Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 31, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

REACHABILITY OF A SECOND-ORDER INTEGRO-DIFFERENTIAL EQUATION ON RIEMANNIAN MANIFOLD FOR A VISCOELASTICITY MODEL

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Communicated by Goong Chen

Abstract. We study a reachability problem for a second-order integro-differential equation on a finite-dimensional Riemannian manifold, which is a model equation for viscoelasiticity. We apply a Carleman estimate for the wave equations on Riemannian manifolds to establish the observability inequality.

1. Introduction

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold with metric $g(\cdot,\cdot)=\langle\cdot,\cdot\rangle$ and squared norm $|X|^2=g(X,X)$. Let Ω be an open bounded, connected, compact set of $\mathcal{M}$ with smooth boundary $\Gamma=\Gamma_0\cup\Gamma_1$, where Γ_0 is nonempty and relatively open in Γ. Let ν denote the outward unit normal field along the boundary Γ. Further, denote by Δ the Laplace-Beltrami operator on the manifold $\mathcal{M}$ and by D the Levi-Civita connection on $\mathcal{M}$, respectively.

Let $T>0$ be given. Set $Q=(0,T)\times\Omega$, $\Sigma=(0,T)\times\Gamma$, and $\Sigma_i=(0,T)\times\Gamma_i$ for $i=0,1$. We consider a reachability problem for the following second-order integro-differential equation

$$
\begin{gathered}
u_{tt}=\Delta u+\langle P(t),Du\rangle+p_0(t,x)u+\int_0^t p(t,s)\Delta u(s)ds \quad \text{in } Q,\\
u=0 \quad \text{on } \Sigma_1,\quad u=\phi \quad \text{on } \Sigma_0,\\
u(0)=u_t(0)=0 \quad \text{in } \Omega.
\end{gathered} \tag{1.1}
$$

Here $P(t)$ is a vector field on $\mathcal{M}$ for $t>0$, and p_0 and p are functions with $p_0\in L^\infty(\mathbb{R}\times\mathcal{M})$ and $p(t,s)\in C^2([0,\infty)^2)$. This equation is a model equation for viscoelasticity (see Prüss [20]). In these physical interpretations, $u(x,t)$ represent displacement from the natural state of the reference configuration at position x and time t. We ask whether the system (1.1) is reachable at time $T>0$ by $L^2(\Sigma_0)$ control. In other words, for given $(u_0,u_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, whether there exists a boundary control function $\phi\in L^2(\Sigma_0)$ that can drive the solution of (1.1) to the final state

$$
u(x,T)=u_0(x),\quad u_t(x,T)=u_1(x) \quad \text{in } \Omega. \tag{1.2}
$$

2010 *Mathematics Subject Classification.* 93B05, 93B27, 35L10, 58J45.
Key words and phrases. Reachability; integro-differential equation; Carleman estimates; Riemannian wave.

Submitted April 11, 2018. Published February 20, 2019.

Let us first review some known results on analogous problems. To begin with, regarding the generality of model (1.1), we notice that such an integro- differential equation on Riemannian manifold includes, in particular, a general second-order hyperbolic equation with a memory term defined on a Euclidean bounded domain, with principal part coefficients $a_{ij}(x)$ variable in space. There are lots of papers studied this second-order hyperbolic equation with a memory for the constant coefficients on a Euclidean domain. A reachability problem for a wave equation with a memory was presented by Lions [17]. Leugering [16] proved reachability for a plate equation with a memory by a harmonic analysis method in a rectangle domain under the assumption that the memory kernel is in the form of convolution. In Loreti and Sforza [18], the authors considered reachability for a wave equation with a special memory kernel $p(t,s) = \beta e^{-\eta(t-s)}$ in an open ball in $\mathbb{R}^n$ by using nonharmonic analysis techniques. For a similar equation in a general domain, Lasiecka [13] resolved reachability by a direct operator method. The argument is valid for a more general kernel that depends on time and space variables. For a general discussion of our problem, Kim [11] obtained a reachability property based on a new kind of unique continuation property and the observability inequalities for wave equations. However, these above conclusions are all constructed for the case of constant coefficients (i.e. $a_{ij}(x) = \delta_{ij}$, $\forall x \in \Omega$). For the variable coefficients case, a Riemannian geometry method was first introduced in Yao [23] to solve the exact controllability of wave equations.

The Riemannian geometry is a necessary tool for control of the wave equation with variable coefficients mainly due to its two virtues: The Bochner technique provides us with a great computations tool to obtain the multiplier identities, and the curvature theory provides the global information on the existence of an escape vector field which guarantees the exact controllability.

Considering the problem on a Riemannian manifold, the unique continuation property presented in Kim [11] is invalid. Moreover as long as equation (1.1) with integral terms is concerned, it is not easy to apply the general geometric multiplier $H(w)$ to obtain the observability inequalities directly.

A Carleman estimate is another important tool for the control problem for a hyperbolic system and was derived by Carleman [5] for proving the unique continuation property. In a word, a Carleman estimate is a L^2-weight estimate with weight function $e^{2\tau\varphi}$ which is valid uniformly for all large parameter $\tau > 0$. There are many works concerning Carleman estimates for hyperbolic equations, as shown, e.g., in Baudouin, Buhan and Ervedoza [1], Fu, Yong and Zhang [8], Fursikov and Imanuvilov [9], Lasiecka and Triggiani [15], Imanuvilov [19] and Triggiani and Yao [21], and the references therein. For general wave equations on Riemannian manifold, Triggiani and Yao [21] established a Carleman estimate with no lower-order terms to obtain the observability inequality by using the geometric method. We apply the above results to establish a Carleman estimate for a wave equation with an integral term on Riemannian manifold. Then we apply the estimate to obtain an observability inequality under some assumption of initial values.

Our paper is organized as follows. In section 2, we introduce the escape vector field and state our main results. In section 3, we use the duality method to change the reachability of problem (1.1) into an observability inequality (3.4). In section 4, we provide a Carleman estimate for wave equations with a memory term on Riemannian manifold and then we prove our main results. In the last section, we give some remarks about our conclusions. The details are presented in the following sections.

2. Main Results

Definition 2.1. Let $(\mathcal{M}, g)$ be a Riemannian manifold and let Ω be a open bounded set of $\mathcal{M}$. A vector field H is said to be an escape vector field on $\bar{\Omega}$ if the covariant differential DH of H in the metric g is a positive tensor field on $\bar{\Omega}$, i.e., there is a constant $\rho_0 > 0$ such that

$$DH(X, X)(x) = \langle D_X X, X\rangle(x) \geq \rho_0 |X|^2, \quad \forall x \in \overline{\Omega},\ X \in \mathcal{M}_x.$$

Remark 2.2. Escape vector fields were introduced by Yao [23] as a checkable assumption for the exact controllability of the wave equation with variable coefficients. The existence of such a vector field is an assumption for our problem (see *(H1)* below).

To state the Carleman estimate, we need the following assumptions.

(H1) There exists a proper function $d : \bar{\Omega} \to \mathbb{R}^n$ of class C^3 that is strictly convex in the metric g. This means $H = Dd$ is an escape vector field on $\overline{\Omega}$, i.e.,

$$DH(X, X)(x) = D^2 d(X, X)(x) \geq \rho_0 |X|^2, \quad \forall x \in \overline{\Omega},\ X \in \mathcal{M}_x. \tag{2.1}$$

(H2) The function d has no critical point on $\overline{\Omega}$:

$$\inf_{x\in\overline{\Omega}} |Dd(x)|^2 > 0.$$

As shown in Lasciecka, Triggiani and Yao [14], by translation and rescaling, we can always achieve $\rho_0 = 2$ in (2.1).

Remark 2.3. The class of escape vector fields for the metric is larger than that which is given by all gradients of strictly convex functions. There is an escape vector field which is not a gradient of any strictly convex function; see Yao [22, Example 2.6].

Remark 2.4. The square of the distance function initiating from a given point $x_0 \in \Omega$ in the metric g is strictly convex in a neighborhood of x_0, which means the escape vector field certainly exists locally. Generally speaking, the sectional curvature of the Riemannian manifold $(\mathcal{M}, g)$ can provide the global information on its existence. For details, please see Yao [22].

We define T_0 by

$$T_0^2 = \max_{x\in\overline{\Omega}} d(x). \tag{2.2}$$

Our reachability result reads as follows.

Theorem 2.5. *Let $T > T_0$ be given. Assume* (H1) *and* (H2) *holds. Then for any $T > T_0$, for given $u_0 \in L^2(\Omega)$, there is a control $\phi \in L^2(0, T; L^2(\Gamma_0))$ such that the solution of system* (1.1) *satisfies*

$$u(x, T) = u_0(x) \quad \text{in } \Omega.$$

This theorem implies system (1.1) being displacement reachable at a time $T > 0$ by $L^2(\Sigma_0)$ control.

3. Observability Inequalities

By the duality method, solving the reachability problem (1.1) and (1.2) amounts to establishing an observability inequality

$$\|v_\nu\|^2_{L^2(\Sigma_0)} \geq C_T(\|y_0\|^2_{H_0^1(\Omega)} + \|y_1\|^2_{L^2(\Omega)}) \tag{3.1}$$

for the dual system

$$\begin{gathered} y_{tt} = \Delta y - \langle P(t), Dy\rangle + (p_0 - \operatorname{div} P)y + \int_t^T p(s,t)\Delta y(s)ds \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(T) = y_0, \quad y_t(T) = y_1 \quad \text{in } \Omega, \end{gathered} \tag{3.2}$$

where div is the divergence operator of the metric g. In (3.1) $v_\nu = \langle Dv, \nu\rangle$ and v is defined by

$$v(x,t) = y(x,t) + \int_t^T p(s,t)y(x,s)\,ds, \quad \forall (x,t) \in Q, \tag{3.3}$$

where y is the solution to problem (3.2).

By following a procedure as in Kim [11] or Yao [22], it is easy to obtain the following result.

Theorem 3.1. *System* (1.1) *is displacement reachable at a time $T > 0$ by $L^2(\Sigma_0)$ control if and only if there is a constant $C_T > 0$, independent of solutions, such that*

$$\|v_\nu\|^2_{L^2(\Sigma_0)} \geq C_T\|y_1\|_{L^2(\Omega)}, \quad \textit{for any } y_0 = 0,\ y_1 \in L^2(\Omega), \tag{3.4}$$

where v and y are given by (3.3) *and* (3.2), *respectively.*

Our task in this article is to obtain the observability inequality (3.4) for system (3.2).

Theorem 3.2. *Let $T > T_0$ be given. Assume* (H1) *and* (H2) *hold. Then there is a constant $C_T > 0$ such that*

$$\|v_\nu\|^2_{L^2(\Sigma_0)} \geq C_T(\|y_0\|^2_{H_0^1(\Omega)} + \|y_1\|^2_{L^2(\Omega)})$$

for all solutions $y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ to (3.2) *with $y_0 = 0$ or $y_1 = -1/2p(T,T)y_0$, where v is defined by* (3.3) *and T_0 is defined in* (2.2) *and*

$$\Gamma_0 = \{x \in \Gamma : d_\nu \equiv \langle Dd, \nu\rangle > 0\}. \tag{3.5}$$

The proof of this theorem will be given in the end of Section 4. Clearly, Theorem 3.1 and Theorem 3.2 imply Theorem 2.5. Next, we present another theorem equivalent to Theorem 3.2. We define an operator $K : C([0,T]; L^2(\Omega)) \to C([0,T]; L^2(\Omega))$ by

$$(Ky)(x,t) = \int_t^T p(s,t)y(x,s)\,ds, \quad \forall (x,t) \in Q. \tag{3.6}$$

The relation (3.3) can be rewritten as

$$v(x,t) = y(x,t) + (Ky)(x,t), \quad \forall (x,t) \in Q. \tag{3.3'}$$

Now we introduce some properties of the above operator K.

Proposition 3.3. *The operator K defined in* (3.6) *has the following properties:*

1. *K is a bounded operator;*

(2) *the operator* $I+K$ *is 1-1, where* I *is an identity operator on the space* $C([0,T];L^2(\Omega))$;
(3) *there is a unique function* $q(s,t)\in C^2(J)$ *such that*

$$y(x,t)=v(x,t)+\int_t^T q(s,t)v(x,s)ds,\ \forall(x,t)\in\Omega\times[0,T], \tag{3.7}$$

where q *is the unique solution of the equation*

$$\int_t^s q(s,r)p(r,t)dr+q(s,t)+p(s,t)=0, \tag{3.8}$$

for all $(s,t)\in J\equiv\{(s,t)|0\le t\le s\le T\}$.

Proof. (1) By Hölder's inequality, we have

$$\begin{aligned}
\|Ky(t)\|^2_{L^2(\Omega)}&=\int_\Omega\big(\int_t^T p(s,t)y(x,s)ds\big)^2dx\\
&\le\int_\Omega\big(\int_t^T p^2(s,t)ds\int_t^T y^2(x,s)ds\big)dx\\
&\le\int_\Omega\big(\int_t^T M^2ds\int_t^T y^2(x,s)ds\big)dx\\
&=M^2(T-t)\int_t^T\|y(s)\|^2_{L^2(\Omega)}ds\\
&\le M^2(T-t)^2\|y\|^2_{C([0,T];L^2(\Omega))},\quad\forall t\in[0,T],
\end{aligned}\tag{3.9}$$

where $M=\sup_{(s,t)\in J}|p(s,t)|$. Taking the maximum over all $t\in[0,T]$, we conclude that K is bounded.

(2) It is sufficient to prove that if

$$y(x,t)+\int_t^T p(s,t)y(x,s)\,ds=0, \tag{3.10}$$

then $y(x,t)\equiv0$ for all $(x,t)\in\Omega\times[0,T]$. In fact, select $\epsilon=\frac{1}{2M}$, then (3.9) and (3.10) imply

$$\|y\|_{C([T-\epsilon,T];L^2(\Omega))}=\|Ky\|_{C([T-\epsilon,T];L^2(\Omega))}\le M\epsilon\|y\|_{C([T-\epsilon,T];L^2(\Omega))},$$

which implies

$$y(x,t)=0\quad\text{in }\Omega\times[T-\epsilon,T].$$

Substituting this into (3.10), we obtain

$$y(x,t)+\int_t^{T-\epsilon}p(s,t)y(x,s)ds=0.$$

By a similar argument, we obtain

$$y(x,t)=0\quad\text{in }\Omega\times[T-2\epsilon,T-\epsilon].$$

Thus $y(x,t)\equiv0$ for all $(x,t)\in\Omega\times[0,T]$ by finite steps.

(3) **Step 1.** Equation (3.8) can be written in the form of an operator equation

$$q=\mathcal{A}q,\quad(\mathcal{A}q)(s,t)=-p(s,t)-\int_t^s q(s,z)p(z,t)dz,\quad\forall(s,t)\in J,$$

where $\mathcal{A}$ is an operator that maps each function q from the Banach space $C(J)$ to a function $\mathcal{A}q$ in the same space. Then for each $q_a, q_b \in C(J)$,

$$\begin{aligned}
|(\mathcal{A}q_a - \mathcal{A}q_b)(s,t)| &= \Big| - \int_t^s (q_a(s,z) - q_b(s,z))p(z,t)dz\Big| \\
&\le M \int_t^s |q_a(s,z) - q_b(s,z)|dz \\
&\le M(T-t)\|q_a - q_b\|_{C(J)} \\
&\le \frac{1}{2}\|q_a - q_b\|_{C(J)}, \quad \forall T-\epsilon \le t \le s \le T.
\end{aligned}$$

Hence $\mathcal{A}$ satisfies a Lipschitz condition with Lipschitz constant $1/2 < 1$. Hence the existence and uniqueness of $q(s,t)$ for all $T-\epsilon \le t \le s \le T$ follows from Banach's fixed point theorem. By similar arguments as used in 2), we obtain a unique $q(s,t)$ which satisfies (3.8) for all $0 \le t \le s \le T$.

Step 2. Define $\tilde{y}$ by (3.7). Combining with (3.8) we have

$$\begin{aligned}
\int_t^T p(s,t)\tilde{y}(x,s)ds &= \int_t^T p(s,t)\Big(v(x,s) + \int_s^T q(z,s)v(x,z)dz\Big)ds \\
&= \int_t^T p(z,t)v(x,z)dz + \int_t^T dz \int_t^z q(z,s)p(s,t)v(x,z)ds \\
&= \int_t^T \big[p(z,t) + \int_t^z q(z,s)p(s,t)ds\big]v(x,z)dz \\
&= -\int_t^T q(z,t)v(x,z)dz \\
&= v(x,t) - \tilde{y}(x,t).
\end{aligned} \tag{3.11}$$

This means that $\tilde{y}$ satisfies (3.3)'. Combining (3.11) and (3.3)', we obtain that

$$(I+K)(y-\tilde{y}) = 0.$$

Thus $y = \tilde{y}$ follows from conclusion 2). The relation (3.7) holds. □

Set

$$w(x,t) = v(x,T-t)\exp\Big(\frac{1}{2}\int_{T-t}^T p(s,s)ds\Big), \quad \forall (x,t) \in Q, \tag{3.12}$$

where v is defined in (3.3) in terms of y, which satisfies (3.2). By Proposition 3.3 and a simple calculation as in Kim [11], we have the following.

Proposition 3.4. *The w, given in* (3.12)*, solves problem*

$$\begin{gathered}
w_{tt} = \Delta w + \langle R(t), Dw\rangle + r_0(x,t)w + \mathcal{B}w \quad \text{in } Q, \\
w = 0 \quad \text{on } \Sigma, \\
w(0) = w_0, \quad w_t(0) = w_1 \quad \text{in } \Omega,
\end{gathered} \tag{3.13}$$

where $w_0 = y_0$, $w_1 = -y_1 - 1/2p(T,T)y_0$, *and*

$$\mathcal{B}w(x,t) = \int_0^t \sum_{i=0}^1 \langle G_i(s,t), D^i w(s)\rangle\, ds, \quad \forall (x,t) \in Q.$$

In addition, $R(t)$ and $G_1(s,t)$ are vector fields for $t \geq s \geq 0$, and r_0 and G_0 are functions which are expressed, in terms of P, p_0 and q, as

$$\begin{aligned}
R(x,t) &= -P(x, T-t), \\
G_1(x,s,t) &= -\exp\Big(\frac{1}{2}\int_{T-s}^{T-t} q(z,z)dz\Big) q(T-s, T-t) P(x, T-t), \\
r_0(x,t) &= p_0(x, T-t) + \frac{3}{2} q_t(T-t, T-t) + \frac{1}{2} q_s(T-t, T-t) \\
&\quad + \frac{1}{4} q^2(T-t, T-t) - \operatorname{div} P(x, T-t), \\
G_0(x,s,t) &= \exp\Big(\frac{1}{2}\int_{T-s}^{T-t} q(z,z)dz\Big)\big[q_{tt}(T-s, T-s) + p_0(T-t) q(T-s, T-t) \\
&\quad - \operatorname{div} P(x, T-t) q(T-s, T-t)\big], \quad \text{for } x \in \Omega \text{ and } t \geq s \geq 0,
\end{aligned}$$

where q is given in by (3.8).

By the relationship (3.12) between w and v and Proposition 3.4, we have the following theorem.

Theorem 3.5. *The observability inequality* (3.4) *holds if and only if there exists a constant $C_T > 0$, such that*

$$\|w_\nu\|^2_{L^2(\Sigma_0)} \geq C_T(\|w_0\|^2_{H_0^1(\Omega)} + \|w_1\|^2_{L^2(\Omega)}), \tag{3.14}$$

where w is a solution to problem (3.13) *with $w_0 = 0$ or $w_1 = 0$.*

4. A Carleman estimate and the proof of Theorem 3.2

In this section we consider the problem

$$\begin{gathered}
w_{tt} = \Delta w + \langle R(t), Dw\rangle + r_0(x,t)w + \mathcal{B}w \quad \text{in } \hat{Q}, \\
w = 0 \quad \text{on } \hat{\Sigma},
\end{gathered} \tag{4.1}$$

where R, r_0 and $\mathcal{B}w$ are the same as presented in (3.13). However, this time in (4.1), $\hat{Q} = (-T,T)\times\Omega$ and $\hat{\Sigma} = (-T,T)\times\Gamma$, that means the time variable t of the unknown w in (4.1) is assumed in $[-T,T]$. We will provide a Carleman estimate for the above system (4.1), which is based on a Carleman estimate for general Riemannian wave equations (Triggiani and Yao [21]).

We define the "*energy*" function

$$E(t) = \frac{1}{2}\int_\Omega \big(w_t^2 + |Dw|^2\big)dx, \quad \text{for all } t \in [-T,T]. \tag{4.2}$$

Proposition 4.1. *Let $T > T_0$ be given. Assume* (H1) *and* (H2) *holds. Let w be a solution to* (4.1) *on $\hat{Q} \equiv (-T,T)\times\Omega$. Then there is a constant $C_T > 0$ such that*

$$\|w_\nu\|^2_{L^2(\hat{\Sigma}_0)} \geq C_T E(0). \tag{4.3}$$

The proof of this proposition will be given after Lemmas 4.2 and 4.3. Define a function $\varphi : \hat{Q} \to \mathbb{R}$ of class C^3 as

$$\varphi(x,t) = d(x) - ct^2, \quad \text{for } (x,t) \in \hat{Q},$$

where $0 < c < 1$ is selected as follows. Let $T > T_0$ be given, where T_0 is given in (2.2). Then there exists $\delta > 0$ such that

$$T^2 > \max_{x\in\overline{\Omega}} d(x) + \delta.$$

For this $\delta > 0$, there exists a constant c, $0 < c < 1$, such that

$$cT^2 > \max_{x\in\overline{\Omega}} d(x) + \delta,$$

which means

$$\varphi(x,-T) = \varphi(x,T) \le \max_{x\in\overline{\Omega}} d(x) - cT^2 \le -\delta$$

uniformly in $x \in \overline{\Omega}$. Since $\varphi(x,0) = d(x) > 0$, there exists $t_0 \in (0,T)$, such that

$$\min_{x\in\overline{\Omega},t\in[-t_0,t_0]} \varphi(x,t) \ge \sigma, \quad 0 < \sigma < \min_{x\in\overline{\Omega}} d(x). \tag{4.4}$$

Define

$$Q(\sigma) = \{(x,t) \in \hat{Q} | \varphi(x,t) \ge \sigma > 0\}. \tag{4.5}$$

It is not difficult to show that $[-t_0,t_0]\times\Omega \subset Q(\sigma) \subset [-T,T]\times\Omega$.

Lemma 4.2. *Assume* (H1) *and* (H2) *hold. Let* $w \in H^{1,1}(\hat{Q}) \equiv L^2(-T,T;H^1(\Omega))\cap H^1(-T,T;L^2(\Omega))$ *be a solution of* (4.1)*. Then there exists constants* $C_T, C_{0T} > 0$ *such that for all* $\tau > 0$ *sufficiently large and any* $\varepsilon > 0$ *small,*

$$\begin{aligned} &2\tau\int_{-T}^{T}\int_{\Gamma_0} e^{2\tau\varphi} w_\nu^2 \langle Dd,\nu\rangle d\Sigma + C_T\int_{\hat{Q}}(w_t^2+|Dw|^2)\,dx\,dt + C_{0T}e^{2\tau\sigma}\int_{\hat{Q}} w^2\,dx\,dt\\ &\ge (\tau\varepsilon\rho - 2C_T)e^{2\tau\sigma}\int_{-t_0}^{t_0}\int_\Omega [w_t^2+|Dw|^2]\,dx\,dt, \end{aligned} \tag{4.6}$$

where $\rho = 1-c$*, and* Γ_0 *is defined by* (3.5)*.*

Proof. We divide our proof into several steps.

Step 1. Consider the problem

$$\begin{gathered} z_{tt} = \Delta z + \langle \hat{P}(t), Dz\rangle + \hat{p}_1 z_t + \hat{p}_0 z + f, \quad \text{in } \hat{Q},\\ z = 0 \quad \text{on } \hat{\Sigma}, \end{gathered} \tag{4.7}$$

where $\hat{P}(t)$ is a vector field on $(\mathcal{M},g)$ for $t\in(-T,T)$ and $\hat{p}_1$, $\hat{p}_0$ are functions on $\hat{Q}$, satisfying

$$|\hat{P}(x,t)|,\ |\hat{p}_1(x,t)|,\ |\hat{p}_0(x,t)| \le C_T, \quad \forall (x,t)\in\hat{Q},$$

and $f \in L^2(\hat{Q})$.

Inserting the boundary formula [21, (8.7) p.358] into the formula [21, (5.1) Theorem 5.1], we obtain, for all $\tau > 0$ sufficiently large and any $\varepsilon > 0$ small,

$$\begin{aligned}
&2\tau \int_{-T}^{T} \int_{\Gamma_0} \mathrm{e}^{2\tau\varphi} z_\nu^2 \langle Dd, \nu\rangle d\Sigma + 2 \int_{\hat{Q}} \mathrm{e}^{2\tau\varphi} f^2 \,dx\,dt + C_{0T}\mathrm{e}^{2\tau\sigma} \int_{\hat{Q}} z^2 \,dx\,dt \\
&\ge (\tau\varepsilon\rho - 2C_T) \int_{\hat{Q}} \mathrm{e}^{2\tau\varphi}(|z_t|^2 + |Dz|^2)\,dx\,dt \\
&\quad + (2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T) \int_{Q(\sigma)} \mathrm{e}^{2\tau\varphi} z^2 \,dx\,dt \\
&\quad - C_T\tau^3 \mathrm{e}^{-2\tau\delta} \int_\Omega (|z_t(-T)|^2 + |Dz(-T)|^2) dx \\
&\quad - C_T\tau^3 \mathrm{e}^{-2\tau\delta} \int_\Omega (|z_t(T)|^2 + |Dz(T)|^2) dx,
\end{aligned} \tag{4.8}$$

with $\rho = 1 - c$ and $\beta > 0$ depending on ε, where $Q(\sigma)$ and Γ_0 are defined by (4.5) and (3.5), respectively.

Let $0 < \eta < T - t_0$ to be fixed. For gaining compact supports in time for functions, we introduce a cut-off function $\chi(t) \in C^\infty(\mathbb{R})$ such that $0 \le \chi(t) \le 1$ defined by

$$\chi(t) = \begin{cases} 1, & \text{if } -T + \eta \le t \le T - \eta, \\ 0, & \text{if } t \le -T \text{ or } t \ge T, \end{cases} \tag{4.9}$$

with $\chi'(\pm T) = 0$. It is easy to check that such function χ exists, for example,

$$\chi(t) = \begin{cases} \frac{1}{2}\cos\frac{\pi}{\eta}(t+T) + \frac{1}{2}, & -T < t < -T + \eta, \\ 1, & -T + \eta \le t < T - \eta, \\ \frac{1}{2}\cos\frac{\pi}{\eta}(t-T) + \frac{1}{2}, & T - \eta \le t < T, \\ 0, & t \le -T \text{ or } t \ge T. \end{cases}$$

Set $z(x,t) = \chi(t)w(x,t)$ in $\hat{Q}$. We calculate that

$$\begin{aligned}
z_{tt} &= (\chi w_t + \chi' w)_t \\
&= \chi w_{tt} + 2\chi' w_t + \chi'' w \\
&= \chi(\Delta w + \langle R(t), Dw\rangle + r_0 w + \mathcal{B}w) + 2\chi' w_t + \chi'' w \\
&= \Delta(\chi w) + \langle R(t), D(\chi w)\rangle + r_0 \cdot \chi w + \chi\mathcal{B}w + 2\chi' w_t + \chi'' w \\
&= \Delta z + \langle R(t), Dz\rangle + r_0 z + [\chi\mathcal{B}w + 2\chi' w_t + \chi'' w], \quad \text{in } \hat{Q},
\end{aligned}$$

and $z = 0$ on $\hat{\Sigma}$. Applying the Carleman estimate (4.8) to z, we obtain

$$\begin{aligned}
&2\tau \int_{\hat{\Sigma}_0} \mathrm{e}^{2\tau\varphi} z_\nu^2 \langle Dd, \nu\rangle d\Sigma + 2 \int_{\hat{Q}} \mathrm{e}^{2\tau\varphi} \chi^2 (\mathcal{B}w)^2 \,dx\,dt \\
&+ 2 \int_{\hat{Q}} \mathrm{e}^{2\tau\varphi} (2\chi' w_t + \chi'' w)^2 \,dx\,dt + C_{0T}\mathrm{e}^{2\tau\sigma} \int_{\hat{Q}} z^2 \,dx\,dt \\
&\ge (\tau\varepsilon\rho - 2C_T) \int_{\hat{Q}} \mathrm{e}^{2\tau\varphi}(z_t^2 + |Dz|^2)\,dx\,dt \\
&\quad + (2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T) \int_{Q(\sigma)} \mathrm{e}^{2\tau\varphi} z^2 \,dx\,dt,
\end{aligned} \tag{4.10}$$

where $\hat{\Sigma}_0 = (-T, T) \times \Gamma_0$, and where we have used

$$z(\pm T) = z_t(\pm T) = 0.$$

We choose $\eta > 0$ small such that $\varphi < 0$ in $[-T, -T+\eta] \cup [T-\eta, T]$ uniformly for $x \in \Omega$ since $\varphi(x, \pm T) \le -\delta < 0$, which means

$$\mathrm{e}^{2\tau\varphi(x,t)} \le 1 \quad \text{for } (x,t) \in (-T, -T+\eta) \cup (T-\eta, T) \times \Omega. \tag{4.11}$$

Noting that the functions χ' and χ'' have compact support in $(-T, -T+\eta) \cup (T-\eta, T)$, we obtain an estimate for the third term on the left-hand side of (4.10):

$$\begin{aligned}
&2\int_{\hat{Q}} \mathrm{e}^{2\tau\varphi}(2\chi' w_t + \chi'' w)^2 \, dx\, dt \\
&= 2\int_{-T}^{-T+\eta}\int_\Omega \mathrm{e}^{2\tau\varphi}(2\chi' w_t + \chi'' w)^2 \, dx\, dt + 2\int_{T-\eta}^{T}\int_\Omega \mathrm{e}^{2\tau\varphi}(2\chi' w_t + \chi'' w)^2 \, dx\, dt \\
&\le 2C_T \int_{-T}^{-T+\eta}\int_\Omega (w_t^2 + w^2)\, dx\, dt + 2C_T\int_{T-\eta}^{T}\int_\Omega (w_t^2 + w^2)\, dx\, dt.
\end{aligned} \tag{4.12}$$

Moreover, since $2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T$ is positive for large $\tau > 0$, we drop the last term on the right-hand side of (4.10). Combining with (4.12), we deduce the following estimate on w:

$$\begin{aligned}
&2\tau\int_{\hat{\Sigma}_0} \mathrm{e}^{2\tau\varphi} w_\nu^2 \langle Dd, \nu\rangle d\Sigma \\
&+ 2\int_{\hat{Q}} \mathrm{e}^{2\tau\varphi}\chi^2(\mathcal{B}w)^2\, dx\, dt + 2C_T\int_{-T}^{-T+\eta}\int_\Omega (w_t^2 + w^2)\, dx\, dt \\
&+ 2C_T\int_{T-\eta}^{T}\int_\Omega (w_t^2 + w^2)\, dx\, dt + C_{0T}\mathrm{e}^{2\tau\sigma}\int_{\hat{Q}} w^2\, dx\, dt \\
&\ge (\tau\varepsilon\rho - 2C_T)\int_{-T+\eta}^{T-\eta}\int_\Omega \mathrm{e}^{2\tau\varphi}[w_t^2 + |Dw|^2]\, dx\, dt.
\end{aligned} \tag{4.13}$$

Step 2. By using a trick from Klibanov [12], we deal with the integral term containing $\mathcal{B}w$ in (4.13). We give the estimate for $t \in (0, T)$. The case for $t \in (-T, 0)$ is similar. Using Hölder's inequality, we have

$$\begin{aligned}
&2\int_0^T\int_\Omega \mathrm{e}^{2\tau\varphi}\chi^2(\mathcal{B}w)^2\, dx\, dt \\
&= 2\int_0^T\int_\Omega \mathrm{e}^{2\tau\varphi}\chi^2\Big(\int_0^t\sum_{i=0}^1 \langle G_i(s,t), D^i w(s)\rangle\, ds\Big)^2\, dx\, dt \\
&\le 2\int_0^T\int_\Omega \mathrm{e}^{2\tau\varphi}\chi^2 t\Big(\int_0^t\sum_{i=0}^1 |\langle G_i(s,t), D^i w(s)\rangle|^2 ds\Big)\, dx\, dt \\
&\le C_T\int_0^T\int_\Omega \mathrm{e}^{2\tau\varphi}\chi^2 t\Big(\int_0^t\sum_{i=0}^1 |D^i w(s)|^2 ds\Big)\, dx\, dt.
\end{aligned} \tag{4.14}$$

Noting that

$$(\mathrm{e}^{2\tau\varphi})' = 2\tau\varphi'\mathrm{e}^{2\tau\varphi} = -4\tau c t \mathrm{e}^{2\tau\varphi},$$

by integration by parts, we have that the right-hand side of (4.14) is equal to

$$
\begin{aligned}
&C_T \int_0^T \int_\Omega \frac{(\mathrm{e}^{2\tau\varphi})'}{-4\tau c}\chi^2\Big(\int_0^t \sum_{i=0}^1 |D^i w(s)|^2 ds\Big)\,dx\,dt \\
&= C_T \int_\Omega \frac{\mathrm{e}^{2\tau\varphi}\chi^2}{-4\tau c}\Big(\int_0^t \sum_{i=0}^1 |D^i w(s)|^2 ds\Big)dx\Big|_{t=0}^{t=T} \\
&\quad + \frac{C_T}{2\tau c}\int_0^T \int_\Omega \mathrm{e}^{2\tau\varphi}\chi\chi'\Big(\int_0^t \sum_{i=0}^1 |D^i w(s)|^2 ds\Big)\,dx\,dt \\
&\quad + \frac{C_T}{4\tau c}\int_0^T \int_\Omega \mathrm{e}^{2\tau\varphi}\chi^2 \sum_{i=0}^1 |D^i w(t)|^2\,dx\,dt \\
&\le C_T\eta\tau^{-1}\int_0^T \int_\Omega \sum_{i=0}^1 |D^i w|^2\,dx\,dt \\
&\quad + C_T\tau^{-1}\int_0^T \int_\Omega \mathrm{e}^{2\tau\varphi}\sum_{i=0}^1 |D^i w(t)|^2\,dx\,dt,
\end{aligned}
\tag{4.15}
$$

where we used the fact (4.11) in the last step and

$$
\int_\Omega \frac{\mathrm{e}^{2\tau\varphi}\chi^2}{-4\tau c}\Big(\int_0^t \sum_{i=0}^1 |D^i w(s)|^2 ds\Big)dx\Big|_{t=0}^{t=T} = 0
$$

by (4.9). Next, we introduce a weighted Poincaré's inequality, given in Buhan and Ervedoza [1, Lemma 2.4], that is, under assumption (H2), there exists $C > 0$ such that

$$
\tau^2 \int_\Omega \mathrm{e}^{2\tau\varphi} w^2 dx \le C \int_\Omega \mathrm{e}^{2\tau\varphi}|Dw|^2 dx, \tag{4.16}
$$

for all $w \in H_0^1(\Omega)$ and $\tau > 0$ sufficiently large. By (4.16), we have an estimate for the last term in the right-hand side of (4.15):

$$
C_T\tau^{-1}\int_0^T \int_\Omega \mathrm{e}^{2\tau\varphi}\sum_{i=0}^1 |D^i w(t)|^2\,dx\,dt \le C_T\tau^{-1}\int_0^T \int_\Omega \mathrm{e}^{2\tau\varphi}|Dw(t)|^2\,dx\,dt. \tag{4.17}
$$

It follows from (4.14), (4.15) and (4.17) that

$$
\begin{aligned}
&2\int_{\hat{Q}} \mathrm{e}^{2\tau\varphi}\chi^2(\mathcal{B}w)^2\,dx\,dt \\
&\le C_T\eta\tau^{-1}\int_{\hat{Q}} \sum_{i=0}^1 |D^i w|^2\,dx\,dt + C_T\tau^{-1}\int_{\hat{Q}} \mathrm{e}^{2\tau\varphi}|Dw|^2\,dx\,dt \\
&\le C_T\tau^{-1}\int_{\hat{Q}} \sum_{i=0}^1 |D^i w|^2\,dx\,dt + C_T\tau^{-1}\int_{-T+\eta}^{T-\eta}\int_\Omega \mathrm{e}^{2\tau\varphi}|Dw|^2\,dx\,dt.
\end{aligned}
\tag{4.18}
$$

Step 3. Substituting the above estimate (4.18) into (4.13), we obtain

$$
\begin{aligned}
&2\tau \int_{\hat{\Sigma}_0} e^{2\tau\varphi} w_\nu^2 \langle Dd, \nu\rangle \, d\Sigma + C_T\tau^{-1} \int_{\hat{Q}} \sum_{i=0}^{1} |D^i w|^2 \, dx \, dt \\
&+ C_T\tau^{-1} \int_{-T+\eta}^{T-\eta} \int_\Omega e^{2\tau\varphi} |Dw|^2 \, dx \, dt + 2C_T \int_{-T}^{-T+\eta} \int_\Omega (w_t^2 + w^2) \, dx \, dt \\
&+ 2C_T \int_{T-\eta}^{T} \int_\Omega (w_t^2 + w^2) \, dx \, dt + C_{0T} e^{2\tau\sigma} \int_{\hat{Q}} w^2 \, dx \, dt \\
&\geq (\tau\varepsilon\rho - 2C_T) \int_{-T+\eta}^{T-\eta} \int_\Omega e^{2\tau\varphi} [w_t^2 + |Dw|^2] \, dx \, dt.
\end{aligned} \tag{4.19}
$$

Choosing $\tau > 0$ sufficiently large, we can absorb the third term on the left-hand side into the right-hand side of (4.19). Then we obtain

$$
\begin{aligned}
&2\tau \int_{\hat{\Sigma}_0} e^{2\tau\varphi} w_\nu^2 \langle Dd, \nu\rangle \, d\Sigma + C_T \int_{\hat{Q}} (w_t^2 + |Dw|^2) \, dx \, dt + C_{0T} e^{2\tau\sigma} \int_{\hat{Q}} w^2 \, dx \, dt \\
&\geq (\tau\varepsilon\rho - 2C_T) \int_{-T+\eta}^{T-\eta} \int_\Omega e^{2\tau\varphi} [w_t^2 + |Dw|^2] \, dx \, dt \\
&\geq (\tau\varepsilon\rho - 2C_T) e^{2\tau\sigma} \int_{-t_0}^{t_0} \int_\Omega [w_t^2 + |Dw|^2] \, dx \, dt,
\end{aligned}
$$

where t_0 is defined as shown in (4.4). The proof is complete. □

To obtain the estimate (4.3), we need the following energy estimates for system (4.1).

Lemma 4.3. *Let w be a solution of* (4.1). *Then there are constants $C_{1T}, C_{2T} > 0$ and $\delta, \tilde{\delta} > 0$ small such that*

1. *$E(t) \leq C_{1T} E(0)$ for $t \in [-T, T]$,*
2. *$E(t) \geq C_{2T} E(0)$ for $-\tilde{\delta} \leq t \leq \delta$,*

where $E(t)$ is defined by (4.2).

Proof. We do the proof only for $t \geq 0$, because the proof for $t < 0$ is similar. On one hand, divergence formula tells us

$$
w_t \Delta w + \langle Dw, Dw_t\rangle = w_t \operatorname{div}(Dw) + \langle Dw, Dw_t\rangle = \operatorname{div}(w_t Dw).
$$

Moreover, using Green's formula, we obtain

$$
\begin{aligned}
E'(t) &= \int_\Omega w_t w_{tt} + \langle Dw, Dw_t\rangle \, dx \\
&= \int_\Omega (w_t \Delta w + \langle Dw, Dw_t\rangle) dx + \int_\Omega w_t(\langle R(t), Dw\rangle + r_0 w + \mathcal{B}w) dx \\
&= \int_\Omega \operatorname{div}(w_t Dw) dx + \int_\Omega w_t(\langle R(t), Dw\rangle + r_0 w + \mathcal{B}w) dx \\
&= \int_\Gamma w_t w_\nu d\Gamma + \int_\Omega w_t(\langle R(t), Dw\rangle + r_0 w + \mathcal{B}w) dx \\
&= \int_\Omega w_t(\langle R(t), Dw\rangle + r_0 w + \mathcal{B}w) dx \\
&\le C_T \int_\Omega (w_t^2 + |Dw|^2 + w^2) dx + C_T \int_\Omega \int_0^t \sum_{i=0}^1 |D^i w(s)|^2 ds dx \\
&\le C_T E(t) + C_T \int_0^t E(s) ds,
\end{aligned}
\tag{4.20}
$$

where we have used Poincaré's inequality

$$
\|w\|_{L^2(\Omega)} \le C \|Dw\|_{L^2(\Omega)} \tag{4.21}
$$

for $w \in H_0^1(\Omega)$. Integrating both side of (4.20) with respect to $t \in [0, T]$, we have

$$
\begin{aligned}
E(t) &\le E(0) + C_T \int_0^t \big(E(s) + \int_0^s E(r) dr\big) ds \\
&= E(0) + C_T \int_0^t E(s) ds + C_T \int_0^t (t-s) E(s) ds \\
&\le E(0) + C_T \int_0^t E(s) ds.
\end{aligned}
$$

Using Gronwall's inequality of integral form, we obtain

$$
E(t) \le E(0)\big(1 + C_T t \mathrm{e}^{C_T t}\big) \le C_{1T} E(0), \quad \text{for } t \in [0, T].
$$

On the other hand, similarly to (4.20), some computations yield

$$
E'(t) \ge -C_T E(t) - C_T \int_0^t E(s) ds.
$$

Noting that

$$
(\mathrm{e}^{C_T t} E(t))' = \mathrm{e}^{C_T t}(C_T E(t) + E'(t)) \ge -C_T \mathrm{e}^{C_T t} \int_0^t E(s) ds,
$$

we have

$$
\mathrm{e}^{C_T t} E(t) \ge E(0) - C_T \int_0^t \Big(\mathrm{e}^{C_T \tau} \int_0^\tau E(s) ds\Big) d\tau,
$$

which means

$$
\begin{aligned}
E(t) &\ge \mathrm{e}^{-C_T t} E(0) - C_T \int_0^t \int_0^\tau E(s) ds d\tau \\
&\ge \mathrm{e}^{-C_T t} E(0) - C_T \int_0^t E(s) ds
\end{aligned}
$$

$$
\begin{aligned}
&\geq \mathrm{e}^{-C_T t} E(0) - C_T C_{1T} t E(0) \\
&= (\mathrm{e}^{-C_T t} - C_T C_{1T} t) E(0), \ \text{for } t \in [0, T],
\end{aligned}
$$

where, in the third step, we have used the conclusion 1). We further choose $0 < \delta < t_0$ small such that $\mathrm{e}^{-C_T t} - C_T C_{1T} t \geq C_{2T} > 0$ for $0 \leq t \leq \delta$. Then $E(t) \geq C_{2T} E(0)$ for $0 \leq t \leq \delta$. Similarly, there exist $\widetilde{\delta} > 0$ such that

$$
E(t) \geq C_{2T} E(0), \quad \text{for } -\widetilde{\delta} \leq t \leq 0.
$$

The proof is complete. □

Proof of Proposition 4.1. Using estimate (4.6) from Lemma 4.2, energy estimates from Lemma 4.3 and Poincaré's inequality (4.21), we obtain

$$
\begin{aligned}
&2\tau \int_{\hat{\Sigma}_0} \mathrm{e}^{2\tau\varphi} w_\nu^2 \langle Dd, \nu \rangle \, d\Sigma + 2(2C_T + \widetilde{C}_{0T} \mathrm{e}^{2\tau\sigma}) C_{1T} T E(0) \\
&\geq 2\delta_1 C_{2T} (\tau\varepsilon\rho - 2C_T) \mathrm{e}^{2\tau\sigma} E(0)
\end{aligned}
$$

for all large $\tau > 0$, where $\delta_1 = \min\{\delta, \tilde{\delta}\} > 0$. Hence

$$
\begin{aligned}
&\tau \int_{\hat{\Sigma}_0} \mathrm{e}^{2\tau\varphi} w_\nu^2 \langle Dd, \nu \rangle \, d\Sigma \\
&\geq \{[\delta_1 C_{2T} (\tau\varepsilon\rho - 2C_T) - \widetilde{C}_{0T} C_{1T} T] \mathrm{e}^{2\tau\sigma} - 2C_T C_{1T} T\} E(0) \\
&\equiv C_{\tau,T} E(0).
\end{aligned}
$$

Choosing $\tau > 0$ sufficiently large such that $C_{\tau,T} > 0$, and noting that $d_\nu > 0$ on Γ_0 by (3.5), we obtain there is a constant $C_T > 0$ such that

$$
\|w_\nu\|^2_{L^2(\hat{\Sigma}_0)} \geq C_T E(0).
$$

The proof is complete. □

Proof of Theorem 3.2. Using Theorem 3.5, we need to prove the inequality (3.14) for system (3.13) with $w_0 = 0$ or $w_1 = 0$.

If $w_0 = 0$, we extend w to $(-T, 0)$, denoted by

$$
\hat{w}(x,t) = \begin{cases} w(x,t), & 0 \leq t < T, \\ -w(x,-t), & -T < t < 0. \end{cases}
$$

It is easy to check that $\hat{w}$ satisfies problem (4.1) on $\hat{Q} = (-T, T) \times \Omega$. Thus the estimate (4.3) yields

$$
\|w_\nu\|_{L^2(\Sigma_0)} \geq C_T \|w_1\|^2_{L^2(\Omega)} \quad \text{for any } w_0 = 0, \ w_1 \in L^2(\Omega). \tag{4.22}
$$

Similarly, if $w_1 = 0$, this time, we extend w to $(-T, 0)$ by

$$
\hat{w}(x,t) = \begin{cases} w(x,t), & 0 \leq t < T, \\ -w(x,-t), & -T < t < 0. \end{cases}
$$

Similarly, we apply the estimate (4.3) to the above $\hat{w}$ on $\hat{Q} = (-T, T) \times \Omega$ to obtain

$$
\|w_\nu\|_{L^2(\Sigma_0)} \geq C_T \|w_0\|^2_{H_0^1(\Omega)} \quad \text{for any } w_0 \in H_0^1(\Omega), \ w_1 = 0. \tag{4.23}
$$

Combining (4.22) with (4.23), we complete the proof. □

5. Final remarks

In the reachability problem that we have solved, the displacement reachability is obtained, but the velocity reachability is unknown. Because in our paper, the equivalent observation inequality (3.14) for system (3.13) holds for the case that $w_0 = 0$ and $w_1 \in L^2(\Omega)$. This means system (1.1) is displacement reachability by the duality method. For the case of constant coefficients, it is easy to apply the general geometric multiplier $H(w)$ to obtain the above observation inequality without any assumption of initial values. However, if we consider our problem on a Riemannian Manifold, this general method is useless because the unique continuation property for system (3.13) is invalid. Moreover, by using the Carleman estimate method, it is difficult to deal with the memory term. By using a trick from Klibanov [12], we extend the solution w to $(-T, 0)$ to obtain the observation inequality (see Lemma 4.2). In this way, we need the assumption $w_0 = 0$ (it means $y_0 = 0$ for system (3.2).) or $w_1 = 0$. We do not know wherever (3.14) is true for any $w_0 \in H_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$. We have tried our best to prove it but, unfortunately, we failed. In our opinion, this is an interesting question worth exploring.

In section 4, the time interval we choose is $[-T, T]$ instead of $[0, T]$. Because it is difficult to deal with the integral term containing $\mathcal{B}w$ in the Carleman estimate inequality (4.13). More specifically, if we considered our problem on $[0, T]$, the weight function φ should be set as

$$\varphi(x,t) = d(x) - c(t - T/2)^2, \quad \text{for} \quad (x,t) \in Q.$$

We did not know how to deal with the integral term

$$\int_0^T \int_\Omega \mathrm{e}^{2\tau\varphi} \chi^2 (\mathcal{B}w)^2 \, dx \, dt$$

to obtain an expected estimate such as (4.18). We extend the time to $[-T, T]$ and set

$$\varphi(x,t) = d(x) - ct^2, \quad \text{for} \quad (x,t) \in \hat{Q}.$$

It is much easier to deal with the above integral term by using a trick from Klibanov [12]. Then we obtain the Carleman estimate (4.6) in $\hat{Q} = (-T, T) \times \Omega$ and use it to prove the equivalent observation inequality (3.14).

Acknowledgements. The author would like to thank the anonymous reviewer for the careful reading of the manuscript, and for the constructive comments. This work is supported by the National Science Foundation of China, grants no. 61473126 and no. 61573342, and Key Research Program of Frontier Sciences, CAS, no. QYZDJ-SSW-SYS011.

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