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BIFURCATION AND STABILITY OF A DIFFUSIVE SIRS EPIDEMIC MODEL WITH TIME DELAY

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Abstract. In this article, we study a reaction-diffusion system for a SIRS epidemic model with time delay and nonlinear incidence rate. On the one hand, we study the existence and stability of the disease-free equilibrium, endemic equilibria and Hopf bifurcation, by analyzing the characteristic equations. On the other hand, we establish formulas determining the direction and stability of the bifurcating periodic solutions.

1. Introduction

Mathematical modelling in epidemiology provides us with an understanding of the mechanisms which impact and influence the spread of diseases and in the process advances the possibilities for control strategies. The earliest mathematical model, concerning epidemic problem, was probably introduced by Bernoullin in 1760 to describe the effect of cow-pox inoculation on the spread of smallpox. Epidemic is caused by the pathogen, which can spread from people to people, people to animals, and animals to animals. Because it can make a range of biological reduce or lode labor, death and spread rapidly in a certain period of time. It caused the scientists and mathematicians widespread attention. Epidemic model [1, 2, 7, 9, 13, 14, 15, 17, 19] is a basic differential equation model describing the interaction between species. To describe the effects of disease latency or immunity, the delay is often incorporated in such models. Kyrychko and Blyuss [10] proposed the following delayed SIRS model which incorporates immunity and a general nonlinear incidence rate

$$
\begin{aligned}
\dot{S}(t) &= \mu - dS(t) - \beta f(I(t))S(t) + \gamma e^{-d\tau} I(t-\tau), \\
\dot{I}(t) &= \beta f(I(t))S(t) - (d+\gamma)I(t), \\
\dot{R}(t) &= \gamma I(t) - \gamma e^{-d\tau} I(t-\tau) - dR(t),
\end{aligned}
\tag{1.1}
$$

where $S(t)$, $I(t)$ and $R(t)$ denote the densities of the susceptible population, infective population and the population who has been removed from the possibility of infection through the temporary immunity, respectively, μ is the recruitment rate of the susceptible population by birth or immigration, d is the natural death rate, β is the infection rate from the suspectable class to the infected class, τ is the length of the immunity period, γ is the recovery rate, the term $\gamma e^{-d\tau} I(t-\tau)$ indicates that the individuals have survived from natural death in a recovery pool

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before becoming susceptible again. It is assumed that all newborns are susceptible and there are no disease-caused deaths. All parameters in system (1.1) are positive except $\tau \geq 0$. The nonlinear incidence rate which is given by the function $f(I)$ is assumed to satisfy the following properties:

$$f(0)=0, \quad \lim_{I\to+\infty} f(I)=c<\infty, \quad f'(I)>0, \quad f''(I)<0 \quad \forall I\geq 0\,. \tag{1.2}$$

In the case of linear incidence rate $f(I)=I$, Kyrychko and Blyuss [10] obtained some sufficient conditions ensuring the globally asymptotical stability of the endemic equilibrium for system (1.1) by choosing some Lyapunov functional, and also revealed the existence of periodic solutions by numerical simulations. Wen and Yang [16] derived some sufficient conditions for the local/global asymptotic stability of the endemic equilibrium of system (1.1) with $f(I)=I$. Jiang and Wei [8] investigated the existence of Hopf bifurcations at the endemic equilibrium for system (1.1) with the nonlinear incidence rate $f(I)=I/(1+I)$. By using the basic reproduction number and an iteration technique, Xu, Ma, and Wang [18] investigated the local asymptotic stability of the disease-free equilibrium and the endemic equilibrium and the global asymptotic stability of the endemic equilibrium for system (1.1) with the nonlinear incidence rate $f(I)=I/(1+\alpha I)$, which was used by Capasso and Serio [3] to represent a *crowding effect* or *protection measure* in modeling the cholera epidemics in Bari in 1973. Obviously, this incidence rate seems more reasonable than the bilinear incidence rate, because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters. In particular, Liu et al. [12] proposed a nonlinear saturated incidence function $f(I)=I^l/(1+\alpha I^h)$ to model the effect of behavioral changes to certain communicable diseases, where βI^l describes the infection force of the disease, $1/(1+\alpha I^h)$ measures the inhibition effect from the behavioral change of the susceptible individuals when the number of infectious individuals increases, l and h are all positive constants, and α is a nonnegative constant.

When the densities of the susceptible population, infective population and removed population are spatially inhomogeneous in a bounded domain with smooth boundary $\Omega\subseteq\mathbb{R}^N$ $(N\geq 1)$, instead of the ordinary delay differential system (1.3), one gets the following reaction-diffusion system

$$\begin{gathered}
S_t-\Delta S=\mu-dS-\beta f(I)S+\gamma e^{-d\tau}I(x,t-\tau), \quad \text{in } \Omega,\\
I_t-\Delta I=\beta f(I)S-(d+\gamma)I, \quad \text{in } \Omega,\\
R_t-\Delta R=\gamma I-\gamma e^{-d\tau}I(x,t-\tau)-dR, \quad \text{in } \Omega,\\
\frac{\partial S}{\partial \mathbf{n}}=\frac{\partial I}{\partial \mathbf{n}}=\frac{\partial R}{\partial \mathbf{n}}=0, \quad \text{on } \partial\Omega,
\end{gathered}\tag{1.3}$$

where parameters μ, d, β, γ, and τ are the same as system (1.1), the nonlinear incidence rate $f(I)$ satisfies (1.2), Δ denotes the Laplacian operator on $\mathbb{R}^N$, $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$. The homogeneous Neumann boundary condition means that the two species have zero flux across the boundary $\partial\Omega$. The initial conditions of system (1.3) are given as

$$\begin{gathered}
S(\cdot,0)=S_0\in C_+(\overline{\Omega}), \quad R(\cdot,0)=R_0\in C_+(\overline{\Omega}),\\
I(\cdot,\theta)=I_0(\cdot,\theta)\in C_+(\overline{\Omega}) \quad \text{for all } \theta\in[-\tau,0],
\end{gathered}\tag{1.4}$$

where $C_+(\overline{\Omega})$ is the space of nonnegative continuous functions. In this paper we can also define a so-called basic reproduction number R_0 such that the disease-free equilibrium is stable when $R_0 < 1$ and that there exists exactly one endemic equilibrium when $R_0 > 1$. Since the first two equations of system (1.3) are independent of the third, in the following, one only considers the subsystem of system (1.3) as follows:

$$\begin{gathered} S_t - \Delta S = \mu - dS - \beta f(I)S + \gamma e^{-d\tau} I(x, t-\tau), \quad \text{in } \Omega, \\ I_t - \Delta I = \beta f(I)S - (d+\gamma)I, \quad \text{in } \Omega, \\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.5}$$

with the initial conditions

$$S(\cdot, 0) = S_0 \in C_+(\Omega), \quad I(\cdot, \theta) = I_0(\cdot, \theta) \in C_+(\overline{\Omega}) \quad \text{for all } \theta \in [-\tau, 0],$$

satisfying $I_0(x, 0) > 0$.

This article is organized as follows. In section 2, we consider the nonnegativity and boundedness of the solutions of system (1.5). In Section 3, we study the global asymptotical stability of the disease-free equilibrium of system (1.5). In Section 4, we investigate the local asymptotical stability of the endemic equilibrium of system (1.5) and the existence of local Hopf bifurcations. In Section 5, we show the Hopf bifurcation direction and the stability of the bifurcating periodic solutions. At last, some numerical simulations are carried out to support our theoretic results.

2. Nonnegativity and boundedness

From biological meaning, it is necessary to show that all solutions of system (1.3) is nonnegative and bounded for all $t \geq 0$. Generally speaking, the existence of a local solution of (1.3) and (1.4) is guaranteed, but the existence of a global solution for (1.3) and (1.4) depends on the fact that the solution does not become infinite in a finite time. Since the growth functions are sufficiently smooth, the standard parabolic equation theory (see Ladyzenskaja, Solonnikov and Uralceva [11]) implies that the solution of (1.3) and (1.4) is unique and continuous for all $t \geq 0$ in $\overline{\Omega}$. Furthermore, we have the following result.

Lemma 2.1. *For each initial value* (S_0, I_0, R_0) *satisfying* (1.4)*, system* (1.3) *has a unique solution* $(S, I, R) \in C^{2,1}(\overline{\Omega} \times \mathbb{R}_+, \mathbb{R}_+^3)$ *satisfying* $\limsup_{t\to+\infty} N(x,t) \leq \mu/d$ *for all* $x \in \Omega$*, where* $N(x,t) = S(x,t) + I(x,t) + R(x,t)$*. In addition,* $\mathbb{D} = \{(S, I, R) \in \mathbb{R}_+^3 : 0 \leq S + I + R \leq \mu/d\}$ *attracts all solutions of system* (1.3) *and is positive forward invariant.*

Proof. Let $[0, t_{\max})$ be the maximal existence interval of the solution (S, I, R). For all $t \in [0, \tau] \cap [0, t_{\max})$, we have

$$\begin{gathered} S_t - \Delta S = \mu - dS - \beta f(I)S + \gamma e^{-d\tau} I_0(x, t-\tau), \quad \text{in } \Omega, \\ I_t - \Delta I = \beta f(I)S - (d+\gamma)I, \quad \text{in } \Omega, \\ R_t - \Delta R = \gamma I - \gamma e^{-d\tau} I_0(x, t-\tau) - dR, \quad \text{in } \Omega, \\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = \frac{\partial R}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

Using the strong maximum principle for parabolic equations, it follows that $S(x,t) \geq 0$, $I(x,t) \geq 0$, and $R(x,t) \geq 0$ for $t \in [0, \tau] \cap [0, t_{\max})$ and $x \in \overline{\Omega}$. Hence, by induction, we have $S(x,t) \geq 0$, $I(x,t) \geq 0$, and $R(x,t) \geq 0$ for $t \in [0, t_{\max})$ and $x \in \overline{\Omega}$.

For $t \in [0, t_{\max})$ and $x \in \overline{\Omega}$,

$$N_t - \Delta N = \mu - dN.$$

This implies that every solution (S, I, R) of system (1.3) with initial value (S_0, I_0, R_0) satisfying (1.4) is bounded for $t \in [0, t_{\max})$. Therefore, it foloows from continuation theorem of solutions for functional differential equations that the maximal existence interval of the solution $(S(x,t), I(x,t), R(x,t))$ of system (1.3) is $[0, +\infty)$. Moreover, $(S(x,t), I(x,t), R(x,t))$ is nonnegative and bounded on $[0, +\infty)$. Clearly, we have $\limsup_{t\to+\infty} N(x,t) \le \mu/d$ for all $x \in \Omega$. Using a similar argument, we see that the solution $(S(x,t), I(x,t), R(x,t))$ of system (1.3) is positive on $[0, +\infty)$, if $S_0 > 0$ and $I_0 > 0$. Finally, it is easy to show that $\mathbb{D}$ is a positively invariant set. This completes the proof. $\square$

3. Disease-free steady-state solution

To investigate the local stability of the positive homogeneous steady-states of model (1.3), it suffices to discuss model (1.5). Let $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n < \cdots$ with $\lim_{n\to\infty} \sigma_n = +\infty$ be the eigenvalues of the linear operator $-\Delta$ subject to the homogeneous boundary condition $\frac{\partial}{\partial \mathbf{n}} u = 0$ on $\partial\Omega$. Let $S(\lambda_j)$ be the eigenspace associated with σ_j with multiplicity $n_j \ge 1$. Let ϕ_{jk}, $1 \le k \le n_j$, be the normalized eigenfunctions corresponding to σ_j then the set $\{\phi_{jk} | j \ge 0, 1 \le k \le n_j\}$ forms a complete orthonormal basis in the Lebesgue space $L^2(\overline{\Omega})$ of integrable functions defined on Ω, $\phi_0(x) > 0$ for all $x \in \Omega$. It follows that there exists an $n_j \times n_j$ matrix $B_j = (b_{jks})$ such that $-\Delta\Phi_j = B_j\Phi_j$, where $\Phi_j = (\phi_{j1}, \phi_{j2}, \dots, \phi_{jn_j})^T$ and the only eigenvalue of B_j is σ_j (also see [4, 20]).

It can be seen that system (1.5) has always a disease-free equilibrium $E_0(\mu/d, 0)$. The linearization of system (1.5) near E_0 is

$$\begin{gathered}
S_t - \Delta S = -dS - \frac{\mu\beta}{d} f'(0) I + \gamma e^{-d\tau} I(x, t-\tau), \quad \text{in } \Omega,\\
I_t - \Delta I = \frac{\mu\beta}{d} f'(0) I - (d+\gamma) I, \quad \text{in } \Omega,\\
\frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = \frac{\partial R}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.1}$$

If (3.1) has a solution of the form $u(x,t) = P(x)e^{\lambda t}$, then we have

$$\begin{bmatrix} \lambda - \Delta + d & \frac{\mu\beta}{d} f'(0) - \gamma e^{-(d+\lambda)\tau} \\ 0 & \lambda - \Delta + d + \gamma - \frac{\mu\beta}{d} f'(0) \end{bmatrix} P = 0 \tag{3.2}$$

Substituting $u = \sum_{j=0}^{\infty} u_j \Phi_j$ with $u_j = (u_{j1}, u_{j2}, \dots, u_{jn_j})$ and $u_{jk} = (u^1_{jk}, u^2_{jk})^T \in \mathbb{C}^2$ for $k \in \{1, 2, \dots, n_j\}$ into (3.2) yields

$$\begin{bmatrix} \lambda - \Delta + d & \frac{\mu\beta}{d} f'(0) - \gamma e^{-(d+\lambda)\tau} \\ 0 & \lambda - \Delta + d + \gamma - \frac{\mu\beta}{d} f'(0) \end{bmatrix} u_j \Phi_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for all $j \in \mathbb{N}_0$; that is, $\mathcal{M}_j(\lambda)\mathbf{u}_j = 0$ for all $j \in \mathbb{N}_0$, where

$$\begin{gathered}
\mathbf{u}_j = (u^1_{j1}, \dots, u^1_{jn_j}, u^2_{j1}, \dots, u^2_{jn_j})^T \in \mathbb{C}^{2n_j},\\
\mathcal{M}_j(\lambda) = \begin{bmatrix} (\lambda + d)\mathrm{Id} + B_j^T & [\frac{\mu\beta}{d} f'(0) - \gamma e^{-(d+\lambda)\tau}]\mathrm{Id} \\ 0 & [\lambda + d + \gamma - \frac{\mu\beta}{d} f'(0)]\mathrm{Id} + B_j^T \end{bmatrix}
\end{gathered}$$

, This implies that for some $j \in \mathbb{N}_0$, $\mathbf{u}_j \neq 0$ if and only if $\det \mathcal{M}_j(\lambda) = 0$; that is, $\mathcal{P}_j(\lambda) = 0$, where

$$\mathcal{P}_j(\lambda) := (\lambda + \sigma_j + d)[\lambda + \sigma_j + d + \gamma - \frac{\mu\beta}{d} f'(0)]. \tag{3.3}$$

Obviously, $\mathcal{P}_j(\cdot)$ has two zeros $-d-\sigma_j$ and $\frac{\mu\beta}{d} f'(0) - d - \gamma - \sigma_j$. Thus, the steady-state solution E_0 is locally asymptotically stable if $\frac{\mu\beta}{d} f'(0) - d - \gamma < 0$ and is unstable if there exists some $n \in \mathbb{N} \cup \{0\}$ such that $\frac{\mu\beta}{d} f'(0) - d - \gamma > 0$. Define the basic reproduction number as

$$R_0 = \frac{\mu\beta f'(0)}{d(d+\gamma)}.$$

Hence one can obtain the following theorem.

Theorem 3.1. *For all $\tau \geq 0$, the disease-free steady-state solution E_0 of system (1.5) is locally asymptotically stable if $R_0 < 1$, and is unstable if $R_0 > 1$.*

In what follows, we consider the global asymptotic stability of E_0 of system (1.5) when $R_0 < 1$. For this purpose, we just need to consider the global attractivity of E_0. Since $\mathbb{D}$ is the attractive set and positively forward invariant for system (1.3), hence we just consider system (1.5) in

$$\mathbb{D}_1 = \{(S, I) \in \mathbb{R}_+^2 : 0 \leq S + I \leq \mu/d\}.$$

Define the Lyapunov functional

$$V(S(x,t), I(x,t)) = \int_\Omega \big[S(x,t) - \frac{\mu}{d} - \frac{\mu}{d} \ln \frac{dS(x,t)}{\mu} + I(x,t)\big] \mathrm{d}x.$$

Obviously, $V(\mu/d, 0) = 0$, and V is positive definite with respect to $(S, I) \in \mathbb{R}_+^2$ and has the property $V(S, I) \to +\infty$ as $\|(S, I)\| \to +\infty$. The derivative of V along the solutions of system (1.5) is

$$\begin{aligned}
&V_t(S(x,t), I(x,t)) \\
&= \int_\Omega [1 - \frac{\mu}{dS(x,t)}][\mu - dS(x,t) - \beta f(I(x,t))S(x,t) + \gamma e^{-d\tau} I(x, t-\tau)] \mathrm{d}x \\
&\quad + \int_\Omega [\beta f(I(x,t))S(x,t) - (d+\gamma)I(x,t)] \mathrm{d}x \\
&\quad + \int_\Omega [1 - \frac{\mu}{dS}] \Delta S(x,t) \mathrm{d}x + \int_\Omega \Delta I(x,t) \mathrm{d}x \\
&= \int_\Omega [1 - \frac{\mu}{dS(x,t)}][\mu - dS(x,t) + \gamma e^{-d\tau} I(x, t-\tau)] \mathrm{d}x \\
&\quad + \frac{1}{d} \int_\Omega [\mu\beta f(I(x,t)) - (d+\gamma) d I(x,t)] \mathrm{d}x \\
&\quad - \int_\Omega \frac{\mu}{dS^2} |\nabla S(x,t)|^2 \mathrm{d}x - \int_\Omega |\nabla I(x,t)|^2 \mathrm{d}x
\end{aligned}$$

It follows from $S(x,t) < \mu/d$ and $f''(I) \leq 0$ for all $I \geq 0$ that $f(I(x,t)) \leq f'(0)I(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0, +\infty)$, and hence that

$$V_t(S(x,t), I(x,t)) \leq \int_\Omega [1 - \frac{\mu}{dS(x,t)}][\gamma e^{-d\tau} I(x, t-\tau)] \mathrm{d}x.$$

This implies that $V_t(S(x,t), I(x,t)) \le 0$ along an orbit $(S(x,t), I(x,t))$ of system (1.5) with any non-negative initial value $\mathbb{D}_1$ when $R_0 < 1$. This, together with Lemma 2.1 and Theorem 3.1, implies that E_0 is globally asymptotically stable if $R_0 < 1$.

Theorem 3.2. *For all $\tau \ge 0$, the disease-free steady-state solution E_0 of system (1.5) is globally asymptotically stable if $R_0 < 1$.*

4. Endemic steady-state solution

In this section, we shall study the existence and stability of the endemic steady-state solution $E^*(S^*_\tau, I^*_\tau)$, where S^*_τ and I^*_τ satisfy

$$S^*_\tau = \frac{(d+\gamma)I^*_\tau}{\beta f(I^*_\tau)}, \quad \frac{d(d+\gamma)I^*_\tau}{\beta f(I^*_\tau)} = \mu + [\gamma e^{-d\tau} - d - \gamma]I^*_\tau.$$

Define $H : \mathbb{R}^2_+ \to \mathbb{R}$ by

$$H(\tau, x) = \mu - \frac{d(d+\gamma)x}{\beta f(x)} + [\gamma e^{-d\tau} - d - \gamma]x.$$

In view of assumption (1.2), we have $H(\tau, +\infty) = -\infty$, $H(\tau, 0) > 0$ when $R_0 > 1$, and $H_x(\tau, x) < 0$ and $H_\tau(\tau, x) < 0$ for all $\tau \ge 0$ and $x \ge 0$. Therefore, we obtain the following result.

Lemma 4.1. *If $R_0 > 1$, system (1.5) has exactly one endemic equilibrium $E^*(S^*_\tau, I^*_\tau)$. Moreover, both $S^*_\tau \in [S^*_0, S^*_\infty)$ and $I^*_\tau \in [I^*_0, I^*_\infty)$ are monophonically increasing with respective to $\tau \ge 0$, where*

$$(S^*_0, I^*_0) = \lim_{\tau\to 0}(S^*_\tau, I^*_\tau), \quad (S^*_\infty, I^*_\infty) = \lim_{\tau\to\infty}(S^*_\tau, I^*_\tau).$$

The linearization of (1.5) at the steady-state solution E^* takes the form

$$\begin{gathered}
S_t - \Delta S = -dS - \beta f'(I^*_\tau)S^*_\tau I - \beta f(I^*_\tau)S + \gamma e^{-d\tau} I(x, t-\tau), \quad \text{in } \Omega, \\
I_t - \Delta I = \beta f'(I^*_\tau)S^*_\tau I + \beta f(I^*_\tau)S - (d+\gamma)I, \quad \text{in } \Omega, \\
\frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{4.1}$$

The characteristic equation of system (4.1) is

$$\mathcal{Q}_n(\tau, \lambda) := \det \begin{bmatrix} \lambda + \sigma_n + d + \beta f(I^*_\tau) & \beta f'(I^*_\tau)S^*_\tau - \gamma e^{-(d+\lambda)\tau} \\ -\beta f(I^*_\tau) & \lambda + \sigma_n + d + \gamma - \beta f'(I^*_\tau)S^*_\tau \end{bmatrix} = 0$$

for $n \in \mathbb{N}_0$. In fact, we have

$$\mathcal{Q}_n(\tau, \lambda) = \lambda^2 + p_n(\tau)\lambda + q_n(\tau) + r(\tau)e^{-\lambda\tau}. \tag{4.2}$$

with

$$\begin{gathered}
p_n(\tau) = 2\sigma_n + 2d + \gamma + \beta f(I^*_\tau) - \beta f'(I^*_\tau)S^*_\tau, \\
q_n(\tau) = [\sigma_n + d + \beta f(I^*_\tau)][\sigma_n + d + \gamma - \beta f'(I^*_\tau)S^*_\tau] + \beta^2 f(I^*_\tau)f'(I^*_\tau)S^*_\tau, \\
r_n(\tau) = -\beta\gamma e^{-d\tau} f(I^*_\tau).
\end{gathered}$$

We need to seek the necessary and sufficient condition ensuring that every zero of $\mathcal{Q}_n(\tau, \cdot)$ has negative real parts. For this purpose, we introduce the following results.

Lemma 4.2. (i) *$d + \gamma + \beta f(I^*_\tau) > \beta f'(I^*_\tau)S^*_\tau$ for all $\tau \ge 0$;*

(ii) *If* $[d+\gamma-\beta f(x)]f(x) > (d+\gamma)xf'(x)$ *for all* $x \in [I_0^*, I_\infty^*)$ *then* $p_n(\tau)$, $q_n(\tau)$, *and* $r_n(\tau)$ *are all monotonically increasing with respect to* $\tau \geq 0$.

Proof. It follows from (1.2) that $f(x) \geq xf'(x)$ for all $x \geq 0$ and hence

$$\frac{d+\gamma+\beta f(x)}{d+\gamma} \cdot \frac{f(x)}{x} > f'(x) \quad \text{for } x > 0,$$

which implies

$$\frac{d+\gamma+\beta f(I_\tau^*)}{\beta S_\tau^*} = \frac{d+\gamma+\beta f(I_\tau^*)}{d+\gamma} \cdot \frac{f(I_\tau^*)}{I_\tau^*} > f'(I_\tau^*);$$

that is, $d+\gamma+\beta f(I_\tau^*) > \beta f'(I_\tau^*)S_\tau^*$ for all $\tau \geq 0$. If $[d+\gamma-\beta f(x)]f(x) > (d+\gamma)xf'(x)$ for all $x \in [I_0^*, I_\infty^*)$ then $\beta f(x) - (d+\gamma)xf'(x)/f(x)$ is monotonically increasing with respect to $x \in [I_0^*, I_\infty^*)$, which implies that $p_n(\tau)$, $q_n(\tau)$, and $r_n(\tau)$ are all monotonically increasing with respect to $\tau \geq 0$. The proof is complete. $\square$

In view of Lemma 4.2, we have

$$q_n(\tau) + r_n(\tau) > 0, \quad q_n(\tau) - r_n(\tau) > 0 \tag{4.3}$$

for all $\tau \geq 0$, which implies that $\mathcal{Q}_n(\tau, \cdot)$ has no zero 0 for all $\tau \geq 0$. Moreover, it follows from $p_n(0) > 0$ and $q_n(0) + r_n(0) > 0$ that $\mathcal{Q}_n(0, \cdot)$ has only zeroes with negative real parts. In the following, we shall investigate the existence of purely imaginary zeroes $\pm i\omega$ of $\mathcal{Q}_n(\tau, \cdot)$. Separating the real and imaginary parts of $\mathcal{Q}_n(\tau, i\omega) = 0$ yields

$$\sin \tau\omega = \frac{\omega p_n(\tau)}{r_n(\tau)}, \quad \cos \tau\omega = \frac{\omega^2 - q_n(\tau)}{r_n(\tau)},$$

and hence

$$\omega^4 + [p_n^2(\tau) - 2q_n(\tau)]\omega^2 + q_n^2(\tau) - r_n^2(\tau) = 0. \tag{4.4}$$

It follows from (4.3) that $q_n^2(\tau) > r_n^2(\tau)$ for all $\tau \geq 0$. Note that

$$[p_n^2(\tau) - 2q_n(\tau)]^2 - 4[q_n^2(\tau) - r_n^2(\tau)] = p_n^4(\tau) - 4p_n^2(\tau)q_n(\tau) + 4r_n^2(\tau).$$

Thus, equation (4.4) has no positive solutions ω if either $p_n^2(\tau) > 2q_n(\tau)$ or $p_n^2(\tau) < 2q_n(\tau)$ and $p_n^4(\tau) + 4r_n^2(\tau) < 4p_n^2(\tau)q_n(\tau)$. If $p_n^2(\tau) < 2q_n(\tau)$ and $p_n^4(\tau) + 4r_n^2(\tau) > 4p_n^2(\tau)q_n(\tau)$; that is,

$$p_n^2(\tau) < 2q_n(\tau) - 2\sqrt{q_n^2(\tau) - r_n^2(\tau)} \tag{4.5}$$

then (4.4) has exactly two positive real solutions $\omega_n^\pm(\tau)$, where

$$\omega_n^\pm(\tau) = \frac{1}{2}\Big(4q_n(\tau) - 2p_n^2(\tau) \pm 2\sqrt{p_n^4(\tau) - 4p_n^2(\tau)q_n(\tau) + 4r_n^2(\tau)}\Big)^{1/2}.$$

Define $\vartheta : \mathbb{R}_+ \to \mathbb{R}$ by

$$\vartheta(\tau) = p_n^2(\tau) - 2q_n(\tau) + 2\sqrt{q_n^2(\tau) - r_n^2(\tau)}$$

Then we have

$$\begin{aligned}
\vartheta'(\tau) &= 2p_n(\tau)p_n'(\tau) - 2q_n'(\tau) + \frac{2q_n(\tau)q_n'(\tau) - 2r_n(\tau)r_n'(\tau)}{\sqrt{q_n^2(\tau) - r_n^2(\tau)}} \\
&= 2p_n(\tau)p_n'(\tau) + \frac{2q_n'(\tau)[q_n(\tau) - \sqrt{q_n^2(\tau) - r_n^2(\tau)}] - 2r_n(\tau)r_n'(\tau)}{\sqrt{q_n^2(\tau) - r_n^2(\tau)}}.
\end{aligned}$$

This and Lemma 4.2, imply that $\vartheta'(\tau) > 0$ for all $\tau \geq 0$ if $[d+\gamma-\beta f(x)]f(x) > (d+\gamma)xf'(x)$ for all $x \in [I_0^*, I_\infty^*)$. In this case, if $\vartheta(\cdot)$ has at least one positive real zero point. Thus, we have the following result.

Lemma 4.3. *Let $\mathbb{I}_n = \{\tau \geq 0 : p_n^2(\tau) < 2q_n(\tau) - 2\sqrt{q_n^2(\tau) - r_n^2(\tau)}\}$ then $\mathbb{I}_{n+1} \subseteq \mathbb{I}_n$ for all $n \in \mathbb{N}_0$. Furthermore, for each $n \in \mathbb{N}_0$, $\mathbb{I}_n$ is either an empty set or a connected subinterval of $\mathbb{R}_+$ if $[d+\gamma-\beta f(x)]f(x) > (d+\gamma)xf'(x)$ for all $x \in [I_0^*, I_\infty^*)$.*

Define $G_n^\pm\colon \mathbb{I}_n \to \mathbb{R}$ as

$$G_n^\pm(\tau) = \frac{\omega^\pm(\tau)p_n(\tau)}{[\omega^\pm(\tau)]^2 - q_n(\tau)} - \tan[\tau\omega^\pm(\tau)]$$

for $\tau \in \mathbb{I}_n$. It follows from (1.2) that $p_n(\tau)$, $q_n(\tau)$, and $r_n(\tau)$ are all bounded for all $\tau \geq 0$. This, implies that $G_n^\pm$ has at least one zero in $\mathbb{I}_n$ when $\mathbb{I}_n \neq \emptyset$ and $[d+\gamma-\beta f(x)]f(x) > (d+\gamma)xf'(x)$ for all $x \in [I_0^*, I_\infty^*)$. Therefore, we obtain the following result on the existence of Hopf bifurcations.

Lemma 4.4.
- (i) *For all $\tau \in \mathbb{R}_+ \setminus \mathbb{I}_n$, $\mathcal{Q}_n(\tau,\cdot)$ has no purely imaginary zeros, and hence all zeros of $\mathcal{Q}_n(\tau,\cdot)$ have negative real parts;*
- (ii) *If $G_n^+(\cdot)$ (or $G_n^-(\cdot)$) has a zero $\tau^* \in \mathbb{I}_n$ then $\mathcal{Q}_n(\tau^*,\cdot)$ has a purely imaginary zero $i\omega^+(\tau^*)$ (respectively, $i\omega^-(\tau^*)$).*

Thus, we obtain the following result.

Theorem 4.5. *Assume that $R_0 > 1$.*
- (i) *The endemic equilibrium E^* of model (1.5) is asymptotically stable for all $\tau \in \mathbb{R}_+ \setminus \mathbb{I}_0$;*
- (ii) *If $\mathbb{I}_0 \neq \emptyset$ and $[d+\gamma-\beta f(x)]f(x) > (d+\gamma)xf'(x)$ for all $x \in [I_0^*, I_\infty^*)$, then the endemic equilibrium E^* is asymptotically stable for all $\tau \in [0,\tau^*)$, and becomes unstable for τ staying in a right neighborhood of τ^*, where τ^* denotes the minimum of all the zeroes of $G_n^\pm$, $n \in \mathbb{N}_0$. In addition, system (1.5) undergoes Hopf bifurcation and a branch of periodic solutions emerge simultaneously from the endemic equilibrium E^*.*

5. Hopf bifurcation

Throughout this section, we assume that

- (H1) $\mathbb{I}_0 \neq \emptyset$, $[d+\gamma-\beta f(x)]f(x) > (d+\gamma)xf'(x)$ for all $x \in [I_0^*, I_\infty^*)$, τ^* denotes the minimum of all the zeroes of $G_n^\pm$, $n \in \mathbb{N}_0$; Moreover, $G_{n^*}^+(\tau^*) = 0$ and σ_{n^*} is a simple eigenvalue σ_n of the linear operator $-\Delta$ subject to the homogeneous boundary condition $\frac{\partial}{\partial \mathbf{n}}u = 0$ on $\partial\Omega$.

Theorem 4.5(ii) tells us that assumption (H1) implies that a family of periodic solutions bifurcate from the endemic equilibrium (S_τ^*, I_τ^*) as the delay τ passes through each critical value τ^*. In this section, we shall study the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions by using the normal theory and the center manifold theorem due to Hassard et al. [6].

We first transform the steady state (S_τ^*, I_τ^*) to the origin via the translation $u_1(\cdot,t) = S(\cdot,\tau t) - S_\tau^*$ and $u_2(\cdot,t) = I(\cdot,\tau t) - I_\tau^*$, and then system (1.5) can be written as

$$u_t(x,t) = B_1(\tau)u(x,t) + B_2(\tau)u(x,t-1) + G(u(x,t),\tau), \tag{5.1}$$

where $u(x,t)=(u_1(x,t),u_2(x,t))^T$,

$$B_1(\tau)=\tau\begin{bmatrix}\Delta-d-\beta f(I^*_\tau) & -\beta S^*_\tau f'(I^*_\tau)\\ \beta f(I^*_\tau) & \Delta+\beta S^*_\tau f'(I^*_\tau)-d-\gamma\end{bmatrix},$$

$$B_2(\tau)=\tau\begin{bmatrix}0 & \gamma e^{-d\tau}\\ 0 & 0\end{bmatrix},$$

$$\begin{aligned}G(u,\tau)&=\tau[\beta f'(I^*_\tau)u_1u_2+\frac{1}{2}\beta f''(I^*_\tau)(S^*_\tau u_2^2+u_1u^2)]\begin{bmatrix}-1\\1\end{bmatrix}\\&\quad+\frac{1}{6}\beta S^*_\tau f'''(I^*_\tau)u_2^3\begin{bmatrix}-1\\1\end{bmatrix}+\text{h.o.t.}\end{aligned}$$

Denote by $\mathcal{C}^k=C^k([-1,0],\mathbb{X}^2)$ the Banach space of k-times continuously differentiable mappings from $[-1,0]$ into $\mathbb{X}^2$ equipped with the supremum norm $\|\phi\|=\sup\{\|\phi^{(j)}(\theta)\|_{\mathbb{X}^2}:\theta\in[-1,0],\ j=0,1,\dots,k\}$ for $\phi\in\mathcal{C}^k$. For convenience, we write $\mathcal{C}^0$ as $\mathcal{C}$. Let $\tau=\tau^*+\upsilon$ then $\upsilon=0$ is the Hopf bifurcation value of system (5.1). Thus, we transform system (5.1) into the following abstract functional differential equation

$$\frac{du(t)}{dt}=L_\upsilon(u_t)+G(u(t),\tau^*+\upsilon),\tag{5.2}$$

where $u_t\in\mathcal{C}^1$, $L_\upsilon(\phi)=B_1(\tau^*+\upsilon)\phi(0)+B_2(\tau^*+\upsilon)\phi(-1)$. For $\phi\in\mathcal{C}$, define

$$\mathcal{A}_\upsilon\phi=\begin{cases}\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ L_\upsilon(\phi), & \theta=0\,,\end{cases}$$

$$\Gamma_\upsilon\phi=\begin{cases}0, & \theta\in[-1,0),\\ G(\phi,\tau^*+\upsilon), & \theta=0\,.\end{cases}$$

Then system (5.2) is equivalent to

$$\dot{u}_t=\mathcal{A}_\upsilon u_t+\Gamma_\upsilon u_t\tag{5.3}$$

where $u_t(\theta)=u(t+\theta)$ for $\theta\in[-1,0]$. For $\psi\in C^1([0,1],\mathbb{X}^2)$, define

$$\mathcal{A}^*\psi(s)=\begin{cases}-\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s\in(0,1],\\ B_1(\tau^*+\upsilon)\psi(0)+B_2(\tau^*+\upsilon)\psi(1), & s=0\,,\end{cases}$$

and a bilinear inner product

$$\langle\langle\psi,\phi\rangle\rangle=\langle\psi(0),\psi(0)\rangle+\int_{-1}^0\langle\psi(\xi+1),B_2(\tau^*)\phi(\xi)\rangle d\xi\tag{5.4}$$

for $\psi\in\mathrm{Dom}(\mathcal{A}_0)$ and $\tilde{\psi}\in\mathrm{Dom}(\mathcal{A}_0^*)$. Then $\mathcal{A}_0$ and $\mathcal{A}_0^*$ satisfy

$$\langle\langle\mathcal{A}_0^*\tilde{\psi},\psi\rangle\rangle=\langle\langle\tilde{\psi},\mathcal{A}_0\psi\rangle\rangle\quad\text{for }\psi\in\mathcal{D}(\mathcal{A}_0)\text{ and }\tilde{\psi}\in\mathcal{D}(\mathcal{A}_0^*).\tag{5.5}$$

Then $\mathcal{A}_0$ and $\mathcal{A}_0^*$ are adjoint operator. By the discussion in section 3, we know that $\pm i\omega^*\tau^*$ are eigenvalues of $\mathcal{A}_0$, where $\omega^*=\omega_n^+(\tau^*)$ or $\omega^*=\omega_n^-(\tau^*)$ for some $n\in\mathbb{N}_0$. Hence, they are also eigenvalues of $\mathcal{A}_0^*$. Now, we need to compute the eigenvector of $\mathcal{A}_0$ and $\mathcal{A}_0^*$ corresponding to $\pm i\omega^*\tau^*$ and $\pm i\omega^*\tau^*$, respectively.

Suppose $q(\theta)=(1,q_1)^T\varphi_n e^{i\omega^*\tau^*\theta}$ is an eigenvector of $\mathcal{A}_0$ corresponding to $i\omega^*\tau^*$ then $\mathcal{A}_0q=i\omega^*\tau^*q$, where φ_n is the normalized eigenfunction associated with the simple eigenvalue σ_n of the linear operator $-\Delta$ subject to the homogeneous boundary condition $\frac{\partial}{\partial\mathbf{n}}u=0$ on $\partial\Omega$. It follows from the definition of $\mathcal{A}_0$ that

$$B_1(\tau^*)q(0)+B_2(\tau^*)q(-1)=i\omega^*\tau^*q(0),\tag{5.6}$$

and hence that

$$q_1=\frac{\beta f(I^*_{\tau^*})}{i\omega^*+\sigma_n+d+\gamma-\beta f'(I^*_{\tau^*})}.$$

Similarly, we can obtain the eigenvector $q^*(s)=\mathcal{D}(1,q_1^*)\varphi_n e^{i\omega^*\tau^* s}$ of $\mathcal{A}^*$ corresponding to $-i\omega^*\tau^*$, where

$$q_1^*=\frac{-i\omega^*+\sigma_n+d+\beta f(I^*_{\tau^*})}{\beta f(I^*_{\tau^*})}$$

and $\mathcal{D}$ is chosen to assure $\langle q^*(s),q(\theta)\rangle=1$. In fact, we have

$$\mathcal{D}=[1+\frac{i\omega^*+\sigma_n+d+\beta f(I^*_{\tau^*})[1+\tau^*\gamma e^{-(d+i\omega^*)\tau^*}]}{i\omega^*+\sigma_n+d+\gamma-\beta f'(I^*_{\tau^*})}]^{-1}.$$

Next we compute the coordinate to describe the center manifold C_0 at $\upsilon=0$. Let u_t be the solution of (5.3) when $\upsilon=0$ define

$$z(t)=\langle\langle q^*,u_t\rangle\rangle,\quad W(t,\theta)=u_t(\theta)-2\operatorname{Re}\{z(t)q(\theta)\}. \tag{5.7}$$

On the center manifold C_0, we have $W(t,\theta)=W(z(t),\bar z(t),\theta)$, where

$$W(z(t),\bar z(t),\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar z+W_{02}(\theta)\frac{\bar z^2}{2}+\cdots. \tag{5.8}$$

Here, z and $\bar z$ are local coordinate for center manifold C_0 in the direction of q^* and $\bar q^*$. Note that W is real if u_t is real. We only consider real solutions. For solution $u_t\in C_0$ of (5.3), since $\upsilon=0$, we have

$$\dot z(t)=i\omega^*\tau^*z+\langle q^*(0),G(W(z,\bar z,0)+2\operatorname{Re}\{zq(\theta)\},\tau^*)\rangle.$$

We rewrite this equation as

$$\dot z(t)=i\omega^*\tau^*z(t)+g(z,\bar z),$$

where

$$\begin{aligned}g(z,\bar z)&=\langle q^*(0),G(W(z,\bar z,0)+2\operatorname{Re}\{zq(\theta)\},\tau^*)\rangle\\&=g_{20}\frac{z^2}{2}+g_{11}z\bar z+g_{02}\frac{\bar z^2}{2}+g_{21}\frac{z^2\bar z}{2}+\dots.\end{aligned} \tag{5.9}$$

It follows from (5.7) and (5.8) that

$$\begin{aligned}u_t(\theta)&=W(t,\theta)+2\operatorname{Re}\{z(t)q(\theta)\}\\&=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar z+W_{02}(\theta)\frac{\bar z^2}{2}+(1,q)^Te^{i\omega^*\tau^*\theta}z+(1,\bar q)^Te^{i\omega^*\tau^*\theta}\bar z+\cdots.\end{aligned} \tag{5.10}$$

Comparing the coefficients with those in (5.9), we have

$$\begin{aligned}g_{20}&=(\bar q_1^*-1)\tau^*\bar{\mathcal{D}}[2\beta f'(I^*_{\tau^*})q_1+\beta f''(I^*_{\tau^*})S^*_{\tau^*}q_1^2],\\g_{11}&=(\bar q_1^*-1)\tau^*\bar{\mathcal{D}}[2\beta f'(I^*_{\tau^*})\operatorname{Re}(q_1)+\beta f''(I^*_{\tau^*})S^*_{\tau^*}|q_1|^2],\\g_{02}&=(\bar q_1^*-1)\tau^*\bar{\mathcal{D}}[2\beta f'(I^*_{\tau^*})\bar q_1+\beta f''(I^*_{\tau^*})S^*_{\tau^*}\bar q_1^2],\\g_{21}&=(\bar q_1^*-1)\tau^*\bar{\mathcal{D}}\beta f'(I^*_{\tau^*})[2W_{11}^{(2)}(0)+W_{20}^{(2)}(0)+\bar q_1W_{20}^{(1)}(0)\\&\quad+2q_1W_{11}^{(1)}(0)]+(\bar q_1^*-1)\tau^*\bar{\mathcal{D}}\beta f''(I^*_{\tau^*})S^*_{\tau^*}[\bar q_1W_{20}^{(2)}(0)+2q_1W_{11}^{(2)}(0)]\\&\quad+(\bar q_1^*-1)\tau^*\bar{\mathcal{D}}[\beta f''(I^*_{\tau^*})\left(2|q_1|^2+q_1^2\right)+\beta f'''(I^*_{\tau^*})S^*_{\tau^*}q_1|q_1|^2].\end{aligned} \tag{5.11}$$

Since $W_{20}(\theta)$ and $W_{11}(\theta)$ are contained in g_{21}, we still need to compute them. From (5.3) and (5.8), we have

$$\begin{aligned}\dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q}\\ &= \begin{cases}\mathcal{A}_0 W - 2\operatorname{Re}\{g(z,\bar{z})q\} + G(W + \operatorname{Re}(zq), \tau^*), & \theta = 0,\\ \mathcal{A}_0 W - 2\operatorname{Re}\{g(z,\bar{z})q\}, & \theta \in [-1, 0),\end{cases}\\ &=: \mathcal{A}_0 W + H(z,\bar{z},\theta),\end{aligned} \tag{5.12}$$

Here,

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \dots. \tag{5.13}$$

Substituting the corresponding series into (5.12) and comparing the coefficients, we have

$$(\mathcal{A}_0 - i\omega^*\tau^*)W_{20}(\theta) = -H_{20}(\theta), \quad \mathcal{A}_0 W_{11}(\theta) = -H_{11}(\theta). \tag{5.14}$$

From (5.12), we know that for $\theta \in [-1, 0)$,

$$H(z,\bar{z},\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta). \tag{5.15}$$

Comparing the coefficients with those in (5.13) gives

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{5.16}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{5.17}$$

From the definition of $\mathcal{A}_0$ and (5.14) and (5.16), we obtain

$$\dot{W}_{20}(\theta) = 2i\omega^*\tau^* W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

and hence

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^*\tau^*}q(0)e^{i\omega^*\tau^*\theta} + \frac{i\bar{g}_{02}}{3\omega^*\tau^*}\bar{q}(0)e^{-i\omega^*\tau^*\theta} + E_1 e^{2i\omega^*\tau^*\theta}, \tag{5.18}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T$ is a constant vector. Similarly, from (5.14) and (5.17), we have

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega^*\tau^*}q(0)e^{i\omega^*\tau^*\theta} + \frac{i\bar{g}_{11}}{\omega^*\tau^*}\bar{q}(0)e^{-i\omega^*\tau^*\theta} + E_2, \tag{5.19}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})^T$ is a constant vector.

In what follows, we shall seek the values of E_1 and E_2. From the definition of $\mathcal{A}_0$ and (5.14), we have

$$L_0 W_{20} = 2i\omega^*\tau^* W_{20}(0) - H_{20}(0), \tag{5.20}$$

$$L_0 W_{11} = -H_{11}(0). \tag{5.21}$$

By (5.12), we know when that $\theta = 0$,

$$H(z,\bar{z},\theta) = -2\operatorname{Re}\{g(z,\bar{z})q\} + G(W + \operatorname{Re}(zq), \tau^*).$$

That is,

$$\begin{aligned}H_{20}(0) &= -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + D_1\varphi_n^2,\\ H_{11}(0) &= -g_{21}q(0) - \bar{g}_{11}\bar{q}(0) + D_2\varphi_n^2,\end{aligned} \tag{5.22}$$

where

$$D_1 = \tau^*[2\beta f'(I^*_{\tau^*})q_1 + \beta f''(I^*_{\tau^*})S^*_{\tau^*}q_1^2]\begin{bmatrix}-1\\1\end{bmatrix},$$

$$D_2 = \tau^*[2\beta f'(I^*_{\tau^*})\operatorname{Re}(q_1) + \beta f''(I^*_{\tau^*})S^*_{\tau^*}|q_1|^2]\begin{bmatrix}-1\\1\end{bmatrix}.$$

Substituting (5.18) and (5.22) into (5.20), and (5.19) and (5.22) into (5.21) we obtain

$$\begin{bmatrix} \Delta - d - \beta f(I_\tau^*) - 2i\omega^* & -\beta S_\tau^* f'(I_\tau^*) + \gamma e^{-d\tau^*} \\ \beta f(I_\tau^*) & \Delta + \beta S_\tau^* f'(I_\tau^*) - d - \gamma - 2i\omega^* \end{bmatrix} E_1 = D_1 \varphi_n^2,$$

and

$$\begin{bmatrix} \Delta - d - \beta f(I_\tau^*) & -\beta S_\tau^* f'(I_\tau^*) + \gamma e^{-d\tau^*} \\ \beta f(I_\tau^*) & \Delta + \beta S_\tau^* f'(I_\tau^*) - d - \gamma \end{bmatrix} E_2 = D_2 \varphi_n^2.$$

It follows that

$$E_1 = \begin{bmatrix} \Delta - d - \beta f(I_\tau^*) - 2i\omega^* & -\beta S_\tau^* f'(I_\tau^*) + \gamma e^{-d\tau^*} \\ \beta f(I_\tau^*) & \Delta + \beta S_\tau^* f'(I_\tau^*) - d - \gamma - 2i\omega^* \end{bmatrix}^{-1} D_1 \varphi_n^2,$$

and

$$E_2 = \begin{bmatrix} \Delta - d - \beta f(I_\tau^*) & -\beta S_\tau^* f'(I_\tau^*) + \gamma e^{-d\tau^*} \\ \beta f(I_\tau^*) & \Delta + \beta S_\tau^* f'(I_\tau^*) - d - \gamma \end{bmatrix}^{-1} D_2 \varphi_n^2.$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (5.18) and (5.19). Furthermore, we can compute g_{21} by (5.11). Thus we can compute the following values [5, 6]:

$$\begin{gathered} C_1(0) = \frac{i}{2\omega^*\tau^*}\big[g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{2}\big] + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\frac{\mathrm{d}\lambda(\tau^*)}{\mathrm{d}\tau}\}}, \quad \beta_2 = 2\,\mathrm{Re}\{C_1(0)\}, \\ T_2 = -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2\,\mathrm{Im}\{\frac{\mathrm{d}\lambda(\tau^*)}{\mathrm{d}\tau}\}}{\omega^*\tau^*}, \end{gathered}$$

which determine the qualities of bifurcating periodic solution on the center manifold at the critical values τ^*. Therefore, we obtain the following results.

Theorem 5.1. *μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ (respectively, $\mu_2 < 0$), then the Hopf bifurcation is supercritical (respectively, subcritical) and the bifurcating periodic solutions exist for $\tau > \tau^*$ (respectively, $\tau < \tau^*$); β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (respectively, unstable) if $\beta_2 < 0$ (respectively, $\beta_2 > 0$); and T_2 determines the period of bifurcating periodic solutions: the period increase (respectively, decrease) if $T_2 > 0$ (respectively, $T_2 < 0$).*

6. Conclusion and numerical simulations

In this article, an SIRS system with time delay and the general nonlinear incidence rate is considered. The positivity and boundedness of solutions are investigated. The basic reproductive number, R_0, is derived. If $R_0 < 1$, then the disease-free equilibrium $E_0(\mu/d, 0)$ is globally asymptotically stable and the disease dies out. If $R_0 > 1$, then there exists a unique endemic equilibrium $E^*(S_\tau^*, I_\tau^*)$ whose locally asymptotical stability and the existence of local Hopf bifurcations are established by analyzing the distribution of the characteristic values. An explicit algorithm for determining the direction of Hopf bifurcations and the stability of the bifurcating periodic solutions is derived by using the center manifold and the normal form theory. In what follows, we present some numerical simulations to support and supplement the our analytic results.

We first consider system (1.5) with $f(I) = I$, $\mu = 0.01$, $d = 0.01$, $\gamma = 18.05$, $\Omega = (0, 3\pi)$, and initial values $S(x,t) = 0.1 + 0.05\cos(x)$, $I(x,t) = 0.1 + 0.05\cos(x)$.

Choose $\beta = 17$, then we have $R_0 = 0.94 < 1$ and hence the disease-free equilibrium E_0 of model (1.5) is asymptotically stable (see Figure 1). Choose $\beta = 20$, then we have $R_0 = 1.11 > 1$ and hence the endemic equilibrium E^* of model (1.5) is asymptotically stable (see Figure 2).

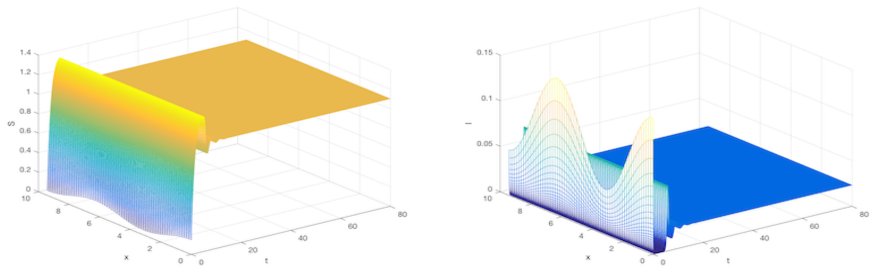

FIGURE 1. The disease-free equilibrium E_0 of model (1.5) is asymptotically stable when $f(I) = I$ and $R_0 = 0.94 < 1$.

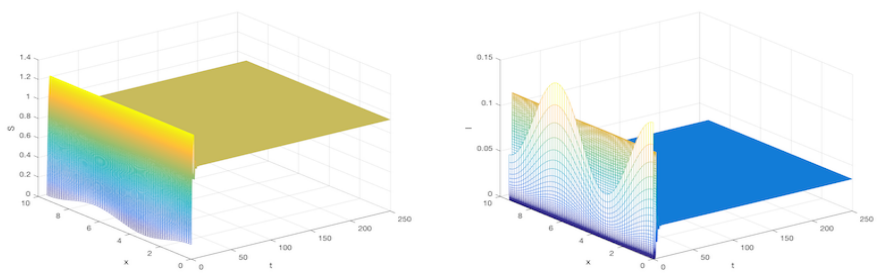

FIGURE 2. The endemic equilibrium E^* of model (1.5) is asymptotically stable when $f(I) = I$ and $R_0 = 1.11 > 1$.

And then we consider the case where $f(I) = \frac{15I}{1+I}$, $\mu = 0.03$, $d = 0.4$, $\gamma = 25.12$, $\Omega = (0, 3\pi)$, and initial values $S(x,t) = 0.01 + 0.05\cos(x)$, $I(x,t) = 0.01 + 0.05\cos(x)$. Choose $\beta = 25$, then we have $R_0 = 1.25 > 1$ and hence the endemic equilibrium E^* of model (1.5) is asymptotically stable (see Figure 3). Choose $\beta = 20$, then we have $R_0 = 0.89 < 1$ and and hence the disease-free equilibrium E_0 of model (1.5) is stable (see Figure 4).

Next we consider system (1.5) for $f(I) = \frac{10I}{1+I}$ with $\mu = -0.9$, $d = 1.7$, $\beta = 0.2$, $\gamma = -1.8$, $\Omega = (0, \pi)$, and initial values $S(x,t) = \frac{\sin^2(x)}{75}$, $I(x,t) = \frac{\sin^2(x)}{125}$, which yields $R_0 = 8.47$ and satisfies all the conditions given in Theorem 4.5. We can obtain the positive critical time delay $\tau^* = 6.47$. Thus, we know that when $\tau \in [0, \tau^*)$, E^* is asymptotically stable. When τ passes through the critical value τ^*, E^* loses its stability and a Hopf bifurcating occurs and a family of periodic solutions bifurcate from E^*, which can be illustrated in Figures 5 and 6.

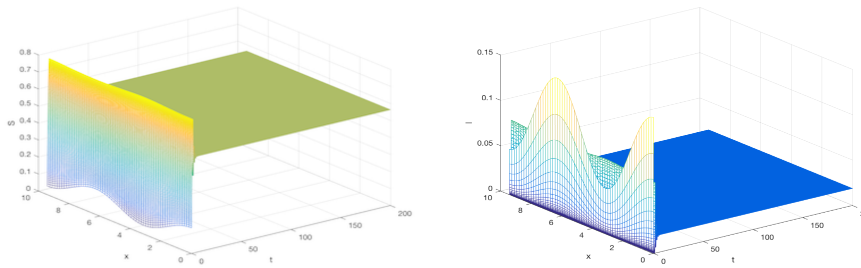

FIGURE 3. The endemic equilibrium E^* of model (1.5) is asymptotically stable when $f(I) = \frac{15I}{1+I}$ and $R_0 = 1.12 > 1$.

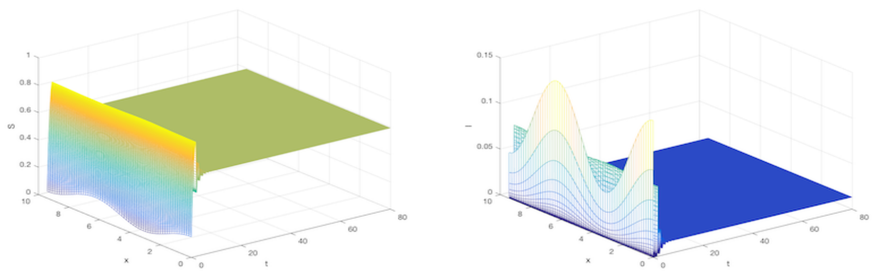

FIGURE 4. The disease-free equilibrium E_0 of model (1.5) is asymptotically stable when $f(I) = \frac{15I}{1+I}$ and $R_0 = 0.89 < 1$.

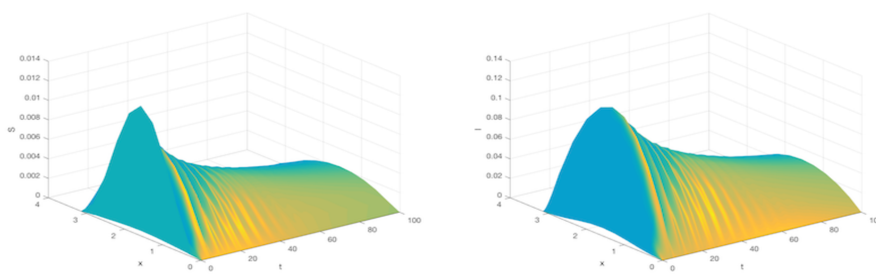

FIGURE 5. Solutions of system (1.5) with $\tau = 5 < \tau^*$ tend to a positive steady state.

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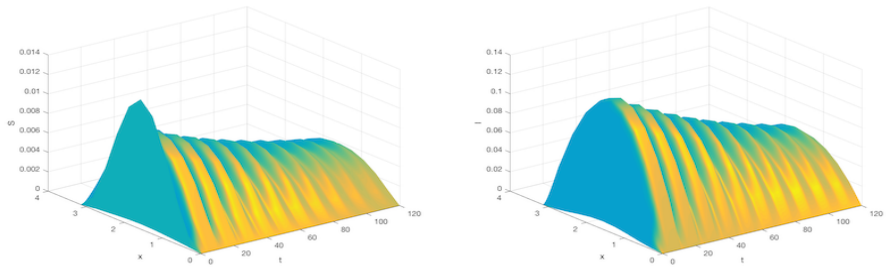

FIGURE 6. Solutions of system (1.5) with $\tau = 11 > \tau^*$ tend to a periodically oscillatory orbit.

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