.3). Then $u = u_1 - u_2$ satisfies $u(x,0) = 0$ and the identity

$$\begin{aligned}
&\int_0^T \int_0^l \left(-\sigma(x)u_t v_t + a(x)u_x v_x - b(x)u_{xt}v_{xt}\right) dx\,dt \\
&+ \int_0^T v(0,t)[\alpha_{11}u(0,t) + \alpha_{12}u(l,t) + \beta_{11}u_t(0,t) + \beta_{12}u_t(l,t)]dt \\
&- \int_0^T v(l,t)[\alpha_{21}u(0,t) + \alpha_{22}u(l,t) + \beta_{21}u_t(0,t) + \beta_{22}u_t(l,t)]dt \\
&- M_1 \int_0^T u_t(0,t)v_t(0,t)dt - M_2 \int_0^T u_t(l,t)v_t(l,t)dt = 0
\end{aligned} \tag{3.1}$$

holds. Let

$$v(x,t)=\begin{cases}\int_\tau^t u(x,\eta)d\eta, & 0\le t\le\tau,\\ 0, \tau\le t\le T\end{cases} \tag{3.2}$$

where $\tau\in[0,T]$ is arbitrary. After integrating (3.1) by parts we obtain

$$\begin{aligned} &\int_0^l [\sigma u^2(x,\tau)+av_x^2(x,0)+bu_x^2(x,\tau)]dx\\ &+\alpha_{11}v^2(0,0)-\alpha_{22}v(0,0)v(l,0)+M_1u_t^2(0,\tau)+M_2u_t^2(l,\tau)\\ &=-2\beta_{11}\int_0^\tau u^2(0,t)dt-2(\beta_{12}-\beta_{21})\int_0^\tau u(0,t)u(l,t)dt\\ &\quad+2\beta_{22}\int_0^\tau u^2(l,t)dt-2(\alpha_{12}+\alpha_{21})\int_0^\tau u(0,t)u(l,t)dt. \end{aligned} \tag{3.3}$$

Under the assumptions of this Theorem,

$$\alpha_{11}v^2(0,0)-\alpha_{22}v^2(l,0)+2\alpha_{12}v(0,0)v(l,0)\ge 0,\quad M_1u_t^2(0,\tau)+M_2u_t^2(l,\tau)>0.$$

We consider the right side of (3.3) and estimate each term. To this end we use Cauchy inequality and obtain

$$2\Big|\int_0^\tau u(0,t)u(l,t)dt\Big|\le\int_0^\tau[u^2(0,t)+u^2(l,t)]dt, \tag{3.4}$$

$$2\Big|\int_0^\tau u(0,t)v(l,t)dt\Big|\le\int_0^\tau[u^2(0,t)+v^2(l,t)]dt. \tag{3.5}$$

Thus from (3.3),

$$\begin{aligned} &\int_0^l [\sigma u^2(x,\tau)+av_x^2(x,0)+bu_x^2(x,\tau)]dx\\ &\le(2|\beta_{11}|+|\beta_{12}|+|\beta_{21}|+|\alpha_{12}|+|\alpha_{21}|)\int_0^\tau u^2(0,t)dt\\ &\quad+(2|\beta_{22}|+|\beta_{12}|+|\beta_{21}|)\int_0^\tau u^2(l,t)dt+(|\alpha_{12}|+|\alpha_{21}|)\int_0^\tau v^2(l,t)dt. \end{aligned}$$

To proceed with the estimate, we derive some inequalities. As for any $u\in W(Q_T)$ representations

$$u(0,t)=\int_x^0 u_\xi(\xi,t)d\xi+u(x,t),\quad u(l,t)=\int_x^l u_\xi(\xi,t)d\xi+u(x,t)$$

hold we easily get the inequalities

$$u^2(0,t)\le 2l\int_0^l u_x^2(x,t)dx+2u^2(x,t),\quad u^2(l,t)\le 2l\int_0^l u_x^2(x,t)dx+2u^2(x,t).$$

Integrating both with respect to x over $(0,l)$ we obtain

$$u^2(z_i,t)\le 2l\int_0^l u_x^2(x,t)dx+\frac{2}{l}\int_0^l u^2(x,t)dx,\quad i=0,1,\ z_0=0,\ z_1=l. \tag{3.6}$$

Using the same procedure we obtain

$$v^2(l,t)\le 2l\int_0^l v_x^2(x,t)dx+\frac{2}{l}\int_0^l v^2(x,t)dx.$$

From ((3.2)) it follows that

$$v^2(x,t) \le \tau \int_0^\tau u^2(x,t)dt, \quad v_x^2(x,t) \le \tau \int_0^\tau u_x^2(x,t)dt,$$

then

$$v^2(l,t) \le 2l\tau \int_0^\tau \int_0^l u_x^2(x,t)\,dx\,dt + \frac{2\tau}{l} \int_0^\tau \int_0^l u^2(x,t)\,dx\,dt. \tag{3.7}$$

Denote $A = |\alpha_{12}| + |\alpha_{21}|$, $B = \sum_{i,j=1}^2 |\beta_{ij}|$,

$$m_0 = \min\{a_0, b_0, \sigma_0\}, \quad M = 2\max\{Bl + AlT, \frac{B + AT}{l}\}.$$

Taking into account (3.6) and (3.7) , from (3.3) we obtain

$$m_0 \int_0^l [u^2(x,\tau) + v_x^2(x,0) + u_x^2(x,\tau)]dx \le M \int_0^\tau \int_0^l (u^2 + u_x^2)\,dx\,dt$$

and therefore

$$m_0 \int_0^l [u^2(x,\tau) + u_x^2(x,\tau)]dx \le M \int_0^\tau \int_0^l (u^2 + u_x^2)\,dx\,dt.$$

Thus from Gronwall's inequality, we have $u(x,\tau) = 0$ for all $\tau \in [0,T]$. Hence there exists at most one weak solution to the problem (2.1)–(2.3).

Existence. We prove the existence part in several steps. First, we construct approximations of the generalized solution by the Faedo-Galerkin method. Second, we obtain a priori estimates to guarantee weak convergence of approximations. Finally, we show that the limit of approximations is the required solution.

Let $w_k(x) \in C^2(\bar\Omega)$ be a basis in $W_2^1(\Omega)$. We define approximations as follows,

$$u^m(x,t) = \sum_{k=1}^m c_k(t) w_k(x) \tag{3.8}$$

and shall seek $c_k(t)$ from relations

$$\begin{aligned}
&\int_0^l \left(\sigma u_{tt}^m w_j + a u_x^m w_j' + b u_{xtt}^m w_j'\right) dx + M_1 u_{tt}^m(0,t) w_j(0) - M_2 u_{tt}^m(l,t)] w_j(l) \\
&+ [\alpha_{11} u^m(0,t) + \alpha_{12} u^m(l,t) + \beta_{11} u_t^m(0,t) + \beta_{12} u_t^m(l,t)] w_j(0) \\
&- [\alpha_{21} u^m(0,t) + \alpha_{22} u^m(l,t) + \beta_{21} u_t^m(0,t) + \beta_{22} u_t^m(l,t)] w_j(l) \\
&= \int_0^l f w_j dx.
\end{aligned} \tag{3.9}$$

For every m the relations (3.9) represent a system of second-order ODE's with respect to $c_k(t)$ and after substituting (3.8) we can rewrite it in the form

$$\sum_{k=1}^m [A_{kj} c_k''(t) + B_{kj} c_k'(t) + D_{kj} c_k(t)] = f_j(t), \tag{3.10}$$

where

$$A_{kj} = \int_0^l (\sigma w_k w_j + b w_k' w_j') dx + M_1 w_k(0) w_j(0) + M_2 w_k(l) w_j(l);$$
$$B_{kj} = \beta_{11} w_k(0) w_j(0) + \beta_{12} w_k(l) w_j(0) - \beta_{21} w_k(0) w_j(l) - \beta_{22} w_k(l) w_j(l);$$

$$
\begin{aligned}
D_{kj} = \int_0^l a w_k' w_j' dx + \alpha_{11} w_k(0) w_j(0) + \alpha_{12} w_k(l) w_j(0) \\
- \alpha_{21} w_k(0) w_j(l) - \alpha_{22} w_k(l) w_j(l);
\end{aligned}
$$

$$
f_j(t) = \int_0^l f(x,t) w_j(x) dx.
$$

Adding the initial data,

$$
c_k(0) = 0, \quad c_k'(0) = 0 \tag{3.11}
$$

we obtain Cauchy problem for (3.10). Now we show that Cauchy problem (3.10)–(3.11) is solvable.

Consider a matrix $\mathcal{A} = (A_{kj})_{k,j=1}^m$ and verify that it is positive definite. To this end we introduce a quadratic form

$$
q = \sum_{k,j=1}^m A_{kj} \xi_k \xi_j,
$$

where ξ_k, ξ_j are coefficients of sums $\xi = \sum_{i=1}^m \xi_i w_i(x)$. Rearrange this quadratic form using representations of the coefficients A_{ij}:

$$
q = \sum_{k,j=1}^m \int_0^l (\sigma w_k w_j \xi_k \xi_j + b w_k' w_j' \xi_k \xi_j) dx + M_1 w_k(0) w_j(0) \xi_k \xi_j + M_2 w_k(l) w_j(l) \xi_k \xi_j.
$$

After changing the order of summing and integrating we obtain

$$
q = \int_0^l \left(\sigma |\xi|^2 + b |\xi_x|^2 \right) dx + M_1 |\xi(0)|^2 + M_2 |\xi(l)|^2.
$$

We know that σ, b, M_1, M_2 are positive. Now note that quadratic form q vanishes only if $\xi = 0$. Hence $\xi_k = 0$ $\forall k = 1, \dots, m$ by virtue of linearity independence of $w_k(x)$. Consequently the matrix $\mathcal{A}$ is positive definite and the system (3.10) is solvable with respect to $c_k''(t)$. The conditions of Theorem imply that the coefficients of (3.10) are bounded and $f \in L_2(Q_T)$. These facts guarantee the solvability of Cauchy problem (3.10)–(3.11). Moreover, $c_k'' \in L_2(0,T)$. Thus, the approximation $\{u^m(x,t)\}$ is constructed.

We need now to derive an a priori estimate. To this end we multiply (3.9) by $c_j'(t)$, sum with respect to $j = 1, \dots, m$, integrate over $(0, \tau)$ and obtain

$$
\begin{aligned}
&\int_0^\tau \int_0^l (\sigma(x) u_{tt}^m u_t^m + a(x) u_x^m u_{xt}^m + b(x) u_{xtt}^m u_{xt}^m) \, dx \, dt \\
&+ \int_0^\tau \Big[\alpha_{11} u^m(0,t) u_t^m(0,t) + \alpha_{12} u^m(l,t) u_t^m(0,t) + \beta_{11} (u_t^m(o,t))^2 \\
&+ \beta_{12} u_t^m(0,t) u_t^m(l,t) \Big] dt - \int_0^\tau [\alpha_{21} u^m(0,t) u_t^m(l,t) + \alpha_{22} u^m(l,t) u_t^m(l,t) \\
&+ \beta_{21} u_t^m(0,t) u_t^m(l,t) + \beta_{22} (u_t^m(l,t))^2] dt + M_1 \int_0^\tau u_{tt}^m(0,t) u_t^m(0,t) dt \\
&- M_2 \int_0^\tau u_{tt}^m(l,t) u_t^m(l,t) dt + (\alpha_{12} + \alpha_{21}) \int_0^\tau u_t^m(l,t) u^m(0,t) dt \\
&= \int_0^\tau \int_0^l f u_t^m \, dx \, dt.
\end{aligned} \tag{3.12}
$$

After integrating by parts in (3.12) we obtain

$$\begin{aligned}
&\int_0^l [\sigma(x)(u_t^m(x,\tau))^2 + a(x)(u_x^m(x,\tau))^2 + b(x)(u_{xt}^m(x,\tau))^2]dx \\
&+ \alpha_{11}(u^m(0,\tau))^2 + 2\alpha_{12}u^m(0,\tau)u^m(l,\tau) - \alpha_{22}(u^m(l,\tau))^2 \\
&+ M_1(u_t^m(0,\tau))^2 + M_2(u_t^m(l,\tau))^2 \\
&= 2\beta_{22}\int_0^\tau (u_t^m(l,t))^2 dt - (\alpha_{12}+\alpha_{21})\int_0^\tau u_t^m(l,t)u^m(0,t)dt \\
&\quad + 2(\beta_{21}-\beta_{12})\int_0^\tau u_t^m(0,t)u_t^m(l,t)dt - 2\beta_{11}\int_0^\tau (u_t^m(0,t))^2 dt \\
&\quad + 2\int_0^\tau\int_0^l f u_t^m \,dx\,dt.
\end{aligned}$$

Under assumption (H1) the left-hand side of this equality is positive. Using Cauchy, Cauchy-Bunyakovskii inequalities and (3.6), (3.7) we derive from the last equality the inequality

$$\begin{aligned}
&m_0\int_0^l [(u_t^m(x,\tau))^2 + (u_x^m(x,\tau))^2 + (u_{xt}^m(x,\tau))^2]dx + \alpha_{11}(u^m(0,\tau))^2 \\
&+ 2\alpha_{12}u^m(0,\tau)u^m(l,\tau) - \alpha_{22}(u^m(l,\tau))^2 + M_1(u_t^m(0,\tau))^2 \\
&+ M_2(u_t^m(l,\tau))^2 \\
&\le M\int_0^\tau\int_0^l [(u_t^m)^2 + (u_x^m)^2 + (u_{xt}^m)^2]\,dx\,dt + \int_0^\tau\int_0^l f^2\,dx\,dt.
\end{aligned} \tag{3.13}$$

Applying Gronwall's inequality to (3.13) we obtain

$$\int_0^l [(u_t^m(x,\tau))^2 + (u_x^m(x,\tau))^2 + (u_{xt}^m(x,\tau))^2]dx \le m_0^{-1}e^{C\tau}\|f\|^2_{L_2(Q_\tau)}$$

where $C = M/m_0$. It is easy to see that from this inequality after integrating over $(0,T)$ we obtain

$$\int_0^T\int_0^l [(u_t^m(x,\tau))^2 + (u_x^m(x,\tau))^2 + (u_{xt}^m(x,\tau))^2]\,dx\,dt \le M^{-1}(e^{CT-1})\|f\|^2_{L_2(Q_T)}.$$

From (3.13) it also follows that

$$M_1\int_0^T (u_t^m(0,t))^2 dt + M_2\int_0^T (u_t^m(l,t))^2 dt \le Te^{CT}\|f\|^2_{L_2(Q_T)}.$$

As $f \in L_2(Q_T)$ then $\|f\|_{L_2(Q_T)}$ is finite: $\|f\|_{L_2(Q_T)} \le k$. Thus the obtained inequalities lead to the required estimate

$$\|u^m\|_{W(Q_T)} \le P \tag{3.14}$$

where $P = k^2\max\{M^{-1}(e^{CT}-1), Te^{CT}\}$ and does not depend on m.

As $W(Q_T)$ is Hilbert space then the estimate (3.14) enables state that we can extract from approximations $u^m(x,t)$ a subsequence weakly convergent in $W(Q_T)$. For technical reasons we do not change notation for it.

At a final step we show that the limit of extracted subsequence is the required weak solution to the problem (2.1)–(2.3).

Multiplying (3.9) by $d_j \in C^2[0,T]$, summing from $j=1$ to $j=m$ and integrating with respect to t from 0 to T we obtain

$$\begin{aligned} &\int_0^T \int_0^l [\sigma u^m_{tt}\eta + a u^m_x \eta_x + b u^m_{xtt}\eta_x]\,dx\,dt + \int_0^T \eta(0,t)\Big[\alpha_{11}u^m(0,t) \\ &+ \alpha_{12}u^m(l,t) + \beta_{11}u^m_t(0,t) + \beta_{12}u^m_t(l,t) + M_1 u^m_{tt}(0,t)\Big]dt \\ &+ \int_0^T \eta(l,t)\Big[\alpha_{21}u^m(0,t) + \alpha_{22}u^m(l,t) + \beta_{21}u^m_t(0,t) + \beta_{22}u^m_t(l,t) \\ &- M_2 u^m_{tt}(l,t)\Big]dt \\ &= \int_0^T \int_0^l f\eta\,dx\,dt \end{aligned} \tag{3.15}$$

where $\eta(x,t) = \sum_{j=1}^m d_j(t)w_j(x)$. Because of obtained estimates we are able to pass to the limit in (3.15) to obtain (3.1) for $v(x,t) = \eta(x,t)$. Taking into account that the set of all functions of the form $\sum_{j=1}^m d_j(t)w_j(x)$ is dense in $V(Q_T)$ we conclude that (3.1) holds for every $v \in V(Q_T)$. This means that $u(x,t)$, weak limit of the subsequence $u^m(x,t)$, is the required solution to the (2.1)–(2.3). The proof is complete. □

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References

[1] Andrews, K. T.; Kuttler, K. L.; Shillor M.; *Second order evolution equations with dynamic boundary conditions*, J. Math. Anal. Appl., 197 (3) (1996), pp. 781-795.
[2] Bažant, Zdeněk, P.; Jir'asek, Milan; *Nonlocal Integral Formulation of Plasticity And Damage: Survey of Progress*, American Society of Civil Engineers. Journal of Engineering Mechanics, 2002, pp. 1119-1149.
[3] Beylin, A. B., Pulkina, L. S.; *A problem on longitudinal vibration in a short bar with dynamical boundary conditions*, Vestnik of Samara State University, 2014, no.1, pp. 9–19.
[4] Beylin, A. B., Pulkina, L. S.; *A problem with nonlocal dynamical conditions for the equation of vibration in a thick bar*, Vestnik of Samara University, 2017, V. 23. no.4, pp. 7–18.
[5] Doronin, G. G., Larkin, N. A., Souza, A. J.; *A hyperbolic problem with nonlinear second-order boundary damping*, Electron. J. Differential Eqns., Vol.1998 (1998), no. 28, pp. 1-10.
[6] Fedotov, I.; Polyanin, A. D.; Shatalov, M.; *Theory of free vibration of rigid rod based on Rayleigh model.* Doklady Physics, 2007. V. 417. P. 56-61.
[7] Gordeziani, D.G .; Avalishvili, G. A.; *Solutions of Nonlocal Problems for One-dimensional Oscillations of the Medium*, Mat. Modelir., 2000, vol.12, no.1, p.94-103.
[8] Ilin, V. A.; Moiseev, E. I.; *Uniqueness of the solution of a mixed problem for the wave equation with nonlocal boundary conditions*, Differen. Equations, 2000, Vol. 36, Issue 5, Pages 728-733.
[9] Korpusov, O. M.; *Blow-up in nonclassical wave equations*, Moscow, URSS. 2010.
[10] Kozhanov, A. I.; Pulkina, L. S.; *On solvability of certain boundary problems with shift for linear hyperbolic equations*, Matem. Journal. Institute of Mathematics and Math. modelling. Almaty, Kazakhstan. 2009. Vol.9(32), p. 78–92.
[11] Kozhanov, A. I.; *On Solvability of Certain Spatially Nonlocal Boundary Problems for Linear Parabolic Equations*, Vestnik of Samara State University, 2008, no.3, pp. 165-174.
[12] Kozhanov, A. I.; *On the solvability of certain spatially nonlocal boundary-value problems for linear hyperbolic equations of second order*, Mathematical Notes, 2011, 90:2, pp. 238–249.
[13] Ladyzhenskaya, O. A.; *Boundary value problems of mathematical physics*, Moscow, 1973.
[14] Lazhetich, N. L.; *On the classical solvability of the mixed problem for a second-order one-dimensional hyperbolic equation*, Diff. Equations, 2006, Vol. 42, Issue 8, Pages 1134-1139.

[15] Pulkina, L. S.; *Solutions to nonlocal problems of pseudohyperbolic equations*, Electron. J. Differential Eqns., 2014 (2014), 116, p.1-9.
[16] Pul'kina, L. S.; *A problem with dynamic nonlocal condition for pseudohyperbolic equation*, Russian Mathematics, 2016, vol. 60, issue 9, p. 38-45.
[17] Rao, J. S.; *Advanced Theory of Vibration*, N.Y.: Wiley, 1992.
[18] Rayleigh, J. W. S.; *Theory of sound*, New York: Dover, 1945. (translated in Russian 1955, Moscow)
[19] Tikhonov, A. N., Samarskii, A. A.; *Equations of mathematical physics*, Moscow, 2004.
[20] Steclov, V. A.; *The problem of cooling in inhomogeneous solid*, Communications of Kharkov mathematical Society. 1896. Vol. 5, issue 3-4, p. 136-181.
[21] Zhang, Zhifei; *Stabilization of the wave equation with variable coefficients and a dynamical boundary control*, Electron. J. Differential. Equ., vol. 2016 (2016), no.27, pp. 1–10.

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