BOUNDARY REGULARITY FOR NONDIVERGENCE ELLIPTIC EQUATION WITH UNBOUNDED DRIFT

YONGPAN HUANG, QIAOZHU ZHAI, SHULIN ZHOU

Communicated by Hongjie Dong

ABSTRACT. We obtain the pointwise boundary differentiability of strong solutions for elliptic equations with the lower order coefficients, the boundary, and the right-hand side term satisfying a Dini type condition. Furthermore, we establish a pointwise estimate of strong solutions and show that the gradients of the strong solutions are continuous along the boundary if the drift term, the boundary, and the right-hand side term satisfy a uniform Dini type condition on the boundary.

1. Introduction

In this article, we will study the boundary regularity of strong solutions of elliptic equation with unbounded lower order coefficients. Suppose that $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\Omega)$ satisfies

$$Lu := -a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x) \quad \text{in } \Omega;$$
$$u(x) = 0 \quad \text{on } \partial \Omega;$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. We use the summation convention over repeated indices and the notations $D_i := \frac{\partial}{\partial x_i}$; $D_{ij} := D_iD_j$. We assume that $a_{ij}$, $b_i$, and $f$ are measurable functions on $\Omega$, the matrix $(a_{ij}(x))_{n \times n}$ is symmetric and satisfies the uniformly elliptic condition

$$\lambda |\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega,$$

(1.2)

with a constant $\lambda \in (0,1]$, and $b_i, f \in L^n(\Omega)$. Throughout this article, the operator $L$ in (1.1) is applied to functions $u$ in the class $W(\Omega) := W^{2,n}_{\text{loc}}(\Omega) \cap C(\Omega)$.

In the following, we extend the results in [15] to elliptic equations with unbounded lower order term. The boundary differentiability is shown for strong solution of nondivergence elliptic equation on $C^{1,\text{Dini}}$ domain with unbounded drift satisfies Dini type condition. Furthermore, we prove that boundary first order derivative is continuous along the boundary.

As for the boundary regularity of nondivergence elliptic equations: If the drift term $|b|$ is bounded, Krylov [8,9] showed that the solution is $C^{1,\alpha}$ along the boundary if $\partial \Omega$ is $C^{1,1}$. Lieberman [13] gave a more general estimates. Wang [19] proved
This paper. Ma and Wang [15] proved a boundary $C^{1,\psi}$ estimate for fully nonlinear elliptic equations on $C^{1,Dini}$ domain. Li and Wang [11, 12] showed the boundary differentiability of solutions of elliptic equations on convex domains. If $|b|$ is unbounded, Ladyzhenskaya and Ural’tseva in [10] proved boundary $C^{1,\alpha}$ estimate of elliptic and parabolic inequalities on $W^{2,q}$ domain with $b \in L^q$, $\Phi \in L^2$, $q > n$ and nonlinear term $\mu_1|Du|^2$. Apushkinskaya and Nazarov [1] proved the boundary $C^{1,\alpha}$ estimate for nondivergence parabolic equation with composite right-hand side and lower order coefficients, and in [2] they gave a counterexample of Hopf-Oleinik Lemma and boundary gradient estimate under minimal restrictions. Dini condition on $|\eta|$ and lower order coefficients, and in [2] they gave a counterexample which indicated that the lemma in the elliptic case. Safonov [18] obtained the Hopf-Oleinik lemma on a flat domain for elliptic equations and gave the counterexample which indicated that the Dini condition on $|b|$ can not be removed for our theorem. Nazarov [16] proved the Hopf-Oleinik Lemma and boundary gradient estimate under minimal restrictions on lower-order coefficients. Braga, Moeira and Wang [3] generalized the elliptic case in [10] to $L^n$ viscosity solutions with $\mu_1 = 0$ and $C^{1,Dini}$ boundary value. Some related results concerning Dini continuity can be found in [4, 6, 7, 17, 20, 21].

Theorem 1.1 ([5, 18]). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and let $u$ be a function in $W(\Omega)$ such that $Lu \leq f$ in $\Omega$. Suppose that the matrix $(a_{ij}(x))_{n \times n}$ is symmetric and satisfies the uniformly elliptic condition (1.2), and $b, f \in L^n(\Omega)$. Then

$$\sup_\Omega u \leq \sup_\Omega u + N \text{diam } \Omega \cdot e^{N \|b\|^n_{L^n(\Omega)}},$$

(1.3)

where

$$\|b\|_{L^n(\Omega)} = \left( \int_\Omega |b|^n \, dx \right)^{1/n}, \quad b = (b_1, b_2, \ldots, b_n),$$

(1.4)

and $N$ is a positive constant depending only on $n$ and $\lambda$.

Theorem 1.2 (Harnack Inequality). Let $u$ be a nonnegative function in $W(B_1)$, $Lu = f$ in $B_1$ and $b, f \in L^n(B_1)$. There exists a positive constant $\epsilon_0$ depending only on $\lambda$ and $n$, such that if $\|b\|_{L^n(B_1)} \leq \epsilon_0$, then

$$\sup_{B_1} u \leq C(\inf_{B_1} u + \|f\|_{L^n(B_1)}),$$

(1.5)

where $C$ is constant depending only on $\lambda$ and $n$.

Theorem 1.2 follows from the the proof in [18] clearly. The most important thing is that the quantity $\|b\|_{L^n}$ is scaling invariant (see [13, Remark 1.4]) and the Harnack constant is invariant in the iteration procedure. Before we state out our main theorem, for convenience, we give the following notation and definitions.

$\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$.

$$|x| := \left( \sum_{i=1}^n x_i^2 \right)^{1/2}$$

is the Euclidean norm of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. $a^+ := \max\{0, a\}$. $B_r := \{x \in \mathbb{R}^n : |x| < r\}$. $B_r(x) := x + B_r$. $\Omega := \Omega \cap B_r$. $\Omega_r(x) := \Omega \cap B_r(x)$. $\text{diam}(\Omega) := \sup_{x, y \in \Omega} |x - y|$. $Q_r := \{x \in \mathbb{R}^n : |x_i| < r, \ i = 1, 2, \ldots, n\}$. $\|f\|_{L^n(\Omega)} := \left( \int_\Omega |f(x)|^n \, dx \right)^{1/n}$. $W(\Omega) := W_{loc}^{2,n}(\Omega) \cap C(\overline{\Omega})$. 


\textbf{Definition 1.3.} We say that \( \partial \Omega \) is \( C^{1, \text{Dini}} \) at \( x \in \partial \Omega \), if there exist a unit vector \( \vec{n} \) and a positive constant \( r_0 \) such that
\[
\frac{1}{r} \sup_{y \in \partial \Omega, |y-x| \leq r} |(y-x) \cdot \vec{n}| \leq \omega(r), \quad \text{for} \ 0 < r \leq r_0,
\]
where \( \omega(r) \) is a nonnegative nondecreasing function and satisfies \( \int_0^{r_0} \frac{\omega(r)}{r} \, dr < \infty \).
We say that \( \partial \Omega \) is \( C^{1, \text{Dini}} \) if for any \( x \in \partial \Omega \), \( \partial \Omega \) is \( C^{1, \text{Dini}} \) at \( x \in \partial \Omega \).

If \( \partial \Omega \) satisfies the pointwise \( C^{1, \text{Dini}} \) condition at any \( x \in \partial \Omega \) with the same \( r_0 \), it follows that \( \partial \Omega \) is \( C^{1, \text{Dini}} \) in the classical sense, i.e., \( \partial \Omega \) can be locally represented as a \( C^1 \) graph with the gradient being Dini continuous.

\textbf{Definition 1.4.} We say that the function \( g \in L^n(\Omega) \) is \( C^{n, \text{Dini}} \) at \( x \in \partial \Omega \), if there exists a positive constant \( r_0 \) such that
\[
\left( \frac{1}{|B_r(x) \cap \Omega|} \int_{B_r(x) \cap \Omega} |g(y)|^n dy \right)^{1/n} \leq r^{-1} \omega(r)
\]
for each \( 0 < r \leq r_0 \), where \( \omega(r) \) is a nonnegative nondecreasing function and satisfies \( \int_0^{r_0} \frac{\omega(r)}{r} \, dr < \infty \). Obviously, we have \( \|g\|_{L^n(\Omega \cap B_r(x))} \leq |B_r(0)|^{1/n} \omega(r) \leq 2 \omega(r) \). We say that \( g \) is \( C^{n, \text{Dini}} \) on \( \partial \Omega \) if for any \( x \in \partial \Omega \), \( g \) is \( C^{n, \text{Dini}} \) at \( x \in \partial \Omega \).

Generally, for any function in \( L^p(\Omega) (1 \leq p \leq \infty) \), we can define the pointwise \( C^{k, \text{Dini}} \) \( (k \in \mathbb{Z}) \). We say that the function \( g \in L^p(\Omega) \) is \( C^{k, \text{Dini}} \) at \( x \in \partial \Omega \), if there exists a positive constant \( r_0 \) and a \( k \)-th order polynomial \( P_k(x) \) \( (P_k(x) \equiv 0 \) if \( k \leq 0) \) such that
\[
\left( \frac{1}{|B_r(x) \cap \Omega|} \int_{B_r(x) \cap \Omega} |g(y) - P_k(y)|^p dy \right)^{1/p} \leq r^k \omega(r)
\]
for each \( 0 < r \leq r_0 \), where \( \omega(r) \) is a nonnegative nondecreasing function and satisfies \( \int_0^{r_0} \frac{\omega(r)}{r} \, dr < \infty \).

The main results of this paper are Theorems 1.5, 1.9 and Corollary 1.7 below.

\textbf{Theorem 1.5.} Assume that
\begin{enumerate}
\item \( 0 \in \partial \Omega \), \( r_0 > 0 \), \( u \in W(\Omega_{r_0}) \), \( u|_{\partial \Omega \setminus B_{r_0}} = 0 \), \( Lu = f \) in \( \Omega_{r_0} \), \( |b|, f \in L^n(\Omega_{r_0}) \) and \( \int_0^{r_0} \frac{|f| L^n(\Omega_{r_0})}{r} \, dr < \infty \);
\item \( \partial \Omega \) is \( C^{1, \text{Dini}} \) at 0 and \( |b| \) is \( C^{n, \text{Dini}} \) at 0 with the modulus of continuity \( \omega(r) \) satisfies
\[
\omega(r_0) \leq \min \left\{ \delta \frac{1}{6}, \frac{1}{2}, \frac{\epsilon_0}{2} \right\} \quad \text{and} \quad \int_0^{r_0} \frac{\omega(r)}{r} \, dr \leq \min \left\{ 1, \frac{\delta \ln \frac{1}{3}}{72 M \sqrt{n} A_2} \right\},
\]
where \( \delta, M \) and \( A_2 \) are constants depending only on \( \lambda \) and \( n \) (see Lemma 2.2), and \( \epsilon_0 \) is the constant in Theorem 1.2.
\end{enumerate}

Then \( u \) is differentiable at 0, furthermore, there exist a linear function \( L(x) \) and constants \( \alpha > 0 \), \( \Lambda > 1 \), \( C > 0 \) such that
\[
|u(x) - L(x)| \leq C \left\{ r^\alpha + \omega(\Lambda r) + \|f\|_{L^n(\Omega_{r_0})} + \int_0^{r_0} \frac{\omega(s)}{s^{1+\alpha}} \, ds \right. \left. + \int_0^\Lambda \frac{\omega(s)}{s} \, ds \right\} r,
\]
which is the main result of this paper. Theorem 1.5, 1.9 and Corollary 1.7 below.
Corollary 1.7. Assume that

1. \( 0 \in \partial \Omega, \ r_0 > 0, \ u \in W(\Omega_{r_0}), \ Lu = f \ in \ \Omega_{r_0}, \ u|_{\partial \Omega \cap B_{r_0}} = 0 \ and \ |b|, \ f \in L^n(\Omega_{r_0}); \)
2. \( \partial \Omega \) is \( C^{1,Dini} \) at 0, \( |b| \) is \( C^{n-1,Dini} \) at 0 and \( f \) is \( C^{n-1,Dini} \) at 0 with the modulus of continuity \( \omega(r) \) satisfies

\[
\omega(r_0) \leq \min \left\{ \frac{\delta}{6}, \frac{1}{2}, \frac{\epsilon_0}{2} \right\}, \quad \int_0^{r_0} \frac{\omega(r)}{r} \, dr \leq \min \left\{ 1, \frac{\delta \ln \frac{1}{r}}{72M \sqrt{n} A_2} \right\},
\]

where \( \delta, M \) and \( A_2 \) are the constants in Lemma 2.2 and \( \epsilon_0 \) is the constant in Theorem 1.2.

Then \( u \) is differentiable at 0, furthermore, there exist a linear function \( L(x) \) and constants \( \hat{\alpha} > 0, \ \Lambda > 1, \ C > 0 \) such that for any \( x \in \Omega_r \) and \( 0 < r \leq r_0/\Lambda, \)

\[
|u(x) - L(x)| \leq C \left( r^{\hat{\alpha}} + \omega(\Lambda r) + r^\alpha \int_r^{r_0} \frac{\omega(s)}{s^{1+\alpha}} \, ds + \int_0^{\Lambda r} \frac{\omega(s)}{s} \, ds \right),
\]

where \( C \) depends on \( \|u\|_{L^\infty(\Omega_{r_0})}, \ r_0, \ \lambda \) and \( n. \)

Remark 1.8. If \( \partial \Omega \) is \( C^{1,\alpha} \) at 0, \( |b| \) is \( C^{n-1,\alpha} \) at 0 and \( f \) is \( C^{n-1,\alpha} \) at 0 with \( \omega(r) = r^\alpha(0 < \alpha < 1) \), then \( u \) is \( C^{1,\hat{\beta}} \) at 0 with \( \hat{\beta} = \min\{\alpha, \hat{\alpha}\} \) if \( \alpha \neq \hat{\alpha} \) and \( 0 < \hat{\beta} < \min\{\alpha, \hat{\alpha}\} \) if \( \alpha = \hat{\alpha}. \)

Theorem 1.9. Assume that

1. \( 0 \in \partial \Omega, \ r_0 > 0, \ u \in W(\Omega_{3r_0}), \ Lu = f \ in \ \Omega_{3r_0}, \ u|_{\partial \Omega \cap B_{r_0}} = 0 \ and \ |b|, \ f \in L^n(\Omega_{3r_0}); \)
2. \( \partial \Omega \) is \( C^{1,Dini} \), \( |b| \) is \( C^{n-1,Dini} \) and \( f \) is \( C^{n-1,Dini} \) on \( \partial \Omega \cap B_{r_0} \) uniformly with the modulus of continuity \( \omega(r) \) satisfies

\[
\omega(r_0) \leq \min \left\{ \frac{\delta}{6}, \frac{1}{2}, \frac{\epsilon_0}{2} \right\}, \quad \int_0^{r_0} \frac{\omega(r)}{r} \, dr \leq \min \left\{ 1, \frac{\delta \ln \frac{1}{r}}{72M \sqrt{n} A_2} \right\},
\]

where \( \delta, M \) and \( A_2 \) are constants in Lemma 2.2 and \( \epsilon_0 \) is the constant in Theorem 1.2.

Then there exist constants \( \hat{\alpha} > 0, \ \Lambda > 1, \ C > 0 \) such that for any \( y, z \in \partial \Omega \cap B_{r_0} \) and \( 0 < |y - z| = r \leq r_0, \)

\[
|\nabla u(y) - \nabla u(z)| \leq C \left( r^{\hat{\alpha}} + \omega(\Lambda r) + r^{\hat{\alpha}} \int_r^{r_0} \frac{\omega(s)}{s^{1+\alpha}} \, ds + \int_0^{\Lambda r} \frac{\omega(s)}{s} \, ds \right),
\]
where $\hat{\alpha}$ and $\Lambda$ are the constants in Corollary 1.4 and $C$ is a constant depending on $\|u\|_{L^\infty(\Omega_{\epsilon,\rho})}$, $r_0$, $\lambda$ and $n$.

**Remark 1.10.** If $\partial \Omega$ is $C^{1,\alpha}$ on $\partial \Omega \cap B_{r_0}$, $|\hat{b}|$ is $C_n^{-1,\alpha}$ on $\partial \Omega \cap B_{r_0}$ and $f$ is $C_n^{-1,\alpha}$ on $\partial \Omega \cap B_{r_0}$ with $\omega(r) = r^\alpha(0 < \alpha < 1)$, then $\nabla u$ is $C^{\hat{\beta}}$ along $\partial \Omega \cap B_{r_0}$ with $\hat{\beta} = \min\{\alpha, \hat{\alpha}\}$ if $\alpha \neq \hat{\alpha}$ and $0 < \hat{\beta} < \min\{\alpha, \hat{\alpha}\}$ if $\alpha = \hat{\alpha}$.

We shall prove Theorems 1.5 and 1.9 in the next section.

2. Boundary estimates

By standard normalization, it is enough to prove Theorem 2.1 below, instead of proving Theorem 1.5. Since $\partial \Omega$ is $C^{1,Dini}$ at $0 \in \partial \Omega$, without loss of generality, we assume $\vec{n} = e_n$ as the inward normal direction in the following Theorem 2.1.

Consider the normalization of solution,

$$
\tilde{u}_\epsilon(x) = \frac{u(r_0x)}{\|u\|_{L^\infty(\Omega_{r_0})} + \epsilon + r_0\|f\|_{L^n(\Omega_{r_0})} + r_0 \int_{r_0}^{r_0} \frac{\|f\|_{L^n(\Omega_{r_0})}}{r} dr},
$$

for $\epsilon > 0$ and $x \in \hat{\Omega} \cap B_1$, with the normalized domain $\hat{\Omega} := \{x \in \mathbb{R}^n : r_0x \in \Omega\}$. Obviously, $\tilde{u}_\epsilon(x)$ satisfies

$$
\|\tilde{u}_\epsilon\|_{L^\infty(\hat{\Omega}_1)} \leq 1 \quad \text{and} \quad -\tilde{a}_{ij}(x)D_{ij}\tilde{u}_\epsilon(x) + \tilde{b}_i(x)D_i\tilde{u}_\epsilon(x) = \tilde{f}(x)
$$

for $x \in \hat{\Omega} \cap B_1$, where

$$
\tilde{a}_{ij}(x) = a_{ij}(r_0x), \quad \tilde{b}_i(x) = r_0b_i(r_0x), \quad \tilde{f}(x) = \frac{r_0^2f(r_0x)}{\|u\|_{L^\infty(\Omega_{r_0})} + \epsilon + r_0\|f\|_{L^n(\Omega_{r_0})} + r_0 \int_{r_0}^{r_0} \frac{\|f\|_{L^n(\Omega_{r_0})}}{r} dr}.
$$

Let $\tilde{\omega}(r) = \omega(r_0r)$. Obviously,

$$
\frac{1}{r} \sup_{y \in \partial \hat{\Omega}, |y| \leq r} |y \cdot e_n| \leq \tilde{\omega}(r), \quad \|\tilde{b}\|_{L^n(\hat{\Omega}_r)} = \|b\|_{L^n(\Omega_{r_0})} \leq 2\tilde{\omega}(r)
$$

for $0 < r \leq 1$, and

$$
\int_0^1 \frac{\tilde{\omega}(r)}{r} dr = \int_0^{r_0} \frac{\omega(r)}{r} dr.
$$

**Theorem 2.1.** Assume that

1. $0 \in \partial \Omega$, $u \in W(\Omega_1)$, $u|_{\partial \Omega \cap B_1} = 0$, $Lu = f$ in $\Omega_1$, and $\|u\|_{L^\infty(\Omega_1)} \leq 1$;
2. $f \in L^n(\Omega_1)$ with $\|f\|_{L^n(\Omega_1)} \leq 1$ and $\int_{r_0}^1 \frac{\|f\|_{L^n(\Omega_{r_0})}}{r} dr \leq 1$;
3. $\partial \Omega$ is $C^{1,Dini}$ at $0$ and $|\hat{b}|$ is $C_n^{-1,Dini}$ at $0$ with the modulus of continuity $\omega(r)$ satisfies the normalized conditions

$$
\omega(1) \leq \min\left\{\frac{\delta}{6}, \frac{1}{2}, \frac{\epsilon_0}{2}\right\}, \quad \int_0^1 \frac{\omega(r)}{r} dr \leq \min\left\{1, \frac{\delta}{72MA_2}\right\},
$$

(2.1)

where $\epsilon_0$ is the constant in Theorem 1.2 and $\delta$, $M$, $A_2$ are constants in Lemma 2.2.
Obviously, we have $	ilde{x}$ for any $x \in \Omega_r$ and $r \leq \frac{1}{\Lambda}$.

We shall establish Theorem 2.1 by an iteration method which is based on Lemmas 2.2 and 2.3 below. For convenience, we define

$$\gamma(r) = \frac{1}{r} \sup_{y \in \partial \Omega, |y| \leq r} |y \cdot e_n| \text{ for } 0 < r \leq 1.$$ 

Obviously,

$$\gamma(r) \leq \omega(r), \quad \|b\|_{L^n(\Omega_1)} \leq 2\omega(r) \text{ for } 0 < r \leq 1.$$ 

**Lemma 2.2.** Suppose that $0 \in \partial \Omega$, $u \in W(\Omega_1)$, $u|_{\partial \Omega \cap B_1} = 0$, $Lu = f$ in $\Omega_1$, $f \in L^n(\Omega_1)$, $\gamma(1) \leq \delta/6$ and $\|b\|_{L^n(\Omega_1)} \leq \min\{\epsilon_0, 1\}$, where $\epsilon_0$ is the constant in Theorem 1.2 and $\delta(<1)$ will be chosen in (2.3). Then there exist positive constants $\mu < 1$, $M$, $A_1$ and $A_2$ depending only on $\lambda$ and $n$. If

$$k x_n - l \leq u(x) \leq K x_n + B \quad \text{in } \Omega_1,$$

(2.3)

for some constants $l \geq 0$, $B(\geq 0)$, $k$ and $K$ with $k \leq K$, then there exist constants $\tilde{k}$ and $\tilde{K}$ such that

$$\tilde{k} x_n - A_1 \|f\|_{L^n(\Omega_1)} - A_2 (|K| + |k| + l) (\gamma(1) + \|b\|_{L^n(\Omega_1)}) \leq u(x) \leq \tilde{K} x_n + A_1 \|f\|_{L^n(\Omega_1)} + A_2 (|K| + |k| + B) (\gamma(1) + \|b\|_{L^n(\Omega_1)})$$

(2.4)

in $\Omega_3$, where either

$$\tilde{k} = k - 3M \sqrt{n}l + \mu(K - k) \quad \text{and} \quad \tilde{K} = K + 3M \sqrt{n}B, \quad (2.5)$$

or

$$\tilde{k} = k - 3M \sqrt{n}l \quad \text{and} \quad \tilde{K} = K + 3M \sqrt{n}B - \mu(K - k).$$

(2.6)

Obviously, we have $\tilde{k} \leq \tilde{K}$.

**Proof of Lemma 2.2.** First we proof the following.

**Claim.** There exist positive constants $M$, $\hat{\delta}$ and $C_1$ depending only on $\lambda$ and $n$, such that

$$(k - 3M \sqrt{n}) x_n - C_1 \|f\|_{L^n(\Omega_1)} - 3M \sqrt{n} (|k| + l) \gamma(1) - C_1 (|k| + l) \|b\|_{L^n(\Omega_1)}$$

$$\leq u(x) \leq (K + 3M \sqrt{n}B) x_n + C_1 \|f\|_{L^n(\Omega_1)} + 3M \sqrt{n} (|K| + B) \gamma(1)$$

$$+ C_1 (|K| + B) \|b\|_{L^n(\Omega_1)} \quad \text{in } \Omega \cap Q_{\hat{\delta}}.$$ 

Proof. Let $M = 1 + \frac{2\sqrt{n} - 1}{\lambda}$ (\geq 3) and $\epsilon (> 0)$ be small enough, such that

$$3 - (1 + \epsilon)(2 + \epsilon)(M - 1)^* \geq 0.$$ 

(2.7)

Let

$$\hat{\delta} = \frac{1}{M \sqrt{n}} \left( \leq \frac{1}{3 \sqrt{n}} \right), \quad \hat{\delta} = \frac{\delta}{2M} = \frac{1}{2\sqrt{n}(1 + \frac{2\sqrt{n} - 1}{\lambda})^2}$$

(2.8)
and
\[ \tilde{\psi}(x) = \frac{4}{3} \left( 2\left( \frac{x_n + \gamma(1)}{\delta} \right) - \frac{(x_n + \gamma(1))^2}{\delta^2} \right) + \frac{\lambda^2}{2(n-1)} \sum_{i=1}^{n-1} \left( \frac{|x_i|}{\delta} - 1 \right)^2 \].

The barrier function \( \tilde{\psi}(x) \) is \( C^2 \) and satisfies the following conditions (observe that \( 1 \leq \frac{\delta + \gamma(1)}{\delta} \leq 3/2 \)):

- \( \tilde{\psi}(x) \geq 1 \) on \( \overline{Q}_{1/\sqrt{n}} \cap \{ x \in \mathbb{R}^n : x_n = \tilde{\delta} \} \);
- \( \tilde{\psi}(x) \geq 0 \) on \( \overline{Q}_{1/\sqrt{n}} \cap \{ x \in \mathbb{R}^n : x_n = -\gamma(1) \} \);
- \( \tilde{\psi}(x) \geq 1 \) on \( \partial Q_{1/\sqrt{n}} \cap \{ x \in \mathbb{R}^n : -\gamma(1) < x_n < \tilde{\delta} \} \);
- \( -a_{ij}(x)D_{ij}\tilde{\psi}(x) \geq 0 \) a.e. in \( Q_{1/\sqrt{n}} \cap \{ x \in \mathbb{R}^n : -\gamma(1) < x_n < \tilde{\delta} \} \cap \Omega \);
- \( \tilde{\psi}(x) \leq \frac{3(x_n + \gamma(1))}{\delta} \) in \( Q_{\frac{\delta}{3}} \cap \{ x : x_n \geq -\gamma(1) \} \).

Combining (??) and (2.4), we have
\[
L(\kappa x_n - l\tilde{\psi}(x) - u(x)) \leq b_i(x)D_i(\kappa x_n - l\tilde{\psi}(x)) - f(x) \quad \text{in } \tilde{\Omega} \cap \Omega;
\]
\[
k\kappa x_n - l\tilde{\psi}(x) - u(x) \leq |k|\gamma(1) \quad \text{on } \partial(\tilde{\Omega} \cap \Omega);
\]
where \( \tilde{\Omega} = Q_{1/\sqrt{n}} \cap \{ x \in \mathbb{R}^n : -\gamma(1) < x_n < \tilde{\delta} \} \).

By the Alexandroff-Bakelman-Pucci maximum principle,
\[
k\kappa x_n - l\tilde{\psi}(x) - u(x) \leq |k|\gamma(1) + C_1(|k| + l)\|b\|_{L^\infty(\Omega_1)} + C_1\|f\|_{L^\infty(\Omega_1)}
\]
in \( \tilde{\Omega} \cap \Omega \), where \( C_1 \) is a constant depending only on \( \lambda \) and \( n \).

By (2.4) (fifth inequality), we have
\[
u(x) \geq \left( k - 3M\sqrt{n}\right)x_n - C_1\|f\|_{L^\infty(\Omega_1)} - 3M\sqrt{n}(|k| + l)\gamma(1)
\]
\[
- C_1(|k| + l)\|b\|_{L^\infty(\Omega_1)}
\]
in \( \Omega \cap Q_{\tilde{\delta}} \). As in (2.5), we have
\[
L\left( u(x) - Kx_n - B\tilde{\psi}(x) \right) \leq f(x) - b_i(x)D_i(Kx_n + B\tilde{\psi}(x)) \quad \text{in } \tilde{\Omega} \cap \Omega;
\]
\[
u(x) - Kx_n - B\tilde{\psi}(x) \leq |K|\gamma(1) \quad \text{on } \partial(\tilde{\Omega} \cap \Omega).
\]

According to the Alexandroff-Bakelman-Pucci maximum principle,
\[
u(x) - Kx_n - B\tilde{\psi}(x) \leq |K|\gamma(1) + C_1\|f\|_{L^\infty(\Omega_1)} + C_1(|K| + B)\|b\|_{L^\infty(\Omega_1)}
\]
in \( \tilde{\Omega} \cap \Omega \), where \( C_1 \) is a constant depending only on \( \lambda \) and \( n \). By (2.4) (fifth inequality), we have
\[
u(x) \leq \left( K + 3M\sqrt{n}B \right)x_n + C_1\|f\|_{L^\infty(\Omega_1)} + 3M\sqrt{n}(|K| + B)\gamma(1)
\]
\[
+ C_1(|K| + B)\|b\|_{L^\infty(\Omega_1)}
\]
in \( \Omega \cap Q_{\tilde{\delta}} \). By (2.7) and (2.8), the claim follows. \( \square \)

Let \( \Gamma = \overline{Q}_{M\delta} \cap \{ x \in \mathbb{R}^n : x_n = \delta \} \). By \( \gamma(1) \leq \delta/6 \), we have
\[
\Gamma \subset \Omega \quad \text{and} \quad \text{dist}(\Gamma, \partial \Omega) \geq \frac{\delta}{2}.
\]

Next, we show (??) for the two cases: \( u(\delta e_n) \geq \frac{1}{2}(K + k)\delta \) and \( u(\delta e_n) < \frac{1}{2}(K + k)\delta \), corresponding to (??) and (??).
Case 1: \( u(\delta \varepsilon_n) \geq \frac{1}{2}(K+k)\delta \). Let
\[
v(x) = u(x) - (k - 3M \sqrt{n}) x_n + C_1 \| f \|_{L^\infty(\Omega_1)} + 3M \sqrt{n} |k| + l) \gamma(1) + C_1(|k| + l) \| b \|_{L^\infty(\Omega_1)}.
\]

Then
\[
v(\delta \varepsilon_n) \geq \left( \frac{K - k}{2} + 3M \sqrt{n} |l| \right) \delta + C_1 \| f \|_{L^\infty(\Omega_1)} + 3M \sqrt{n} |k| + l) \gamma(1) + C_1(|k| + l) \| b \|_{L^\infty(\Omega_1)}.
\] (2.15)

Since \( v(x) \geq 0 \) for \( x \in \Omega \cap Q_{\delta} \), from (2.9) and the interior Harnack inequality, it follows that
\[
\sup_{\Gamma} v(x) \leq C_2 \left( \inf_{\Gamma} v(x) + \| f \|_{L^\infty(\Omega_1)} + (|k| + l) \| b \|_{L^\infty(\Omega_1)} \right),
\] (2.16)

where \( C_2 (\geq 1) \) is a constant depending only on \( \lambda \) and \( n \). Combining (2.10), (2.11) and \( v(x) \geq 0 \), we have
\[
\inf_{\Gamma} v(x) \geq \left\{ \frac{1}{C_2} \left( \frac{K - k}{2} + 3M \sqrt{n} |l| \right) \delta + C_1 \| f \|_{L^\infty(\Omega_1)} + 3M \sqrt{n} |k| + l) \gamma(1) + \right\}^+: = a.
\]

Let
\[
\psi(x) = \frac{3}{8} \left( \left( \frac{x_n + \gamma(1)}{\delta} \right) + \left( \frac{x_n + \gamma(1)}{\delta} \right)^2 \right) - \frac{\lambda^2}{4(n-1)} \sum_{i=1}^{n-1} \left( \frac{|x_i|}{\delta} - 1 \right)^{2+\epsilon},
\] (2.17)

where \( \epsilon \) satisfies (2.2).

The barrier function \( \psi(x) \) is \( C^2 \) and satisfies the following conditions (observe that \( 1 \leq \frac{\delta + \gamma(1)}{\delta} \leq 7/6 \):
\[
\begin{align*}
\psi(x) & \leq 1 \quad \text{on } Q_{M\delta} \cap \{ x \in \mathbb{R}^n : x_n = \delta \}; \\
\psi(x) & \leq 0 \quad \text{on } Q_{M\delta} \cap \{ x \in \mathbb{R}^n : x_n = -\gamma(1) \}; \\
\psi(x) & \leq 0 \quad \text{on } \partial Q_{M\delta} \cap \{ x \in \mathbb{R}^n : -\gamma(1) \leq x_n \leq \delta \}; \\
- a_{ij}(x) D_{ij} \psi(x) & \leq 0 \quad \text{a.e. in } Q_{M\delta} \cap \{ x \in \mathbb{R}^n : -\gamma(1) < x_n < \delta \} \cap \Omega; \\
\psi(x) & \geq \frac{x_n + \gamma(1)}{3\delta} \quad \text{in } Q_{\delta} \cap \{ x : x_n \geq -\gamma(1) \}; \\
\psi(x) & \leq \frac{x_n + \gamma(1)}{\delta} \quad \text{in } Q_{M\delta} \cap \{ x \in \mathbb{R}^n : -\gamma(1) \leq x_n \leq \delta \}.
\end{align*}
\] (2.18)

We claim that
\[
L(a \psi(x) - v(x)) \leq b_i(x) D_i (a \psi(x) + (k - 3M \sqrt{n}) x_n) - f \quad \text{in } \tilde{Q} \cap \Omega;
\]
\[
a \psi(x) - v(x) \leq \frac{2 + 9M \sqrt{n}}{C_2} (|K| + |k| + l) \gamma(1) \quad \text{on } \partial(\tilde{Q} \cap \Omega);
\] (2.19)

where \( \tilde{Q} = Q_{M\delta} \cap \{ x \in \mathbb{R}^n : -\gamma(1) < x_n < \delta \} \).

In fact, the first inequality is clear. For the second inequality, we separate the boundary \( \partial(\tilde{Q} \cap \Omega) \) into three parts:
\[
\partial \tilde{Q} \cap \{ x \in \mathbb{R}^n : x_n = \delta \}, \quad \partial \tilde{Q} \cap \{ x \in \mathbb{R}^n : -\delta < x_n < \delta \} \cap \overline{\Omega}, \quad \partial \Omega \cap \tilde{Q}.
\]
The first part is just $\Gamma$ where $v(x) \geq a$ and $\psi(x) \leq 1$, then $a\psi(x) - v(x) \leq 0$ on it. On the second part, since $v(x) \geq 0$ and $\psi(x) \leq 0$, we have $a\psi(x) - v(x) \leq 0$ on them. On the last part, since $\psi(x) \leq \frac{x_n + \gamma(1)}{\delta} \leq 1$ on it by 2.13(6), we have

$$a\psi(x) - v(x) \leq \frac{1}{C_2} \left( \frac{K - k}{2} + 3M\sqrt{n}\delta + 3M\sqrt{n}(|k| + l)\gamma(1) \right) x_n + \frac{\gamma(1)}{\delta}$$

$$\leq \frac{1}{C_2} \left( \frac{|K| + |k|}{2} + 3M\sqrt{n}(x_n + \gamma(1)) + 3M\sqrt{n}(|k| + l)\gamma(1) \right)$$

$$\leq \frac{2 + 9M\sqrt{n}}{C_2}(|K| + |k| + l)\gamma(1),$$

where we have used $-\gamma(1) \leq x_n \leq \gamma(1)$ for $x \in \partial \Omega \cap \tilde{Q}$. By the Alexandroff-Bakelman-Pucci maximum principle,

$$a\psi(x) - v(x) \leq C_3(|K| + |k| + l)(\gamma(1) + \|b\|_{L^n(\Omega_1)}) + C_3\|f\|_{L^n(\Omega_1)} \quad \text{in} \quad \tilde{Q} \cap \Omega,$$

where we have used $\|b\|_{L^n(\Omega_1)} \leq 1$ and $C_3$ is a constant depending only on $\lambda$ and $n$.

From 2.13 (fifth inequality), it follows that for all $x \in \Omega \cap Q_\delta$,

$$a\psi(x) \geq \frac{a}{3\delta}(x_n + \gamma(1))$$

$$\geq \frac{(K - k)\delta}{2C_2} \|f\|_{L^n(\Omega_1)} - (|k| + l)\|b\|_{L^n(\Omega_1)} \left( x_n + \gamma(1) \right)$$

$$\geq \frac{K - k}{6C_2} x_n - \|f\|_{L^n(\Omega_1)} - (|k| + l)\|b\|_{L^n(\Omega_1)},$$

where we have used $K - k \geq 0$.

Therefore, for all $x \in \Omega_\delta$,

$$u(x) \geq a\psi(x) + (k - 3M\sqrt{n})x_n - (C_1 + C_3)\|f\|_{L^n(\Omega_1)}$$

$$- (C_3 + 3M\sqrt{n} + C_1)(|K| + |k| + l)(\gamma(1) + \|b\|_{L^n(\Omega_1)})$$

$$\geq \left( k - 3M\sqrt{n} + \frac{1}{6C_2}(K - k) \right) x_n - (C_1 + C_3 + 1)\|f\|_{L^n(\Omega_1)}$$

$$- (C_3 + 3M\sqrt{n} + C_1 + 1)(|K| + |k| + l)(\gamma(1) + \|b\|_{L^n(\Omega_1)}).$$

Let

$$\mu = \frac{1}{6C_2}, \quad A_1 = C_1 + C_3 + 1, \quad A_2 = C_1 + C_3 + 3M\sqrt{n} + 1.$$  \hfill (2.21)

Combining (2.8), (2.13) and (2.16), we have (2?) and (2?).

**Case 2:** $u(\delta e_n) < \frac{1}{2}(K + k)\delta$. The proof is similar to that of Case 1. Let

$$v(x) = (K + 3M\sqrt{n}B)x_n + C_1\|f\|_{L^n(\Omega_1)} + 3M\sqrt{n}(|K| + B)\gamma(1)$$

$$+ C_1(|K| + B)\|b\|_{L^n(\Omega_1)} - u(x)$$

for $x \in \Omega \cap Q_\delta$. Then

$$v(\delta e_n) > \left( \frac{K - k}{2} + 3M\sqrt{n}B \right) \delta + C_1\|f\|_{L^n(\Omega_1)} + 3M\sqrt{n}(|K| + B)\gamma(1)$$

$$+ C_1(|K| + B)\|b\|_{L^n(\Omega_1)}.$$
By the interior Harnack inequality, we have
\[ \sup_v v \leq C_2 \left( \inf_v v + \|f\|_{L^n(\Omega_1)} + (|K| + B)\|b\|_{L^n(\Omega_1)} \right), \]
where \( C_2 (\geq 1) \) is a constant depending only on \( \lambda \) and \( n \). Then
\[ \inf_v v \geq \left\{ \frac{1}{C_2} \left( \frac{K - k}{2} + 3M\sqrt{n}B\delta + 3M\sqrt{n}(|K| + B)\gamma(1) \right) \\
+ \frac{C_1}{C_2} \frac{1}{\delta} \left( \|f\|_{L^n(\Omega_1)} + (|K| + B)\|b\|_{L^n(\Omega_1)} \right) \right\}^+ \]
(2.22)
Let \( \psi(x) \) be defined by (2.11). As in (2.14), we have
\[ L(a\psi(x) - v(x)) \leq b_i D_i (a\psi(x)) - (|K| + 3M\sqrt{n}B)x_n + f(x) \quad \text{in} \quad \tilde{Q} \cap \Omega; \]
(2.23)
where \( \tilde{Q} = Q_{M_\delta} \cap \{ x \in \mathbb{R}^n : -\gamma(1) < x_n < \delta \} \).

Therefore, by the Alexandrov-Bakelman-Pucci maximum principle,
\[ a\psi(x) - v(x) \leq C_3 (|K| + |k| + B)(\gamma(1) + \|b\|_{L^n(\Omega_1)}) + C_3 \|f\|_{L^n(\Omega_1)}, \]
(2.24)
in \( \tilde{Q} \cap \Omega \), where we have used \( \|b\|_{L^n(\Omega_1)} \leq 1 \), and \( C_3 \) is a constant depending only on \( \lambda \) and \( n \).

By (2.13) (fifth inequality), we have that for any \( x \in \Omega \cap Q_\delta \),
\[ \frac{a}{\delta^3} (x_n + \gamma(1)) - v(x) \leq C_3 (|K| + |k| + B)(\gamma(1) + \|b\|_{L^n(\Omega_1)}) + C_3 \|f\|_{L^n(\Omega_1)}. \]

Combining (2.17) with (2.19), we have that for all \( x \in \Omega_\delta \),
\[ u(x) \leq (K + 3M\sqrt{n}B)x_n - \frac{a}{\delta^3} (x_n + \gamma(1)) + (C_1 + C_3) \|f\|_{L^n(\Omega_1)} \\
+ (C_1 + C_3 + 3M\sqrt{n})(|K| + |k| + B)(\gamma(1) + \|b\|_{L^n(\Omega_1)}) \]
\[ \leq (K + 3M\sqrt{n}B - \frac{1}{6C_2^2} (K - k))x_n + (C_1 + C_3 + 1) \|f\|_{L^n(\Omega_1)} \\
+ (C_1 + C_3 + 3M\sqrt{n} + 1)(|K| + |k| + B)(\gamma(1) + \|b\|_{L^n(\Omega_1)}). \]
(2.25)

Let \( \mu = \frac{1}{6C_2^2} \): \( A_1 = C_1 + C_3 + 1 \) and \( A_2 = C_1 + C_3 + 3M\sqrt{n} + 1 \). Combining (2.7) and (2.20), we have that (??) and (??) hold.

Using induction, the following lemma is a direct consequence of Lemma 2.2.

**Lemma 2.3.** Suppose that \( 0 \in \partial \Omega, u \in W(\Omega_1), u|_{\partial \Omega \cap B_1} = 0, Lu = f \) in \( \Omega_1 \), \( \|u\|_{L^\infty(\Omega_1)} \leq 1, f \in L^n(\Omega_1) \) and \( \omega(1) \leq \min\{ \epsilon_0/2, 1/2, \delta/6 \} \). Then there exist nonnegative sequences \( \{l_m\}_{m=0}^\infty \), \( \{B_m\}_{m=0}^\infty \) and sequences \( \{k_m\}_{m=0}^\infty \), \( \{K_m\}_{m=0}^\infty \) with \( k_0 = K_0 = 0 \), \( l_0 = B_0 = 1 \), and for \( m = 0, 1, 2, \ldots \),
\[ l_{m+1} = A_1 \delta^m \|f\|_{L^n(\Omega_{l_m})} + A_2 \delta^m (|K_m| + |k_m| + \frac{l_m}{\delta^m} \gamma(\delta^m) + \|b\|_{L^n(\Omega_{l_m})}); \]
\[ B_{m+1} = A_1 \delta^m \|f\|_{L^n(\Omega_{l_m})} + A_2 \delta^m (|K_m| + |k_m| + \frac{B_m}{\delta^m} \gamma(\delta^m) + \|b\|_{L^n(\Omega_{l_m})}); \]
and
\[ k_{m+1} = k_m - 3M\sqrt{n} \frac{l_m}{\delta^m} + \mu (K_m - k_m) \quad \text{and} \quad K_{m+1} = K_m + 3M\sqrt{n} \frac{B_m}{\delta^m}. \]
\[ k_{m+1} = k_m - 3M \sqrt{n} \frac{l_m}{\delta_m} \quad \text{and} \quad K_{m+1} = K_m + 3M \sqrt{n} \frac{B_m}{\delta_m} - \mu(K_m - k_m), \]

such that

\[ k_m x_n - l_m \leq u(x) \leq K_m x_n + B_m \quad \text{in} \quad \Omega_{\delta_m}, \quad \text{(2.26)} \]

where \( \delta, \mu, M, A_1 \) and \( A_2 \) are positive constants given by Lemma 2.3.

**Proof of Theorem 2.1.** Let \( \{l_m\}_{m=0}^{\infty}, \{B_m\}_{m=0}^{\infty}, \{k_m\}_{m=0}^{\infty} \) and \( \{K_m\}_{m=0}^{\infty} \) be defined by Lemma 2.3. We prove the following claim first.

**Claim.** There exists a constant \( C_1 \) depending only \( \lambda \) and \( n \) such that for all \( m = 0, 1, 2, \ldots, \)

\[ |K_m|, |k_m|, \frac{B_m}{\delta_m}, \frac{l_m}{\delta_m} \leq C_1. \quad \text{(2.27)} \]

**Proof.** Firstly, notice that we take \( K_0 = k_0 = 0 \) and \( l_0 = B_0 = 1 \), then by induction, we have \( K_m \geq k_m \) for all \( m \geq 0 \). For \( m \geq 0 \), we define \( S_m = \sum_{i=0}^{m} \left( \frac{B_i}{\delta_i} + \frac{l_i}{\delta_i} \right) \). For any \( m \geq 0 \), since

\[ K_{m+1} \leq K_m + 3M \sqrt{n} \frac{B_m}{\delta_m} \quad \text{and} \quad K_0 = 0, \]

we have

\[ K_{m+1} \leq 3M \sqrt{n} S_m \quad \text{for} \quad m \geq 0. \]

Similarly, we have

\[ k_{m+1} \geq -3M \sqrt{n} S_m \quad \text{for} \quad m \geq 0. \]

It follows that

\[ |k_{m+1}| + |K_{m+1}| \leq 6M \sqrt{n} S_m \quad \text{for} \quad m \geq 0. \quad \text{(2.28)} \]

Since

\[
\frac{B_{m+1} + l_{m+1}}{\delta_{m+1}} = \frac{A_2}{\delta} \left( \gamma(\delta^m) + \|b\|_{L^n(\Omega_{\delta^m})}(2|K_m| + 2|k_m| + \frac{B_m + l_m}{\delta_m}) \right) + \frac{2A_1}{\delta} \|f\|_{L^n(\Omega_{\delta^m})},
\]

for \( m \geq 1 \), combining the above identity with (2.23), we obtain

\[
\frac{B_{m+1} + l_{m+1}}{\delta_{m+1}} \leq \frac{A_2}{\delta} \left( \gamma(\delta^m) + \|b\|_{L^n(\Omega_{\delta^m})} \left( 12M \sqrt{n} S_{m-1} + \frac{B_m + l_m}{\delta_m} \right) \right) + \frac{2A_1}{\delta} \|f\|_{L^n(\Omega_{\delta^m})},
\]

\[ \leq \frac{12M \sqrt{n} A_2}{\delta} \left( \gamma(\delta^m) + \|b\|_{L^n(\Omega_{\delta^m})} S_m \right) + \frac{2A_1}{\delta} \|f\|_{L^n(\Omega_{\delta^m})}. \quad \text{(2.29)} \]

By the normalized condition, we have

\[
\sum_{i=1}^{\infty} \frac{12M \sqrt{n} A_2}{\delta} \gamma(\delta^i) + \|b\|_{L^n(\Omega_{\delta^i})} \leq \sum_{i=1}^{\infty} \frac{36M \sqrt{n} A_2}{\delta} \omega(\delta^i)
\]

\[ \leq \frac{36M \sqrt{n} A_2}{\delta \ln \frac{1}{\delta}} \frac{1}{r} \int_0^1 \omega(r) \frac{dr}{r} \leq \frac{1}{2}, \quad \text{(2.30)} \]

and

\[
\frac{2A_1}{\delta} \sum_{i=1}^{\infty} \|f\|_{L^n(\Omega_{\delta^i})} \leq \frac{2A_1}{\delta \ln \frac{1}{\delta}} \frac{1}{r} \int_0^1 \frac{dr}{r} \leq \frac{2A_1}{\delta \ln \frac{1}{\delta}}, \quad \text{(2.31)}
\]
From (2.24)-(2.26), it follows that for any \( m \geq 1 \),
\[
S_{m+1} - S_1 = \sum_{i=1}^{m} \frac{B_{i+1} + l_{i+1}}{\delta^{i+1}} \\
\leq S_{m+1} \sum_{i=1}^{m} \frac{12M \sqrt{n} A_2}{\delta} (\gamma(\delta^i) + \| b \|_{L^\infty(\Omega_{\delta_i})}) + \frac{2A_1}{\delta} \sum_{i=1}^{m} \| f \|_{L^\infty(\Omega_{\delta_i})} \\
\leq \frac{1}{2} S_{m+1} + \frac{2A_1}{\delta \ln \frac{\delta}{2}}.
\]

Therefore, for all \( m \geq 1 \),
\[
S_{m+1} \leq \frac{4A_1}{\delta \ln(1/\delta)} + 2S_1.
\]

Since \( S_0 = 2 \), \( 0 \leq S_1 \leq A_1 + A_2 \), we have
\[
0 \leq S_m \leq 2A_1 + 2A_2 + 2 + \frac{4A_1}{\delta \ln \frac{\delta}{2}} \quad \text{for all } m \geq 0.
\]

Let \( C_1 = 3M \sqrt{n}(2A_1 + 2A_2 + 2 + \frac{4A_1}{\delta \ln \frac{\delta}{2}}) \). This completes the proof of the claim. \( \square \)

Next we show estimate (2.32). By Lemma 2.3, we have that for all \( m \geq 1 \),
\[
0 \leq K_{m+1} - k_{m+1} \leq (1 - \mu)(K_m - k_m) + 3M \sqrt{n} \frac{l_m}{\delta^m} + B_m \delta^m
\]

or
\[
|K_{m+1} - k_{m+1}| \leq (1 - \mu)|K_m - k_m| + C_2(\| f \|_{L^\infty(\Omega_{\delta^m})} + \omega(\delta^{m-1})),
\]
where \( C_2 = (3M \sqrt{n}(A_1 + 6A_2 C_1))/\delta \).

Let \( 1 - \mu = \delta^{\hat{\alpha}}(\hat{\alpha} > 0) \). By iteration, we have that for all \( m \geq 1 \),
\[
|K_{m+1} - k_{m+1}| \leq C_3 \delta^{\hat{\alpha}m} \left( 1 + \int_{\delta}^{1} \frac{\omega(r) + \| f \|_{L^\infty(\Omega_{r})}}{r^{1+\hat{\alpha}}} dr \right),
\]
where \( C_3 \) is a constant depending only on \( \lambda \) and \( n \).

For any \( m \geq 1 \),
\[
K_{m+1} + k_{m+1} \leq K_m + k_m + \mu(K_m - k_m) + 3M \sqrt{n} \frac{l_m}{\delta^m},
\]
\[
K_{m+1} + k_{m+1} \geq K_m + k_m - \mu(K_m - k_m) - 3M \sqrt{n} \frac{l_m}{\delta^m}.
\]

Hence,
\[
|(K_{m+1} + k_{m+1}) - (K_m + k_m)| \\
\leq \mu|K_m - k_m| + 3M \sqrt{n} \frac{l_m + B_m}{\delta^m}
\]

(2.32)
where \( C_4 \) is a constant depending only on \( \lambda \) and \( n \). It follows that

\[
\sum_{j=m}^{\infty} |(K_{j+1} + k_{j+1}) - (K_j + k_j)| 
\leq C_3 \mu \sum_{j=m}^{\infty} (\delta^{j-1})^\hat{\alpha} \left( 1 + \int_{\delta_{j-1}}^{1} \frac{\omega(r) + \|f\|_{L^n(\Omega_r)}}{r^{1+\hat{\alpha}}} dr \right) 
+ C_4 \sum_{j=m}^{\infty} (\omega(\delta^{j-1}) + \|f\|_{L^n(\Omega_{\delta_{j-1}})}),
\]

Let

\[ F_r := \int_r^1 \omega(s) \frac{\|f\|_{L^n(\Omega_s)}}{s^{1+\hat{\alpha}}} ds. \]

By

\[
\sum_{j=m}^{\infty} (\delta^{j-1})^\hat{\alpha} \int_{\delta_{j-1}}^{1} \frac{\omega(r) + \|f\|_{L^n(\Omega_r)}}{r^{1+\hat{\alpha}}} dr 
= \sum_{j=m-1}^{\infty} (\delta^{j})^\hat{\alpha} F_{\delta^j} 
\leq \frac{1}{\delta^\hat{\alpha}(1-\delta)} \sum_{j=m-1}^{\infty} (\delta^{j+1})^\hat{\alpha} F_{\delta^j} \cdot \frac{\delta^j - \delta^{j+1}}{\delta^j} 
\leq \frac{1}{\delta^\hat{\alpha}(1-\delta)} \sum_{j=m-1}^{\infty} \int_{\delta^j}^{\delta^{j+1}} r^{\hat{\alpha}-1} F_r dr 
= \frac{1}{\delta^\hat{\alpha}(1-\delta)} \int_0^{\delta^{m-1}} r^{\hat{\alpha}-1} F_r dr 
= \frac{1}{\delta^\hat{\alpha}(1-\delta)^\hat{\alpha}} \left( \int_0^{\delta^{m-1}} \frac{\omega(r) + \|f\|_{L^n(\Omega_r)}}{r^{1+\hat{\alpha}}} dr \right) 
+ (\delta^{m-1})^\hat{\alpha} \int_{\delta^{m-1}}^1 \frac{\omega(r) + \|f\|_{L^n(\Omega_r)}}{r^{1+\hat{\alpha}}} dr \]

and

\[
\sum_{j=m}^{\infty} (\omega(\delta^{j-1}) + \|f\|_{L^n(\Omega_{\delta_{j-1}})}) \leq \frac{1}{1-\delta} \int_0^{\delta^{m-2}} \frac{\omega(r) + \|f\|_{L^n(\Omega_r)}}{r^{1+\hat{\alpha}}} dr,
\]

it follows that

\[
\sum_{j=m}^{\infty} |(K_{j+1} + k_{j+1}) - (K_j + k_j)| 
\leq C_5 \left\{ (\delta^{m-1})^\hat{\alpha} + (\delta^{m-1})^\hat{\alpha} \int_{\delta^{m-1}}^{1} \frac{\omega(r) + \|f\|_{L^n(\Omega_r)}}{r^{1+\hat{\alpha}}} dr \right. 
\left. + \int_0^{\delta^{m-2}} \frac{\omega(r) + \|f\|_{L^n(\Omega_r)}}{r} dr \right\},
\]

where \( C_5 \) is a constant depending only on \( \lambda \) and \( n \).
While \( m \to \infty \), by \( \lim_{m \to 0^+} \omega(r) = 0 \) and L’Hospital rule, we have the righthand side of (2.29) tends to 0. Hence \( \{K_m + k_m\}_{m=0}^\infty \) is convergent. Let \( \lim_{m \to \infty} \frac{K_m + k_m}{2} = \theta \). Then for all \( m \geq 2 \),

\[
\left| \theta - \frac{K_m + k_m}{2} \right| \leq \sum_{j=m}^{\infty} \left| \frac{K_m + k_m}{2} - \frac{K_j + k_j}{2} \right|
\]

\[
\leq C_5 \left\{ \left( \delta^{m-1} \right)^{\hat{\alpha}} + \left( \delta^{m-1} \right)^{\hat{\alpha}} \int_{\delta^{m-1}}^{1} \frac{\omega(r) + \|f\|_{L^\infty(\Omega_r)}}{r^{1+\hat{\alpha}}} \, dr \right\}
\]

(2.35)

For any \( m \geq 0 \) and any \( x \in \Omega_{\delta^m} \), we have

\[
|u(x) - \theta x_n| \leq |u(x) - \frac{K_m + k_m}{2} x_n| + \left| \frac{(K_m + k_m) - \theta}{2} x_n \right|.
\]

(2.36)

From (2.21), it follows that

\[
-\frac{|K_m - k_m|}{m} |x_n| - l_m \leq u(x) - \frac{K_m + k_m}{2} x_n \leq \frac{|K_m - k_m|}{m} |x_n| + B_m.
\]

Then for any \( m \geq 0 \) and any \( x \in \Omega_{\delta^m} \),

\[
|u(x) - \frac{K_m + k_m}{2} x_n| \leq (|K_m - k_m| + \frac{l_m + B_m}{\delta^m}) \delta^m.
\]

(2.37)

By (2.30) and the inequality above, for all \( x \in \Omega_{\delta^m}, \ m = 2, 3, \ldots \),

\[
|u(x) - \theta x_n| 
\leq |u(x) - \frac{K_m + k_m}{2} x_n| + \left| \frac{(K_m + k_m) - \theta}{2} x_n \right|
\]

\[
\leq \left( |K_m - k_m| + \frac{B_m + l_m}{\delta^m} + \left| \frac{K_m + k_m}{2} - \theta \right| \right) \delta^m
\]

(2.38)

\[
\leq C_6 \left\{ \left( \delta^{m-1} \right)^{\hat{\alpha}} + \left( \delta^{m-1} \right)^{\hat{\alpha}} \int_{\delta^{m-1}}^{1} \frac{\omega(r) + \|f\|_{L^\infty(\Omega_r)}}{r^{1+\hat{\alpha}}} \, dr \right\} + \omega(\delta^{m-1}) + \|f\|_{L^\infty(\Omega_{\delta^{m-1}})} + \int_{0}^{\delta^{m-1}} \frac{\omega(r) + \|f\|_{L^\infty(\Omega_r)}}{r} \, dr
\]

\[
\delta^m,
\]

where \( C_6 \) is a constant depending only on \( \lambda \) and \( n \).

Let \( \Lambda = 1/\delta^2 \) (\( \geq 324n \)). By (2.33), we have that for all \( x \in \Omega_r \) and \( r \leq 1/\Lambda \),

\[
|u(x) - \theta x_n| \leq C_7 \left\{ \left. r^{\hat{\alpha}} + \omega(A r) + r^{\hat{\alpha}} \int_{r}^{1} \frac{\omega(s)}{s^{1+\hat{\alpha}}} \, ds \right| + \left. \int_{0}^{s} \frac{\omega(s)}{s} \, ds \right| \right\} r.
\]

This completes the proof of Theorem 2.1. \( \square \)

\textbf{Proof of Theorem 1.5.} Consider \( |\nabla u(y) - \nabla u(z)| \), where \( y, z \in \partial \Omega \cap B_{r_0} \) and \( 0 < |y - z| = r \leq \frac{r_0}{2} \). By Corollary 1.7, we have

\[
\|u(x) - L_y(x)\|_{L^\infty(\Omega_r(y))} \leq C \left\{ \left. r^{\hat{\alpha}} + \omega(A r) + r^{\hat{\alpha}} \int_{r}^{r_0} \frac{\omega(s)}{s^{1+\hat{\alpha}}} \, ds \right| + \left. \int_{0}^{r_0} \frac{\omega(s)}{s} \, ds \right| \right\} r,
\]

\[
\|u(x) - L_z(x)\|_{L^\infty(\Omega_r(z))} \leq C \left\{ \left. r^{\hat{\alpha}} + \omega(A r) + r^{\hat{\alpha}} \int_{r}^{r_0} \frac{\omega(s)}{s^{1+\hat{\alpha}}} \, ds \right| + \left. \int_{0}^{r_0} \frac{\omega(s)}{s} \, ds \right| \right\} r.
\]
Hence, for $y, z \in B_{\eta r}(p)$ and a small positive constant $\eta(<\frac{1}{\lambda})$ such that $B_{\eta r}(p) \subset \Omega_r(y) \cap \Omega_r(z)$. Then by the triangle inequality, we have
\[||L_y(x) - L_z(x)||_{L^\infty(B_{\eta r}(p))} \leq 2C \left( r^{\hat{\alpha}} + \omega(\Lambda r) + r^\hat{\alpha} \int_r^{r_0} \omega(s) ds \right) r.\]

Since $L_y(x) - L_z(x)$ is an affine function, we obtain
\[|\nabla L_y(x) - \nabla L_z(x)| \leq \frac{1}{\eta^r} ||L_y(x) - L_z(x)||_{L^\infty(B_{\eta r}(p))}.\]

It follows that
\[|\nabla u(y) - \nabla u(z)| \leq \frac{2C}{\eta^r} \left( r^{\hat{\alpha}} + \omega(\Lambda r) + r^\hat{\alpha} \int_r^{r_0} \omega(s) ds \right).\]

Hence, for $y, z \in \partial \Omega \cap B_{\eta r}, 0 < |y - z| = r \leq \frac{\eta}{\lambda}$, we have
\[|\nabla u(y) - \nabla u(z)| \leq \frac{2C}{\eta^r} \left( r^{\hat{\alpha}} + \omega(\Lambda r) + r^\hat{\alpha} \int_r^{r_0} \omega(s) ds \right).\]

This completes the proof. \hfill \Box

**Acknowledgements.** The authors would like to thank the anonymous referee for the carefully reading and for the useful comments and suggestions. This work was supported by NSFC 11401460. This work was done while Y. Huang was visiting School of Mathematical Sciences, Peking University. He would like to thank Peking University for the hospitality.

**References**


**Yongpan Huang**

Department of Mathematics, Xi’an Polytechnic University, Xi’an 710048, China.

School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China

Email address: yongpanhuang@xjtu.edu.cn, huangyongpan@gmail.com

**Qiaozhu Zhai**

Systems Engineering Institute, Xi’an Jiaotong University, Xi’an 710049, China

Email address: qzzhai@sei.xjtu.edu.cn

**Shulin Zhou**

School of Mathematical Sciences, Peking University, Beijing, 100871, China

Email address: szhou@math.pku.edu.cn