NONEMPTYNESS AND COMPACTNESS OF THE SOLUTION SET FOR FRACTIONAL EVOLUTION INCLUSIONS WITH NON-INSTANTANEOUS IMPULSES

JINRONG WANG, AHMED G. IBRAHIM, DONAL O’REGAN

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Abstract. In this article, we consider the nonemptyness and compactness of the solution set for a class of fractional semilinear evolution inclusions with non-instantaneous impulses in Banach spaces. To achieve this we use fixed point theorems with semigroup theory, upper semicontinuous multi-functions and measures of noncompactness.

1. Introduction

The qualitative theory of differential equations and evolution inclusions involving various fractional derivatives was considered in the monographs [8, 10, 15, 40], and in a series of papers [3, 4, 17, 19, 20, 23, 24, 34, 35, 36, 37, 38, 39]. Recently research in mathematical modelling described by differential equations with non-instantaneous impulses was considered in [13, 14, 25, 26, 27], and these equations are suitable to characterize the dynamics of evolution processes in pharmacotherapy. In particular, existence, topological structure, stability and controllability theory in the fractional order case was investigated in [1, 2, 5, 28, 29, 30, 31, 32, 33]. However, there seems to be very little available in the literature concerning the existence of mild solutions to evolution inclusions with not instantaneous impulses (with integer order or fractional order). This is the main motivation in this paper.

Let \( J = [0, l], \quad l > 0, \quad \alpha \in (0, 1) \) and \( E \) be a Banach space. Denote \( A \) by the infinitesimal generator of a \( C_0 \)-semigroup \( \{T(t) : t \geq 0\} \) on \( E \). Inspired by the references mentioned above, in this work, we consider the nonemptyness and compactness of the solution set to the following fractional semilinear evolution inclusions with non-instantaneous impulses:

\[
\mathcal{D}_{a_i}^\alpha u(t) \in Au(t) + F(t, u(t)), \\
\text{a.e. } t \in \cup_{i=0}^{m} (a_i, b_{i+1}] \subset J, \quad a_0 := 0, \quad b_{m+1} := l > 0, \\
u(t) = g_i(t, u(b_i^{-})), \quad t \in (b_i, a_i] \subset [0, l], \quad i = 1, 2, \ldots, m, \\
u(0) = u_0 \in E, \tag{1.1}
\]

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where $^{C}D_{a_{i}}^{\alpha}$ denotes the Caputo derivative [15] of order $\alpha$ from the lower limit $a_{i}$ to the upper limit $t$, $F : [0,l] \times E \to 2^{E} - \{\phi\}$ is a multi-function, the sequences $\{a_{i}\}$ and $\{b_{i+1}\}$ satisfy $a_{i} < b_{i+1}$, $i = 0,1,\ldots,m$, and moreover, $g_{i} : [b_{i},a_{i}] \times E \to E$, $i = 1,2,\ldots,m$, and $u(b_{i}^{-})$ denotes the left limit of $u$ at $b_{i}$.

The article is organized as follows. In Section 2, we collect some background material concerning multi-functions and establish necessary lemmas on the operator semigroup and a generalized Cauchy operator. In Section 3, we show that the solution set $\Sigma_{D_{\alpha}^{x_{0}}}^{T} [0,l]$ is nonempty and compact under mild conditions on $\{T(t) : t \geq 0\}$ and $F$. An example is given in the final section to illustrate our theory.

2. Notation and preliminaries

Denote $L^{1}(J,E) = \{v : v : J \to E \text{ is Bochner integrable}\}$ endowed with the norm $\|v\|_{L^{1}(J,E)} = \int_{0}^{1} \|v(t)\|\,dt$. Denote

\[ P_{b}(E) = \{B \subseteq E : B \text{ is nonempty and bounded}\}, \]
\[ P_{cl}(E) = \{B \subseteq E : B \text{ is nonempty and closed}\}, \]
\[ P_{b}(E) = \{B \subseteq E : B \text{ is nonempty and compact}\}, \]
\[ P_{cl,cv}(E) = \{B \subseteq E : B \text{ is nonempty, closed and convex}\}, \]
\[ P(E) = \{B \subseteq E : B \text{ is nonempty}\}, \]
\[ P_{ck}(E) = \{B \subseteq E : B \text{ is nonempty, convex and compact}\}. \]

$\text{conv}(B)$ (respect., $\text{clconv}(B)$) is the convex hull (respect., convex closed hull in $E$) of a subset $B$.

Let $C(J,E) = \{f : f : J \to E \text{ is continuous}\}$ be endowed with the supremum norm. We consider the set of functions $PC(J,E) = \{u : J \to E : u_{|J_{i}} \in C(J_{i},E), J_{i} := (b_{i},b_{i+1}], \, i = 0,1,2,\ldots,m \text{ and } u(b_{i}^{+}) \text{ and } u(b_{i}^{-}) \text{ exist for each } i = 1,2,\ldots,m\}$. It is easy to check that $PC(J,E)$ is a Banach space endowed with the Chebyshev $PC$-norm: $\|u\|_{PC(J,E)} = \max_{1 \leq i \leq m} \|u(t)\| : t \in J$.

Let $G : J \to P(E)$ be a multifunction and $S_{G}^{1} = \{z \in L^{1}(J,E) : z(t) \in G(t) \text{ a.e.}\}$. This set may be empty. For $P_{cl}(E)$-valued measurable multi-function, it is nonempty if and only if $t \to \inf\{\|x\| : x \in G(t)\} \in L^{1}(J,\mathbb{R}^{+})$. In particular, this is the case if $t \to \sup\{\|x\| : x \in G(t)\} \in L^{1}(J,\mathbb{R}^{+})$ (such a multifunction is said to be integrably bounded). Note that $S_{G}^{1} \subseteq L^{1}(J,E)$ is closed and it is convex if and only if for almost all $t \in J, G(t)$ is a convex set in $E$. The following definitions on multivalued mappings can be found in [6] [11] [10].

**Definition 2.1.** Let $X$ and $Y$ be two topological spaces. A multifunction $G : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) if $G^{-1}(V) = \{x \in X : G(x) \subseteq V\}$ is an open subset of $X$ for every open $V \subseteq Y$. The map $G$ is said to be closed if its graph $\Gamma_{G} = \{(x,y) \in X \times Y : y \in G(x)\}$ is closed subset of the topological space $X \times Y$. The map $G$ is said to be compact if $G(B)$ is relatively compact for every bounded subset $B$ of $X$. The map $G$ is said to be locally compact if for every point $x \in X$ has a neighborhood $V(x)$ such that $\cup\{F(z) : z \in V(x)\}$ is relatively compact. Finally the map $G$ is said to be quasicompact, if its restriction to any compact subset $A \subseteq X$ is compact.

**Remark 2.2.** Let $X$ and $Y$ be two topological spaces and $G : X \to P(Y)$. If $Y$ is regular and the multifunction $G$ is u.s.c. with nonempty closed values, then it is
closed. If $G$ is closed, quasi-compact and has nonempty compact values, then it is u.s.c. (see [6, 16]).

**Definition 2.3.** A sequence $\{f_n\}_{n=1}^{\infty} \subseteq L^P(J,E)(P \geq 1)$ is called $P$-time integrably bounded if there is a function $h \in L^P(J,\mathbb{R})$ such that $\|f_n(t)\| \leq h(t)$, a.e. on $J$.

**Definition 2.4.** A sequence $\{f_n\}_{n=1}^{\infty} \subseteq L^P(J,E)(P \geq 1)$ is called $P$-time semicompact if it is $P$-time integrably bounded and the set $\{f_n(t) : n \geq 1\}$ is relatively compact for a.e. $t \in J$.

Next, we recall the definition of a mild solution. For any fixed $t \geq 0$, define

$$K_1(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha \theta)d\theta, \quad K_2(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta)T(t^\alpha \theta)d\theta,$$

where $\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{\alpha - \frac{1}{\alpha}} w_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0$, and

$$w_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n - \frac{1}{\alpha}} \frac{\Gamma(n \alpha + 1)}{n!} \sin(n \pi \alpha), \quad \theta \in (0, \infty).$$

**Definition 2.5.** A function $u \in PC(J,E)$ is called a $PC$-mild solution of (1.1) if

$$u(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [0, b_1], \\ K_1(t-a_i)g_i(a_i, u(b_i^-)) + \int_{a_i}^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [a_i, b_{i+1}], i = 1, 2, \ldots, m, \\ \end{cases}$$

where $f \in S^P_C = \{f \in L^P(J,E) : f(t) \in F(t, u(t)) \text{ a.e.}, P > 1/\alpha\}$.

The Hausdorff measure of noncompactness on $E$ is defined on bounded subsets as

$$\chi(B) = \inf\{\epsilon > 0 : B \text{ can be covered by finitely many balls of radius } \epsilon\}.$$ 

Next, the map $\chi_{PC} : B_0(PC(J,E)) \to [0, \infty)$ is defined as

$$\chi_{PC}(B) = \max_{i=0,1,2,\ldots,m} \chi_i(B|_{\mathcal{T}_i}) = \max_{i=0,1,2,\ldots,m} \chi_i(B|_{[b_i, b_{i+1}]})$$

where $\chi_i$ is the Hausdorff measure of noncompactness on the Banach space $C(\mathcal{T}_i, E)$ and $B|_{\mathcal{T}_i} = \{u^* : \mathcal{T}_i \to E : u^*(t) = u(t), t \in J_i \text{ and } u^*(b_i) = u(b_i^-), u \in B\}$, $i = 0, 1, \ldots, m$. Of course $B|_{\mathcal{T}_0} = \{u|_{\mathcal{T}_0} : u \in B\}$. It is easily seen that $\chi_{PC}$ is the Hausdorff measure of noncompactness on $PC(J,E)$.

We assume the following conditions:

(A1) $A : D(A) \subseteq E \to E$ is a linear closed (not necessarily bounded) operator generating a $C_0$-semigroup $\{T(t) : t \geq 0\}$ of bounded linear operators and there exists a $M \geq 1$ such that $\sup_{t \geq 0} \|T(t)\| \leq M$.

(A2) for every $u \in E$, $t \mapsto F(t, u)$ is strongly measurable, and for almost every $t \in J$, $u \mapsto F(t, u)$ is upper semicontinuous.

(A3) for any bounded subset $\Omega$ there exists a function $\varphi_{\Omega} \in L^P(J, \mathbb{R}^+)(P \geq 1)$ such that for any $u \in E$,

$$\|F(t, u)\| \leq \varphi_{\Omega}(t), \quad \forall u \in \Omega \text{ and for a.e. } t \in J.$$
Proof.\par Such that Lemma 2.7.

\[ \chi(F(t, D)) \leq \beta(t)\chi(D), \quad \text{for a.e. } t \in J, \]

for every bounded subset \( D \subseteq E. \)

**Lemma 2.6 (24).** Assume (A1) holds. Then we have

(i) \( K_1(t) \) and \( K_2(t) \) are linear bounded operators for \( t \geq 0. \)

(ii) \( \int_0^{\infty} \theta^\gamma \xi_\alpha(\theta)d\theta = \frac{(1+\gamma)}{(1+\alpha\gamma)}, \gamma \in [0, 1]. \)

(iii) \( \|K_1(t)u\| \leq M\|u\| \) and \( \|K_2(t)u\| \leq \frac{M}{\Gamma(\alpha)}\|u\| \) for any \( u \in E. \)

(iv) \( K_1(t) \) and \( K_2(t) \) are strongly continuous for any \( t \geq 0. \)

(v) \( K_1(t) \) and \( K_2(t) \) are compact if \( T(t), t > 0 \) is compact.

**Lemma 2.7.** Assume that (A1) holds. Let \( K \) be a compact subset of \( E. \) For any \( s \in (0, \infty) \) and \( t \in (s, \infty), \) one has

\[ \lim_{t \to s, x \in K} \|T(t)x - T(s)x\| = 0. \]

Proof. Let \( \epsilon > 0. \) Note from the compactness of \( K, \) there exist \( x_1, x_2, \ldots, x_r \) such that \( K \subseteq \cup \{B(x_i, \frac{\epsilon}{2^{2^r}}) : i = 1, 2, \ldots, r\}. \) Next, the strong continuity of \( \{T(t) : t \geq 0\} \) implies that for any \( x_i, i = 1, 2, \ldots, r \) there exits \( \delta_i > 0 \) such that for any \( |t| < \delta_i, i = 1, 2, \ldots, r \) we have \( \|T(t)x_i - T(0)x_i\| < \epsilon/(2M). \)

Put \( \delta = \min\{\delta_i : i = 1, 2, \ldots, r\} \) and set \( x \in K. \) There exists \( x_i, i = 1, 2, \ldots, r \) such that \( \|x - x_i\| < \frac{\epsilon}{2M^2}. \) For any \( s \in (0, \infty) \) and \( t \in (s, s + \delta) \) we have

\[ \|T(t)x - T(s)x\| = \|T(s)(T(t-s)x - T(0)x)\| \]

\[ \leq \|T(s)\|\|T(t-s)x - T(0)x\| \]

\[ \leq M(\|T(t-s)x_i - T(0)x_i\| + \|T(t-s)x - T(t-s)x_i\|) \]

\[ + \|T(0)x_i - T(0)x\|) \]

\[ \leq \frac{\epsilon}{2} + 2M^2\|x - x_i\| < \epsilon. \]

The proof is complete. \( \square \)

**Lemma 2.8 ([18], Lemma 5.1.1, for \( P > 1, \) [23] for \( P = 1)).** The conditions (A2)–(A4) imply that the superposition multi-operator \( P_F : C(J, E) \to P(L^P(J, E)), P \in [1, \infty), P_F(x) = S_{F(x)}^P, \) generated by \( F \) is well defined, and is weakly closed in the following sense: if \( \{x_n\}_{n=1}^\infty \subset C(J, E), \) \( \{f_n\}_{n=1}^\infty \subset L^P(J, E), \) \( f_n \in P_F(x_n), \) \( n \geq 1 \) are such that \( x_n \to x, \) \( f_n \to f \) (weakly), then \( f \in P_F(x). \)

**Lemma 2.9 ([18] Lemmas 3.4.3.5).** Let \( P \in [1, \infty) \) and \( S : L^P(J, X) \to C(J, X) \) be an operator satisfying the conditions:

(A5) there exists a \( \zeta \geq 0 \) such that

\[ \|Sf(t) - Sh(t)\|_E \leq \zeta \left( \int_0^t \|f(s) - h(s)\|^{\frac{p}{p}} ds \right)^{1/p}, \quad t \in J, \]

(2.1)

for every \( f, h \in L^P(J, E). \)

(A6) for any compact \( K \subseteq E \) and sequence \( \{f_n\}_{n=1}^{\infty} \subset L^P(J, E) \) such that for all \( n \geq 1, f_n(t) \in K, \) a.e. \( t \in J, \) the weak convergence \( f_n \to f_0 \) in \( L^P(J, E) \) implies the convergence \( Sf_n \to Sf_0. \)
prove (A6) we note that for every compact set \( \{f_n\}_{n=1}^{+\infty} \subset L^P(J,E) \) the set \( \{Sf_n\}_{n=1}^{+\infty} \) is relatively compact in \( C(J,E) \). Moreover, if \( (f_n)_{n \geq 1} \) converges weakly to \( f_0 \) in \( L^P(J,E) \) then \( Sf_n \to Sf_0 \) in \( C(J,E) \).

**Definition 2.10.** Let \( P > 1/\alpha \), \( \alpha \in (0,1) \). Then the operator \( G : L^P(J,X) \to C(J,X) \) defined by

\[
Gf(t) = \int_0^t (t-s)^{\alpha-1} K_2(t-s)f(s)ds, \quad (2.2)
\]

is called the generalized Cauchy operator.

**Lemma 2.11.** Let \( P > 1/\alpha \), \( \alpha \in (0,1) \). Then

(i) \( G \) satisfies property (A5), with \( \zeta = \frac{M}{\Gamma(\alpha)} \left( \frac{P-1}{P-1}\right)^{\frac{1}{P-1}} \).  

(ii) If condition (A1) holds then \( G \) satisfies (A6).

**Proof.** (i) The proof of the first assertion is exactly as in [IS Lemma 3.6]. (ii) To prove (A6) we note that for every compact \( K \subset \mathbb{E} \) the set

\[
Q_t = \int_0^t (t-s)^{\alpha-1} K_2(t-s)Kds, \quad t \in J,
\]

is relatively compact. Let \( \{f_n\}_{n=1}^{+\infty} \subset L^P(J,E) \) be a sequence such that \( \forall n \geq 1, f_n(t) \in K \), a.e. \( t \in J \) and \( f_n \to f_0 \) in \( L^P(J,E) \). Note that for \( t \in J \),

\[
\{Gf_n(t) : n \geq 1\} \subseteq Q_t.
\]

Then \( \{Gf_n(t) : n \geq 1\} \) is relatively compact for every \( t \in J \). In order to apply the Arzela-Ascoli theorem we show that the set of functions \( \{Gf_n(t) : n \geq 1\} \) is equi-continuous. Since \( f_n(t) \in K \), for \( n \geq 1 \), and a.e. \( t \in J \), there is a \( N > 0 \) such that

\[
\|f_n(t)\| \leq N, \quad \text{for } n \geq 1 \text{ and a.e. } t \in J.
\]

Let \( n \geq 1 \) be fixed and \( t_1, t_2 \) (\( t_1 < t_2 \)) be two points in \( J \). Then

\[
\|Gf_n(t_2) - Gf_n(t_1)\|
\]

\[
\leq \left\| \int_0^{t_2} (t_2-s)^{\alpha-1} K_2(t_2-s)f_n(s)ds - \int_0^{t_1} (t_1-s)^{\alpha-1} K_2(t_1-s)f_n(s)ds \right\|
\]

\[
\leq I_1 + I_2 + I_3,
\]

where

\[
I_1 = \int_0^{t_1} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|\|K_2(t_2-s)f_n(s)\|ds,
\]

\[
I_2 = \int_0^{t_1} |(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}|\|K_2(t_1-s)f_n(s) - K_2(t_2-s)f_n(s)\|ds,
\]

\[
I_3 = \int_{t_1}^{t_2} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|\|K_2(t_2-s)f_n(s)\|ds.
\]

Note that

\[
\lim_{t_2 \to t_1} I_1 \leq \lim_{t_2 \to t_1} \frac{NM}{\Gamma(\alpha)} \int_0^{t_1} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|ds = 0.
\]

Then \( \lim_{t_2 \to t_1} I_1 = 0 \). Similarly, \( \lim_{t_2 \to t_1} I_3 = 0 \).
For $I_2$, it follows from Lemma 2.7 that for any $s \in (0, \infty)$ and $t \in (s, \infty)$, that
\[
\lim_{t \to s} \|T(t)x - T(s)x\| = 0,
\]
indpendently of $x \in K$. Then, by applying the Lebesgue dominated convergence theorem, we obtain
\[
\lim_{t_2 \to t_1} I_2 \leq \int_0^{t_1} \int_0^\infty \theta(t_1 - s)^{\alpha - 1}\zeta_\alpha(\theta) \left[ \lim_{t_2 \to t_1} \|T((t_2 - s)^\alpha\theta) - T((t_1 - s)^\alpha\theta)\| \right] d\theta ds
\]
\[
= \int_0^{t_1} \int_0^\infty \theta(t_1 - s)^{\alpha - 1}\zeta_\alpha(\theta) \left[ \lim_{t_2 \to t_1} \|T((t_1 - s)^\alpha\theta)\|\right] d\theta ds
\]
\[
\leq M \int_0^{t_1} \int_0^\infty \theta(t_1 - s)^{\alpha - 1}\zeta_\alpha(\theta) \left[ \lim_{t_2 \to t_1} \|T((t_2 - s)^\alpha\theta)\|\right] d\theta ds = 0.
\]

Then $\{Gf_n : n \geq 1\}$ is equicontinuous. The relative compactness of the set $\{Gf_n : n \geq 1\}$ follows from the Arzela-Ascoli theorem.

Now, condition (A5) implies that $G : L^p(J, X) \to C(J, X)$ is a linear bounded operator. Hence $f_n \to f_0$ in $L^p(J, E)$ implies the convergence $Sf_n \to Sf_0$ in $C(J, E)$. However the relative compactness of $\{Gf_n : n \geq 1\}$ implies that this last convergence is in the norm of the space $C(J, E)$.

Applying Lemmas 2.9 and 2.11 we have the following corollary.

**Corollary 2.12.** For every $P$-time semicompact set $\{f_n\}_{n=1}^{+\infty} \subset L^p(J, E)$ the set $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J, E)$. Moreover, if $(f_n)_{n\geq 1}$ converges weakly to $f_0$ in $L^p(J, E)$ then $Sf_n \to Sf_0$ in $C(J, E)$.

**Lemma 2.13 ([11] Lemma 4).** Let $\{f_n : n \in \mathbb{N}\} \subset L^p(J, E)$, $p \geq 1$ be an integrably bounded sequence such that
\[
\chi\{f_n : n \geq 1\} \leq \gamma(t), \quad \text{a.e. } t \in J,
\]
where $\gamma \in L^1(J, \mathbb{R}^+)$. Then for each $\epsilon > 0$ there exists a compact $K_\epsilon \subset E$, a measurable set $I_\epsilon \subset J$, with measure less than $\epsilon$, and a sequence of functions $\{g_n^\epsilon\} \subset L^p(J, E)$ such that $\{g_n^\epsilon(t) : n \geq 1\} \subset K_\epsilon$, for all $t \in J$ and
\[
\|f_n(t) - g_n^\epsilon(t)\| < 2\gamma(t) + \epsilon, \quad \text{for every } n \geq 1 \text{ and every } t \in J - I_\epsilon.
\]

**Lemma 2.14 ([18] Lemma 3.9).** Let the set of function $\{f_n\}_{n=1}^{\infty}$ be integrably bounded in $L^p(J, E)$ with the property $\chi(\{f_n(t) : n \geq 1\}) \leq \eta(t)$, for a.e. $t \in J$, where $\eta(\cdot) \in L^p_\infty(J, \mathbb{R}^+)$. Then
\[
\chi(\{Gf_n(t) : n \geq 1\}) \leq 2^{1+\frac{1}{p}} \zeta \int_0^t \eta(s) ds,
\]
where $\zeta$ is the constant in relation (2.1).

The following fixed point theorems for multi-functions are crucial in the proof of our results.
Lemma 2.15 (Kakutani-Glicksberg-Fan theorem [16]). Let $W$ be a nonempty compact and convex subset of a locally convex topological vector space. If $\Phi : W \to P_{cl,cv}(W)$ is an u.s.c. multi-function, then it has a fixed point.

Lemma 2.16 ([16] Prop.3.5.1]). Let $W$ be a closed subset of $E$ and $\Phi : W \to P_{cl}(E)$ be a closed multi-function which is $\gamma$-condensing on every bounded subset of $W$, where $\gamma$ is a monotone measure of noncompactness defined on $E$. If the set of fixed points for $\Phi$ is a bounded subset of $E$ then it is compact.

3. Nonemptiness and compactness of solution set

By the symbol $\Sigma^E_{\rho_0}[0,l]$, we denote the set of mild solutions to (1.1). In this section, we show that $\Sigma^E_{\rho_0}[0,l]$ is nonempty and compact in $PC(J,E)$. First we prove that $\Sigma^E_{\rho_0}[0,l]$ is nonempty.

Theorem 3.1. Assume (A1), (A2), (A4) and the following conditions:

(A7) there exists a function $\varphi \in L^P(J,R^+)(P > \frac{1}{\alpha})$ such that for any $u \in E$

$$\|F(t,u)\| \leq \varphi(t)(1 + \|u\|) \quad a.e. \ t \in J.$$  

(A8) for every $i = 1, 2, \ldots, m$, $g_i : [b_i, a_i] \times E \to E$ is continuous and there exists a positive constant $h_i$ such that

$$\|g_i(t,u)\| \leq h_i \|u\|, \quad t \in [b_i, a_i], \ u \in E.$$  

Then the solution set of mild solutions of (1.1) is nonempty provided that

$$Mh + \zeta \|\varphi\|_{L^P(J,R^+)} < 1, \quad h = \sum_{i=1}^{m} h_i. \quad (3.1)$$

Proof. From Lemma 2.8 the superposition multi operator

$$P_F : C(J,E) \to P(L^P(J,E)),$$

$$P_F(\cdot)(u) = S^P_F(\cdot,u(\cdot)),$$

generated by $F$ is well defined. Therefore, we can define a multi operator $\Phi : PC(J,E) \to P(PC(J,E))$ as follows: let $u \in PC(J,E)$, and a function $y \in \Phi(u)$ if and only if

$$y(t) = \begin{cases} 
K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [0,b_1], \\
g_i(t,u(b_i^-)), & t \in (b_i,b_i^+), \ i = 1, 2, \ldots, m, \\
K_1(t-a_i)g_i(a_i,u(b_i^-)) + \int_{a_i}^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [a_i,b_{i+1}], \ i = 1, 2, \ldots, m, 
\end{cases} \quad (3.2)$$

where $f \in S^P_F(\cdot,u(\cdot))$.

It is clear that any fixed point for $\Phi$ is a mild solution for (1.1). We prove using Lemma 2.15 that $\Phi$ has a fixed point. We divide the proof into several steps.

Step 1. $\Phi$ is closed with compact values. Let $\{u_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ be two sequences in $PC(J,E)$ such that $u_n \to u, y_n \to y$ and

$$y_n(t) = \begin{cases} 
K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f_n(s)ds, & t \in [0,b_1], \\
g_i(t,u_n(b_i^-)), & t \in (b_i,b_i^+), \ i = 1, 2, \ldots, m, \\
K_1(t-a_i)g_i(a_i,u_n(b_i^-)) + \int_{a_i}^t (t-s)^{\alpha-1}K_2(t-s)f_n(s)ds, & t \in [a_i,b_{i+1}], \ i = 1, 2, \ldots, m, 
\end{cases}$$

where $f_n \in S^P_F(\cdot,u_n(\cdot))$.  

3. Nonemptiness and compactness of solution set
where \( f_n \in S_p^{P(\cdot,u_n(\cdot))} \).

For \( i = 0, 1, 2, \ldots, m \), consider \( G_i : L^p([a_i, b_{i+1}], E) \to C([a_i, b_{i+1}], E) \) defined by

\[
G_i f(t) = \int_{a_i}^{t} (t - s)^{\alpha - 1} K_2(t - s)f_n(s)ds.
\]

As in Lemma 2.11 we see that \( G_i(i = 0, 1, 2, \ldots, m) \) satisfies (A5) and (A6). Because \( u_n \to u \) in \( PC(J,E) \) we can find a positive constant \( \omega \) such that \( \|u_n\|_{PC(J,E)} \leq \omega \). Therefore, (A7) implies

\[
\|f_n(t)\| \leq \varphi(t)(1 + \omega), \quad \text{a.e. } t \in J.
\]

Then \( \{f_n : n \geq 1\} \) is bounded in \( L^p(J, E) \), and hence it is weakly compact and we may assume without generality that \( f_n \to f_0 \) in \( L^p(J, E) \). Moreover, (A4) implies

\[
\chi\{f_n(t) : n \geq 1\} \leq \beta(t)\chi\{u_n(t) : n \geq 1\} = 0, \quad \text{a.e. } t \in J. \tag{3.4}
\]

Lemma 2.9 implies that \( G_if_n \to G_if_0 \) in \( C([a_i, b_{i+1}], E) \). Moreover the continuity of \( g_i(t, u_n(b_i^-)) \) implies that

\[
\lim_{n \to \infty} g_i(t, u_n(b_i^-)) = g_i(t, u(b_i^-)), \quad t \in (b_i, a_i).
\]

Then \( y_n \to z \) in \( PC(J, E) \), where

\[
z(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t - s)^{\alpha - 1} K_2(t - s)f_0(s)ds, \quad t \in [0, b_1], \\ g_i(t, u(b_i^-)), \quad t \in (b_i, a_i), \ i = 1, 2, \ldots, m, \\ K_1(t - a_i)g_i(a_i, u(b_i^-)) + \int_{a_i}^{t} (t - s)^{\alpha - 1} K_2(t - s)f_0(s)ds, \quad t \in [a_i, b_{i+1}], \ i = 1, 2, \ldots, m. \end{cases}
\]

It follows from this and the fact that \( P_P \) is weakly closed that \( f_0 \in S_p^{P(\cdot,u(\cdot)))} \). Therefore, \( z \in \Phi(u) \). Since \( y_n \to z \) in \( PC(J, E) \) then \( z = y \). This shows that \( \Phi \) is closed.

To show that the values of \( \Phi \) are compact let \( u \in PC(J, E) \) and \( y_n \in \Phi(u), \ n \geq 1 \). The same argument as above implies that \( \{y_n : n \geq 1\} \) has a convergent subsequence. Thus, \( \Phi(u) \) is relatively compact. Notice arguing as above we see that \( \Phi(u) \) is closed. Then \( \Phi(u) \) is compact.

**Step 2.** \( \Phi \) is upper semicontinuous. Since \( \Phi \) is closed with compact values, it is enough to show that \( \Phi \) is quasicompact (see Definition 2.1). Let \( U \) be a compact subset in \( PC(J,E) \) and \( \{y_n\}_{n=1}^{\infty} \) be a sequence in \( \Phi(U) \). Then there exists a sequence \( \{u_n\}_{n=1}^{\infty} \) in \( U \) such that \( y_n \in \Phi(u_n) \), \( n \geq 1 \). The compactness of \( U \) implies that we can assume without loss of generality that \( u_n \to u \) in \( U \). Let \( f_n \in S_p^{P(\cdot,u_n(\cdot)))} \) be such that

\[
y_n(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t - s)^{\alpha - 1} K_2(t - s)f_n(s)ds, \quad t \in [0, b_1], \\ g_i(t, u_n(b_i^-)), \quad t \in (b_i, a_i), \ i = 1, 2, \ldots, m, \\ K_1(t - a_i)g_i(a_i, u_n(b_i^-)) + \int_{a_i}^{t} (t - s)^{\alpha - 1} K_2(t - s)f_n(s)ds, \quad t \in [a_i, b_{i+1}], \ i = 1, 2, \ldots, m. \end{cases}
\]

The same argument as above implies that we can assume without loss of generality that \( f_n \to f_0 \) in \( L^p(J, E) \) and \( y_n \to z \) in \( PC(J, E) \), where \( z \) is given in \( (3.5) \) and \( f_0 \in S_p^{P(\cdot,u(\cdot)))} \). Therefore \( \{y_n : n \geq 1\} \) converges to an element in \( \Phi(u) \).
Step 3. Let \( B_r = B(0, r) = \{ u \in PC(J, E) : \| u \| \leq r \} \), where
\[
    r = \frac{M\| u_0 \| + \zeta \| \varphi \|_{L^p(J,R^+)} }{1 - [Mh + \zeta \| \varphi \|_{L^p(J,R^+) ]}}. \tag{3.6}
\]

Note that from \((3.1)\), \( r \) is well defined. Obviously, \( B_r \) is a bounded, closed and convex subset of \( PC(J, E) \). We claim that \( \Phi(B_r) \subseteq B_r \). Let \( u \in B_r \) and \( y \in \Phi(u) \). By using \((3.2)\), \((A7)\) and Hölder’s inequality we obtain for \( t \in [0, b_1] \),
\[
    \| y(t) \| \leq M\| u_0 \| + \frac{M}{\Gamma(\alpha)} (1 + r) \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \\
    \leq M\| u_0 \| + \zeta(1 + r) \| \varphi \|_{L^p(J,R^+)}.
\]

If \( t \in (b_i, a_i] \), \( i = 1, 2, \ldots, m \), then
\[
    \| y(t) \| \leq \| g_i(t, u(b_i^-)) \| \leq h_i u(b_i^-) \| \leq h_i r \leq Mh r.
\]

Similarly, we obtain for \( t \in [a_i, b_{i+1}] \), \( i = 1, 2, \ldots, m \),
\[
    \| y(t) \| \leq K_1 (t-a_i) g_i(a_i, u(b_i^-)) + \int_{a_i}^t (t-s)^{\alpha-1} K_2 (t-s) f(s) ds \\
    \leq Mh r + \zeta(1 + r) \| \varphi \|_{L^p(J,R^+)}.\]

Therefore,
\[
    \| y \|_{PC(J,E)} \leq M(\| u_0 \| + hr) + \zeta(1 + r) \| \varphi \|_{L^p(J,R^+)} \leq r.
\]

It follows that \( \Phi(B_r) \subseteq B_r \). By applying Lemma \((2.15)\), we see that there is a function \( u \in PC(J, E) \) such that \( u \in \Phi(u) \). Clearly the function \( u \) is a solution for \((1.1)\). \( \square \)

In the following theorem we prove the compactness of \( \Sigma_u^F(0, l) \).

Theorem 3.2. Replace the conditions \((A1)\) and \((A8)\) in Theorem \((3.1)\) with the following two conditions:

\( \text{(A9)} \) the \( C_0 \)-semigroup \( \{ T(t) : t \geq 0 \} \) is equicontinuous.

\( \text{(A10)} \) for every \( i = 1, 2, \ldots, m \), \( g_i : [b_i, a_i] \times E \rightarrow E \) is uniformly continuous on bounded sets and for any \( t \in [b_i, a_i] \), \( g_i(t, \cdot) \) maps any bounded subset of \( E \) into a relatively compact subset of \( E \) and there exists a positive constant \( h_i \) such that
\[
    \| g_i(t, u) \| \leq h_i \| u \|, \quad t \in [b_i, a_i], \ u \in E.
\]

Then \( \Sigma_u^F(0, l) \) is compact provided that \((3.1)\) holds and
\[
    4\zeta \| \beta \|_{L^p(J,R^+)} < 1. \tag{3.7}
\]

Proof. From Theorem \((3.1)\) \( \Phi \) is a closed multifunction from \( B_r \) to \( P_{ck}(B_r) \), where \( r \) is given in \((3.6)\). We will use Lemma \((2.16)\) and we divide the proof into two steps.

Step 1. Set \( B_1 = \text{conv} \Phi(B_r) \) and \( B_n = \text{conv} \Phi(B_{n-1}) \), \( n \geq 2 \). From Theorem \((3.1)\) \( B_n \) is a nonempty, closed and convex subset of \( PC(J, E) \). Moreover, \( B_1 = \text{conv} \Phi(B_r) \subseteq B_r \). Also \( B_2 = \text{conv} \Phi(B_1) \subseteq \text{conv} \Phi(B_r) = B_1 \). By induction, the sequence \( (B_n) \), \( n \geq 1 \) is a decreasing sequence of nonempty, closed and bounded subsets of \( PC(J, E) \). Set \( B = \cap_{n=1}^\infty B_n \). Notice that every \( B_n \) being bounded, closed and convex, \( B \) is also bounded closed and convex. We now show \( \Phi(B) \subseteq B \).
Indeed, Φ(B) ⊆ Φ(B_n) ⊆ convΦ(B_n) = B_{n+1}, for every n ≥ 1. Therefore, Φ(B) ⊆ \bigcap_{n=2}^{\infty} B_n. On the other hand B_n ⊆ B_1 for every n ≥ 1. Thus,

\[ \Phi(B) \subseteq \bigcap_{n=2}^{\infty} B_n = \bigcap_{n=1}^{\infty} B_n = B \subseteq B_r. \]

We now show B is compact. According to the generalized Cantor’s intersection property (see [10]) to ensure the compactness of B, it is enough to show that

\[ \lim_{n \to \infty} \chi_{PC}(B_n) = 0, \quad (3.8) \]

where \( \chi_{PC} \) is the Hausdorff measure of noncompactness on \( PC(J, E) \).

First we verify that \( Z|_{\bar{J}} \) is equicontinuous for every \( i = 0, 1, 2, \ldots, m \), where \( Z = \Phi(B_r) \) and

\[ Z|_{\bar{J}} = \{ y^* \in C(\bar{J}, E) : y^*(t) = y(t), \quad t \in J_i, \quad y^*(b_i) = y(b_i^*), \quad y \in Z \}. \]

Let \( y \in Z \). Then there is a \( u \in B_r \) and \( f \in S^p_{P_E(t, u(\cdot))} \) such that

\[ y(t) = \begin{cases} 
K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [0, b_1], \\
g_i(t, u(b_i^-)), & t \in (b_i, a_i], \quad i = 1, 2, \ldots, m, \\
K_1(t-a_i)g_i(a_i, u(b_i^-)) + \int_{a_i}^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [a_i, b_{i+1}], \quad i = 1, 2, \ldots, m.
\end{cases} \]

We consider the following cases:

**Case 1.** Let \( i = 0 \) and \( t, t+\delta \) be two points in \( \bar{J}_0 = [0, b_1] \). Then

\[
\|y^*(t+\delta) - y^*(t)\|
= \|y(t+\delta) - y(t)\|
\leq \|K_1(t+\delta)(u_0) - K_1(t)(u_0)\| + \left\| \int_0^{t+\delta} (t+\delta-s)^{\alpha-1}K_2(t+\delta-s)f(s)ds \right\|
- \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds \right\|
= G_1 + G_2 + G_3 + G_4,
\]

where

\[ G_1 = \|K_1(t+\delta)u_0 - K_1(t)u_0\|, \]

\[ G_2 = \left\| \int_t^{t+\delta} (t+\delta-s)^{\alpha-1}K_2(t+\delta-s)f(s)ds \right\|, \]

\[ G_3 = \left\| \int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}] K_2(t+\delta-s)f(s)ds \right\|, \]

\[ G_4 = \left\| \int_0^t (t-s)^{\alpha-1}[K_2(t+\delta-s) - K_2(t-s)]f(s)ds \right\|. \]

By Lemma 2.6, \( K_1(t), t \in J \) is strongly continuous, and hence \( \lim_{\delta \to 0} G_1 = 0 \). For \( G_2 \), note that by Lemma 2.6 (A7) and H"older’s inequality it follows that

\[ \lim_{\delta \to 0} G_2 = \lim_{\delta \to 0} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1}K_2(t+\delta-s)f(s)ds \leq \frac{M}{\Gamma(\alpha)} \lim_{\delta \to 0} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1}\|f(s)\|ds. \]
there exists a sequence $(y_k)$. It is easy to show that

$$
\lim_{\delta \rightarrow 0} G_3 \leq \frac{M(r+1)}{\Gamma(\alpha)} \lim_{\delta \rightarrow 0} \left\| \int_t^t (t+\delta-s)^{\alpha-1} \varphi(s) ds \right\|_{L^p_{(\alpha,\beta)}} = 0.
$$

For $G_4$, the equicontinuity of $\{T(t): t \geq 0\}$ means that

$$
\lim_{t_2 \rightarrow t_1} \|T(t_2) - T(t_1)\| = 0, \quad t_2, t_1 \in (0, \infty).
$$

Therefore by the Lebesgue dominated convergence theorem,

$$
\lim_{\delta \rightarrow 0} G_4 \leq \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \zeta_\alpha(\theta)
\times \lim_{\delta \rightarrow 0} \|T((t+\delta-s)^{\alpha} - T((t-s)^{\alpha})\| \left| (1+r)^{\varphi(s)} \right| d\theta ds = 0.
$$

**Case 2.** Let $i = 1$. If $t$ and $t+\delta$ are two points in $(b_1, a_1]$ then invoking the definition of $\Phi$ and by the uniform continuity of $g_1$ on bounded sets it follows that

$$
\lim_{\delta \rightarrow 0} \|y^*(t+\delta) - y^*(t)\| = \lim_{\delta \rightarrow 0} \|y(t+\delta) - y(t)\|
\leq \lim_{\delta \rightarrow 0} \|g_1(t+\delta, u(b_1^-)) - g_1(t, u(b_1^-))\| = 0,
$$

independently of $x$.

If $t$ and $t+\delta$ are two points in $(a_1, b_2]$ then invoking the definition of $\Phi$ we have

$$
\lim_{\delta \rightarrow 0} \|y^*(t+\delta) - y^*(t)\| = \lim_{\delta \rightarrow 0} \|y(t+\delta) - y(t)\|
\leq \|K_1(t+\delta-a_1)g_1(a_1, u(b_1^-)) - K_1(t-a_1)g_1(a_1, u(b_1^-))\|
\quad + \| \int_{a_1}^{t+\delta} (t+\delta-s)^{\alpha-1} K_2(t+\delta-s) f(s) ds
\quad - \int_{a_1}^{t+\delta} (t-s)^{\alpha-1} K_2(t+\delta-s) f(s) ds\|.
$$

By arguing as in the first case we obtain $\lim_{\delta \rightarrow 0} \|y^*(t+\delta) - y^*(t)\| = 0$.

If $t = b_1$ and $t+\delta \in (b_1, a_1]$ then

$$
\|y^*(b_1+\delta) - y^*(b_1)\| = \lim_{\sigma \rightarrow b_1^+} \|y(b_1+\delta) - y(\sigma)\|
\leq \lim_{\sigma \rightarrow b_1^+} \|g_1(b_1+\delta, u(b_1^-)) - g_1(\sigma, u(b_1^-))\|.
$$

Again by the uniform continuity of $g_1$, we obtain

$$
\lim_{\delta \rightarrow 0, \sigma \rightarrow b_1^+} \|y(b_1+\delta) - y(\sigma)\| = 0.
$$

Therefore, $Z|_{\mathcal{F}_1}$ is equicontinuous. Similarly $Z|_{\mathcal{F}_i}$ is equicontinuous for $i = 2, \ldots, m$.

Next, let $n \geq 1$ be a fixed natural number and $\varepsilon > 0$. In view of [9] Lemma 2.9], there exists a sequence $(y_k)$, $k \geq 1$ in $\Phi(B_{n-1})$ such that

$$
\chi_{PC} \Phi(B_{n-1}) \leq 2 \chi_{PC} \{ y_k : k \geq 1 \} + \varepsilon.
$$
From the definition of $\chi_{PC}$, the above inequality becomes

$$\chi_{PC}(B_n) = \chi_{PC}\Phi(B_{n-1}) \leq 2 \max_{i=0,1,...,m} \chi_i(S|J) + \varepsilon, \quad (3.9)$$

where $S = \{y_k : k \geq 1\}$ and $\chi_i$ is the Hausdorff measure of noncompactness on $C(J_i, E)$. Since $B_{n|J_i}$, $i = 0, 1, \ldots, m$, is equicontinuous, then (see [12]),

$$\chi_i(S|J) = \sup_{t \in J} \chi(S(t)).$$

Therefore, by using the nonsingularity of $\chi$, (see [12]) we have

$$\chi_{PC}(B_n) \leq 2 \max_{i \leq 0,1,\ldots,m} [\sup_{t \in J} \chi(S(t))] + \varepsilon \tag{3.10}$$

$$= 2 \sup_{t \in J} \chi(S(t)) + \varepsilon$$

$$= 2 \sup_{t \in J} \chi\{y_k(t) : k \geq 1\} + \varepsilon.$$

Now, since $y_k \in \Phi(B_{n-1})$, $k \geq 1$ there is $x_k \in B_{n-1}$ such that $y_k \in \Phi(x_k)$, $k \geq 1$. By recalling the definition of $\Phi$ for every $k \geq 1$ there is $f_k \in S_{F(x_k(\cdot))}^P$ such that for every $t \in J$,

$$\chi\{y_k(t) : k \geq 1\}$$

$$\leq \chi\left\{\int_{0}^{t} (t-s)^{\alpha-1} K_2(t-s)f_k(s) ds : k \geq 1\right\}, \quad \text{if } t \in [0,b_1],$$

$$\chi\left\{g_i(t,x_k(b_i^-)) : k \geq 1\right\}, \quad \text{if } t \in (b_i,a_i), \quad i = 1,2,\ldots,m,$$

$$\chi\left\{K_1(t-a_i)g_i(a_i,x_k(b_i^-)) : k \geq 1\right\}$$

$$+ \chi\left\{\int_{a_i}^{t} (t-s)^{\alpha-1} K_2(t-s)f_k(s) ds : k \geq 1\right\}, \quad \text{if } t \in [a_i,b_{i+1}], \quad i = 1,2,\ldots,m. \quad (3.11)$$

Note that, by the compactness of $g_i$, $i = 1,2,\ldots,m$, we obtain

$$\chi\{g_i(t,x_k(b_i^-)) : k \geq 1\} = 0. \quad (3.12)$$

Notice that from (A4) we have for a.e. $t \in J$ that

$$\chi\{f_k(t) : k \geq 1\} \leq \chi\{F(s,x_k(t)) : k \geq 1\}$$

$$\leq \beta(t)\chi\{x_k(t) : k \geq 1\}$$

$$\leq \beta(t)\chi\{B_{n-1}(t)\}$$

$$\leq \beta(t)\chi_{PC}(B_{n-1}) = \gamma(t). \quad (3.13)$$

Furthermore, by (A7), for any $k \geq 1$, and for almost $t \in J$, we have $\|f_k(t)\| \leq \varphi(t)(r+1)$. Consequently, $f_k \in L^P(J,E), k \geq 1$, and hence $\gamma \in L^P(J,R^+).$ Then, from Lemma 2.13, there exist a compact $K_e \subseteq E$, a measurable set $J_e \subseteq J$, with measure less than $\varepsilon$, and a sequence of functions $\{z_k^e\} \subseteq L^P(J,E)$ such that for all $s \in J$, $\{z_k^e(s) : k \geq 1\} \subseteq K$ and

$$\|f_k(s) - z_k^e(s)\| < 2\gamma(s) + \varepsilon, \quad \text{for every } k \geq 1 \text{ and every } s \in J - J_e. \quad (3.14)$$
Let $J_0 = [0, t_1]$. Then, using (3.14) and Hölder’s inequality, it follows for $k \geq 1$,

$$
\| \int_{J_0} (t-s)^{\alpha-1} K_2(t-s)(f_k(s) - z_k^\xi(s))ds \|
\leq \frac{M}{\Gamma(\alpha)} \| \int_{J_0} (t-s)^{\alpha-1}(f_k(s) - z_k^\xi(s))ds \|
\leq \frac{M}{\Gamma(\alpha)} \| f_k - z_k^\xi \|_{L^p(J_0, J_1)} \left( \int_{J_0} (t-s)^{\alpha/p} ds \right)^{\frac{p-1}{p}}
\leq \zeta \| f_k - z_k^\xi \|_{L^p(J_0, J_1)}
\leq \zeta (2\| \gamma \|_{L^p(J_0, J_1, R^+)} + \epsilon l^{1/p})
= \zeta (2\| \beta \|_{L^p(J_1, R^+)} \chi_{PC}(B_{n-1}) + \epsilon l^{1/p}),
$$

(3.15)

Also, by Hölder’s inequality, for $k \geq 1$, we obtain

$$
\| \int_{J_0} (t-s)^{\alpha-1} K_2(t-s)f_k(s)ds \|
\leq \frac{M}{\Gamma(\alpha)} (r+1) \int_{J_0} (t-s)^{\alpha-1}\varphi(s)ds
\leq \frac{M}{\Gamma(\alpha)} (r+1) \| \varphi \|_{L^p(J_0, J_1, R^+)} \left( \int_{J_0} (t-s)^{\alpha/p} ds \right)^{\frac{p-1}{p}}.
$$

From this inequality, (3.13) and (3.15), for $t \in [0, t_1]$, it follows that

$$
\chi \left( \left\{ \int_0^t (t-s)^{\alpha-1} K_2(t-s)f_k(s)ds : k \geq 1 \right\} \right)
\leq \chi \left( \left\{ \int_{J_0-J_1} (t-s)^{\alpha-1} K_2(t-s)f_k(s)ds : k \geq 1 \right\} \right)
+ \chi \left( \left\{ \int_{J_0-J_1} (t-s)^{\alpha-1}(f_k(s) - z_k^\xi(s))ds : k \geq 1 \right\} \right)
\leq \chi \left( \left\{ \int_{J_0-J_1} (t-s)^{\alpha-1}(f_k(s) - z_k^\xi(s))ds : k \geq 1 \right\} \right)
+ \chi \left( \left\{ \int_{J_0-J_1} (t-s)^{\alpha-1}z_k^\xi(s)ds : k \geq 1 \right\} \right)
+ \chi \left( \left\{ \int_{J_0-J_1} (t-s)^{\alpha-1}f_k(s)ds : k \geq 1 \right\} \right)
\leq \zeta (2\| \beta \|_{L^p(J_1, R^+)} \chi_{PC}(B_{n-1}) + \epsilon l^{1/p})
+ \frac{M}{\Gamma(\alpha)} (r+1) \| \varphi \|_{L^p(J_1, R^+)} \left( \int_{J_0-J_1} (t-s)^{\alpha/p} ds \right)^{\frac{p-1}{p}}.
$$

Taking into account that $\epsilon$ is arbitrary, the above inequality becomes

$$
\chi \left( \left\{ \int_0^t (t-s)^{\alpha-1} K_2(t-s)f_k(s)ds : k \geq 1 \right\} \right) \leq 2\zeta \| \beta \|_{L^p(J_1, R^+)} \chi_{PC}(B_{n-1}).
$$

(3.16)

Similarly, we can show that if $t \in [a_i, b_{i+1}]$, $i = 1, 2, \ldots, m$, then

$$
\chi \left( \left\{ \int_{a_i}^{b_{i+1}} (t-s)^{\alpha-1} K_2(t-s)f_k(s)ds : k \geq 1 \right\} \right) \leq 2\zeta \| \beta \|_{L^p(J_1, R^+)} \chi_{PC}(B_{n-1}).
$$

(3.17)
Then, by (3.11), (3.12), (3.16) and (3.17) for every $t \in J$, 
\[ \chi \{ y_k(t) : k \geq 1 \} \leq 4\zeta \| \beta \|_{L^p(J,R^+)} \chi_{PC}(B_{n-1}). \]
From this inequality, (3.10) and the fact that $\varepsilon$ is arbitrary it follows that 
\[ \chi_{PC}(B_n) \leq 4\zeta \| \beta \|_{L^p(J,R^+)} \chi_{PC}(B_{n-1}). \]
From a finite number of steps, we obtain 
\[ 0 \leq \chi_{PC}(B_n) \leq (4\zeta \| \beta \|_{L^p(J,R^+)})^{n-1} \chi_{PC}(B_1), \quad \forall n \geq 1. \]
Since this inequality is true for every $n \in \mathbb{N}$, then by (3.7) and by passing to the limit as $n \to +\infty$, we obtain (3.8). Hence $B$ is a nonempty and compact subset of $PC(J,E)$. Then $\Phi : B \to P_{ck}(B)$ is compact.

**Step 2.** The set of fixed points of $\Phi$ is a bounded subset of $PC(J,E)$. Let $u \in \Phi(u)$, $u \in B$ and $f \in S^F_{\gamma(u)}$ such that 
\[ u(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1} K_2(t-s)f(s)ds, & t \in [0,b_1], \\ g_i(t,u(b_i^{-})) & t \in (b_i,a_i], \quad i = 1,2,\ldots,m, \\ K_1(t-a_i)g_i(a_i,u(b_i^{+})) & t \in [a_i,b_{i+1}], \quad i = 1,2,\ldots,m. \\ + \int_{a_i}^t (t-s)^{\alpha-1} K_2(t-s)f(s)ds, & t \in [a_i,b_{i+1}], \quad i = 1,2,\ldots,m. \end{cases} \]
By arguing as in Step 3 in Theorem 3.1 we can show that $\|u\|_{PC(J,E)} \leq r$, where $r$ is given in (3.6). Applying Lemma 2.16 and we conclude that $\Sigma^F_{u\in[0,l]}$ is compact in $PC(J,E)$. \hfill $\square$

4. **An example**

In this section, we give an example to illustrate our results. Let $J = [0,1]$, $K$ be a non-empty convex compact subset in a Banach space $E$ and $F : J \times E \to P_{ck}(E)$ be a multi-valued function defined by 
\[ F(t,u) = \frac{e^{-\gamma t} \|u\|}{\lambda(1+\|u\|)} K, \quad (4.1) \]
where $\gamma \in (1,\infty)$ and $\lambda$ is a constant such that $\sup \{ \|z\| : z \in K \} \leq \lambda$. Clearly for every $u \in E$, $t \to F(t,u)$ is measurable. Moreover, for any $u,v \in E$ and any $t \in J$, we have 
\[ H(F(t,u),F(t,v)) \leq e^{-\gamma t} \|u-v\|. \]
Then, for almost all $t \in J$, $u \to F(t,u)$ is upper semicontinuous and so (A2) holds. Moreover, for every bounded subset $D \subseteq E$, $\chi(F(t,D)) \leq \beta(t)\chi(D)$ for a.e. $t \in J$, holds with $\beta(t) = e^{-\gamma t}$. Therefore (A4) is satisfied. Also, for any $(t,u) \in J \times E$, 
\[ \|F(t,u)\| \leq e^{-\gamma t} \leq e^{-\gamma t}(1+\|u\|). \]
Then (A7) is satisfied with $\varphi(t) = e^{-\gamma t}$.

Now, for any $i = 1,2,\ldots,m$, let $g_i : [t_i,s_i] \times E \to E$, defined by 
\[ g_i(t,u) = tu \quad (4.2) \]
Note that (A8) is satisfied.

Assume that $A : D(A) \subseteq E \to E$ is a linear closed operator generating a $C_0$-semigroup $\{T(t) : t \geq 0\}$ of bounded linear operator and there is $M \geq 1$, such that $\sup_{t \geq 0} \|T(t)\| \leq M$, see for example, [25] Example 3.12. By applying Theorem 3.1
problem (1.1), where $F$ and $g_i$, $i = 1, 2, \ldots, m$, are given by (4.1) and (4.2), has a mild solution, provided that
\[ Mh + \zeta \| \varphi \|_{L^p(J, R^+)} < 1, \]
where $h = \text{Im}$, $\zeta = \frac{M}{\Gamma(\alpha)} \left( \frac{P-1}{P} \right)^{P-1} \frac{1}{l^{\alpha-1}}$ and $\varphi(t) = e^{-\gamma t}$, $t \in J$.

If, in addition, the $C_0$-semigroup $\{ T(t) : t \geq 0 \}$ is equicontinuous then, by Theorem 3.2 the solution set of (1.1) is compact provided that $4\zeta \| \beta \|_{L^p(J, R^+)} < 1$, where $\beta(t) = e^{-\gamma t}$, $t \in J$.

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References


Jinrong Wang  
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China.  
School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China  
Email address: wjr9668@126.com

Ahmed G. Ibrahim  
Department of Mathematics, Faculty of Science, King Faisal University, Al-Ahasa 31982, Saudi Arabia  
Email address: agamal@kfup.edu.sa

Donal O’Regan  
School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland  
Email address: donal.oregan@nuigalway.ie