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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NON-DEGENERATE KIRCHHOFF TYPE PROBLEM WITH NONLINEAR BOUNDARY CONDITION

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Abstract. We show the existence of solutions for nonlinear elliptic partial differential equations with Steklov nonlinear boundary conditions involving a Kirchhoff type operator. By using variational and topological methods, we prove the existence and multiplicity of solutions. The results obtained are new even for the standard stationary Kirchhoff equation with nonlinear boundary condition involving the p-Laplacian operator.

1. Introduction

In this paper, we study the existence of nontrivial weak solutions to a Kirchhoff type problem with Steklov nonlinear boundary conditions:

$$\begin{gathered} \mathbb{M}(u) = f(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \eta} = g(x,u) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where

$$\mathbb{M}(u) = -\Big[M\Big(\int_\Omega |\nabla u|^p dx + \int_\Omega c(x)|u|^p dx\Big)\Big][\operatorname{div}(|\nabla u|^{p-2}\nabla u) + c(x)|u|^{p-2}u],$$

and $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with class $C^{0,1}$-boundary, $\partial\Omega$. In the boundary conditions, the symbol $\frac{\partial u}{\partial \eta}$ denotes the directional derivative

$$\frac{\partial u}{\partial \eta} = (\nabla u, \eta)_{\mathbb{R}^N},$$

where η is the outward (unit) normal derivative on $\partial\Omega$ and $p > 1$.

The Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is assumed to be continuous and to satisfy the structural assumption

(A1) There exists $m_0 > 0$, such that $M(t) \geq m_0$ for all $t > 0$.

Problem (1.1) is said to be degenerate when $M(0) = 0$, and non-degenerate when $M(0) > 0$. In this paper we cover only the non-degenerate case. Throughout this paper we shall assume that the weight function $c : \Omega \to \mathbb{R}$, and the continuous functions $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

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(A2) $c \geq 0$ a.e. on Ω, $c \in L^\infty(\Omega)$ and $\int_\Omega c(x)dx > 0$.

(A3) There exist constants $b_1, b_2 > 0$ such that
$$|f(x,u)| \leq b_1 + b_2|u|^r \quad \text{for all } (x,u) \in \overline{\Omega} \times \mathbb{R},$$
with $0 < r < p_*(N) - 1$, and the critical exponent
$$p_*(N) = \begin{cases} \frac{Np}{N-p}, & \text{if } p < N, \\ +\infty, & \text{if } p \geq N. \end{cases}$$

(A4) There exists $b_3 > 0$ such that $b_3|u|^\alpha \leq f(x,u)$ for all $(x,u) \in \overline{\Omega} \times \mathbb{R}^+$, with $1 < \alpha + 1 < p$.

(A5) There exist positive constants a_1 and a_2 such that
$$|g(x,u)| \leq a_1 + a_2|u|^s \quad \text{for all } (x,u) \in \overline{\Omega} \times \mathbb{R},$$
with $0 < s < p_*^1(N) - 1$, and the critical exponent
$$p_*^1(N) = \begin{cases} \frac{(N-1)p}{N-p}, & \text{if } p < N, \\ +\infty, & \text{if } p \geq N. \end{cases}$$

Problem (1.1) has its origin in the canonical model of Kirchhoff and Carrier which describes small vibrations of an elastic stretched string. The interest in recent years has been focused on the study of Kirchhoff type problems
$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$
where L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. Equation (1.2) extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations.

We stress that because of the presence of the term
$$M\Big(\int_\Omega |\nabla u|^p dx + \int_\Omega c(x)|u|^p dx\Big),$$
the first equation of (1.1) is no longer a pointwise equation, therefore it is often called a nonlocal problem. This is a source of mathematical difficulties in the analysis of (1.1), which makes the study of such class of problem, particularly interesting.

In the previous years many authors studied the nonlocal problem
$$-M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) \text{i n} Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.3}$$
Problems of the type (1.3) may be used to model several physical and biological problems, see for example [2]. Many interesting results for problems of the Kirchhoff type have already been obtained, see [2, 7, 15], and the references therein. The study of Kirchhoff-type equations was extended to the case involving the p-Laplacian operator, see [9, 11, 16]. Systems of Kirchhoff-type equations where dealt for example in [8, 10].

In [24] the authors studied a problem involving a Kirchhoff-type operator and a nonlinear boundary condition. We also refer to [10], which involve a system of equations with Neumann boundary condition. Problems involving a nonlinear boundary condition were studied, for example, in [13, 23]. These works, however, do not deal with a Kirchhoff-type operator.

Let us consider μ_1 and λ_1 the first eigenvalues of the Steklov and Neumann eigenvalues problems (we refer the reader to [4, 14]). The authors studied the existence and multiplicity of solution to the elliptic equations, involving the p-Laplacian,

$$\begin{gathered} -\Delta_p u + c(x)|u|^{p-2}u = 0, \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \eta} = \mu |u|^{p-2}u, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

and

$$\begin{gathered} -\Delta_p u + c(x)|u|^{p-2}u = \lambda |u|^{p-2}u, \text{ in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

Mavinga and Nkashama [23] established a similar result for problem (1.1) with $M = 1$, $p = 2$, and the hypothesis:

(A6) there exist constants $\lambda, \mu \in \mathbb{R}$ such that

$$\limsup_{|u|\to+\infty} \frac{2F(x,u)}{|u|^2} \le \lambda < \lambda_1, \quad \limsup_{|u|\to+\infty} \frac{2G(x,u)}{|u|^2} \le \mu < \mu_1$$

uniformly for $x \in \overline{\Omega}$, with $\lambda_1\mu + \mu_1\lambda < \mu_1\lambda_1$. See Figure 1.

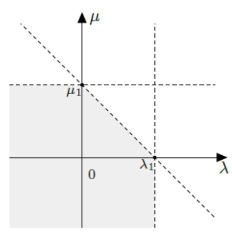

FIGURE 1. Region of solutions in the plane $\lambda\mu$ obtained in [23].

In [13] the authors improve some results of [23] for the p-Laplacian operator with the same relationship on the Steklov and Neumann eigenvalues. Motivated by the above papers, especially [4, 10, 11, 15, 16, 23, 24], we consider problem (1.1) which combines the Kirchhoff model. In this sense, one of the novelty of this paper is that the problem (1.1) involve a nondegenerate nonlocal term once in the majority of works in the literature the authors study the degenerate case. We also note that we have to deal with nonlinear terms at the border. This generates another difficulty for which we use the Theorem Trace and its properties of compactness. Moreover, our results are established when the nonlinearity interacts with Steklov and Neumann eigenvalues for p-Laplacian operator.

In this work we extend the results of [23] and [13] for Kirchhoff-type problems, when the nonlinearities f and g have a p-sublinear growth. Moreover, in our results, we obtain an independence between the eigenvalues λ_1 and μ_1, by eliminating the necessity of the condition $\lambda_1\mu + \mu_1\lambda < \mu_1\lambda_1$. Thus, we get two results that extend the area of solutions in the Cartesian plane $\lambda\mu$, as shown in Figures 2 and 3. Furthermore, it was demonstrated that our solutions are nontrivial. Moreover, supposing f and g odd functions and by using the Krasnoselskii's genus theory, we established two results of multiplicity of solution to problem (1.1).

The structure of this article is as follows. First, in Section 2, we present our main results. In Section 3, we introduce some basic preliminary results and lemmas. In Section 4, we prove that the Euler Lagrange functional satisfies the Palais-Smale condition. n the Section 5 we give the proofs of our main results. To the best of our knowledge, this is the first study of the existence of infinite many solutions to non-degenerate Kirchhoff type problems with nonlinear boundary conditions.

2. Existence results

Theorem 2.1. *Suppose that assumptions* (A1)–(A5) *hold, and suppose that* $s+1 < p$. *Let the potential* $F(x,u) = \int_0^u f(x,t)dt$ *be such that*

(A7) *there exists* $\lambda \in \mathbb{R}$ *such that*

$$\limsup_{|u|\to+\infty} \frac{pF(x,u)}{|u|^p} \leq \lambda < \lambda_1 m_0,$$

uniformly for $x \in \overline{\Omega}$. *Then* (1.1) *has at least one nontrivial solution.*

Theorem 2.2. *Suppose that assumptions* (A1)–(A5) *hold, and suppose that* $s+1 < p$. *Additionally suppose that*

(A8) $f(x,-t) = -f(x,t)$, *for all* $(x,t) \in \overline{\Omega} \times \mathbb{R}$;

(A9) $g(x,-t) = -g(x,t)$, *for all* $(x,t) \in \overline{\Omega} \times \mathbb{R}$,

and the potential $F(x,u) = \int_0^u f(x,t)dt$ *satisfies* (A7). *Then* (1.1) *has infinitely many solutions.*

The region of the plane $\lambda\mu$ obtained in the Theorems 2.1 and 2.2 is depicted in figure 2.

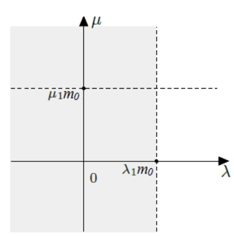

Figure 2. Region obtained in Theorems 2.1 and 2.2.

Theorem 2.3. *Suppose that assumptions* (A1)–(A5) *hold, and suppose that* $r+1 < p$. *Let the potential* $G(x,u) = \int_0^u g(x,t)dt$ *be such that*

(A10) *there exists* $\mu \in \mathbb{R}$ *such that*

$$\limsup_{|u|\to+\infty} \frac{pG(x,u)}{|u|^p} \leq \mu < \mu_1 m_0,$$

uniformly for $x \in \overline{\Omega}$.

Then (1.1) *has at least one nontrivial solution.*

Theorem 2.4. *Suppose that assumptions* (A1)–(A5), (A8)–(A10) *hold, and suppose that* $r+1 < p$. *Then* (1.1) *has infinitely many solutions.*

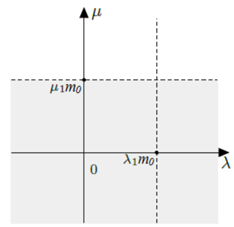

FIGURE 3. Region obtained in Theorems 2.3 and 2.4.

The region in the plane $\lambda\mu$ obtained in the Theorems 2.3 and 2.4 is depicted in Figure 3.

In Theorem 2.1 we prove the existence of at least one weak solution via minimization methods. With additional hypotheses of symmetry on the non-linearities, in Theorem 2.2, we obtain infinite weak solutions to (1.1) via Krasnoselskii's genus. The proof Theorem 2.3 is analogous to Theorem 2.1 with a different growth hypothesis (A10). The proof of Theorem 2.4 is analogous to Theorem 2.2.

3. Preliminary results and variational framework

Consider $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded domain with boundary, $\partial\Omega$, of class $C^{0,1}$ e $p > 1$. Let $W^{1,p}(\Omega)$ be the Sobolev space with respect to the norm

$$\|u\|_{1,p} = \Big(\int_\Omega [|\nabla u|^p + |u|^p] dx\Big)^{1/p},$$

for all $u \in W^{1,p}(\Omega)$. Since $c : \Omega \to \mathbb{R}$ satisfies (A2), we have

$$\|u\|_c = \Big(\int_\Omega [\ |\nabla u|^p + c(x)|u|^p] dx\Big)^{1/p}$$

is a norm in $\in W^{1,p}(\Omega)$ (see [18, Theorem 25]) and is equivalent to$\|\cdot\|_{1,p}$. In our work we consider $W^{1,p}(\Omega)$ with respect to the norm $\|\cdot\|_c$. As we can see in [22], the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous if $1 \leq p < N$ and $1 \leq q \leq \frac{Np}{N-p}$ or $p \geq N$ and $q \in [1, +\infty)$. Moreover, if $1 \leq p < N$ and $1 \leq q < \frac{Np}{N-p}$ or $p \geq N$ and $q \in [1, +\infty)$, this embedding is compact. The Compact Trace Theorem (see [1, 25]) establishes that there exists a unique continuous operator

$$\Gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega),$$

if $p < N$ and $1 \leq q \leq \frac{(N-1)p}{N-p}$ or $p \geq N$ and $q \in [1, +\infty)$. Furthermore, if $p < N$ and $1 \leq q < \frac{(N-1)p}{N-p}$ or $p \geq N$ and $q \in [1, +\infty)$, the operator Γ is compact. We denote the norm of $L^q(\partial\Omega)$ by $\|\cdot\|_{q,\partial}$. The following inequalities are related to the first eigenvalues, μ_1 and λ_1, respectively to the problems (1.4) and (1.5), and are of our particular interest (see [13]):

$$\|u\|_c^p \geq \mu_1 \|u\|_{p,\partial}^p, \ \forall u \in W^{1,p}(\Omega), \tag{3.1}$$

$$\|u\|_c^p \geq \lambda_1 \|u\|_p^p, \ \forall u \in W^{1,p}(\Omega). \tag{3.2}$$

Since our approach is variational, we define the Euler-Lagrange functional, $I_p : W^{1,p}(\Omega) \to \mathbb{R}$, as

$$I_p(u) = \frac{1}{p}\widehat{M}(\|u\|_c^p) - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma, \tag{3.3}$$

for all $u \in W^{1,p}(\Omega)$, where $\widehat{M}(t) := \int_0^t M(s)ds$, $F(x,u) = \int_0^u f(x,s)ds$, and $G(x,u) = \int_0^u g(x,s)ds$. The functional I_p is well defined in $W^{1,p}(\Omega)$ and $I_p \in C^1(W^{1,p}(\Omega),\mathbb{R})$. Moreover, I_p has Fréchet derivative in $u \in W^{1,p}(\Omega)$, given by

$$\begin{aligned} I_p'(u)(v) &= M(\|u\|_c^p)\int_\Omega [|\nabla u|^{p-2}\nabla u \nabla v + c(x)|u|^{p-2}uv]dx \\ &\quad - \int_\Omega f(x,u)vdx - \int_{\partial\Omega} g(x,u)vd\sigma, \end{aligned}$$

for all $v \in W^{1,p}(\Omega)$. Next define weak solutions of problem (1.1).

Definition 3.1. We say that $u \in W^{1,p}(\Omega)$ is a weak solution of problem (1.1), if

$$M(\|u\|_c^p)\int_\Omega [|\nabla u|^{p-2}\nabla u \nabla v + c(x)|u|^{p-2}uv]dx = \int_\Omega f(x,u)vdx + \int_{\partial\Omega} g(x,u)vd\sigma,$$

for all $v \in W^{1,p}(\Omega)$.

To prove results of multiplicity of solutions on the main theorems, our main tool is a result due to Clark. We will give some basic notion on the Krasnoselskii genus that we will use in the proof of Theorems 2.2 and 2.4.

Let E be a real Banach space. We denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin; that is, $u \in A$ implies $-u \in A$.

Definition 3.2. Let $A \in \mathfrak{A}$. The Krasnoselskii genus of A, $\gamma(A)$, is defined as the least positive integer k such that there is an odd mapping $\phi \in C(A,\mathbb{R}^k)$ such that $\phi(u) \neq 0$, for all $u \in A$. If k does not exists we set $\gamma(A) = +\infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

In the sequel we enunciate some results on the Krasnoselskii genus that can be found in [3, 6, 12, 20].

Proposition 3.3. *Suppose that $E = \mathbb{R}^N$ and $\partial\Omega$ the boundary of an open, symmetric, and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Corollary 3.4. *If $\partial\Omega = S^{N-1}$ is the unit sphere in $\mathbb{R}^N$, then $\gamma(S^{N-1}) = N$.*

We now establish a result due to Clark [21].

Proposition 3.5 (Clark's proposition)**.** *Consider $\Phi \in C^1(X,\mathbb{R})$ a functional satisfying the Palais-Smale condition and suppose that*

(i) *Φ is bounded from below, and even;*
(ii) *there is a compact set $K \in \mathfrak{A}$ such that $\gamma(K) = k$ and $\sup_{u\in K}\Phi(u) < \Phi(0)$.*

Then Φ possesses at least k distinct pairs of critical points and their corresponding critical values are less than $\Phi(0)$.

We point out that this result is a consequence of a basic multiplicity theorem involving an invariant functional under the action of a compact topological group. The following theorem will be used in the proof of Theorems 2.1 and 2.3 (see [19]):

Theorem 3.6. *Let E be a Banach space. If $I \in C^1(E,\mathbb{R})$ satisfies the Palais-Smale condition and is bounded from below, then $l = \inf_E I$ is a critical point of I.*

For using the above results, it is necessary that the functional I_p be coercive in $W^{1,p}(\Omega)$. The next two lemmas are about this.

Lemma 3.7. *Suppose that* (A1), (A2), (A5), (A7) *hold, and suppose that $s+1 < p$. Then I_p is coercive in $W^{1,p}(\Omega)$.*

Proof. It follows from (A7) that for $\varepsilon > 0$, there exists $R = R(\epsilon) > 0$ such that

$$\frac{pF(x,u)}{|u|^p} \le \lambda + \epsilon, \tag{3.4}$$

for $x \in \overline{\Omega}$ and $|u| > R$. Moreover, since Ω is bounded in $\mathbb{R}^N$, we have $\overline{\Omega}$ compact in $\mathbb{R}^N$. Thus $\overline{\Omega} \times [-R,R]$ is compact in $\mathbb{R}^{N+1}$. Since that $F \in C^1(\overline{\Omega} \times \mathbb{R},\mathbb{R})$, we can conclude that F achieve a maximum in $\overline{\Omega} \times [-R,R]$. So, there exists $M_\epsilon > 0$ such that, if $x \in \overline{\Omega}$ and $|u| \le R$, we have

$$F(x,u) \le M_\epsilon. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$F(x,u(x)) \le \frac{1}{p}(\lambda+\epsilon)\,|u(x)|^p + M_\epsilon, \tag{3.6}$$

for all $x \in \overline{\Omega}$ and for all $u \in W^{1,p}(\Omega)$. By using (A5), (3.2), (3.6), and the Compact Trace Theorem, we obtain

$$\begin{aligned}
I_p(u) &= \frac{1}{p}\widehat{M}(\|u\|_c^p) - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma \\
&\ge \frac{m_0}{p}\|u\|_c^p - \frac{1}{p}(\lambda+\epsilon)\,\|u\|_p^p - M_\epsilon|\Omega| - a_1\int_{\partial\Omega}|u|d\sigma - \frac{a_2}{s+1}\int_{\partial\Omega}|u|^{s+1}d\sigma \\
&\ge \frac{m_0}{p}\|u\|_c^p - \frac{1}{p}\frac{(\lambda+\epsilon)}{\lambda_1}\|u\|_c^p - a_1\|u\|_{1,\partial} - \frac{a_2}{s+1}\|u\|_{s+1,\partial}^{s+1} - M_\epsilon|\Omega| \\
&\ge \frac{1}{p}\Big(m_0 - \frac{\lambda}{\lambda_1} - \frac{\epsilon}{\lambda_1}\Big)\|u\|_c^p - a_1\bar{C}\|u\|_c - \frac{a_2}{s+1}\tilde{C}\|u\|_c^{s+1} - M_\epsilon|\Omega|,
\end{aligned}$$

where $|\Omega|$ is the measure of Ω. Since $\lambda < \lambda_1 m_0$, we have $m_0 - \frac{\lambda}{\lambda_1} > 0$. For $\epsilon < \lambda_1 m_0 - \lambda$, we obtain $\big(m_0 - \frac{\lambda}{\lambda_1} - \frac{\epsilon}{\lambda_1}\big) > 0$. Since that $s+1 < p$,

$$\lim_{\|u\|_c \to +\infty} I_p(u) = +\infty.$$

It follows that I_p is coercive in $W^{1,p}(\Omega)$. □

Lemma 3.8. *Suppose that* (A1)–(A3), (A10) *hold, and suppose that $r+1 < p$. Then I_p is coercive in $W^{1,p}(\Omega)$.*

The proof of the above lemma is analogous to that of Lemma 3.7.; we omit it here. Let $\mathcal{A} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the map defined by

$$\langle \mathcal{A}(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx, \quad \forall u, v \in W^{1,p}(\Omega). \tag{3.7}$$

From [17, Proposition 3.1], we have the following lemma.

Lemma 3.9 (S_+ condition)**.** *If the map $\mathcal{A} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is defined by* (**??**)*, then $\mathcal{A}$ is bounded, continuous, monotone and of type $(S)_+$, that is, if $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, as $n \to +\infty$, and $\limsup_{n\to+\infty}\langle \mathcal{A}(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W^{1,p}(\Omega)$.*

4. Palais-Smale condition

In this section we show that the functional I_p satisfies the Palais-Smale condition if we assume (A7) or (A10).

Lemma 4.1. *Suppose that* (A1)–(A3), (A5), (A7) *hold, and suppose that $s+1 < p$. Then I_p satisfies the Palais-Smale condition.*

Proof. Let $(u_n) \subset W^{1,p}(\Omega)$ be a Palais-Smale sequence at level l; that is, $I_p(u_n) \to l$ and $I_p'(u_n) \to 0$ (in the dual of $W^{1,p}(\Omega)$), as $n \to +\infty$. It follows from Lemma 3.7 that I_p is coercive in $W^{1,p}(\Omega)$. Thus $(u_n) \subset W^{1,p}(\Omega)$ is bounded in $W^{1,p}(\Omega)$. Since the norms $\|\cdot\|_c$ and $\|\cdot\|_{1,p}$ are equivalents, $(W^{1,p}(\Omega), \|\cdot\|_{1,p})$ is reflexive and (u_n) is bounded in $W^{1,p}(\Omega)$, up to a subsequence, there exists $u \in W^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega) \text{ as } n \to +\infty.$$

Moreover, as the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we have

$$\begin{gathered} u_n \to u \quad \text{in } L^p(\Omega), \\ u_n(x) \to u(x) \quad \text{a.e. in } \Omega, \\ \|u_n\| \to t_0 \ge 0, \end{gathered} \tag{4.1}$$

as $n \to +\infty$. From the Compact Trace Theorem, we obtain

$$\begin{gathered} u_n \to u \quad \text{in } L^p(\partial\Omega), \\ u_n(x) \to u(x) \quad \text{a.e. in } \partial\Omega, \end{gathered} \tag{4.2}$$

as $n \to +\infty$. Since (u_n) is bounded in $W^{1,p}(\Omega)$ and $I_p'(u_n) \to 0$ (in the dual of $W^{1,p}(\Omega)$), we have $I_p'(u_n)(u_n - u) = o_n(1)$, where $\lim_{n\to+\infty} o_n(1) = 0$; that is,

$$\begin{aligned} &M(\|u_n\|_c^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx + \int_\Omega c(x)|u_n|^{p-2}u_n(u_n - u)dx \\ &- \int_\Omega f(x,u_n)(u_n - u)dx - \int_{\partial\Omega} g(x,u_n)(u_n - u)dx = o_n(1). \end{aligned}$$

Using Hölder's inequality, (**??**), (**??**), and the Dominated Convergence Theorem, we obtain that $f(x,u_n)(u_n - u)$ and $c(x)|u_n|^{p-2}u_n(u_n - u)$ converges to 0 in $L^1(\Omega)$, and $g(x,u_n)(u_n - u)$ converges to 0 in $L^1(\partial\Omega)$. Thus, we have

$$M(\|u_n\|_c^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx = o_n(1).$$

Since that M is continuous and (u_n) is bounded in $W^{1,p}(\Omega)$, there exists $C > 0$ such that $M(\|u_n\|_c^p) \le C$. It follows from (A1) that

$$0 < m_0 \le M(\|u_n\|_c^p) \le C,$$

for all $n \in \mathbb{N}$. Thus, we achieve

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx = o_n(1).$$

Using Lemma 3.9, the proof is complete. □

Lemma 4.2. *Suppose that* (A1)–(A3), (A5), (A10) *hold, and suppose that* $r+1 < p$. *Then* I_p *satisfies the Palais-Smale condition.*

The proof of this lemma is analogous to that of Lemma 4.1, changing Lemma 3.7 by Lemma 3.8. We omit it here.

5. Proof of main results

Proof of Theorem 2.1. It follows from Lemma 3.7 that I_p is bounded from below in $W^{1,p}(\Omega)$. Moreover, from Lemma 4.1, I_p satisfies the Palais-Smale condition. Since $I_p \in C^1(W^{1,p}(\Omega), \mathbb{R})$, it follows from Theorem 3.6 that $l = \inf_{W^{1,p}(\Omega)} I_p$ is a critical value of I_p, that is, I_p has a critical point $u_0 \in W^{1,p}(\Omega)$ such that $I_p(u_0) = l$. Thus we have $I_p'(u_0)(v) = 0$, for all $v \in W^{1,p}(\Omega)$; that is, u_0 is a weak solution of problem (1.1). The next lemma shows that u_0 is nontrivial.

Lemma 5.1. *Suppose that* (A1)–(A5), (A7) *hold and that* $s + 1 < p$. *Then* $l < 0$.

Proof. Let φ_1 be the first positive engeinfunction of the problem
$$
\begin{gathered}
-\Delta_p u = |u|^{p-2}u \text{ in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$
(see [5]). Since $\varphi_1 = 0$ in $\partial\Omega$, we have $G(x, t\varphi_1) = 0$, for all $t > 0$ and for all $x \in \partial\Omega$. For $t < 1$, it follows from (A4) that there exists $C > 0$ (since M is continuous) such that
$$
\begin{aligned}
I_p(t\varphi_1) &= \frac{1}{p}\widehat{M}(\|t\varphi_1\|_c^p) - \int_\Omega F(x, t\varphi_1)dx - \int_{\partial\Omega} G(x, t\varphi_1)d\sigma \\
&\le \frac{1}{p}\int_0^{\|t\varphi_1\|_c^p} M(s)ds - \frac{b_3 t^{\alpha+1}}{\alpha+1}\int_\Omega |\varphi_1|^{\alpha+1}dx \\
&\le \frac{C}{p}t^p\|\varphi_1\|_c^p - \frac{b_3 t^{\alpha+1}}{\alpha+1}\int_\Omega |\varphi_1|^{\alpha+1}dx \\
&= C_1 t^p - C_2 t^{\alpha+1},
\end{aligned}
$$
where $C_1 = \frac{C}{p}\|\varphi_1\|_c^p > 0$ and $C_2 = \frac{b_3}{\alpha+1}\int_\Omega |\varphi_1|^{\alpha+1}dx > 0$. Thus we obtain
$$
I_p(t\varphi_1) < 0, \quad \forall t < \min\big\{1, \big(\frac{C_2}{C_1}\big)^{\frac{1}{p-(\alpha+1)}}\big\},
$$
which implies $l = \inf_{W^{1,p}(\Omega)} I_p < 0$. ∎

By using Lemma 5.1 we have $I_p(u_0) = l < 0$. Once that $I_p(0) = 0$, we obtain $u_0 \neq 0$ and the proof Teorema 2.1 is complete.

Proof of Theorem 2.2. For the proof of Theorem 2.2, we will use Proposition 3.5. From the definition of F and G, and by using (A8) and (A9) we have that I_p is even and $I_p(0) = 0$. Moreover, $I_p \in C^1(W^{1,p}(\Omega), \mathbb{R})$, and by Lemmas 3.7 and 4.1, we get I_p bounded from below in $W^{1,p}(\Omega)$ and I_p satisfies the Palais-Smale condition. The next lemma shows that I_p verifies the condition *ii*) of Proposition 3.5.

Lemma 5.2. *Suppose that* (A2),– (A5) *hold. Then, there exists a compact set* $\mathcal{S} \subset W^{1,p}(\Omega)$ *such that* $\gamma(\mathcal{S}) = k < \infty$ *and* $\sup_{u\in\mathcal{S}} I_p(u) < I_p(0) = 0$.

Proof. Since $C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ has infinite di mension, for each $k \in \mathbb{N}$, there exists a k-dimensional linear subspace of $W_0^{1,p}(\Omega)$, $\mathcal{X}_k$, such that $\mathcal{X}_k \subset C_0^\infty(\Omega)$. Thus, all norms in $\mathcal{X}_k$ are equivalent; that is, there exists a positive constant $C(k)$, which depends on k, such that

$$C(k)\|u\|_c^{\alpha+1} \leq \frac{b_3}{\alpha+1}\int_\Omega |u|^{\alpha+1}dx,$$

for all $u \in \mathcal{X}_k$. Thus, if $u \in \mathcal{X}_k$, it follows from (A4) that

$$\int_\Omega F(x,u)dx \geq \frac{b_3}{\alpha+1}\int_\Omega |u|^{\alpha+1}dx \geq C(k)\|u\|_c^{\alpha+1}.$$

Let $u \in \mathcal{X}_k$ be such that $\|u\|_c \leq 1$. Since $\mathcal{X}_k \subset W_0^{1,p}(\Omega)$, we have $u = 0$ in $\partial\Omega$. From the continuity of M, there exists $C > 0$ such that

$$\begin{aligned}
I_p(u) &= \frac{1}{p}\widehat{M}(\|u\|_c^p) - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma \\
&\leq \frac{1}{p}\int_0^{\|u\|_c^p} M(s)ds - C(k)\|u\|_c^{\alpha+1} \\
&\leq \frac{C}{p}\|u\|_c^p - C(k)\|u\|_c^{\alpha+1},
\end{aligned}$$

for all $u \in \mathcal{X}_k$. Take

$$R = \min\big\{1, \big(\frac{pC(k)}{C}\big)^{\frac{1}{p-(\alpha+1)}}\big\}$$

and consider $\mathcal{S} = \{u \in \mathcal{X}_k : \|u\| = s\}$, with $0 < s < R$. Once $1 \leq \alpha+1 < p$, for all $u \in \mathcal{S}$, we obtain

$$I_p(u) \leq s^{\alpha+1}\big[\frac{C}{p}s^{p-(\alpha+1)} - C(k)\big] < 0 = I_p(0),$$

which implies $\sup_{\mathcal{S}} I_p(u) < 0 = I_p(0)$. As $\mathcal{X}_k$ and $\mathbb{R}^k$ are isomorphic and $\mathcal{S}$ and S^{k-1} are homeomorphic, we conclude that $\gamma(\mathcal{S}) = k$. □

Since $I_p \in C^1(W^{1,p}(\Omega), \mathbb{R})$ is coercive, even, satisfies the Palais-Smale condition, and, from Lemma 5.2, satisfies the condition (ii) of Proposition 3.5, we conclude that I_p has at least k pairs of different critical points. It follows from the arbitrary of k that I_p infinitely many critical points.

Proof of Theorem 2.3. The proof is analogous to that of Theorem 2.1. By using Lemmas 3.8 and 4.2 we can apply Theorem 3.6 to obtain a weak solution of (1.1). Similar to proof of the Lemma 5.1, we have that the solution is nontrivial. □

Proof of Theorem 2.4. By using Lemmas 3.8, 4.2, 5.2, and Proposition 3.5, the proof of Theorem 2.4 follows analogously to the proof of Theorem 2.2. □

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