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QUALITATIVE PROPERTIES OF TRAVELING WAVEFRONTS FOR A THREE-COMPONENT LATTICE DYNAMICAL SYSTEM WITH DELAY

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Abstract. This article concerns a three-component delayed lattice dynamical system arising in competition models. In such models, traveling wave solutions serve an important tool to understand the competition mechanism, i.e. which species will survive or die out eventually. We first prove the existence of the minimal wave speed of the traveling wavefronts connecting two equilibria $(1,0,1)$ and $(0,1,0)$. Then, for sufficiently small intra-specific competitive delays, we establish the asymptotic behavior of the traveling wave solutions at minus/plus infinity. Finally the strict monotonicity and uniqueness of all traveling wave solutions are obtained for the case where intra-specific competitive delays are zeros. In particular, the effect of the delays on the minimal wave speed and the decay rate of the traveling profiles at minus/plus infinity is also investigated.

1. Introduction

In this article, we study the traveling wave solutions of the three-component delayed lattice dynamical system (LDS for short)

$$\begin{gathered}
\frac{\partial u_j(t)}{\partial t} = d_1 D[u_j](t) + r_1 u_j(t)[1 - u_j(t-\tau_1) - a_{12} v_j(t-\tau_2)], \\
\begin{aligned}\frac{\partial v_j(t)}{\partial t} &= d_2 D[v_j](t) + r_2 v_j(t)\big[1 - a_{21} u_j(t-\tau_3) \\ &\quad - v_j(t-\tau_4) - a_{23} w_j(t-\tau_5)\big],\end{aligned} \\
\frac{\partial w_j(t)}{\partial t} = d_3 D[w_j](t) + r_3 w_j(t)[1 - a_{32} v_j(t-\tau_6) - w_j(t-\tau_7)],
\end{gathered} \tag{1.1}$$

where $j \in \mathbb{Z}$, $t \in \mathbb{R}$, $d_i > 0$, $r_i > 0$, $a_{ij} > 0$, $\tau_i \ge 0$ $(i = 1, \dots, 7)$ are given constants. This system arises in the study of competition between three species with diffusion and time delays when the habitat is of one-dimensional and is divided into niches or regions. In this model, u, v, w are the population densities of species $1, 2, 3$, respectively, a_{ij} is the competition coefficient of species j to species i, r_i and d_i are the growth rate and diffusion coefficient of species i, respectively, $\tau_i \ge 0$ $(i = 1, 4, 7)$ and $\tau_j \ge 0$ $(j = 2, 3, 5, 6)$ are the intra-specific and inter-specific competitive delays,

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respectively. Also, the carrying capacity of each species is normalized to be 1. For more biological meanings on this model, we refer to [7, 8, 23].

In this system, species u and w have different preferences of food resource, while species v has both preferences so that it needs to compete with both species u and w. In such competition system, it is very important to investigate which species will survive or die out eventually. Throughout this paper, we shall always assume that

(H0) $a_{32} > a_{12} > 1$, $a_{21} + a_{23} < 1$,

which means that the species u, w are weak competitors to the species v. Therefore, one shall expect that the species v shall win the competition eventually. So, we are interested in the traveling wave solution of (1.1) connecting two equilibria $(1, 0, 1)$ and $(0, 1, 0)$.

When $\tau_i = 0$, $i = 1, \dots, 7$, system (1.1) reduces the lattice dynamical system

$$\begin{aligned} \frac{\partial u_j(t)}{\partial t} &= d_1 D[u_j](t) + r_1 u_j(t)[1 - u_j(t) - a_{12} v_j(t)], \\ \frac{\partial v_j(t)}{\partial t} &= d_2 D[v_j](t) + r_2 v_j(t)[1 - a_{21} u_j(t) - v_j(t) - a_{23} w_j(t)], \\ \frac{\partial w_j(t)}{\partial t} &= d_3 D[w_j](t) + r_3 w_j(t)[1 - a_{32} v_j(t) - w_j(t)]. \end{aligned} \tag{1.2}$$

Under the bistable condition: $0 < a_{21}, a_{23} < 1 < a_{12}$ and $a_{21} + a_{23} > 1$, Guo and Wu [7] proved the existence of bistable traveling wavefront of (1.2) connecting $(1, 0, 1)$ and $(0, 1, 0)$. Under the monostable condition (A), Guo et al. [8] showed that there exists a positive constants $c_{\min}$ such that (1.2) has a traveling wavefront if and only if $c \geq c_{\min}$. They also provided some conditions on the parameters of the competition system such that the linear determinacy is assured. Wu [23] further obtained the asymptotic behavior of the traveling wave solutions of (1.2) at $+\infty$ and constructed some entire solutions which behave as two traveling wavefronts moving towards each other from both sides of x-axis.

It should be mentioned that, in the past decades, there have been many works devoted to the traveling wavefronts of the following two-component competition systems with or without delay, see e.g. [3, 4, 5, 6, 9, 11, 12, 13, 14, 15, 16, 17, 25],

$$\begin{aligned} \frac{\partial u_1(x,t)}{\partial t} &= d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t)[1 - a_1 u_1(x, t - \tau_1) - b_1 u_2(x, t - \tau_2)], \\ \frac{\partial u_2(x,t)}{\partial t} &= d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t)[1 - b_2 u_1(x, t - \tau_3) - a_2 u_2(x, t - \tau_4)], \end{aligned} \tag{1.3}$$

and its corresponding spatial discrete version

$$\begin{aligned} \frac{du_j}{dt} &= d_1[u_{j+1} - 2u_j + u_{j-1}] + r_1 u_j(t)[1 - a_1 u_j(t - \tau_1) - b_1 v_j(t - \tau_2)], \\ \frac{dv_j}{dt} &= d_2[v_{j+1} - 2v_j + v_{j-1}] + r_2 v_j(t)[1 - b_2 u_j(t - \tau_3) - a_2 v_j(t - \tau_4)]. \end{aligned} \tag{1.4}$$

For instance, Lin and Li [14] considered the existence of traveling wave solutions of (1.3) connecting the trivial equilibrium with the coexistence equilibrium. Guo and Wu [6] studied the existence, asymptotic behavior, monotonicity and uniqueness of monostable traveling wave solutions of (1.4) connecting two semi-positive equilibria when $\tau_i = 0$, $i = 1, \dots, 4$. In [5], they further established the existence of the bistable traveling wavefronts of (1.4) in strong competition case. Lin and his collaborators [17, 19, 20, 18] considered the monostable and bistable traveling wavefronts of two-component competition systems with nonlocal or distributed delays. More precisely, using the method in [6], Li et al. [13] and Li and Li [14] established the existence, asymptotic behavior, monotonicity and uniqueness of monostable traveling wave solutions of (1.3) and (1.4), respectively.

As mentioned before, traveling wave solutions serve an important object to understand the competition mechanism, i.e. which species will survive or die out eventually. For the models (1.1), the condition (A) means that the species u and w are weaker competitors than the species v. Intuitively, species v should win the competition. However, v must compete with u and w. Thus, it is interesting to determine which species will win the competition. In this article, we shall give an affirmative answer under certain conditions.

More precisely, we shall study various qualitative properties of traveling wavefronts of (1.1) connecting $(1, 0, 1)$ and $(0, 1, 0)$. By applying Schauder's fixed point theorem as in [21, 10, 15], we first prove the existence of traveling wavefronts $\Phi(\cdot) = (\phi(\cdot), \psi(\cdot), \theta(\cdot))$ of (1.1) connecting $(1, 0, 1)$ and $(0, 1, 0)$ via constructing a pair of upper and lower solution. Then, we establish the asymptotic behavior of the traveling wavefronts of (1.1) at $\pm\infty$ by using Ikehara's theorem. Finally, we prove the strict monotonicity and uniqueness of all traveling wave solutions provided that the intra-specific competitive delays are zeros. Of particular interest is the effect of the delays on the wave propagation. We find that the minimal wave speed does not depend on all delays under some assumptions (see Theorem 2.7). The decay rate of $\psi(\cdot)$ at $-\infty$ and the decay rate of $\phi(\cdot)$ and $\theta(\cdot)$ at $+\infty$ also does not depend on the all delays. When $\lambda_3(\tau_1) < \Lambda$ (see Lemma 3.6), the delay τ_1 slow down the decay rate of $\phi(\cdot)$ at $-\infty$; when $\lambda_4(\tau_7) < \Lambda$, the delay τ_7 slow down the decay rate of $\theta(\cdot)$ at $-\infty$. Furthermore, when $\lambda_7(\tau_4) \geq \Pi$ (see Lemma 3.10), the delay τ_4 slow down the decay rate of $1 - \psi(\cdot)$ at $+\infty$.

This article is organized as follows. Section 2 and 3 are devoted to the existence and asymptotic behavior of the traveling wavefronts of (2.1), respectively. In section 4, we study that the monotonicity and uniqueness of the traveling wavefronts.

2. Existence of minimal wave speed

In this section, we prove the existence of the minimal wave speed c_* of the traveling fronts of (1.1) connecting $(1, 0, 1)$ and $(0, 1, 0)$. Using Schauder's fixed point theorem, the proof of the existence of the traveling fronts with speed $c > c_*$ is similar to that of [21, 10, 15]. Here, we only indicate the differences which may appear.

Throughout this paper, we use the usual notations for the standard ordering in $\mathbb{R}^3$. That is, for $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, we denote $u \leq v$ if $u_i \leq v_i$, $i = 1, 2, 3$; $u < v$ if $u \leq v$ but $u \neq v$; and $u \ll v$ if $u \leq v$ but $u_i \neq v_i$, $i = 1, 2, 3$. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^3$ and $\|\cdot\|$ denote the supremum norm in $C([-c\tau, 0], \mathbb{R}^3)$.

Letting $u_j^* = 1 - u_j$, $v_j^* = v_j$, $w_j^* = 1 - w_j$ and dropping the star, system (1.1) is converted to the cooperative system

$$
\begin{aligned}
\frac{\partial u_j(t)}{\partial t} &= d_1 D[u_j](t) + r_1(1-u_j(t))[-u_j(t-\tau_1) + a_{12}v_j(t-\tau_2)],\\
\frac{\partial v_j(t)}{\partial t} &= d_2 D[v_j](t) + r_2 v_j(t)\big[r_0 - v_j(t-\tau_4)\\
&\quad + a_{21}u_j(t-\tau_3) + a_{23}w_j(t-\tau_5)\big],\\
\frac{\partial w_j(t)}{\partial t} &= d_3 D[w_j](t) + r_3(1-w_j(t))[a_{32}v_j(t-\tau_6) - w_j(t-\tau_7)],
\end{aligned}
\tag{2.1}
$$

where $r_0 = 1 - a_{21} - a_{23}$. Obviously, the equilibria $(1,0,1)$ and $(0,1,0)$ of (1.1) become the equilibria $\mathbf{0} := (0,0,0)$ and $\mathbf{1} := (1,1,1)$ of system (2.1). In the following, we shall deal with the traveling wave solution of system (2.1) connecting $\mathbf{0}$ and $\mathbf{1}$, because of its equivalence to system (1.1).

For convenience, we denote $\tau = \max_{i=1,2,\dots,7}\{\tau_i\}$, and $C_{[\mathbf{0},\mathbf{1}]}(\mathbb{R},\mathbb{R}^3) = \{\Psi \in C(\mathbb{R},\mathbb{R}^3) : \mathbf{0} \le \Psi(\xi) \le \mathbf{1}, \xi \in \mathbb{R}\}$. As usual, for any $\phi \in C(\mathbb{R},\mathbb{R})$, we define $\phi_\xi(\cdot) = \phi(\xi + \cdot) \in C([-c\tau, 0],\mathbb{R})$. We further define

$$
\begin{aligned}
f_1(\psi_1,\psi_2,\psi_3) &= r_1(1-\psi_1(0))[-\psi_1(-c\tau_1) + a_{12}\psi_2(-c\tau_2)],\\
f_2(\psi_1,\psi_2,\psi_3) &= r_2\psi_2(0)[r_0 - \psi_2(-c\tau_4) + a_{21}\psi_1(-c\tau_3) + a_{23}\psi_3(-c\tau_5)],\\
f_3(\psi_1,\psi_2,\psi_3) &= r_3(1-\psi_3(0))[-\psi_3(-c\tau_7) + a_{32}\psi_2(-c\tau_6)].
\end{aligned}
$$

for $(\psi_1,\psi_2,\psi_3) \in C([-c\tau,0],\mathbb{R}^3)$.

A solution $(u_j(t), v_j(t), w_j(t)) : \mathbb{Z}\times\mathbb{R} \to [\mathbf{0},\mathbf{1}]$ of (2.1) is called a traveling wave solution connecting $\mathbf{0}$ and $\mathbf{1}$ if there exist $c \in \mathbb{R}$ and a smooth function $\Phi(\cdot) = (\phi_1(\cdot),\phi_2(\cdot),\phi_3(\cdot)) : \mathbb{R} \to [\mathbf{0},\mathbf{1}]$ such that

$$
(u_j(t), v_j(t), w_j(t)) = (\phi_1(j+ct), \phi_2(j+ct), \phi_3(j+ct)) \tag{2.2}
$$

and

$$
\lim_{\xi\to-\infty}\Phi(\xi) = \mathbf{0}, \quad \lim_{\xi\to+\infty}\Phi(\xi) = \mathbf{1}. \tag{2.3}
$$

According to (2.2), it is obvious that $(\phi_1(\xi),\phi_2(\xi),\phi_3(\xi))$ with $\xi := j + ct$ satisfies the equations

$$
c\phi_i'(\xi) - d_i[\phi_i(\xi+1) - 2\phi_i(\xi) + \phi_i(\xi-1)] - f_i(\phi_{1,\xi},\phi_{2,\xi},\phi_{3,\xi}) = 0, \quad i = 1,2,3. \tag{2.4}
$$

Lemma 2.1. *If $\tau_1,\tau_4,\tau_7 \ge 0$ are small enough, then f satisfies the quasi-monotone condition: there exist positive constants l_1, l_2 and l_3 such that*

$$
f_i(\phi_1,\phi_2,\phi_3) - f_i(\psi_1,\psi_2,\psi_3) + l_i[\phi_i(0) - \psi_i(0)] \ge 2d_i[\phi_i(0) - \psi_i(0)], \quad i = 1,2,3
$$

for $\Phi = (\phi_1,\phi_2,\phi_3), \Psi = (\psi_1,\psi_2,\psi_3) \in C([-c\tau,0],\mathbb{R}^3)$, where $\mathbf{0} \le \Phi(s) \le \Psi(s) \le \mathbf{1}$; $e^{\frac{l_i}{c}s}[\phi_i(s) - \psi_i(s)]$ are nondecreasing in $s \in [-c\tau, 0]$, $i = 1,2,3$.

Definition 2.2. A continuous function $\bar\Phi = (\bar\phi_1,\bar\phi_2,\bar\phi_3) \in C_{[\mathbf{0},\mathbf{1}]}(\mathbb{R},\mathbb{R}^3)$ is called an upper solution of (2.4) if it satisfies

$$
c\bar\phi_i'(\xi) - d_i[\bar\phi_i(\xi+1) - 2\bar\phi_i(\xi) + \bar\phi_i(\xi+1)] - f_i(\bar\Phi_\xi) \ge 0 \quad \text{for } \xi \in \mathbb{R},\ i = 1,2,3. \tag{2.5}
$$

A lower solution of (2.4) can be similarly defined by only reversing the inequality in (2.5).

In what follows, we assume that there exist an upper solution $\bar\Phi = (\bar\phi_1,\bar\phi_2,\bar\phi_3)$ and a lower solution $\underline{\Phi} = (\underline{\phi}_1,\underline{\phi}_2,\underline{\phi}_3)$ of (2.4) satisfying the hypotheses

(H1) $\mathbf{0} \le \underline{\Phi}(\xi) \le \bar{\Phi}(\xi) \le \mathbf{1}$, $\underline{\phi}_2(\xi) \ne 0$, $\xi \in \mathbb{R}$;
(H2) $\lim_{\xi\to-\infty} \underline{\Phi}(\xi) = \mathbf{0}$, $\lim_{\xi\to\infty} \bar{\Phi}(\xi) = \mathbf{1}$, $\lim_{\xi\to-\infty} \bar{\phi}_1(\xi) = 0$, $\lim_{\xi\to-\infty} \bar{\phi}_3(\xi) = 0$;
(H3) The set $\Gamma := \Gamma(\underline{\Phi}, \bar{\Phi}) \in C(\mathbb{R}, \mathbb{R}^3)$ is nonempty, where

$$
\begin{aligned}
\Gamma = \Big\{ & \Psi = (\psi_1, \psi_2, \psi_3) : \text{(i) } \underline{\Phi}(\xi) \le \Psi(\xi) \le \bar{\Phi}(\xi) \text{ for } \xi \in \mathbb{R}; \\
& \text{(ii) } \Psi(\xi) \text{ is nondecreasing in } \xi \in \mathbb{R}; \\
& \text{(iii) } e^{\frac{L_i}{c}\xi}[\bar{\phi}_i(\xi) - \psi_i(\xi)] \text{ and } e^{\frac{L_i}{c}\xi}[\psi_i(\xi) - \underline{\phi}_i(\xi)] \text{ are nondecreasing} \\
& \quad \text{in } \xi \in \mathbb{R}, \text{for } i = 1, 2, 3; \\
& \text{(iv) } e^{\frac{L_i}{c}\xi}[\psi_i(\xi + s) - \psi_i(\xi)] \text{ are nondecreasing in } \xi \in \mathbb{R} \text{ for } s > 0, \\
& \quad i = 1, 2, 3. \Big\}
\end{aligned}
$$

Theorem 2.3. *Assume that $\tau_1, \tau_4, \tau_7 \ge 0$ are small enough. If there is an upper solution $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \in C_{[\mathbf{0},\mathbf{1}]}(\mathbb{R}, \mathbb{R}^3)$ and a lower solution $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3) \in C_{[\mathbf{0},\mathbf{1}]}(\mathbb{R}, \mathbb{R}^3)$ of* (2.4) *and satisfying* (H1)–(H3)*, then* (2.1) *admits a traveling wavefront satisfying* (2.3).

Proof. By Schauder's fixed point theorem and the method in [21, 10, 15], one can easily show that (2.4) admits a solution $\Phi = (\phi_1, \phi_2, \phi_3) \in \Gamma$.

Next, we verify the boundary conditions (2.3). Since (ϕ_1, ϕ_2, ϕ_3) is monotone and bounded, $\lim_{\xi\to\pm\infty}(\phi_1, \phi_2, \phi_3)$ exists, denoting $(\phi_{1\pm}, \phi_{2\pm}, \phi_{3\pm})$. Taking the limit in (2.4) as $\xi \to \pm\infty$, we have $f(\phi_{1\pm}, \phi_{2\pm}, \phi_{3\pm}) = 0$. That is, $(\phi_{1\pm}, \phi_{2\pm}, \phi_{3\pm})$ are two equilibria of (2.4). Moreover, $(\phi_{1\pm}, \phi_{2\pm}, \phi_{3\pm}) \in [\mathbf{0}, \mathbf{1}]$ with $(\phi_{1-}, \phi_{2-}, \phi_{3-}) \le (\phi_{1+}, \phi_{2+}, \phi_{3+})$. So $(\phi_{1\pm}, \phi_{2\pm}, \phi_{3\pm})$ may be $(1, 0, 1), (0, 0, 1), (1, 1, 1), (1, 0, 0)$ and $(0, 0, 0)$ by (A). Since $0 < \sup_{\xi\in\mathbb{R}} \underline{\phi}_2(\xi) \le \phi_{2+}$ by (H1), it must be $(\phi_{1+}, \phi_{2+}, \phi_{3+}) = (1, 1, 1)$. Since $\lim_{\xi\to-\infty} \bar{\phi}_1(\xi) = 0$ and $\lim_{\xi\to-\infty} \bar{\phi}_3(\xi) = 0$ by (H2), we have $(\phi_{1-}, \phi_{2-}, \phi_{3-}) = (0, 0, 0)$. The proof is complete. $\square$

Theorem 2.3 implies that the existence of traveling wave solutions could be transformed to the existence of upper and lower solutions. Next, we construct a pair of upper and lower solution of (2.4) satisfying the condition in Theorem 2.3.

By linearizing the second equation in (2.4) around the unstable equilibrium $(0, 0, 0)$, we could get the following characteristic equation

$$
\Delta_2(\lambda, c) = d_2(e^{\lambda} + e^{-\lambda} - 2) - c\lambda + r_2(1 - a_{21} - a_{23}). \tag{2.6}
$$

Thus, we can easily obtain the following Lemma.

Lemma 2.4. *Assume* (H0) *holds. Then there exist $c_* > 0$ and $\lambda_* > 0$ such that $\Delta_2(\lambda_*, c_*) = 0$ and $\frac{\partial \Delta_2(\lambda,c)}{\partial\lambda}|_{(\lambda_*,c_*)} = 0$. Moreover, for $c > c_*$, $\Delta_2(\lambda, c)$ has two positive roots λ_1, λ_2 with $0 < \lambda_1 < \lambda_* < \lambda_2$, and*

$$
\Delta_2(\lambda, c) \begin{cases} > 0, & \lambda < \lambda_1, \\ = 0, & \lambda_1 < \lambda < \lambda_2, \\ < 0, & \lambda > \lambda_2, \end{cases}
$$

for $0 < c < c_*, \Delta_2(\lambda, c) > 0$ for $\lambda \in \mathbb{R}$.

Take $c > c_*$. For $\eta \in (1, \min\{2, \frac{\lambda_2}{\lambda_1}\})$, we define the continuous functions

$$\bar{\phi}_1(\xi) = \begin{cases} e^{\lambda_1 \xi}, & \xi \le 0, \\ 1, & \xi > 0, \end{cases} \quad \bar{\phi}_2(\xi) = \begin{cases} e^{\lambda_1 \xi}, & \xi \le 0, \\ 1, & \xi > 0, \end{cases}$$

$$\bar{\phi}_3(\xi) = \begin{cases} e^{\lambda_1 \xi}, & \xi \le 0, \\ 1, & \xi > 0, \end{cases} \quad \underline{\phi}_1(\xi) \equiv 0,\ \xi \in \mathbb{R},$$

$$\underline{\phi}_2(\xi) = \begin{cases} e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi}, & \xi \le \xi_0, \\ 0, & \xi > \xi_0, \end{cases} \quad \underline{\phi}_3(\xi) \equiv 0,\ \xi \in \mathbb{R}.$$

where $q > 0$ is large enough, $\xi_0(q) = \frac{1}{(\eta-1)\lambda_0} \ln \frac{1}{q}$, $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$, $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ satisfy (H1) and (H2) for large enough q. By the definition of η, it follows that $\Delta_1(\eta\lambda_1, c) < 0$.

By simple calculations, we have the following results.

Lemma 2.5. *Let* $(\phi_1, \phi_2, \phi_3) = (\frac{e^{\lambda_1 \xi}}{1+e^{\lambda_1 \xi}}, \frac{e^{\lambda_1 \xi}}{1+e^{\lambda_1 \xi}}, \frac{e^{\lambda_1 \xi}}{1+e^{\lambda_1 \xi}})$. *Then* $(\phi_1, \phi_2, \phi_3) \in \Gamma$.

Lemma 2.6. *Let* (H0) *hold. Assume* $\tau_1 < \tau_2$, $\tau_7 < \tau_6$ *and* $\tau_4 < \min\{\tau_3, \tau_5\}$. *Assume further that* $d_2 \ge \max\{d_1, d_3\}$ *and* $r_2(1 - a_{21} - a_{23}) \ge \max\{r_1(a_{12} - 1), r_3(a_{32} - 1)\}$ *hold. Then* $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ *is an upper solution and* $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ *is a lower solution of* (2.4).

Proof. For simplicity, denote

$$T_i(\Phi)(\xi) = d_i[\phi_i(\xi+1) - 2\phi_i(\xi) + \phi_i(\xi-1)] - c\phi_i'(\xi) + f_i(\phi_{1,\xi}, \phi_{2,\xi}, \phi_{3,\xi}), \quad i = 1, 2, 3,$$

for $\Phi = (\phi_1, \phi_2, \phi_3)$. We now show that $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ is an upper solution. First, we prove that $T_1(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \le 0$. We distinguish two cases.

Case (i): $\xi \le 0$. Noting that $\bar{\phi}_2(\xi - c\tau_2) = \bar{\phi}_1(\xi - c\tau_2) \le \bar{\phi}_1(\xi - c\tau_1)$, $r_1(a_{12} - 1) \le r_2(1 - a_{21} - a_{23})$ and $\bar{\phi}_1(\xi \pm 1) \le e^{\lambda_1(\xi \pm 1)}$, we have

$$\begin{aligned} T_1(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)(\xi) &\le d_1 D[\bar{\phi}_1] - c\bar{\phi}_1'(\xi) + r_1(1 - \bar{\phi}_1(\xi))(a_{12} - 1)\bar{\phi}_1(\xi - c\tau_1) \\ &\le d_1 D[\bar{\phi}_1] - c\bar{\phi}_1'(\xi) + r_1(a_{12} - 1)\bar{\phi}_1(\xi) \\ &\le e^{\lambda_1 \xi}[d_1(e^{\lambda_1} + e^{-\lambda_1} - 2) - c\lambda_1 + r_2(1 - a_{21} - a_{23})] \\ &\le e^{\lambda_1 \xi}[d_2(e^{\lambda_1} + e^{-\lambda_1} - 2) - c\lambda_1 + r_2(1 - a_{21} - a_{23})] = 0. \end{aligned}$$

Case (ii): $\xi > 0$. Noting that $\bar{\phi}_1(\xi - 1) \le 1$, we have $T_1(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) = d_1[-1 + \bar{\phi}_1(\xi - 1)] \le 0$. Similarly, we can verify that $T_3(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \le 0$.

Next, we show that $T_2(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \le 0$. Now, we distinguish three cases.

Case (i): $\xi \le 0$. In this case, by $\tau_4 < \min\{\tau_3, \tau_5\}$, we have

$$\begin{aligned} &a_{21}\bar{\phi}_1(\xi - c\tau_3) + a_{23}\bar{\phi}_3(\xi - c\tau_5) - \bar{\phi}_2(\xi - c\tau_4) \\ &= a_{21}\bar{\phi}_2(\xi - c\tau_3) + a_{23}\bar{\phi}_2(\xi - c\tau_5) - \bar{\phi}_2(\xi - c\tau_4) \\ &\le (a_{21} + a_{23} - 1)\bar{\phi}_2(\xi - c\tau_4) \le 0. \end{aligned}$$

Noting that $\bar{\phi}_2(\xi \pm 1) \le e^{\lambda_1(\xi \pm 1)}$, we have

$$T_2(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)(\xi) \le d_2 D[\bar{\phi}_2] - c\bar{\phi}_2'(\xi) + r_2(1 - a_{21} - a_{23})\bar{\phi}_2(\xi) \le e^{\lambda_1 \xi}\Delta_2(\lambda_1, c) = 0.$$

Case (ii): $0 < \xi \le c\tau_4$. Noting that $\bar{\phi}_2(c\tau_4 - 1) \le 1$, $\bar{\phi}_2(0) = 1$ and $\tau_4 < \min\{\tau_3, \tau_5\}$, we have

$$T_2(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)(c\tau_4) \le r_2\bar{\phi}_2(c\tau_4)[r_0 + a_{21}\bar{\phi}_1(c\tau_4 - c\tau_3) + a_{23}\bar{\phi}_3(c\tau_4 - c\tau_5) - \bar{\phi}_2(0)]$$

$$= -r_2(a_{21} - a_{21}e^{\lambda_1 c(\tau_4-\tau_3)} + a_{23} - a_{23}e^{\lambda_1 c(\tau_4-\tau_5)}) < 0.$$

Since τ_4 is small enough and independent of $\bar\phi_1(\xi), \bar\phi_2(\xi), \bar\phi_3(\xi)$, $\bar\phi_2'(\xi), \bar\phi_1(\xi), \bar\phi_2(\xi)$ and $\bar\phi_3(\xi)$ are uniformly bounded and uniformly continuous for $\xi \in \mathbb{R}\setminus\{0\}$, it follows that $T_2(\bar\phi_1, \bar\phi_2, \bar\phi_3)(\xi) \le 0$ for $0 < \xi < c\tau_4$.

Case (iii): $\xi > c\tau_4$. In this case $\bar\phi_2(\xi - c\tau_4) = \bar\phi_2(\xi+1) = 1$. Since $\bar\phi_2(\xi-1) \le 1, \bar\phi_1(\xi - c\tau_3) \le 1$ and $\bar\phi_3(\xi - c\tau_5) \le 1$, we have

$$\begin{aligned}
T_2(\bar\phi_1, \bar\phi_2, \bar\phi_3)(\xi) &= d_2[-1 + \bar\phi_2(\xi-1)] + r_2\bar\phi_2(\xi)\big[-a_{21} - a_{23} \\
&\quad + a_{21}\bar\phi_1(\xi - c\tau_3) + a_{23}\bar\phi_3(\xi - c\tau_5)\big] \le 0.
\end{aligned}$$

Therefore, $(\bar\phi_1, \bar\phi_2, \bar\phi_3)$ is an upper solution. The verification of the lower solution is similar and omitted. The proof is completed. □

Lemmas 2.5 and 2.6 imply that Γ is nonempty and there exist a pair of upper and lower solutions for (2.4). Now we state the existence result on traveling wave solutions of (2.1) connecting $\mathbf{0}$ and $\mathbf{1}$.

Theorem 2.7. *Let* (H0) *hold. Assume* $\tau_1 < \tau_2$, $\tau_7 < \tau_6$ *and* $\tau_4 < \min\{\tau_3, \tau_5\}$. *Assume further that* $d_2 \ge \max\{d_1, d_3\}$ *and* $r_2(1 - a_{21} - a_{23}) \ge \max\{r_1(a_{12} - 1), r_3(a_{32} - 1)\}$ *hold. Then, the following result holds:*

(i) *For each* $c \ge c_*$, (2.1) *has a traveling wavefront* $\Phi(\xi) = \Phi(j + ct) = (\phi_1(j + ct), \phi_2(j + ct), \phi_3(j + ct))$ *with the wave speed* c *which connects* $\mathbf{0}$ *and* $\mathbf{1}$. *Moreover, for* $c > c_*$,
$$\lim_{\xi\to-\infty} \phi_2(\xi)e^{-\lambda_1\xi} = 1, \quad \lim_{\xi\to-\infty} \phi_2'(\xi)e^{-\lambda_1\xi} = \lambda_1. \tag{2.7}$$

(ii) *For* $0 < c < c_*$, (2.1) *has no traveling wave solution* $\Psi(\xi)$ *with* $\Psi(-\infty) = \mathbf{0}$.

Proof. (i) For $c > c_*$, the existence of the traveling wavefront sandwiched by the upper and lower solutions follows from Theorem 2.3 and Lemmas 2.5 and 2.6. From the definition of the upper and lower solutions, one can easily see that $\lim_{\xi\to-\infty} \phi_2(\xi)e^{-\lambda_1\xi} = 1$. Moreover, using $\lim_{\xi\to-\infty} \phi_2(\xi)e^{-\lambda_1\xi} = 1$ and $\lim_{\xi\to-\infty}(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) = (0, 0, 0)$, we obtain

$$\begin{aligned}
&\lim_{\xi\to-\infty} \phi_2'(\xi)e^{-\lambda_1\xi} \\
&= \frac{1}{c}\lim_{\xi\to-\infty} \Big\{d_2[\phi_2(\xi+1) - 2\phi_2(\xi) + \phi_2(\xi-1)] + r_2\phi_2(\xi)\big[1 - a_{21} - a_{23} \\
&\quad + a_{21}\phi_1(\xi - c\tau_3) - \phi_2(\xi - c\tau_4) + a_{23}\phi_3(\xi - c\tau_5)\big]\Big\}e^{-\lambda_1\xi} \\
&= \frac{1}{c}[d_2(e^{\lambda_1} - 2 + e^{-\lambda_1}) + r_2(1 - a_{21} - a_{23})] = \lambda_1.
\end{aligned}$$

For $c = c_*$, the existence of the traveling wavefront can be established by a limiting argument as in [27, Theorem 3.1]. We omit the details here.

(ii) For $c < c_*$, the non-existence of traveling wave solutions will be proved in Section 3. The proof is complete. □

3. Asymptotic behavior of traveling fronts

In this section, we shall study the asymptotic behavior of wave profile as $\xi \to \pm\infty$ with $\tau_1 = \tau_4 = \tau_7 = 0$ or τ_1, τ_4 and τ_7 small enough. The non-existence of the traveling wave solutions with $c \in (0, c_*)$ is also proved. Throughout this section, we assume that (H0) holds.

We first give some properties of solutions to (2.4).

Lemma 3.1. *Let $(\phi(\xi), \psi(\xi), \theta(\xi)) : \mathbb{R} \to [\mathbf{0}, \mathbf{1}]$ be a solution of (2.4) and (2.3) with $c > 0$. Then $\psi(\xi) > 0$, $\phi(\xi) < 1$ and $\theta(\xi) < 1$ for $\xi \in \mathbb{R}$.*

Proof. We first prove that $\psi(\xi) > 0$ for $\xi \in \mathbb{R}$. Suppose for the contrary that there exists ξ_0 such that $\psi(\xi_0) = 0$. Since $\psi(+\infty) = 1$, we may assume ξ_0 is the right-most point such that $\psi(\xi_0) = 0$. Then $\psi(\xi_0) = 0$ is a minimum point, so we have $\psi'(\xi_0) = 0$. From the second equation of (2.4), it follows that $\psi(\xi_0 + 1) = \psi(\xi_0 - 1) = 0$, a contradiction with the definition of ξ_0. Thus $\psi(\xi) > 0$ in $\mathbb{R}$.

Similarly, one can easily prove that $\phi(\xi) < 1$ and $\theta(\xi) < 1$ in $\mathbb{R}$. This completes the proof. $\square$

The following Proposition and the Ikehara's theorem will play an important role in proving the asymptotic behavior at infinity.

Proposition 3.2 ([1, 2])**.** *Let $c > 0$ be a constant and $B(\cdot)$ be a continuous function having finite $B(\pm\infty) := \lim_{x\to\pm\infty} B(x)$. Let $z(\cdot)$ be a measurable function satisfying*

$$cz(x) = e^{\int_x^{x+1} z(s)ds} + e^{-\int_{x-1}^{x} z(s)ds} + B(x), \quad \forall x \in \mathbb{R}.$$

Then z is uniformly continuous and bounded. In addition, $w^{\pm} = \lim_{x\to\pm\infty} z(x)$ exist and are real roots of the characteristic equation

$$cw = e^{w} + e^{-w} + B(\pm\infty).$$

Lemma 3.3 (Ikehara's theorem)**.** *Let φ be a positive nondecreasing function on $\mathbb{R}$, and define $F(\lambda) := \int_{-\infty}^{0} e^{-\lambda\xi}\varphi(\xi)d\xi$. If F can be written as $F(\lambda) := \frac{H(\lambda)}{(\alpha-\lambda)^{\nu+1}}$ for some $\nu > -1, \alpha > 0$, and H is analytic in the strip $0 < \Re\lambda \leq \alpha$, then*

$$\lim_{\xi\to-\infty} \frac{\varphi(\xi)}{|\xi|^{\nu} e^{\alpha\xi}} = \frac{H(\alpha)}{\Gamma(\alpha + 1)}.$$

By Lemmas 2.4 and 3.1 and Proposition 3.2, we could obtain the following theorem on the asymptotic behavior of ψ at $-\infty$.

Theorem 3.4. *Let $(\phi(\xi), \psi(\xi), \theta(\xi)) : \mathbb{R} \to [\mathbf{0}, \mathbf{1}]$ be a solution of (2.4) and (2.3) with $c > 0$. Then $\lim_{\xi\to-\infty} \frac{\psi'(\xi)}{\psi(\xi)} = \Lambda \in \{\lambda_1, \lambda_2\}$, where λ_1, λ_2 are given in Lemma 2.4.*

Proof. Set $z(\xi) = \frac{\psi'(\xi)}{\psi(\xi)}$. By the second equation of (2.4), $z(\xi)$ satisfies

$$\begin{aligned} &d_2\Big(e^{\int_\xi^{\xi+1} z(s)ds} + e^{\int_\xi^{\xi-1} z(s)ds} - 2\Big) - cz(\xi) \\ &+ r_2[1 - a_{21} - a_{23} - \psi(\xi - c\tau_4) + a_{21}\phi(\xi - c\tau_3) + a_{23}\theta(\xi - c\tau_5)] = 0. \end{aligned}$$

Thus, since $\lim_{\xi\to-\infty}(\phi(\xi), \psi(\xi), \theta(\xi)) = (0, 0, 0)$, the assertion follows from Lemma 2.4 and Proposition 3.2. The proof is complete. $\square$

Based on Theorem 3.4, we now give the proof of the non-existence of the traveling wave solutions with $c \in (0, c_*)$.

Proof of Theorem 2.7 (ii). Suppose for the contrary that (2.1) has a traveling wave solution $\Psi(\xi) = (\phi_1(\xi), \psi_1(\xi), \theta_1(\xi))$ with speed $c \in (0, c_*)$ and $\Psi(-\infty) = \mathbf{0}$. By Lemma 2.4, we see that $\Delta_2(\Lambda, c) > 0$. On the other hand, from the second equation of (2.4), we have

$$\begin{aligned} &d_2\Big(e^{\int_\xi^{\xi+1}\frac{\psi_1'(s)}{\psi_1(s)}ds} + e^{\int_\xi^{\xi-1}\frac{\psi_1'(s)}{\psi_1(s)}ds} - 2\Big) - c\frac{\psi_1'(\xi)}{\psi_1(\xi)} + r_2\big[1 - a_{21} - a_{23} \\ &- \psi_1(\xi - c\tau_4) + a_{21}\phi_1(\xi - c\tau_3) + a_{23}\theta_1(\xi - c\tau_5)\big] = 0. \end{aligned} \tag{3.1}$$

By Theorem 3.4, $\lim_{t\to-\infty}\frac{\psi_1'(\xi)}{\psi_1(\xi)} = \Lambda$. It then follows from (3.1) and the fact $\Psi(-\infty) = \mathbf{0}$ that

$$\Delta_2(\Lambda, c) = d_2(e^\Lambda + e^{-\Lambda} - 2) - c\Lambda + r_2(1 - a_{21} - a_{23}) = 0,$$

which contradicts to $\Delta_2(\Lambda, c) > 0$. This completes the proof of Theorem 2.7 (ii). $\square$

Given the continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, define the bilateral Laplace transform

$$L(\lambda, \varphi) = \int_{-\infty}^{\infty} e^{-\lambda\xi}\varphi(\xi)d\xi.$$

Lemma 3.5. *Let $(\phi(\xi), \psi(\xi), \theta(\xi)) : \mathbb{R} \to [\mathbf{0}, \mathbf{1}]$ be a solution of (2.4) and (2.3) with $c \ge c_*$. Then $L(\lambda, \psi) < \infty, \lambda \in (0, \Lambda)$ and $L(\lambda, \psi) = \infty, \lambda \in \mathbb{R} \setminus (0, \Lambda)$.*

Proof. From the definition of bilateral Laplace transform, we have

$$\begin{aligned} L(\lambda, \psi) &= \int_{-\infty}^{\infty} \psi(\xi)e^{-\lambda\xi}d\xi \\ &= \int_{-\infty}^{0} \psi(\xi)e^{-\lambda\xi}d\xi + \int_{0}^{\infty} \psi(\xi)e^{-\lambda\xi}d\xi. \end{aligned}$$

Then the assertion follows from Theorem 3.4 and the fact $\psi(-\infty) = 0$. We omit it here. $\square$

Let

$$\Delta_1(\lambda, c, \tau_1) = d_1(e^\lambda + e^{-\lambda} - 2) - c\lambda - r_1e^{-\lambda c\tau_1}, \tag{3.2}$$

$$\Delta_3(\lambda, c, \tau_7) = d_3(e^\lambda + e^{-\lambda} - 2) - c\lambda - r_3e^{-\lambda c\tau_7}. \tag{3.3}$$

It is easy to verify that the following results hold, see e.g. [13].

Lemma 3.6. (i) *$\Delta_1(\lambda, c, \tau_1) = 0$ has a unique positive root $\lambda_3(\tau_1) > 0$ for $c > 0$ and $\Delta_1(\lambda, c, \tau_1) < 0$ for $\lambda \in (0, \lambda_3(\tau_1))$. Moreover, $\lambda_3(\tau_1)$ is decreasing with respect to τ_1.*

(ii) *$\Delta_3(\lambda, c, \tau_7) = 0$ has a unique positive root $\lambda_4(\tau_7) > 0$ for $c > 0$ and $\Delta_3(\lambda, c, \tau_7) < 0$ for $\lambda \in (0, \lambda_4(\tau_7))$. Moreover, $\lambda_4(\tau_7)$ is decreasing with respect to τ_7.*

(iii) *For $\tau_1 = 0$ or small enough $\tau_1 > 0$, $\lambda = \lambda_3(\tau_1)$ is a unique root with $\Re\lambda = \lambda_3(\tau_1)$ of $\Delta_1(\lambda, c, \tau_1) = 0$.*

(iv) *For $\tau_7 = 0$ or small enough $\tau_7 > 0$, $\lambda = \lambda_4(\tau_7)$ is a unique root with $\Re\lambda = \lambda_4(\tau_7)$ of $\Delta_3(\lambda, c, \tau_7) = 0$.*

The following Lemma shows that $\phi(\xi)$ and $\theta(\xi)$ have similar properties as those of $\psi(\xi)$ described in Lemma 3.5.

Lemma 3.7. *Let $(\phi(\xi),\psi(\xi),\theta(\xi)):\mathbb{R}\to[\mathbf{0},\mathbf{1}]$ be a solution of (2.4) and (2.3) with $c\geq c_*$. Then the following results hold:*

- (i) *$L(\lambda,\phi)<\infty$, $\lambda\in(0,\gamma(\tau_1))$ and $L(\lambda,\phi)=\infty$, $\lambda\in\mathbb{R}\setminus(0,\gamma(\tau_1))$, where $\gamma(\tau_1)=\min\{\Lambda,\lambda_3(\tau_1)\}$.*
- (ii) *$L(\lambda,\theta)<\infty$, $\lambda\in(0,\alpha(\tau_7))$ and $L(\lambda,\theta)=\infty$, $\lambda\in\mathbb{R}\setminus(0,\alpha(\tau_7))$, where $\alpha(\tau_7)=\min\{\Lambda,\lambda_4(\tau_7)\}$.*

Proof. We only prove the assertion (i), since the proof of assertion (ii) is similar. We divide the proof into two steps.

Step 1. We first show that there exists $\nu>0$ such that $L(\lambda,\phi)<\infty$, $\lambda\in(0,\nu)$. Since $d_1(e^{\lambda}-2+e^{-\lambda})-c\lambda=0$ has only two real roots 0 and $\tilde{\Lambda}$ for $c>0$,

$$d_1(e^{\lambda}-2+e^{-\lambda})-c\lambda<0,\ \lambda\in(0,\tilde{\Lambda}).$$

For $\lambda\in(0,\Lambda_0)$, where $\Lambda_0:=\min\{\Lambda,\tilde{\Lambda},\frac{1}{c\tau_1},\frac{1}{2e^2(d_1+c)}\}$, we choose $a_0<0$ small enough satisfying $\phi(\xi)\leq\frac{1}{2}$ for $\xi\leq a_0+c\tau_1+\frac{1}{\lambda}$. Multiplying the first equation of (2.4) by $e^{-\lambda\xi}$, integrating from $a\leq a_0$ to ∞, we have

$$\begin{aligned}
&r_1a_{12}\int_a^\infty(1-\phi(\xi))\psi(\xi-c\tau_2)e^{-\lambda\xi}d\xi\\
&=-d_1\int_a^\infty[\phi(\xi+1)-2\phi(\xi)+\phi(\xi-1)]e^{-\lambda\xi}d\xi+c\int_a^\infty\phi'(\xi)e^{-\lambda\xi}d\xi\\
&\quad+r_1\int_a^\infty(1-\phi(\xi))\phi(\xi-c\tau_1)e^{-\lambda\xi}d\xi.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
r_1a_{12}\int_a^\infty(1-\phi(\xi))\psi(\xi-c\tau_2)e^{-\lambda\xi}d\xi&\leq r_1a_{12}\int_{-\infty}^\infty(1-\phi(\xi))\psi(\xi-c\tau_2)e^{-\lambda\xi}d\xi\\
&\leq r_1a_{12}\int_{-\infty}^\infty\psi(\xi-c\tau_2)e^{-\lambda\xi}d\xi\\
&\leq r_1a_{12}\int_{-\infty}^\infty\psi(\xi)e^{-\lambda\xi}d\xi\\
&=r_1a_{12}L(\lambda,\psi).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&-d_1\int_a^\infty[\phi(\xi+1)-2\phi(\xi)+\phi(\xi-1)]e^{-\lambda\xi}d\xi+c\int_a^\infty\phi'(\xi)e^{-\lambda\xi}d\xi\\
&\quad+r_1\int_a^\infty(1-\phi(\xi))\phi(\xi-c\tau_1)e^{-\lambda\xi}d\xi\\
&=-d_1\Big\{e^{\lambda}\int_{a+1}^\infty-2\int_a^\infty+e^{-\lambda}\int_{a-1}^\infty\Big\}\phi(\xi)e^{-\lambda\xi}d\xi-c\phi(a)e^{-\lambda a}\\
&\quad+c\lambda\int_a^\infty\phi(\xi)e^{-\lambda\xi}d\xi+r_1\int_a^\infty(1-\phi(\xi))\phi(\xi-c\tau_1)e^{-\lambda\xi}d\xi\\
&=-d_1\Big\{e^{\lambda}\int_{a+1}^a+e^{-\lambda}\int_{a-1}^a\Big\}\phi(\xi)e^{-\lambda\xi}d\xi-c\phi(a)e^{-\lambda a}\\
&\quad+[c\lambda-d_1(e^{\lambda}-2+e^{-\lambda})]\int_a^\infty\phi(\xi)e^{-\lambda\xi}d\xi
\end{aligned}$$

$$
\begin{aligned}
&+ r_1 \int_a^{\infty} (1-\phi(\xi))\phi(\xi - c\tau_1) e^{-\lambda\xi} d\xi \\
&\geq -d_1 e^{-\lambda} \int_{a-1}^{a} \phi(\xi) e^{-\lambda\xi} d\xi - c\phi(a)e^{-\lambda a} + r_1 \int_a^{\infty} (1-\phi(\xi))\phi(\xi - c\tau_1) e^{-\lambda\xi} d\xi \\
&\geq -d_1 e^{-\lambda} \int_{a-1}^{a} \phi(a) e^{-\lambda(a-1)} d\xi - c\phi(a)e^{-\lambda a} + r_1 \int_a^{\infty} (1-\phi(\xi))\phi(\xi - c\tau_1) e^{-\lambda\xi} d\xi \\
&\geq -d_1 \phi(a) e^{-\lambda a} - c\phi(a)e^{-\lambda a} + r_1 \int_{a+c\tau_1}^{a+c\tau_1+\frac{1}{\lambda}} (1-\phi(\xi))\phi(\xi - c\tau_1) e^{-\lambda\xi} d\xi \\
&\geq -d_1 \phi(a) e^{-\lambda a} - c\phi(a)e^{-\lambda a} + \frac{r_1}{2} \int_{a+c\tau_1}^{a+c\tau_1+\frac{1}{\lambda}} \phi(\xi - c\tau_1) e^{-\lambda\xi} d\xi \\
&\geq -d_1 \phi(a) e^{-\lambda a} - c\phi(a)e^{-\lambda a} + \frac{r_1}{2} \int_{a+c\tau_1}^{a+c\tau_1+\frac{1}{\lambda}} \phi(a + c\tau_1 - c\tau_1) e^{-\lambda(a+c\tau_1+\frac{1}{\lambda})} d\xi \\
&= \big(-d_1 - c + \frac{r_1}{2\lambda} e^{-(\lambda c\tau_1 + 1)} \big) \phi(a) e^{-\lambda a} \\
&\geq \big(-d_1 - c + \frac{r_1}{2\lambda} e^{-2} \big) \phi(a) e^{-\lambda a}.
\end{aligned}
$$

Thus, using Lemma 3.5, for any $\lambda \in (0, \Lambda_0)$, we obtain $0 < \sup_{a \leq a_0} \phi(a) e^{-\lambda a} < \infty$. Take $\nu \in (0, \Lambda_0)$. Then $L(\lambda, \phi)$ well defined for $\lambda \in (0, \nu)$.

Step 2. We prove $\max \nu = \gamma(\tau_1) = \min\{\Lambda, \lambda_3(\tau_1)\}$. Multiplying the first equation of (2.4) by $e^{-\lambda\xi}$ with $\lambda > 0$ and integrating from $-\infty$ to ∞, we obtain

$$
\begin{aligned}
\Delta_1(\lambda, c, \tau_1) L(\lambda, \phi) = &-r_1 a_{12} \int_{-\infty}^{\infty} (1-\phi(\xi))\psi(\xi - c\tau_2) e^{-\lambda\xi} d\xi \\
&- r_1 \int_{-\infty}^{\infty} \phi(\xi)\phi(\xi - c\tau_1) e^{-\lambda\xi} d\xi.
\end{aligned}
\tag{3.4}
$$

Obviously, the right side of (3.4) is well defined for $\lambda \in (0, \min\{2\max\nu, \Lambda\})$ and $\max\nu \leq \Lambda$. We claim that $\max\nu \leq \lambda_3(\tau_1)$. Otherwise, if $\max\nu > \lambda_3(\tau_1)$, then $L(\lambda_3(\tau_1), \phi) < \infty$. Taking $\lambda = \lambda_3(\tau_1)$ in (3.4), the left side of (3.4) equals to 0 and the right side of (3.4) is always negative by $\phi(\xi) \leq 1$, which leads to a contradiction. It also follows easily from (3.4) that $\gamma(\tau_1) = \Lambda$ if $\Lambda < \lambda_3(\tau_1)$ and $\gamma(\tau_1) = \lambda_3(\tau_1)$ if $\Lambda \geq \lambda_3(\tau_1)$, i.e. $\gamma(\tau_1) = \min\{\Lambda, \lambda_3(\tau_1)\}$. The proof is complete. □

Define

$$
\begin{gathered}
Q_1(\lambda) := \int_{-\infty}^{\infty} \phi(\xi)[-\phi(\xi - c\tau_1) + a_{12}\psi(\xi - c\tau_2)] e^{-\lambda\xi} d\xi, \\
Q_2(\lambda) := \int_{-\infty}^{\infty} \psi(\xi)[\psi(\xi - c\tau_4) - a_{21}\phi(\xi - c\tau_3) - a_{23}\theta(\xi - c\tau_5)] e^{-\lambda\xi} d\xi, \\
Q_3(\lambda) := \int_{-\infty}^{\infty} \theta(\xi)[-\theta(\xi - c\tau_7) + a_{32}\psi(\xi - c\tau_6)] e^{-\lambda\xi} d\xi.
\end{gathered}
$$

Let $q = 1$ when $Q_2(\Lambda) \neq 0$ and $q = 0$ when $Q_2(\Lambda) = 0$. Then, the asymptotic behavior of $(\phi(\xi), \psi(\xi), \theta(\xi))$ at $-\infty$ are as follows.

Theorem 3.8. *Assume that τ_1, τ_4, τ_7 are small enough. Let $(\phi(\xi), \psi(\xi), \theta(\xi)) : \mathbb{R} \to [\mathbf{0}, \mathbf{1}]$ be a solution of (2.4) and (2.3) with $c \geq c_*$. Then there exist $\eta_i > 0 (i = 1, \dots, 8)$ and $\omega_i > 0$ $(i = 3, \dots, 8)$ such that the solution exhibits the asymptotic behavior as $\xi \to -\infty$ as follows:*

(i) $\psi(\xi) = \eta_1 e^{\Lambda\xi} + h.o.t.$ *for* $c > c_*$, $\psi(\xi) = -\eta_2 \xi^q e^{\Lambda\xi} + h.o.t.$ *for* $c = c_*$.

(ii) *For* $c > c_*$,

$$\phi(\xi) = \begin{cases} \eta_3 e^{\Lambda\xi} + h.o.t., & \text{if } \lambda_3(\tau_1) > \Lambda, \\ -\eta_4 \xi e^{\Lambda\xi} + h.o.t., & \text{if } \lambda_3(\tau_1) = \Lambda, \\ \eta_5 e^{\lambda_3(\tau_1)\xi} + h.o.t., & \text{if } \lambda_3(\tau_1) < \Lambda, \end{cases}$$

and

$$\theta(\xi) = \begin{cases} \omega_3 e^{\Lambda\xi} + h.o.t., & \text{if } \lambda_4(\tau_7) > \Lambda, \\ -\omega_4 \xi e^{\Lambda\xi} + h.o.t., & \text{if } \lambda_4(\tau_7) = \Lambda, \\ \omega_5 e^{\lambda_4(\tau_7)\xi} + h.o.t., & \text{if } \lambda_4(\tau_7) < \Lambda. \end{cases}$$

(iii) *For* $c = c_*$,

$$\phi(\xi) = \begin{cases} -\eta_6 \xi^q e^{\Lambda\xi} + h.o.t., & \text{if } \lambda_3(\tau_1) > \Lambda, \\ -\eta_7 \xi^{q+1} e^{\Lambda\xi} + h.o.t., & \text{if } \lambda_3(\tau_1) = \Lambda, \\ \eta_8 e^{\lambda_3(\tau_1)\xi} + h.o.t., & \text{if } \lambda_3(\tau_1) < \Lambda, \end{cases}$$

and

$$\theta(\xi) = \begin{cases} -\omega_6 \xi^q e^{\Lambda\xi} + h.o.t., & \text{if } \lambda_4(\tau_7) > \Lambda, \\ -\omega_7 \xi^{q+1} e^{\Lambda\xi} + h.o.t., & \text{if } \lambda_4(\tau_7) = \Lambda, \\ \omega_8 e^{\lambda_4(\tau_7)\xi} + h.o.t., & \text{if } \lambda_4(\tau_7) < \Lambda. \end{cases}$$

Proof. The idea of the proof follows from [6]. From Lemmas 3.5 and 3.7, $L(\lambda, \phi)$, $L(\lambda, \psi)$ and $L(\lambda, \theta)$ are well defined for $\lambda \in \mathbb{C}$ with $\Re\lambda \in (0, \gamma(\tau_1))$, $\Re\lambda \in (0, \Lambda)$ and $\Re\lambda \in (0, \alpha(\tau_7))$, respectively. It follows from (2.4) that

$$\Delta_2(\lambda, c) \int_{-\infty}^{\infty} \psi(\xi) e^{-\lambda\xi} d\xi = r_2 Q_2(\lambda) \tag{3.5}$$

for $\lambda \in \mathbb{C}$ with $0 < \Re\lambda < \Lambda$,

$$\Delta_1(\lambda, c, \tau_1) \int_{-\infty}^{\infty} \phi(\xi) e^{-\lambda\xi} d\xi = -r_1 r_2 a_{12} e^{-\lambda c\tau_2} \frac{Q_2(\lambda)}{\Delta_2(\lambda, c)} + r_1 Q_1(\lambda) \tag{3.6}$$

for $\lambda \in \mathbb{C}$ with $0 < \Re\lambda < \gamma(\tau_1)$ and

$$\Delta_3(\lambda, c, \tau_7) \int_{-\infty}^{\infty} \theta(\xi) e^{-\lambda\xi} d\xi = -r_3 r_2 a_{32} e^{-\lambda c\tau_6} \frac{Q_2(\lambda)}{\Delta_2(\lambda, c)} + r_3 Q_3(\lambda) \tag{3.7}$$

for $\lambda \in \mathbb{C}$ with $0 < \Re\lambda < \alpha(\tau_7)$. By calculating directly, we could easily obtain that $Q_1(\lambda)$, $Q_2(\lambda)$ and $Q_3(\lambda)$ are analytic in the strip $0 < \Re\lambda < 2\gamma(\tau_1)$, $0 < \Re\lambda < \Lambda + \gamma(\tau_1)$ and $0 < \Re\lambda < 2\alpha(\tau_7)$ by Lemmas 3.5 and 3.7, respectively. Let

$$F(\lambda) = \int_{-\infty}^{0} \psi(\xi) e^{-\lambda\xi} d\xi = \frac{r_2 Q_2(\lambda)}{\Delta_2(\lambda, c)} - \int_0^{\infty} \psi(\xi) e^{-\lambda\xi} d\xi, \tag{3.8}$$

$$H(\lambda) = \frac{r_2 Q_2(\lambda)}{\frac{\Delta_2(\lambda, c)}{(\Lambda - \lambda)^{p+1}}} - (\Lambda - \lambda)^{p+1} \int_0^{\infty} \psi(\xi) e^{-\lambda\xi} d\xi \tag{3.9}$$

where $p = 0$ when $c > c_*$ and $p = q$ when $c = c_*$. Then $H(\lambda)$ is analytic in the strip $0 < \Re\lambda < \Lambda$. Moreover, it is easy to see that $H(\lambda)$ is analytic in the strip $\{\lambda | \Re\lambda = \Lambda\}$. So $H(\lambda)$ is analytic in the strip $0 < \Re\lambda \leq \Lambda$.

It follows from Lemma 3.3 that

$$\lim_{\xi \to -\infty} \frac{\psi(\xi)}{|\xi|^p e^{\Lambda\xi}} = \frac{H(\Lambda)}{\Gamma(\Lambda + 1)},$$

where $p = 0$ when $c > c_*$ and $p = q$ when $c = c_*$. It is easy to see that (i) holds when $H(\Lambda) \neq 0$. We next claim that $H(\Lambda) \neq 0$.

For $c > c_*$, since Λ is a simple root of $\Delta_2(\lambda, c)$ and $p = 0$, it follows that the denominator of the first term of the right side of (3.9) does not equal to zero. We claim that $Q_2(\Lambda) \neq 0$. If not, we obtain that $L(\Lambda, \psi)$ exists by (3.8), which leads to a contradiction of Lemma 3.5. Thus $Q_2(\Lambda) \neq 0$ which implies that $H(\Lambda) \neq 0$ by (3.9).

For $c = c_*$, Λ is a double root of $\Delta_2(\lambda, c)$. If $Q_2(\Lambda) \neq 0$, we take $q = 1$ such that $H(\Lambda) \neq 0$ by (3.9). If $Q_2(\Lambda) = 0$, then Λ must be a simple root of $Q_2(\lambda) = 0$, otherwise, $L(\Lambda, \psi)$ exists by (3.8), which leads to a contradiction for Lemma 3.5. Then we can take $q = 0$ such that $H(\Lambda) \neq 0$ by (3.9).

Next we only need to prove the assertion (ii), since the assertion (iii) can be discussed similarly. Define

$$
\begin{aligned}
F_0(\lambda) &= \int_{-\infty}^{0} \phi(\xi) e^{-\lambda\xi} d\xi \\
&= -\int_{0}^{\infty} \phi(\xi) e^{-\lambda\xi} d\xi - \frac{r_1 r_2 a_{12} e^{-\lambda c \tau_2} Q_2(\lambda)}{\Delta_2(\lambda, c)\Delta_1(\lambda, c, \tau_1)} + \frac{r_1 Q_1(\lambda)}{\Delta_1(\lambda, c, \tau_1)},
\end{aligned} \tag{3.10}
$$

$$
H_0(\lambda) = (\gamma(\tau_1) - \lambda)^{p+1} F_0(\lambda) \tag{3.11}
$$

in the strip $0 < \Re\lambda \leq \gamma(\tau_1)$, where $p = 0$ when $\lambda_3(\tau_1) \neq \Lambda$, $p = 1$ when $\lambda_3(\tau_1) = \Lambda$. By using a similar argument as (i), $H_0(\lambda)$ is also analytic in the strip $0 < \Re\lambda \leq \gamma(\tau_1)$. It follows from Lemma 3.3 that

$$
\lim_{\xi \to -\infty} \frac{\phi(\xi)}{|\xi|^p e^{\gamma\xi}} = \frac{H_0(\gamma(\tau_1))}{\Gamma(\gamma(\tau_1) + 1)},
$$

where $q = 0$ when $\lambda_3(\tau_1) \neq \Lambda$, $q = 1$ when $\lambda_3(\tau_1) = \Lambda$. Next we claim that $H_0(\gamma(\tau_1)) \neq 0$.

If $\lambda_3(\tau_1) \geq \Lambda$, then $\gamma(\tau_1) = \Lambda$. Combining (3.10) and (3.11), we can easily obtain $H_0(\gamma(\tau_1)) \neq 0$ by $Q_2(\lambda) \neq 0$. If $\lambda_3(\tau_1) < \Lambda$, then $\gamma(\tau_1) = \lambda_3(\tau_1)$. Since

$$
\begin{aligned}
H_0(\lambda) = {} & \frac{-r_1 \int_{-\infty}^{\infty} [\phi(\xi)\phi(\xi - c\tau_1) + a_{12}(1 - \phi(\xi))\psi(\xi - c\tau_2)] e^{-\lambda\xi} d\xi}{\frac{\Delta_1(\lambda, c, \tau_1)}{(\lambda_3(\tau_1) - \lambda)}} \\
& - (\lambda_3(\tau_1) - \lambda) \int_0^{\infty} \phi(\xi) e^{-\lambda\xi} d\xi,
\end{aligned}
$$

we see that $H_0(\lambda_3(\tau_1)) \neq 0$. Indeed, if $H_0(\lambda_3(\tau_1)) = 0$, then

$$
\int_{-\infty}^{\infty} [\phi(\xi)\phi(\xi - c\tau_1) + a_{12}(1 - \phi(\xi))\psi(\xi - c\tau_2)] e^{-\lambda\xi} d\xi = 0,
$$

which implies that $\phi(\xi) \equiv \psi(\xi) \equiv 0$, for all $t \in \mathbb{R}$. This contradiction yields that $H_0(\lambda_3(\tau_1)) \neq 0$. The proof is complete. $\square$

From Theorem 3.8, it is not difficult to get the following corollary.

Corollary 3.9. *If $(\phi(\xi), \psi(\xi), \theta(\xi))$ is described as Theorem 3.8, then*

$$
\lim_{\xi \to -\infty} \frac{\phi'(\xi)}{\phi(\xi)} = \gamma(\tau_1), \quad \lim_{\xi \to -\infty} \frac{\theta'(\xi)}{\theta(\xi)} = \alpha(\tau_7), \quad \lim_{\xi \to -\infty} \frac{\psi'(\xi)}{\psi(\xi)} = \Lambda.
$$

Next we establish the asymptotic behavior of $(\phi(\xi),\psi(\xi),\theta(\xi))$ at $+\infty$. For this purpose, let $\tilde\phi=1-\phi$, $\tilde\psi=1-\psi$, $\tilde\theta=1-\theta$, substituting $\tilde\phi$, $\tilde\psi$ and $\tilde\theta$ into (2.4), we have

$$\begin{aligned}
&d_1[\tilde\phi(\xi+1)-2\tilde\phi(\xi)+\tilde\phi(\xi-1)]-c\tilde\phi'(\xi)+\tilde f_1(\tilde\phi_\xi,\tilde\psi_\xi,\tilde\theta_\xi)=0,\\
&d_2[\tilde\psi(\xi+1)-2\tilde\psi(\xi)+\tilde\psi(\xi-1)]-c\tilde\psi'(\xi)+\tilde f_2(\tilde\phi_\xi,\tilde\psi_\xi,\tilde\theta_\xi)=0,\\
&d_3[\tilde\theta(\xi+1)-2\tilde\theta(\xi)+\tilde\theta(\xi-1)]-c\tilde\theta'(\xi)+\tilde f_3(\tilde\phi_\xi,\tilde\psi_\xi,\tilde\theta_\xi)=0
\end{aligned}\tag{3.12}$$

with the asymptotic boundary conditions

$$\lim_{\xi\to-\infty}(\tilde\phi(\xi),\tilde\psi(\xi),\tilde\theta(\xi))=(1,1,1),\quad \lim_{\xi\to\infty}(\tilde\phi(\xi),\tilde\psi(\xi),\tilde\theta(\xi))=(0,0,0),\tag{3.13}$$

where

$$\begin{aligned}
\tilde f_1(\tilde\phi_\xi,\tilde\psi_\xi,\tilde\theta_\xi)&=r_1\tilde\phi(\xi)[1-a_{12}-\tilde\phi(\xi-c\tau_1)+a_{12}\tilde\psi(\xi-c\tau_2)],\\
\tilde f_2(\tilde\phi_\xi,\tilde\psi_\xi,\tilde\theta_\xi)&=r_2(1-\tilde\psi(\xi))[a_{23}\tilde\theta(\xi-c\tau_5)+a_{21}\tilde\phi(\xi-c\tau_3)-\tilde\psi(\xi-c\tau_4)],\\
\tilde f_3(\tilde\phi_\xi,\tilde\psi_\xi,\tilde\theta_\xi)&=r_3\tilde\theta(\xi)[1-a_{32}-\tilde\theta(\xi-c\tau_7)+a_{32}\tilde\psi(\xi-c\tau_6)].
\end{aligned}$$

Define

$$\begin{aligned}
\Delta_4(\lambda,c)&:=d_1(e^\lambda-2+e^{-\lambda})-c\lambda+r_1(1-a_{12}),\\
\Delta_5(\lambda,c,\tau_4)&:=d_2(e^\lambda-2+e^{-\lambda})-c\lambda-r_2e^{-\lambda c\tau_4},\\
\Delta_6(\lambda,c)&:=d_3(e^\lambda-2+e^{-\lambda})-c\lambda+r_3(1-a_{32}).
\end{aligned}\tag{3.14}$$

Lemma 3.10. $\Delta_4(\lambda,c)$, $\Delta_5(\lambda,c,\tau_4)$ *and* $\Delta_6(\lambda,c)$ *are described as* (3.14). *Then*

(i) $\Delta_4(\lambda,c)=0$ *has a unique negative root* $\lambda_5<0$ *for* $c>0$, *and* $\Delta_6(\lambda,c)=0$ *has a unique negative root* $\lambda_6<0$ *for* $c>0$;

(ii) *for* $\tau_4=0$ *or small enough* τ_4, $\Delta_5(\lambda,c,\tau_4)=0$ *has a unique negative root* $\lambda_7(\tau_4)<0$ *for* $c>0$ *and* $\Delta_5(\lambda,c,\tau_4)<0$ *for* $\lambda\in(\lambda_7(\tau_4),0)$. *Moreover,* $\lambda_7(\tau_4)$ *is decreasing with respect to* τ_4.

Proof. The proof of the assertion (i) is direct and is omitted. Also, it is easy to see that the assertion (ii) holds for $\tau_4=0$. For small enough $\tau_4>0$, we assume that $0<c\tau_4\ll 1$. Since $\Delta_5(0,c,\tau_4)=-r_2<0$ and $\Delta_5(-\infty,c,\tau_4)=\infty$, it follows that $\Delta_5(\lambda,c,\tau_4)=0$ has a negative root $\lambda_7(\tau_4)<0$. Furthermore, we can take τ_4 small enough such that

$$\frac{\partial\Delta_5(\lambda,c,\tau_4)}{\partial\lambda}=d_2(e^\lambda-e^{-\lambda})-c+c\tau_4r_2e^{-\lambda c\tau_4}<0,\quad \lambda\in(-\infty,0].$$

Thus $\Delta_5(\lambda,c,\tau_4)=0$ has a unique negative root $\lambda_7(\tau_4)<0$. This completes the proof. $\square$

Similar to Lemma 3.6, we have the following result.

Lemma 3.11. *For* $\tau_4=0$ *or small enough* τ_4, $\lambda=\lambda_7(\tau_4)$ *is a unique root with* $\Re\lambda=\lambda_7(\tau_4)$ *of* $\Delta_5(\lambda,c,\tau_4)=0$.

By Proposition 3.2 and the fact $\lim_{\xi\to+\infty}(\tilde\phi(\xi),\tilde\psi(\xi),\tilde\theta(\xi))=(0,0,0)$, we have the following result.

Theorem 3.12. *Let* $(\phi(\xi),\psi(\xi),\theta(\xi)):\mathbb{R}\to[\mathbf{0},\mathbf{1}]$ *be a solution of* (2.4) *and* (2.3) *with* $c\ge c_*$. *Then*

$$\lim_{\xi\to\infty}\frac{\phi'(\xi)}{1-\phi(\xi)}=-\lambda_5>0,\ \lim_{\xi\to\infty}\frac{\theta'(\xi)}{1-\theta(\xi)}=-\lambda_6>0.$$

Lemma 3.13. *Let $(\phi(\xi),\psi(\xi),\theta(\xi)) : \mathbb{R} \to [\mathbf{0},\mathbf{1}]$ be a solution of* (2.4) *and* (2.3) *with $c \geq c_*$. Then*

(i) $L(\lambda,\hat{\phi}) < \infty$, $\lambda \in (\lambda_5, 0)$ *and* $L(\lambda,\hat{\phi}) = \infty$, $\lambda \in \mathbb{R} \setminus (\lambda_5, 0)$;
(ii) $L(\lambda,\hat{\theta}) < \infty$, $\lambda \in (\lambda_6, 0)$ *and* $L(\lambda,\hat{\theta}) = \infty$, $\lambda \in \mathbb{R} \setminus (\lambda_6, 0)$;
(iii) $L(\lambda,\hat{\psi}) < \infty$, $\lambda \in (\rho(\tau_4), 0)$ *and* $L(\lambda,\hat{\psi}) = \infty$, $\lambda \in \mathbb{R} \setminus (\rho(\tau_4), 0)$, *where* $\rho(\tau_4) = \max\{\lambda_5, \lambda_6, \lambda_7(\tau_4)\} < 0$.

Proof. The proof is similar to that of Lemmas 3.5 and 3.7. So we omit it here. $\square$

Using a similar argument as Theorem 3.8, we have the following exponential asymptotic behavior of $(\phi(\xi),\psi(\xi),\theta(\xi))$ at $+\infty$.

Theorem 3.14. *Assume that $\tau_1 = \tau_4 = \tau_7 = 0$ or τ_1, τ_4 and τ_7 are small enough. Let $(\phi(\xi),\psi(\xi),\theta(\xi)) : \mathbb{R} \to [\mathbf{0},\mathbf{1}]$ be a solution of* (2.4) *and* (2.3) *with $c \geq c_*$. Then the solution exhibits the asymptotic behavior as $\xi \to \infty$ as follows:*

(i) *There exist $\eta_9 > 0$ and $\omega_9 > 0$ such that*
$$1 - \phi(\xi) = \eta_9 e^{\lambda_5 \xi} + h.o.t., \quad 1 - \theta(\xi) = \omega_9 e^{\lambda_6 \xi} + h.o.t.$$
(ii) *For $\Pi \in \{\lambda_5, \lambda_6\}$, there exist $\eta_i > 0 (i = 10, 11, 12)$ such that*
$$1 - \psi(\xi) = \begin{cases} \eta_{10} e^{\lambda_7(\tau_4)\xi} + h.o.t., & \text{if } \lambda_7(\tau_4) > \Pi, \\ \eta_{11} \xi e^{\lambda_7(\tau_4)\xi} + h.o.t., & \text{if } \lambda_7(\tau_4) = \Pi, \\ \eta_{12} e^{\Pi \xi} + h.o.t., & \text{if } \lambda_7(\tau_4) < \Pi. \end{cases}$$

The following corollary follows directly from Theorem 3.14.

Corollary 3.15. *Let $(\phi(\xi),\psi(\xi),\theta(\xi))$ be as described in Theorem 3.14. Then*
$$\lim_{\xi \to \infty} \frac{\psi'(\xi)}{1 - \psi(\xi)} = -\rho(\tau_4),$$
where $\rho(\tau_4) = \max\{\lambda_5, \lambda_6, \lambda_7(\tau_4)\}$.

4. Monotonicity and uniqueness of wave profiles

In this section, we adopt the strong comparing principle and sliding method to prove the strict monotonicity and uniqueness of traveling wavefronts of (2.1) when $\tau_1 = \tau_4 = \tau_7 = 0$. We first give the strong comparison principle as follows.

Lemma 4.1. *Assume* (H0) *and $\tau_1 = \tau_4 = \tau_7 = 0$ hold. Let (c, Φ) and (c, Ψ) be two traveling wave solutions of* (2.4) *satisfying $\Phi(\xi) \leq \Psi(\xi)$ for $\xi \in \mathbb{R}$. Then we have either (i)$\Phi(\xi) < \Psi(\xi)$ for $\xi \in \mathbb{R}$, or (ii) $\Phi(\xi) \equiv \Psi(\xi)$ for $\xi \in \mathbb{R}$.*

Proof. For given constants $L_1, L_2, L_3 > 0$, we denote
$$\begin{aligned} H_1(\Phi)(\xi) &= d_1 D[\phi](\xi) + (L_1 + 2)\phi(\xi) + r_1(1 - \phi(\xi))[-\phi(\xi) + a_{12}\psi(\xi - c\tau_2)], \\ H_2(\Phi)(\xi) &= d_2 D[\psi](\xi) + (L_2 + 2)\psi(\xi) + r_2\psi(\xi)\big[1 - a_{21} - a_{23} - \psi(\xi) \\ &\quad + a_{21}\phi(\xi - c\tau_3) + a_{23}\theta(\xi - c\tau_5)\big], \\ H_3(\Phi)(\xi) &= d_3 D[\theta](\xi) + (L_3 + 2)\theta(\xi) + r_3(1 - \theta(\xi))[a_{32}\psi(\xi - c\tau_6) - \theta(\xi)]. \end{aligned}$$
where $\Phi(\xi) = (\phi(\xi),\psi(\xi),\theta(\xi))$. It is clear that one can choose sufficiently large $L_1, L_2, L_3 > 0$ such that $H_i(\Phi)$ is non-decreasing with respect to Φ. When $\tau_1 =$

$\tau_4=\tau_7=0$, $\Phi(\xi)=(\phi(\xi),\psi(\xi),\theta(\xi))$ satisfies

$$\begin{aligned} c\phi'(\xi)+(L_1+2)\phi(\xi)&=H_1(\phi(\xi),\psi(\xi),\theta(\xi)),\\ c\psi'(\xi)+(L_2+2)\psi(\xi)&=H_2(\phi(\xi),\psi(\xi),\theta(\xi)),\\ c\theta'(\xi)+(L_3+2)\theta(\xi)&=H_3(\phi(\xi),\psi(\xi),\theta(\xi)), \end{aligned} \tag{4.1}$$

which implies

$$\begin{aligned} \phi(\xi)&=\frac{1}{c}e^{-\frac{L_1+2}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{(L_1+2)s}{c}}H_1(\phi,\psi,\theta)(s)ds,\\ \psi(\xi)&=\frac{1}{c}e^{-\frac{L_2+2}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{(L_2+2)s}{c}}H_2(\phi,\psi,\theta)(s)ds,\\ \theta(\xi)&=\frac{1}{c}e^{-\frac{L_3+2}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{(L_3+2)s}{c}}H_3(\phi,\psi,\theta)(s)ds\,. \end{aligned} \tag{4.2}$$

Using (4.2) and the method in [6, Lemma 4.1], one can easily prove the assertion. This completes the proof. □

Theorem 4.2. *Assume that* (H0) *holds and* $\tau_1=\tau_4=\tau_7=0$. *Let* $(c,\Phi):=(c,\phi,\psi,\theta)$ *be any traveling wavefront of* (2.1) *with the wave speed* $c\geq c_*$. *Then,* $\Phi(\xi)$ *is strictly monotone.*

Proof. By Theorems 3.4, 3.8, 3.12 and 3.14, one can choose large enough $N>0$ such that $\phi'(\xi)>0$, $\psi'(\xi)>0$ and $\theta'(\xi)>0$ for any $\xi\in\mathbb{R}\setminus[-N,N]$. By (2.3), the set

$$I:=\{\eta>0|\phi(\xi+s)\geq\phi(\xi),\psi(\xi+s)\geq\psi(\xi),\theta(\xi+s)\geq\theta(\xi),\forall s\geq\eta,\xi\in\mathbb{R}\}$$

is not empty. Thus $\eta^*=\inf I$ is well defined. By the continuity, we have

$$\phi(\xi+\eta^*)\geq\phi(\xi),\quad \psi(\xi+\eta^*)\geq\psi(\xi),\quad \theta(\xi+\eta^*)\geq\theta(\xi),\ \forall\xi\in\mathbb{R}.$$

Next, we prove $\eta^*=0$. Suppose for the contrary that $\eta^*>0$. Then, by Lemma 4.1, we have

$$\phi(\xi+\eta^*)>\phi(\xi),\quad \psi(\xi+\eta^*)>\psi(\xi),\quad \theta(\xi+\eta^*)>\theta(\xi),\ \forall\xi\in\mathbb{R}.$$

By the continuity of (ϕ,ψ,θ), there exists $\eta_0\in(0,\eta^*)$ satisfying

$$\phi(\xi+\eta)>\phi(\xi),\quad \psi(\xi+\eta)>\psi(\xi),\quad \theta(\xi+\eta)>\theta(\xi),\quad \eta\in[\eta_0,\eta^*],$$

for $\xi\in[-N-\eta^*,N]$. Note that

$$\phi'(\xi)>0,\quad \psi'(\xi)>0,\quad \theta'(\xi)>0,\quad \xi\in\mathbb{R}\setminus[-N,N].$$

Then

$$\phi(\xi+\eta)>\phi(\xi),\quad \psi(\xi+\eta)>\psi(\xi),\quad \theta(\xi+\eta)>\theta(\xi),\quad \eta\in[\eta_0,\eta^*],$$

for $\xi\in\mathbb{R}\setminus[-N-\eta^*,N]$. Thus,

$$\phi(\xi+\eta)\geq\phi(\xi),\quad \psi(\xi+\eta)\geq\psi(\xi),\quad \theta(\xi+\eta)\geq\theta(\xi),\quad \forall\eta>\eta_0,\ \xi\in\mathbb{R},$$

which leads to a contradiction for the definition of η^*. Thus $\eta^*=0$. Then $\phi'(\xi)\geq0$, $\psi'(\xi)\geq0$ and $\theta'(\xi)\geq0$ in $\mathbb{R}$. By differentiating the two side of (4.2) and using the mononicity of H_i, one can easily verify that $\phi'(\xi)>0$, $\psi'(\xi)>0$ and $\theta'(\xi)>0$, $\xi\in\mathbb{R}$. The proof is complete. □

Now we give the uniqueness of traveling wavefronts of (2.1).

Theorem 4.3. *Assume that* (H0), $\tau_1 = \tau_4 = \tau_7 = 0$ *and* $d_2 \geq \max\{d_1, d_3\}$ *hold. Let*

$$\Phi_1(\xi) = (\phi_1(\xi), \psi_1(\xi), \theta_1(\xi)), \quad \Phi_2(\xi) = (\phi_2(\xi), \psi_2(\xi), \theta_2(\xi))$$

be any two traveling wavefronts of (2.1) *with the speed* $c \geq c_*$ *which connects* $\mathbf{0}$ *with* $\mathbf{1}$. *Then there exists* $\eta_0 \in \mathbb{R}$ *such that* $\Phi_1(\xi + \eta_0) = \Phi_2(\xi)$.

Proof. From Theorem 3.8 and $\Lambda = \lambda_1$ or λ_2, there exist $\omega_i = \omega_i(\phi_i, \psi_i, \theta_i)(i = 1.2)$ such that at least one of the following holds:

(i) $\lim_{\xi\to-\infty} \frac{\psi_i(\xi+\omega_i)}{|\xi|^{\upsilon} e^{\lambda_1\xi}} = 1, i = 1, 2;$

(ii) $\lim_{\xi\to-\infty} \frac{\psi_i(\xi+\omega_i)}{|\xi|^{\upsilon} e^{\lambda_2\xi}} = 1, i = 1, 2;$

(iii) $\lim_{\xi\to-\infty} \frac{\psi_1(\xi+\omega_1)}{|\xi|^{\upsilon} e^{\lambda_1\xi}} = 1,\ \lim_{\xi\to-\infty} \frac{\psi_2(\xi+\omega_2)}{|\xi|^{\upsilon} e^{\lambda_2\xi}} = 1;$

(iv) $\lim_{\xi\to-\infty} \frac{\psi_1(\xi+\omega_1)}{|\xi|^{\upsilon} e^{\lambda_2\xi}} = 1,\ \lim_{\xi\to-\infty} \frac{\psi_2(\xi+\omega_2)}{|\xi|^{\upsilon} e^{\lambda_1\xi}} = 1,$

where $\upsilon = 0$ if $c > c_*$ and $\upsilon = q$ if $c = c_*$.

Note that $\lambda_1 \leq \lambda_2$. Then there exists $\eta_1 = \eta_1(\phi_1, \psi_1, \theta_1, \phi_2, \psi_2, \theta_2)$ such that one of the following is true, which corresponds to the above cases (i)-(iv):

(i) $\lim_{\xi\to-\infty} \frac{\psi_1(\xi+\eta_1)}{\psi_2(\xi)} = 1$, and $\lim_{\xi\to-\infty} \frac{\psi_1(\xi+\bar\xi)}{\psi_2(\xi)} = e^{\lambda_1(\bar\xi-\eta_1)} > 1$ for all $\bar\xi > \max\{\eta_1, 0\}$;

(ii) $\lim_{\xi\to-\infty} \frac{\psi_1(\xi+\eta_1)}{\psi_2(\xi)} = 1$, and $\lim_{\xi\to-\infty} \frac{\psi_1(\xi+\bar\xi)}{\psi_2(\xi)} = e^{\lambda_2(\bar\xi-\eta_1)} > 1$ for all $\bar\xi > \max\{\eta_1, 0\}$;

(iii) $\lim_{\xi\to-\infty} \frac{\psi_1(\xi+\eta_1)}{\psi_2(\xi)} = 1$ or ∞, and $\lim_{\xi\to-\infty} \frac{\psi_1(\xi+\bar\xi)}{\psi_2(\xi)} > 1$ or ∞ for all $\bar\xi > \max\{\eta_1, 0\}$;

(iv) $\lim_{\xi\to-\infty} \frac{\psi_1(\xi)}{\psi_2(\xi+\eta_1)} = 1$ or 0, and $\lim_{\xi\to-\infty} \frac{\psi_1(\xi)}{\psi_2(\xi+\bar\xi)} < 1$ or 0 for all $\bar\xi < \min\{\eta_1, 0\}$.

Suppose that (i) happens. From (2.6), (3.2) and (3.3), and the condition $d_2 \geq \max\{d_1, d_3\}$, it is easy to see that $\lambda_3(\tau_1) > \Lambda$ and $\lambda_4(\tau_7) > \Lambda$. By Theorem 3.8 again, there exist

$$\eta_2 = \eta_2(\phi_1, \psi_1, \theta_1, \phi_2, \psi_2, \theta_2), \quad \eta_3 = \eta_3(\phi_1, \psi_1, \theta_1, \phi_2, \psi_2, \theta_2)$$

such that

$$\lim_{\xi\to-\infty} \frac{\phi_1(\xi + \eta_2)}{\phi_2(\xi)} = 1, \quad \lim_{\xi\to-\infty} \frac{\theta_1(\xi + \eta_3)}{\theta_2(\xi)} = 1.$$

Then

$$\lim_{\xi\to-\infty} \frac{\phi_1(\xi + \bar\xi)}{\phi_2(\xi)} = e^{\lambda_1(\bar\xi-\eta_2)} > 1, \quad \lim_{\xi\to-\infty} \frac{\theta_1(\xi + \bar\xi)}{\theta_2(\xi)} = e^{\lambda_1(\bar\xi-\eta_3)} > 1$$

for all $\bar\xi > \max\{\eta_2, \eta_3, 0\}$. Thus, for $\bar\xi_1 > \max\{\eta_1, \eta_2, \eta_3, 0\}$, there exists $M_1 \gg 1$ such that

$$\frac{\phi_1(\xi + \bar\xi_1)}{\phi_2(\xi)} \geq 1, \quad \frac{\theta_1(\xi + \bar\xi_1)}{\theta_2(\xi)} \geq 1, \quad \frac{\psi_1(\xi + \bar\xi_1)}{\psi_2(\xi)} \geq 1, \quad \forall \xi \in (-\infty, -M_1].$$

By the monotonicity of $(\phi_i(\xi), \psi_i(\xi), \theta_i(\xi))(i = 1, 2)$, for any $\bar\xi \geq \bar\xi_1$,

$$\phi_1(\xi + \bar\xi) \geq \phi_2(\xi), \quad \psi_1(\xi + \bar\xi) \geq \psi_2(\xi), \quad \theta_1(\xi + \bar\xi) \geq \theta_2(\xi), \quad \forall \xi \in (-\infty, -M_1].$$

From Theorem 3.14, by a similar argument as the one for $-\infty$, we can get that there exist η_4, η_5, η_6 and $M_2 \gg 1$ such that

$$\phi_1(\xi + \bar\xi) \geq \phi_2(\xi), \quad \psi_1(\xi + \bar\xi) \geq \psi_2(\xi), \quad \theta_1(\xi + \bar\xi) \geq \theta_2(\xi), \quad \forall \xi \in [M_2, \infty).$$

for $\bar{\xi} > \bar{\xi}_2 > \max\{\eta_4, \eta_5, \eta_6, 0\}$.

Take $M = \max\{M_1, M_2\}$. By $(\phi_i(+\infty), \psi_i(+\infty), \theta_i(+\infty)) = \mathbf{1}$, we can choose $\bar{\xi}_0 > \max\{\bar{\xi}_1, \bar{\xi}_2\}$ suitable large satisfying

$$\phi_1(\xi + \bar{\xi}_0) \geq \phi_2(\xi), \quad \psi_1(\xi + \bar{\xi}_0) \geq \psi_2(\xi), \quad \theta_1(\xi + \bar{\xi}_0) \geq \theta_2(\xi),\ \forall \xi \in [-M, M].$$

Thus,

$$\phi_1(\xi + \bar{\xi}_0) \geq \phi_2(\xi), \quad \psi_1(\xi + \bar{\xi}_0) \geq \psi_2(\xi), \quad \theta_1(\xi + \bar{\xi}_0) \geq \theta_2(\xi), \quad \forall \xi \in \mathbb{R}.$$

Then there exists $\xi_0 \leq \bar{\xi}_0$ (by translation) such that at least one of the following is true

(a) $\phi_1(\hat{\xi}+\xi_0) = \phi_2(\hat{\xi})$ for some $\hat{\xi} \in \mathbb{R}$, $\theta_1(\xi+\xi_0) \geq \theta_2(\xi)$ and $\psi_1(\xi+\xi_0) \geq \psi_2(\xi)$, $\xi \in \mathbb{R}$;
(b) $\psi_1(\hat{\xi}+\xi_0) = \psi_2(\hat{\xi})$ for some $\hat{\xi} \in \mathbb{R}$, $\phi_1(\xi+\xi_0) \geq \phi_2(\xi)$ and $\psi_1(\xi+\xi_0) \geq \psi_2(\xi)$, $\xi \in \mathbb{R}$;
(c) $\theta_1(\hat{\xi}+\xi_0) = \theta_2(\hat{\xi})$ for some $\hat{\xi} \in \mathbb{R}$, $\phi_1(\xi+\xi_0) \geq \phi_2(\xi)$ and $\psi_1(\xi+\xi_0) \geq \psi_2(\xi)$, $\xi \in \mathbb{R}$.

Without loss of generality, we assume (a) is true, since the traveling wavefronts of (2.4) are translation invariant, $(\phi_1(\xi+\xi_0), \psi_1(\xi+\xi_0), \theta_1(\xi+\xi_0))$ is also a traveling wavefront of (2.4). By Lemma 4.1,

$$\phi_1(\xi + \xi_0) \equiv \phi_2(\xi), \quad \psi_1(\xi + \xi_0) \equiv \psi_2(\xi), \quad \theta_1(\xi + \xi_0) \equiv \theta_2(\xi).$$

The proof of the assertion (ii) is similar and omitted here. Next we prove that the case (iii) and (iv) cannot happen. For instance, if (iii) happens, by a similar argument as above, there exists $\xi_0 \in \mathbb{R}$ satisfying

$$(\phi_1(\xi + \xi_0), \psi_1(\xi + \xi_0), \theta_1(\xi + \xi_0)) = (\phi_2(\xi), \psi_2(\xi), \theta_2(\xi)),\ \xi \in \mathbb{R},$$

which contradicts to the asymptotic behavior of $\psi_1(\cdot)$ and $\psi_2(\cdot)$ at $-\infty$. Similarly, (iv) cannot happen. This completes the proof. □

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